

p -ADIC LIMIT OF THE FOURIER COEFFICIENTS OF WEAKLY HOLOMORPHIC MODULAR FORMS OF HALF INTEGRAL WEIGHT

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ABSTRACT. Serre obtained the p -adic limit of the integral Fourier coefficients of modular forms on $SL_2(\mathbb{Z})$ for $p = 2, 3, 5, 7$. In this paper, we extend the result of Serre to weakly holomorphic modular forms of half integral weight on $\Gamma_0(4N)$ for $N = 1, 2, 4$. The proof is based on linear relations among Fourier coefficients of modular forms of half integral weight. As applications of our main result, we obtain congruences on various modular objects, such as those for Borcherds exponents, for Fourier coefficients of quotients of Eisenstein series and for Fourier coefficients of Siegel modular forms on the Maass Space.

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1. Introduction and Statement of Main Results

Serre obtained the p -adic limits of the integral Fourier coefficients of modular forms on $SL_2(\mathbb{Z})$ for $p = 2, 3, 5, 7$ (see Théorème 7 and Lemma 8 in [20]). In this paper, we extend the result of Serre to weakly holomorphic modular forms of half integral weight on $\Gamma_0(4N)$ for $N = 1, 2, 4$. The proof is based on linear relations among Fourier coefficients of modular forms of half integral weight. As applications of our main result, we obtain congruences for various modular objects, such as those for Borcherds exponents, for Fourier coefficients of quotients of Eisenstein series and for Fourier coefficients of Siegel modular forms on the Maass Space.

For odd d , let

$$\langle \left(\begin{smallmatrix} 1 & h_t \\ 0 & 1 \end{smallmatrix} \right) \rangle := \gamma_t \Gamma_0(4N)_t \gamma_t^{-1},$$

where $\gamma_t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ and $\gamma_t(t) = \infty$. We denote the q -expansion of a modular form $f \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ at each cusp t of $\Gamma_0(4N)$ by

$$(1.1) \quad (f |_{\lambda+\frac{1}{2}} \gamma_t)(z) = (cz + d)^{-\lambda-\frac{1}{2}} f\left(\frac{az + b}{cz + d}\right) = q_t^{r(t)} \sum_{n=b_t}^{\infty} a_f^t(n) q_t^n, \quad q_t := q^{\frac{1}{h_t}},$$

where

$$(1.2) \quad r(t) \in \left\{ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \right\}.$$

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When $t \sim \infty$, we denote $a_f^t(n)$ by $a_f(n)$. Note that the number $r(t)$ is independent of the choice of $f \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ and λ . We call t a regular cusp if $r(t) = 0$ (see Chapter IV. §1. of [15] for a more general definition of a λ -regular cusp).

Remark 1.1. Our definition of a regular cusp is different from the usual one.

Let $U_{4N} := \{t_1, \dots, t_{\nu(4N)}\}$ be the set of all inequivalent regular cusps of $\Gamma_0(4N)$. Note that the genus of $\Gamma_0(4N)$ is zero if and only if $1 \leq N \leq 4$. Let $\mathcal{M}_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ be the space of weakly holomorphic modular forms of weight $\lambda + \frac{1}{2}$ on $\Gamma_0(4N)$ and let $\mathcal{M}_{\lambda+\frac{1}{2}}^0(\Gamma_0(4N))$ denote the set of $f(z) \in \mathcal{M}_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ such that the constant term of its q -expansion at each cusp is zero. Let U_p be the operator defined by

$$(f|U_p)(z) := \sum_{n=n_0}^{\infty} a_f(pn)q^n.$$

Let \mathcal{O}_L be the ring of integers of a number field L with a prime ideal $\mathfrak{p} \subset \mathcal{O}_L$. For $f(z) := \sum a_f(n)q^n$ and $g(z) := \sum a_g(n)q^n \in L[[q^{-1}, q]]$ we write

$$f(z) \equiv g(z) \pmod{\mathfrak{p}}$$

if and only if $a_f(n) - a_g(n) \in \mathfrak{p}$ for every integer n .

With these notations we state the following theorem.

Theorem 1. *For $N = 1, 2, 4$ consider*

$$f(z) := \sum_{n=n_0}^{\infty} a_f(n)q^n \in \mathcal{M}_{\lambda+\frac{1}{2}}^0(\Gamma_0(4N)) \cap L[[q^{-1}, q]].$$

Suppose that $\mathfrak{p} \subset \mathcal{O}_L$ is any prime ideal such that $\mathfrak{p}|p$, p prime, and that $a_f(n)$ is \mathfrak{p} -integral for every integer $n \geq n_0$.

(1) *If $p = 2$ and $a_f(0) = 0$, then there exists a positive integer b such that*

$$(f|(U_p)^b)(z) \equiv 0 \pmod{\mathfrak{p}^j} \text{ for each } j \in \mathbb{N}.$$

(2) *If $p \geq 3$ and $f(z) \in \mathcal{M}_{\lambda+\frac{1}{2}}^0(\Gamma_0(4N))$ with $\lambda \equiv 2$ or $2 + [\frac{1}{N}] \pmod{\frac{p-1}{2}}$, then there exists a positive integer b such that*

$$(f|(U_p)^b)(z) \equiv 0 \pmod{\mathfrak{p}^j} \text{ for each } j \in \mathbb{N}.$$

Remark 1.2. The p -adic limit of a sum of Fourier coefficients of $f \in M_{\frac{3}{2}}(\Gamma_0(4N))$ was studied in [13].

Our method only allows to prove a weaker result if $f(z) \notin \mathcal{M}_{\lambda+\frac{1}{2}}^0(\Gamma_0(4N))$.

Theorem 2. For $N = 1, 2$ or 4 , let

$$f(z) := \sum_{n=n_0}^{\infty} a_f(n)q^n \in \mathcal{M}_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) \cap L[[q^{-1}, q]].$$

Suppose that $\mathfrak{p} \subset \mathcal{O}_L$ is any prime ideal with $\mathfrak{p}|p$, p prime, $p \geq 5$, and that $a_f(n)$ is \mathfrak{p} -integral for every integer $n \geq n_0$. If $\lambda \equiv 2$ or $2 + \left[\frac{1}{N}\right] \pmod{\frac{p-1}{2}}$, then there exists a positive integer b_0 such that

$$a_f(p^{2b-m(p:\lambda)}) \equiv - \sum_{t \in U_{4N}} h_t a_t^{\frac{\Delta_{4N, 3-\alpha(p:\lambda)}(z)}{R_{4N}(z)^{e \cdot \omega(4N)}}}(0) a_f^t(0) \pmod{\mathfrak{p}}$$

for every positive integer $b > b_0$ (see Section 3 for detailed notation).

Example 1.3. Recall that the generating function of the overpartition $\bar{P}(n)$ of n (see [11])

$$\sum_{n=0}^{\infty} \bar{P}(n)q^n = \frac{\eta(2z)}{\eta(z)^2}$$

is in $\mathcal{M}_{-\frac{1}{2}}(\Gamma_0(16))$, where $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$. Therefore, theorem 2 implies that

$$\bar{P}(5^{2b}) \equiv 1 \pmod{5}, \forall b \in \mathbb{N}.$$

2. Applications: More Congruences

In this section, we study congruences for various modular objects such as those for Borcherds exponents and for quotients of Eisenstein series.

2.1. p -adic Limits of Borcherds Exponents. Let \mathcal{M}_H denote the set of meromorphic modular forms of integral weight on $SL_2(\mathbb{Z})$ with Heegner divisor, integer coefficients and leading coefficient 1. Let

$$\mathcal{M}_{\frac{1}{2}}^+(\Gamma_0(4)) := \left\{ f(z) = \sum_{n=m}^{\infty} a_f(n)q^n \in \mathcal{M}_{\frac{1}{2}}(\Gamma_0(4)) \mid a(n) = 0 \text{ for } n \equiv 2, 3 \pmod{4} \right\}.$$

If $f(z) = \sum_{n=n_0}^{\infty} a_f(n)q^n \in \mathcal{M}_{\frac{1}{2}}^+(\Gamma_0(4))$, then define $\Psi(f(z))$ by

$$\Psi(f(z)) := q^{-h} \prod_{n=1}^{\infty} (1 - q^n)^{a_f(n^2)},$$

where $h = -\frac{1}{12}a_f(0) + \sum_{1 < n \equiv 0, 1 \pmod{4}} a_f(-n)H(-n)$. Here $H(-n)$ denotes the usual Hurwitz class number of discriminant $-n$. The following was proved by Borcherds.

Theorem 2.1 ([4]). *The map Ψ is an isomorphism from $\mathcal{M}_{\frac{1}{2}}^+(\Gamma_0(4))$ to \mathcal{M}_H , and the weight of $\Psi(f(z))$ is $a_f(0)$.*

Let $j(z)$ be the usual j -invariant function with the product expansion

$$j(z) = q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{A(n)}.$$

Let $F(z) := q^{-h} \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}$ be a meromorphic modular form of weight k in \mathcal{M}_H . The p -adic limit of $\sum_{d|n} d \cdot c(d)$ was studied in [5] for $p = 2, 3, 5, 7$. Here we obtain the p -adic limit of $c(d)$ for $p = 2, 3, 5, 7$.

Theorem 3. *Let $F(z) := q^{-h} \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}$ be a meromorphic modular form of weight k in \mathcal{M}_H .*

(1) *If $p = 2$, then for each $j \in \mathbb{N}$ there exists a positive integer b such that*

$$c(mp^b) \equiv 2k \pmod{p^j}$$

for every positive integer m .

(2) *If $p \in \{3, 5, 7\}$, then, for each $j \in \mathbb{N}$ there exists a positive integer b such that*

$$5c(mp^b) - \varpi(F)A(mp^b) \equiv 10k \pmod{p^j}$$

for every positive integer m . Here, $\varpi(F)$ is a constant determined by the constant term of the q -expansion of $\Psi^{-1}(F)$ at 0.

2.2. Sums of n -Squares. For $u \in \mathbb{Z}_{>0}$, let

$$r_n(u) := \#\{(s_1, \dots, s_n) \in \mathbb{Z}^n : s_1^2 + \dots + s_n^2 = u\}.$$

Theorem 4. *Suppose that $p \geq 5$ is a prime. If $\lambda \equiv 2$ or $3 \pmod{\frac{p-1}{2}}$, then there exists a positive integer C_0 such that*

$$r_{2\lambda+1}(p^{2b-m(p:\lambda)}) \equiv -(14 - 4\alpha(p:\lambda)) + 16 \left(\frac{-1}{p}\right)^{\left[\frac{\lambda}{p-1}\right] + \alpha(p:\lambda)\mathbf{m}(p:\lambda)} \pmod{p},$$

for every $b > C_0$.

Remark 2.2. As for an example, if $\lambda \equiv 2 \pmod{p-1}$ and p is an odd prime, then there exists a positive integer C_0 such that

$$r_{2\lambda+1}(p^{2b}) \equiv 10 \pmod{p}, \forall b > C_0$$

2.3. Quotients of Eisenstein Series. Congruences for the coefficients of quotients of elliptic Eisenstein series have been studied in [3]. Let us consider the Cohen Eisenstein series $H_{r+\frac{1}{2}}(z) := \sum_{N=0}^{\infty} H(r, N)q^N$ of weight $r + \frac{1}{2}$, $r \geq 2$ (see [7]). We derive congruences for the coefficients of quotients of $H_{r+\frac{1}{2}}(z)$ and Eisenstein series.

Theorem 5. *Let*

$$F(z) := \frac{H_{\frac{5}{2}}(z)}{E_4(z)} = \sum_{n=0}^{\infty} a_F(n)q^n,$$

$$G(z) := \frac{H_{\frac{7}{2}}(z)}{E_6(z)} = \sum_{n=0}^{\infty} a_G(n)q^n$$

and

$$W(z) := \frac{H_{\frac{9}{2}}(z)}{E_6(z)} = \sum_{n=0}^{\infty} a_W(n)q^n.$$

Then there exists a positive integer C_0 such that

$$\begin{aligned} a_F(11^{2b+1}) &\equiv 1 \pmod{11}, \\ a_G(11^{2b+1}) &\equiv 6 \pmod{11}, \\ a_W(11^{2b+1}) &\equiv 2 \pmod{11} \end{aligned}$$

for every integer $b > C_0$.

2.4. The Maass Space. Next we deal with congruences for the Fourier coefficients of a Siegel modular form in the Maass space. To define the Maass space, let us introduce notations given in [17]: let $T \in M_{2g}(\mathbb{Q})$ be a rational, half-integral, symmetric, non-degenerate matrix of size $2g$ with discriminant

$$D_T := (-1)^g \det(2T).$$

Let $D_T = D_{T,0}f_T^2$, where $D_{T,0}$ is the corresponding fundamental discriminant. Furthermore, let

$$G_8 := \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and G_7 be the upper $(7, 7)$ -submatrix of G_8 . Define

$$S_g := \begin{cases} G_8^{\oplus (g-1)/8} \oplus 2, & \text{if } g \equiv 1 \pmod{8}, \\ G_8^{\oplus (g-7)/8} \oplus G_7, & \text{if } g \equiv -1 \pmod{8}. \end{cases}$$

For each $m \in \mathbb{N}$ such that $(-1)^g m \equiv 0, 1 \pmod{4}$, define a rational, half-integral, symmetric, positive definite matrix T_m of size $2g$ by

$$T_m := \begin{cases} \begin{pmatrix} \frac{1}{2}S_g & 0 \\ 0 & m/4 \end{pmatrix}, & \text{if } m \equiv 0 \pmod{4}, \\ \begin{pmatrix} \frac{1}{2}S_g & \frac{1}{2}e_{2g-1} \\ \frac{1}{2}e'_{2g-1} & [m + 2 + (-1)^g]/4 \end{pmatrix}, & \text{if } m \equiv (-1)^g \pmod{4} \end{cases}$$

Here $e_{2g-1} \in \mathbb{Z}^{(2n-1,1)}$ is the standard column vector and e'_{2g-1} is its transpose.

Definition 2.3. (The Maass Space) Take $g, k \in \mathbb{N}$ such that $g \equiv 0, 1 \pmod{4}$ and $g \equiv k \pmod{2}$. Let

$$S_{k+g}^{Maass}(\Gamma_{2g}) := \left\{ F(Z) = \sum_{T>0} A(T)q^{tr(TZ)} \in S_{k+g}(\Gamma_{2g}) \mid A(T) = \sum_{a|f_T} a^{k-1}\phi(a; T)A(T|_{D_T|/a^2}) \right\}$$

(see (6.2) for details). This space is called the Maass space of genus $2g$ and weight $g + k$.

In [17] it was proved that the Maass space is the same as the image of the Ikeda lifting when $g \equiv 0, 1 \pmod{4}$. Using this fact together with Theorem 1, we derive the following congruences for the Fourier coefficients of $F(Z)$ in $S_{k+g}^{Maass}(\Gamma_{2g})$.

Theorem 6. For $g \equiv 0, 1 \pmod{4}$, let

$$F(Z) := \sum_{T>0} A(T)q^{tr(TZ)} \in S_{k+g}^{Maass}(\Gamma_{2g})$$

with integral coefficients $A(T)$, $T > 0$. If $k \equiv 2$ or $3 \pmod{\frac{p-1}{2}}$ for some prime p , then, for each $j \in \mathbb{N}$, there exists a positive integer b for which

$$A(T) \equiv 0 \pmod{p^j}$$

for every $T > 0$, $\det(2T) \equiv 0 \pmod{p^b}$.

This paper is organized as follows. Section 3 gives a linear relation among Fourier coefficients of modular forms of half integral weight. The remaining sections contain detailed proofs of the main theorems.

3. Linear Relation among Fourier Coefficients of modular forms of Half Integral Weight

Let $V(N; k, n)$ be the subspace of \mathbb{C}^n generated by the first n coefficients of the q -expansion of f at ∞ for $f \in S_k(\Gamma_0(N))$, where $S_k(\Gamma_0(N))$ denotes the space of cusp forms of weight $k \in \mathbb{Z}$ on $\Gamma_0(N)$. Let $L(N; k, n)$ be the orthogonal complement of $V(N; k, n)$

in \mathbb{C}^n with the usual inner product of \mathbb{C}^n . The vector space $L(1; k, d(k) + 1)$, $d(k) = \dim(S_k(\Gamma(1)))$, was studied by Siegel to evaluate the value of the Dedekind zeta function at a certain point. The vector space $L(1; k, n)$ is explicitly described in terms of the principal part of negative weight modular forms in [9]. These results were extended in [8] to the groups $\Gamma_0(N)$ of genus zero. For $1 \leq N \leq 4$, let

$$EV \left(4N, \lambda + \frac{1}{2}; n \right) := \left\{ \left(a_f^{t_1}(0), \dots, a_f^{t_{\nu(4N)}}(0), a_f(1), \dots, a_f(n) \right) \in \mathbb{C}^{n+\nu(4N)} \mid f \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) \right\},$$

where $U_{4N} := \{t_1, \dots, t_{\nu(4N)}\}$ is the set of all inequivalent regular cusps of $\Gamma_0(4N)$. We define $EL(4N, \lambda + \frac{1}{2}; n)$ to be the orthogonal complement of $EV(4N, \lambda + \frac{1}{2}; n)$ in $\mathbb{C}^{n+\nu(4N)}$.

Let $\Delta_{4N, \lambda} := q^{\delta\lambda(4N)} + O(q^{\delta\lambda(4N)+1})$ be in $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ with the maximum order at ∞ , that is, its order at ∞ is bigger than that of any other modular form of the same level and weight. Furthermore, let

$$R_4(z) := \frac{\eta(4z)^8}{\eta(2z)^4}, \quad R_8(z) := \frac{\eta(8z)^8}{\eta(4z)^4},$$

$$R_{12}(z) := \frac{\eta(12z)^{12}\eta(2z)^2}{\eta(6z)^6\eta(4z)^4} \quad \text{and} \quad R_{16}(z) := \frac{\eta(16z)^8}{\eta(8z)^4}.$$

For $\ell, n \in \mathbb{N}$, define

$$\mathbf{m}(\ell : n) := \begin{cases} 0 & \text{if } \left[\frac{2n}{\ell-1} \right] \equiv 0 \pmod{2} \\ 1 & \text{if } \left[\frac{2n}{\ell-1} \right] \equiv 1 \pmod{2} \end{cases}$$

and

$$\alpha(\ell : n) := n - \frac{\ell-1}{2} \left[\frac{2n}{\ell-1} \right].$$

Let $\omega(4N)$ be the order of zero of $R_{4N}(z)$ at ∞ . Note that $R_{4N}(z) \in M_2(\Gamma_0(4N))$ has its only zero at ∞ . So, using the definition of $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$, we find that

$$(3.1) \quad \omega(4) = 1, \omega(8) = 2, \omega(12) = 4, \omega(16) = 4.$$

For each $g \in M_{r+\frac{1}{2}}(\Gamma_0(4N))$ and $e \in \mathbb{N}$, let

$$(3.2) \quad \frac{g(z)}{R_{4N}(z)^e} = \sum_{\nu=1}^{e \cdot \omega(4N)} b(4N, e, g; \nu) q^{-\nu} + O(1) \text{ at } \infty.$$

With these notations we state the following theorem:

Theorem 3.1. *Suppose that $\lambda \geq 0$ is an integer and $1 \leq N \leq 4$. For each $e \in \mathbb{N}$ such that $e \geq \frac{\lambda}{2} - 1$, take $r = 2e - \lambda + 1$. The linear map $\Phi_{r,e}(4N) : M_{r+\frac{1}{2}}(\Gamma_0(4N)) \rightarrow$*

$EL(4N, \lambda + \frac{1}{2}; e \cdot \omega(4N))$, defined by

$$\Phi_{r,e}(4N)(g) = \left(h_{t_1} a_{\frac{g(z)}{R_{4N}(z)^e}}^{t_1}(0), \dots, h_{t_{\nu(4N)}} a_{\frac{g(z)}{R_{4N}(z)^e}}^{t_{\nu(4N)}}(0), b(4N, e, g; 1), \dots, b(4N, e, g; e \cdot \omega(4N)) \right),$$

is an isomorphism.

Proof of Theorem 3.1. Suppose that $G(z)$ is a meromorphic modular form of weight 2 on $\Gamma_0(4N)$. For $\tau \in \mathbb{H} \cup C_{4N}$, let D_τ be the image of τ under the canonical map from $\mathbb{H} \cup C_{4N}$ to a compact Riemann surface $X_0(4N)$. Here \mathbb{H} is the usual complex upper half plane, and C_{4N} denotes the set of all inequivalent cusps of $\Gamma_0(4N)$. The residue $\text{Res}_{D_\tau} G dz$ of $G(z)$ at $D_\tau \in X_0(4N)$ is well-defined since we have a canonical correspondence between a meromorphic modular form of weight 2 on $\Gamma_0(4N)$ and a meromorphic 1-form of $X_0(4N)$. If $\text{Res}_\tau G$ denotes the residue of G at τ on \mathbb{H} , then

$$\text{Res}_{D_\tau} G dz = \frac{1}{l_\tau} \text{Res}_\tau G.$$

Here l_τ is the order of the isotropy group at τ . The residue of G at each cusp $t \in C_{4N}$ is

$$(3.3) \quad \text{Res}_{D_t} G dz = h_t \cdot \frac{a_G^t(0)}{2\pi i}.$$

Now we give a proof of Theorem 3.1.

To prove Theorem 3.1, take

$$G(z) = \frac{g(z)}{R_{4N}(z)^e} f(z),$$

where $g \in M_{r+\frac{1}{2}}(\Gamma_0(4N))$ and $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$. Note that $G(z)$ is holomorphic on \mathbb{H} . Since $g(z)$, $R_{4N}(z)$ and $f(z)$ are holomorphic and $R_{4N}(z)$ has no zero on \mathbb{H} , it is enough to compute the residues of $G(z)$ only at all inequivalent cusps to apply the Residue Theorem. The q -expansion of $\frac{g(z)}{R_{4N}(z)^e} f(z)$ at ∞ is

$$\frac{g(z)}{R_{4N}(z)^e} f(z) = \left(\sum_{\nu=1}^{e \cdot \omega(4N)} b(4N, e, g; \nu) q^{-\nu} + a_{\frac{g(z)}{R_{4N}(z)^e}}(0) + O(q) \right) \left(\sum_{n=0}^{\infty} a_f(n) q^n \right).$$

Since $R_{4N}(z)$ has no zero at $t \approx \infty$, we have

$$\left. \frac{g(z)}{R_{4N}(z)^e} f(z) \right|_2 \gamma_t = a_{\frac{g(z)}{R_{4N}(z)^e}}^t(0) a_f(0) + O(q_t).$$

Further note that, for an irregular cusp t ,

$$a_{\frac{g(z)}{R_{4N}(z)^e}}^t(0) a_f(0) = 0.$$

So the Residue Theorem and (3.3) imply that

$$(3.4) \quad \sum_{t \in U_{4N}} h_t a_{\frac{g}{R^{e \cdot \omega(4N)}}}^t(0) a_f^t(0) + \sum_{\nu=1}^{e \cdot \omega(4N)} b(4N, e, g; \nu) a_f(\nu) = 0.$$

This shows that $\Phi_{r,e}(4N)$ is well-defined. The linearity of the map $\Phi_{r,e}(4N)$ is clear.

It remains to check that $\Phi_{r,e}(4N)$ is an isomorphism. Since there exists no holomorphic modular form of negative weight except the zero function, we obtain the injectivity of $\Phi_{r,e}(4N)$. Note that for $e \geq \frac{\lambda-1}{2}$,

$$\dim_{\mathbb{C}} \left(EL \left(4N; \lambda + \frac{1}{2}, e \cdot \omega(4N) \right) \right) = e \cdot \omega(4N) + \nu(4N) - \dim_{\mathbb{C}} \left(M_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) \right).$$

However, the set $C_{4N}, 1 \leq N \leq 4$, of all inequivalent cusps of $\Gamma_0(4N)$ are

$$\begin{aligned} C_4 &= \left\{ \infty, 0, \frac{1}{2} \right\}, \\ C_8 &= \left\{ \infty, 0, \frac{1}{2}, \frac{1}{4} \right\}, \\ C_{12} &= \left\{ \infty, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6} \right\}, \\ C_{16} &= \left\{ \infty, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8} \right\} \end{aligned}$$

and it can be checked that

$$(3.5) \quad \nu(4) = 2, \nu(8) = 3, \nu(12) = 4, \nu(16) = 6$$

(see §1 of Chapter 4. in [15] for details). The dimension formula of $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ (see Table 1) together with the results in (3.1) and (3.5), implies that

$$\dim_{\mathbb{C}} \left(EL \left(4N, \lambda + \frac{1}{2}; e \cdot \omega(N) \right) \right) = \dim_{\mathbb{C}}(M_{r+\frac{1}{2}}(\Gamma_0(4N)))$$

since $r = 2e - \lambda + 1$.

TABLE 1. Dimension Formula for $M_k(\Gamma_0(4N))$

N	$k = 2n + \frac{1}{2}$	$k = 2n + \frac{3}{2}$	$k = 2n$
$N = 1$	$n + 1$	$n + 1$	$n + 1$
$N = 2$	$2n + 1$	$2n + 2$	$2n + 1$
$N = 3$	$4n + 1$	$4n + 3$	$4n + 1$
$N = 4$	$4n + 2$	$4n + 4$	$4n + 1$

So $\Phi_{r,e}(4N)$ is surjective since the map $\Phi_{r,e}(4N)$ is injective. This completes our claim. \square

4. Proofs of Theorem 1 and 2

4.1. **Proof of Theorem 1.** First, we obtain linear relations among Fourier coefficients of modular forms of half integral weight modulo \mathfrak{p} . Let

$$\mathcal{O}_{\mathfrak{p}} := \{\alpha \in L \mid \alpha \text{ is } \mathfrak{p}\text{-integral}\}.$$

Let

$$\begin{aligned} \widetilde{M}_{\lambda+\frac{1}{2}, \mathfrak{p}}(\Gamma_0(4N)) &:= \{H(z) = \sum_{n=0}^{\infty} a_H(n)q^n \in \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}[[q^{-1}, q]] \mid \\ &H \equiv h \pmod{\mathfrak{p}} \text{ for some } h \in \mathcal{O}_{\mathfrak{p}}[[q^{-1}, q]] \cap M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))\}. \end{aligned}$$

and

$$\begin{aligned} \widetilde{S}_{\lambda+\frac{1}{2}, \mathfrak{p}}(\Gamma_0(4N)) &:= \{H(z) = \sum_{n=0}^{\infty} a_H(n)q^n \in \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}[[q^{-1}, q]] \mid \\ &H \equiv h \pmod{\mathfrak{p}} \text{ for some } h \in \mathcal{O}_{\mathfrak{p}}[[q^{-1}, q]] \cap S_{\lambda+\frac{1}{2}}(\Gamma_0(4N))\}. \end{aligned}$$

The following lemma gives the dimension of $\widetilde{M}_{\lambda+\frac{1}{2}, \mathfrak{p}}(\Gamma_0(4N))$.

Lemma 4.1. *Take $\lambda \in \mathbb{N}$, $1 \leq N \leq 4$ and a prime p such that*

$$\begin{cases} p \geq 3 & \text{if } N = 1, 2, 4, \\ p \geq 5 & \text{if } N = 3. \end{cases}$$

Now take any prime ideal $\mathfrak{p} \subset \mathcal{O}_L$, $\mathfrak{p} \mid p$. Then

$$\dim \widetilde{M}_{\lambda+\frac{1}{2}, \mathfrak{p}}(\Gamma_0(4N)) = \dim M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$$

and

$$\dim \widetilde{S}_{\lambda+\frac{1}{2}, \mathfrak{p}}(\Gamma_0(4N)) = \dim S_{\lambda+\frac{1}{2}}(\Gamma_0(4N)).$$

Proof. Let

$$j_{4N}(z) = q^{-1} + O(q)$$

be a meromorphic modular function with a pole only at ∞ . Explicitly, these functions are

$$\begin{aligned} j_4(z) &= \frac{\eta(z)^8}{\eta(4z)^8} + 8, & j_8(z) &= \frac{\eta(4z)^{12}}{\eta(2z)^4\eta(8z)^8}, \\ j_{12}(z) &= \frac{\eta(4z)^4\eta(6z)^2}{\eta(2z)^2\eta(12z)^4}, & j_{16}(z) &= \frac{\eta^2(z)\eta(8z)}{\eta(2z)\eta^2(16z)} + 2. \end{aligned}$$

Since the Fourier coefficients of $\eta(z)$ and $\frac{1}{\eta(z)}$ are integral, the q -expansion of $j_{4N}(z)$ has integral coefficients.

Recall that $\Delta_{4N, \lambda} = q^{\delta_{\lambda}(4N)} + O(q^{\delta_{\lambda}(4N)+1})$ is the modular form of weight $\lambda + \frac{1}{2}$ on $\Gamma_0(4N)$ such that the order of its zero at ∞ is higher than that of any other modular form

of the same level and weight. Denote the order of zero of $\Delta_{4N,\lambda}$ at ∞ by $\delta_\lambda(4N)$. Then the basis of $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ can be chosen as

$$(4.1) \quad \{\Delta_{4N,\lambda}(z)j_{4N}(z)^e \mid 0 \leq e \leq \delta_\lambda(4N)\}.$$

If $\Delta_{4N,\lambda}(z)$ is \mathfrak{p} -integral, then $\{\Delta_{4N,\lambda}(z)j_{4N}(z)^e \mid 0 \leq e \leq \delta_\lambda(4N)\}$ also forms a basis of $\widetilde{M}_{\lambda+\frac{1}{2},\mathfrak{p}}(\Gamma_0(4N))$. Note that $\delta_\lambda(4N) = \dim M_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) - 1$. So from Table 1 we have

$$(4.2) \quad \Delta_{4N,\lambda}(z) = \Delta_{4N,j}(z)R_{4N}(z)^{\frac{\lambda-j}{2}},$$

where $\lambda \equiv j \pmod{2}, j \in \{0, 1\}$. More precisely, one can choose $\Delta_{4N,j}(z)$ as followings:

$$\begin{aligned} \Delta_{4,0}(z) &= \theta(z), \quad \Delta_{4,1}(z) = \theta(z)^3, \\ \Delta_{8,0}(z) &= \theta(z), \quad \Delta_{8,1}(z) = \frac{1}{4}(\theta(z)^3 - \theta(z)\theta(2z)^2), \\ \Delta_{12,0}(z) &= \theta(z), \quad \Delta_{12,1}(z) = \frac{1}{6}\left(\sum_{x,y,z \in \mathbb{Z}} q^{3x^2+2(y^2+z^2+yz)} - \sum_{x,y,z \in \mathbb{Z}} q^{3x^2+4y^2+4z^2+4yz}\right), \\ \Delta_{16,0}(z) &= \frac{1}{2}(\theta(z) - \theta(4z)), \quad \Delta_{16,1}(z) = \frac{1}{8}(\theta(z)^3 - 3\theta(z)^2\theta(4z) + 3\theta(z)\theta(4z)^2 - \theta(4z)^3). \end{aligned}$$

Since $\theta(z) = 1 + 2 \sum_{n=1}^{\infty} q^n$, the coefficients of the q -expansion of $\Delta_{4N,j}(z)$, $j \in \{0, 1\}$, are \mathfrak{p} -integral. This completes the proof. \square

Remark 4.2. The proof of Lemma 4.1 implies that the spaces of $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ for $N = 1, 2, 4$ are generated by eta-quotients since $\theta(z) = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2}$.

For $1 \leq N \leq 4$ set

$$\widetilde{V}_S\left(4N, \lambda + \frac{1}{2}; n\right) := \left\{ (a_f(1), \dots, a_f(n)) \in \mathbb{F}_{\mathfrak{p}}^n \mid f \in \widetilde{S}_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) \right\}, \mathbb{F}_{\mathfrak{p}} := \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}.$$

We define $\widetilde{L}_S(4N, \lambda + \frac{1}{2}; n)$ to be the orthogonal complement of $\widetilde{V}_S(4N, \lambda + \frac{1}{2}; n)$ in $\mathbb{F}_{\mathfrak{p}}^n$. Using Lemma 4.1, we obtain the following proposition.

Proposition 4.3. *Suppose that λ is a positive integer and $1 \leq N \leq 4$. For each $e \in \mathbb{N}$, $e \geq \frac{\lambda}{2} - 1$, take $r = 2e - \lambda + 1$. The linear map $\widetilde{\psi}_{r,e}(4N) : \widetilde{M}_{r+\frac{1}{2},\mathfrak{p}}(\Gamma_0(4N)) \rightarrow \widetilde{L}_S(4N, \lambda + \frac{1}{2}; e \cdot \omega(4N))$, defined by*

$$\widetilde{\psi}_{r,e}(4N)(g) = (b(4N, e, g; 1), \dots, b(N, e, g; e \cdot \omega(4N))),$$

is an isomorphism. Here $b(4N, e, g; \nu)$ is defined in (3.2).

Proof. Note that $\dim S_{\frac{3}{2}}(4N) = 0$ and that

$$\dim S_{\lambda+\frac{1}{2}}(4N) + N + 1 + \left\lceil \frac{N}{4} \right\rceil = \dim M_{\lambda+\frac{1}{2}}(4N)$$

(see [10]). So, from Lemma 4.1 and Table 1, it is enough to show that $\psi_{r,e}(4N)$ is injective. If g is in the kernel of $\psi_{r,e}(4N)$, then $\frac{g(z)}{R_{4N}(z)^e} \cdot R_{4N}(z)^e \equiv 0 \pmod{\mathfrak{p}}$ by Sturm's formula (see [21]). So we have $g(z) \equiv 0 \pmod{\mathfrak{p}}$ since $R_{4N}(z)^e \not\equiv 0 \pmod{\mathfrak{p}}$. This completes the proof. \square

Theorem 4.4. *Take a prime p , $N = 1, 2, 4$ and*

$$f(z) := \sum_{n=n_0}^{\infty} a_f(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) \cap L[[q]].$$

Suppose that $\mathfrak{p} \subset \mathcal{O}_L$ is any prime ideal with $\mathfrak{p}|p$ and that $a_f(n)$ is \mathfrak{p} -integral for every integer $n \geq n_0$. If $\lambda \equiv 2$ or $2 + \left[\frac{1}{N}\right] \pmod{\frac{p-1}{2}}$ or $p = 2$, then there exists a positive integer b such that

$$a_f(np^b) \equiv 0 \pmod{\mathfrak{p}}, \forall n \in \mathbb{N}.$$

Proof of Theorem 4.4. i) First, suppose that $p \geq 3$: Take positive integers ℓ and b such that

$$(4.3) \quad \frac{3 - 2\alpha(p : \lambda)}{2} p^{2b} + \left(\lambda + \frac{1}{2}\right) p^{\mathfrak{m}(p:\lambda)} + \ell(p-1) = 2.$$

Note that if b is large enough, that is, $b > \log_p \left(\frac{2}{3-2\alpha(p:\lambda)} \left(\lambda + \frac{1}{2}\right) p^{\mathfrak{m}(p:\lambda)} - 2 \right)$, then there exists a positive integer ℓ satisfying (4.3). Also note that $a_f^t(0) = 0$ for every cusp t of $\Gamma_0(4N)$ since $f(z)$ is a cusp form. So, if $r = 2e - \alpha(p : \lambda) + 1$, then Theorem 3.1 implies that, for $g(z) \in \widetilde{M}_{r+\frac{1}{2}}(\Gamma_0(4N))$,

$$\sum_{\nu=1}^{e \cdot \omega(4N)} b(4N, e, g; \nu) a_f(\nu p^{2b - \mathfrak{m}(p:\lambda)}) \equiv 0 \pmod{\mathfrak{p}},$$

since

$$\begin{aligned} & \left(\frac{g(z)}{R_{4N}(z)^e} \right)^{p^{2b}} f(z)^{p^{\mathfrak{m}(p:\lambda)}} E_{p-1}^\ell(z) \\ & \equiv \left(\sum_{\nu=1}^{e \cdot \omega(4N)} b(4N, e, g; \nu) q^{-\nu p^{2b}} + a_{\frac{g(z)}{R_{4N}(z)^e}}(0) + \sum_{n=1}^{\infty} a_{\frac{g(z)}{R_{4N}(z)^e}}(n) q^{np^{2b}} \right) \\ & \quad \cdot \left(\sum_{n=0}^{\infty} a_f(n) q^{np^{\mathfrak{m}(p:\lambda)}} \right) \pmod{p}. \end{aligned}$$

So Proposition 4.3 implies that

$$\begin{aligned} & (a(p^{2b - \mathfrak{m}(p:\lambda)}), a(2p^{2b - \mathfrak{m}(p:\lambda)}), \dots, a(e \cdot \omega(4N)p^{2b - \mathfrak{m}(p:\lambda)})) \\ & \in \widetilde{V}_S(4N, \alpha(p : \lambda) + \frac{1}{2}; n). \end{aligned}$$

If $\alpha(p : \lambda) = 2$ or $2 + \left[\frac{1}{N}\right]$, then

$$\dim S_{\alpha(p:\lambda)+\frac{1}{2}}(\Gamma_0(4N)) = \dim \widetilde{V}_S \left(4N, \alpha(p : \lambda) + \frac{1}{2}; n \right) = 0.$$

ii) $p = 2$: Note that $\frac{\Delta_{4N,1}(z)}{R_{4N}(z)} = q^{-1} + O(1)$ for $N = 1, 2, 4$. So, there exists a polynomial $F(X) \in \mathbb{Z}[X]$ such that

$$F(j_{4N}(z)) \frac{\Delta_{4N,1}(z)}{R_{4N}(z)} = q^{-n} + O(1).$$

For an integer b , $2^{2b} > \lambda + 2$, let

$$G(z) := \left(F(j_{4N}(z)) \frac{\Delta_{4N,1}(z)}{R_{4N}(z)} \right)^{2b} f(z) \theta(z)^{2^{1+2b} - 2\lambda + 3}.$$

Since $\theta(z) \equiv 1 \pmod{2}$, Theorem 3.1 implies that $a_f(2^b \cdot n) \equiv 0 \pmod{\mathfrak{p}}$. \square

To apply Theorem 4.4, we need the following two propositions.

Proposition 4.5 (Proposition 3.2 in [22]). *Suppose that p is an odd prime, k and N are integers with $(N, p) = 1$. Let*

$$f(z) = \sum a(n)q^n \in \mathcal{M}_{\lambda+\frac{1}{2}}(\Gamma_0(4N)).$$

Suppose that $\xi := \begin{pmatrix} a & b \\ cp^2 & d \end{pmatrix}$, with $ac > 0$. Then there exist $n_0, h_0 \in \mathbb{N}$ with $h_0 | N$, a sequence $\{a_0(n)\}_{n \geq n_0}$ and $r_0 \in \{0, 1, 2, 3\}$ such that

$$(f|U_{p^m}|_{\lambda+\frac{1}{2}}\xi)(z) = \sum_{\substack{n \geq n_0 \\ 4n+r_0 \equiv 0 \pmod{p^m}}} a_0(n) q^{\frac{4n+r_0}{4h_0p^m}}, \quad \forall m \geq 1.$$

Proposition 4.6 (Proposition 5.1 in [1]). *Suppose that p is an odd prime such that $p \nmid N$ and consider*

$$g(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4Np^j)) \cap L[[q]], \quad \text{for each } j \in \mathbb{N}.$$

Suppose further that $\mathfrak{p} \subset \mathcal{O}_L$ is any prime ideal with $\mathfrak{p}|p$ and that $a(n)$ is \mathfrak{p} -integral for every integer $n \geq 1$. Then there exists $G(z) \in S_{\lambda'+\frac{1}{2}}(\Gamma_0(4N)) \cap \mathcal{O}_L[[q]]$ such that

$$G(z) \equiv g(z) \pmod{\mathfrak{p}},$$

where $\lambda' + \frac{1}{2} = (\lambda + \frac{1}{2})p^j + p^e(p-1)$ with $e \in \mathbb{N}$ large.

Remark 4.7. Proposition 4.6 was proved for $p \geq 5$ in [1]. One can check that this holds also for $p = 3$.

Now we prove Theorem 1.

Proof of Theorem 1. Take

$$G_p(z) := \begin{cases} \frac{\eta(8z)^{48}}{\eta(16z)^{24}} \in M_{12}(\Gamma_0(16)) & \text{if } p = 2, \\ \frac{\eta(z)^{27}}{\eta(9z)^3} \in M_{12}(\Gamma_0(9)) & \text{if } p = 3, \\ \frac{\eta(4z)^{p^2}}{\eta(4p^2z)} \in M_{\frac{p^2-1}{2}}(\Gamma_0(p^2)) & \text{if } p \geq 5. \end{cases}$$

Using properties of eta-quotients (see [12]), note that $G_p(z)$ vanishes at every cusp of $\Gamma_0(16)$ except ∞ if $p = 2$, and vanishes at every cusp $\frac{a}{c}$ of $\Gamma_0(4Np^2)$ with $p^2 \nmid N$ if $p \geq 3$. Thus, Proposition 4.5 implies that there exist positive integers ℓ, m, k such that

$$\begin{cases} (f|U_{p^m})(z)G_p(z)^\ell \in S_{k+\frac{1}{2}}(\Gamma_0(16)) & \text{if } p = 2, \\ (f|U_{p^m})(z)G_p(z)^\ell \in S_{k+\frac{1}{2}}(\Gamma_0(4p^2N)) & \text{if } p \geq 3. \end{cases}$$

Note that $k \equiv \lambda \pmod{p-1}$. Using Proposition 4.6, we can find

$$F(z) \in S_{k'+\frac{1}{2}}(\Gamma_0(4N))$$

such that $F(z) \equiv (f(z)|U_{p^m})G_p(z)^\ell \equiv (f|U_{p^m})(z) \pmod{\mathfrak{p}}$ and $k' \equiv k \pmod{p-1}$. Theorem 4.4 implies that there exists a positive integer b such that $(F|U_{p^{2b}})(z) \equiv 0 \pmod{\mathfrak{p}}$. Thus, we have shown so far that if $\rho \in \mathfrak{p} \setminus \mathfrak{p}^2$, all the Fourier coefficients of $\frac{1}{\rho} \cdot F(z)|U_{p^{m+2b}}$ are \mathfrak{p} -integral. Repeat this argument to complete our claim. \square

4.2. Proof of Theorem 2. Theorem 2 can be derived from Theorem 3.1 by taking a special modular form.

Proof of Theorem 2. Take a positive integer ℓ and a positive even integer u such that

$$\frac{3 - 2\alpha(p : \lambda)}{2} p^u + \left(\lambda + \frac{1}{2} \right) p^{\mathbf{m}(p:\lambda)} + \ell(p-1) = 2.$$

Let $F(z) := \left(\frac{\Delta_{4N, 3-\alpha(p:\lambda)}(z)}{R_{4N}(z)} \right)^{p^u}$ and $G(z) := E_{p-1}(z)^\ell f(z)^{p^{\mathbf{m}(p:\lambda)}}$. Since $E_{p-1}(z) \equiv 1 \pmod{p}$, we have

$$F(z)G(z) \equiv \left(\sum_{n=-1}^{\infty} a_{\frac{\Delta_{4N, 3-\alpha(p:\lambda)}(z)}{R_{4N}(z)}}(n) q^{np^u} \right) \left(\sum_{n=m_\infty}^{\infty} a_f(n) q^{n\mathbf{m}(p:\lambda)} \right) \pmod{\mathfrak{p}}.$$

If Fourier coefficients of $f(z)$ at each cusp are \mathfrak{p} -integral, then

$$\begin{aligned} ((F \cdot G)|_{2\gamma_t})(z) &\equiv \left(q_t^r \sum_{n=m_t}^{\infty} a_F^t(n) q_t^n \right) \left(q_t^r \sum_{n=0}^{\infty} a_G^t(n) q_t^n \right) \\ &\equiv \left(q_t^r \sum_{n=m_t}^{\infty} a_f^t(n) q_t^n \right) \left(q_t^{p^u} \sum_{n=0}^{\infty} a_{\frac{\Delta_{4N, 3-\alpha(p:\lambda)}(z)}{R_{4N}(z)}}(n) q_t^{p^u} \right) \pmod{\mathfrak{p}} \end{aligned}$$

for $t \asymp \infty$. Since

$$\begin{aligned} a_{F(z)G(z)}(0) &\equiv a_{\frac{\Delta_{4N, 3-\alpha(p:\lambda)}(z)}{R_{4N}(z)}}(0) a_f(0) + a_f(p^{u-\mathbf{m}(p:\lambda)}) \pmod{\mathfrak{p}}, \\ a_{F(z)G(z)}^t(0) &\equiv a_{\frac{\Delta_{4N, 3-\alpha(p:\lambda)}(z)}{R_{4N}(z)}}^t(0) a_f^t(0) \pmod{\mathfrak{p}} \quad \text{for } t \asymp \infty, \end{aligned}$$

for large u , the Residue Theorem implies Theorem 2 by letting $u = 2b$. Therefore it is enough to check a \mathfrak{p} -integral property of Fourier coefficients of $f(z)$ at each cusp: take a positive integer e such that $\Delta(z)^e f(z)$ is a holomorphic modular form, where

$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}$. Note that the q -expansions of $j_{4N}(z)$ and $\Delta_{4N,12e+\lambda}(z)$ at each cusp are p -integral. Thus (4.1) implies that

$$\Delta(z)^e f(z) = \sum_{n=0}^{\delta_{12e+\lambda}(4N)} c_n j_{4N}(z)^n \Delta_{4N,12e+\lambda}(z).$$

Moreover, c_n is \mathfrak{p} -integral since

$$j_{4N}(z)^n \Delta_{4N,12e+\lambda}(z) = q^{\delta_{12e+\lambda}(4N)-n} + O(q^{\delta_{12e+\lambda}(4N)-n+1})$$

and $f(z) \in \mathcal{O}_L[[q, q^{-1}]]$. Note that $p \nmid 4N$ since $1 \leq N \leq 4$ and $p \geq 5$ is a prime. So Fourier coefficients of $j_{4N}(z)$, $\Delta_{4N,12e+\lambda}(z)$ and $\frac{1}{\Delta(z)}$ at each cusp are \mathfrak{p} -integral. This completes our claim. \square

5. Proof of Theorem 3

Theorem 3 follows from Theorem 1 and Theorem 2.1.

Proof of Theorem 3. Note that $j(z) \in \mathcal{M}_H$. Let

$$g(z) := \Psi^{-1}(j(z)) \text{ and } f(z) := \Psi^{-1}(F(z)) = \sum_{n=n_0}^{\infty} a_f(n) q^n.$$

It is known (see §14 in [4]) that

$$\frac{1}{3}g(z) = \frac{\frac{d}{dz}(\theta(z))E_{10}(4z)}{4\pi i \Delta(4z)} - \frac{\theta(z) \frac{d}{dz}(E_{10}(4z))}{80\pi i \Delta(4z)} - \frac{152}{5}\theta(z).$$

Since the constant terms of the q -expansions at ∞ of $f(z)$, $\theta(z)$ and $g(z)$ are 0, $a_{\theta(z)}^0(0) = \frac{1-i}{2}$ and $a_g^0(0) = \frac{1-i}{2} \cdot \frac{456}{5}$, respectively, we have

$$f(z) - k\theta(z) - \frac{a_f^0(0) + k(1-i)/2}{a_g^0(0)}g(z) \in \mathcal{M}_{\frac{1}{2}}^0(\Gamma_0(4)).$$

Applying Theorem 1, one obtains the result. \square

6. Proofs of Theorem 4 and 5

We begin with the following proposition.

Proposition 6.1. *Let p be an odd prime and*

$$f(z) := \sum_{n=0}^{\infty} a_f(n) q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4)) \cap \mathbb{Z}_p[[q]].$$

If $\lambda \equiv 2$ or $3 \pmod{\frac{p-1}{2}}$, then

$$\begin{aligned} & a_f(p^{2b-\mathfrak{m}(p:\lambda)}) \\ & \equiv -(14 - 4\alpha(p:\lambda))a_f(0) + 2^8 (2^{-1} - 2^{-1}i)^{p^b(7-2\alpha(p:\lambda))} a_f^0(0) \pmod{p} \end{aligned}$$

for every integer $b > \log_p \left(\frac{2}{2\alpha(p:\lambda)-3} \left(\lambda + \frac{1}{2} \right) p^{\mathbf{m}(p:\lambda)} + 2 \right)$.

Proof of Proposition 6.1. For $\nu \in \mathbb{Z}_{\geq 0}$,

$$\left(\lambda + \frac{1}{2} \right) p^{\mathbf{m}(p:\lambda)} := \nu \cdot (p-1) + \alpha(p:\lambda) + \frac{1}{2}.$$

For an integer b with

$$b > \frac{1}{2} \log_p \left(\frac{2}{3-2\alpha(p:\lambda)} \left(\left(\lambda + \frac{1}{2} \right) p^{\mathbf{m}(p:\lambda)} - 2 \right) \right),$$

there exists an $\ell \in \mathbb{N}$ such that

$$\frac{3-2\alpha(p:\lambda)}{2} p^{2b} + \left(\lambda + \frac{1}{2} \right) p^{\mathbf{m}(p:\lambda)} + \ell(p-1) = 2,$$

since

$$\frac{3-2\alpha(p:\lambda)}{2} p^{2b} + \left(\lambda + \frac{1}{2} \right) p^{\mathbf{m}(p:\lambda)} - 2 = \frac{3-2\alpha(p:\lambda)}{2} (p^{2b} - 1) + \nu(p-1).$$

We have

$$\begin{aligned} F(z) &\equiv \sum_{n=0}^{\infty} a_f(n) q^{np^{\mathbf{m}(p:\lambda)}} \pmod{p}, \\ G(z) &\equiv q^{-p^b} + 14 - 4\alpha(p:\lambda) + a_G(1)q + \cdots \pmod{p}. \end{aligned}$$

Note that $a_G(n)$ is p -integral for every integer n . Moreover, we obtain

$$F(z)G(z)|_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv (a_f^0(0) + \cdots) \left(-2^{6p^b} \left(\frac{1}{2} - \frac{i}{2} \right)^{p^b(7-2\alpha(p:\lambda))} + \cdots \right) \pmod{p},$$

where $a_f^0(0)$ is given in (1.1). Note that $\{\infty, 0, \frac{1}{2}\}$ is the set of cusps of $\Gamma_0(4)$, so Theorem 2 implies that

$$a_f(p^{2b-\mathbf{m}(p:n)}) + (14 - 4\alpha(p:\lambda))a_f(0) - 2^8 a_f^0(0) \left(\frac{1}{2} - \frac{i}{2} \right)^{p^b(7-2\alpha(p:\lambda))} \equiv 0 \pmod{p}.$$

This proves Proposition 6.1. □

6.1. Proof of Theorem 4. Now we prove Theorem 4.

Proof of Theorem 4. Take

$$f(z) := \theta^{2\lambda+1}(z) = 1 + \sum_{\ell=1}^{\infty} r_{2\lambda+1}(\ell) q^\ell = \sum_{n=0}^{\infty} a_f(n) q^n.$$

Note that $f(z) \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4))$. Since $(\theta|_{\frac{1}{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})(z) = \frac{1-i}{2} + O\left(q^{\frac{1}{4}}\right)$, we obtain

$$a_f(0) = 1 \text{ and } a_f^0(0) = \left(\frac{1-i}{2} \right)^{2\lambda+1}.$$

Since $\lambda \equiv 2, 3 \pmod{\frac{p-1}{2}}$ and $\left(\frac{1-i}{2}\right)^8 = \frac{1}{16}$, we have

$$\begin{aligned}
& \left(\frac{1}{2} - \frac{i}{2}\right)^{p^{2u}(7-2\alpha(p:\lambda))} a_f^0(0) p^{\mathbf{m}(p:\lambda)} \\
& \equiv \left(\frac{1}{2} - \frac{i}{2}\right)^{p^{2u}(7-2\alpha(p:\lambda))} \left(\frac{1}{2} - \frac{i}{2}\right)^{p^{\mathbf{m}(p:\lambda)}(2\alpha(p:\lambda)+(p-1)(2\lceil\frac{\lambda}{p-1}\rceil+\mathbf{m}(p:\lambda))+1)} \\
& \equiv \left(\frac{1}{2} - \frac{i}{2}\right)^{(7-2\alpha(p:\lambda))(p^{2u}-1)} \left(\frac{1}{2} - \frac{i}{2}\right)^{8+2(p-1)\lceil\frac{\lambda}{p-1}\rceil+\mathbf{m}(p:\lambda)p^{\mathbf{m}(p:\lambda)}(p-1)+(p^{\mathbf{m}(p:\lambda)}-1)(1+2\alpha(p:\lambda))} \\
& \equiv \left(\frac{1}{2} - \frac{i}{2}\right)^{8+2\lceil\frac{\lambda}{p-1}\rceil(p-1)+2\alpha(p:\lambda)(p^{\mathbf{m}(p:\lambda)}-1)} \equiv \frac{1}{16} \left(\frac{-1}{p}\right)^{\lceil\frac{\lambda}{p-1}\rceil+\alpha(p:\lambda)\mathbf{m}(p:\lambda)} \pmod{p},
\end{aligned}$$

for some $u \in \mathbf{N}$. Applying Proposition 6.1, we obtain the result. \square

6.2. Proof of Theorem 5. Consider the Cohen Eisenstein series $H_{r+\frac{1}{2}}(z) := \sum_{N=0}^{\infty} H(r, N)q^N$ of weight $r + \frac{1}{2}$, where $r \geq 2$ is an integer. If $(-1)^r N \equiv 0, 1 \pmod{4}$, then $H(r, N) = 0$. If $N = 0$, then $H(r, 0) = \frac{-B_{2r}}{2r}$. If N is a positive integer and $Df^2 = (-1)^r N$, where D is a fundamental discriminant, then

$$(6.1) \quad H(r, N) = L(1-r, \chi_D) \sum_{df} \mu(d) \chi_D(d) d^{r-1} \sigma_{2r-1}(f/d).$$

Here $\mu(d)$ is the Möbius function. The following theorem implies that the Fourier coefficients of $H_{r+\frac{1}{2}}(z)$ are p -integral if $\frac{p-1}{2} \nmid r$.

Theorem 6.2 ([6]). *Let D be a fundamental discriminant. If D is divisible by at least two different primes, then $L(1-n, \chi_D)$ is an integer for every positive integer n . If $D = p$, $p > 2$, then $L(1-n, \chi_D)$ is an integer for every positive integer n unless $\gcd(p, 1-\chi_D(g)g^n) \neq 1$, where g is a primitive root \pmod{p} .*

Proof of Theorem 5. Note that $E_{10}(z) = E_4(z)E_6(z)$. So, $E_{10}(z)F(z)$, $E_{10}(z)G(z)$ and $E_{10}(z)W(z)$ are modular forms of weights, $8 \cdot \frac{1}{2}$, $7 \cdot \frac{1}{2}$ and $8 \cdot \frac{1}{2}$ respectively. Moreover, the Fourier coefficients of those modular forms are 11-integral, since the Fourier coefficients of $H_{\frac{5}{2}}(z)$, $H_{\frac{7}{2}}(z)$ and $H_{\frac{9}{2}}(z)$ are 11-integral by Theorem 6.2. We have

$$\begin{aligned}
E_{10}(z)F(z) &= \frac{B_4}{4} + O(q), \\
E_{10}(z)F(z)|_{\frac{17}{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= \frac{B_4}{4}(1+i)(2i)^{-5} + O\left(q^{\frac{1}{4}}\right), \\
E_{10}(z)G(z) &= \frac{B_6}{6} + O(q), \\
E_{10}(z)G(z)|_{\frac{15}{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= \frac{B_6}{6}(1-i)(2i)^{-7} + O\left(q^{\frac{1}{4}}\right), \\
E_{10}(z)W(z) &= \frac{B_8}{8} + O(q), \\
E_{10}(z)W(z)|_{\frac{17}{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= \frac{B_8}{8}(1+i)(2i)^{-9} + O\left(q^{\frac{1}{4}}\right),
\end{aligned}$$

where B_{2r} is the $2r$ th Bernoulli number. The conclusion now follows from Proposition 6.1. \square

6.3. Proof of Theorem 6. We begin by introducing some notations (see [17]). Let $V := (\mathbb{F}_p^{2n}, Q)$ be the quadratic space over \mathbb{F}_p , where Q is the quadratic form obtained from a quadratic form $x \mapsto T[x](x \in \mathbb{Z}_p^{2n})$ by reducing modulo p . We denote by $\langle x, y \rangle := Q(x, y) - Q(x) - Q(y)$, $x, y \in \mathbb{F}_p^{2n}$, the associated bilinear form and let

$$R(V) := \{x \in \mathbb{F}_p^{2n} : \langle x, y \rangle = 0, \forall y \in \mathbb{F}_p^{2n}, Q(x) = 0\}$$

be the radical of $R(V)$. Following [14], define a polynomial

$$H_{n,p}(T; X) := \begin{cases} 1 & \text{if } s_p = 0, \\ \prod_{j=1}^{[(s_p-1)/2]} (1 - p^{2j-1} X^2) & \text{if } s_p > 0, s_p \text{ odd,} \\ (1 + \lambda_p(T) p^{(s_p-1)/2} X) \prod_{j=1}^{[(s_p-1)/2]} (1 - p^{2j-1} X^2) & \text{if } s_p > 0, s_p \text{ even,} \end{cases}$$

where for even s_p we denote

$$\lambda_p(T) := \begin{cases} 1 & \text{if } W \text{ is a hyperbolic space or } s_p = 2n, \\ -1 & \text{otherwise.} \end{cases}$$

Following [16], for a nonnegative integer μ , define $\rho_T(p^\mu)$ by

$$\sum_{\mu \geq 0} \rho_T(p^\mu) X^\mu := \begin{cases} (1 - X^2) H_{n,p}(T; X), & \text{if } p|f_T, \\ 1 & \text{otherwise.} \end{cases}$$

We extend the functions ρ_T multiplicatively to natural numbers \mathbb{N} by defining

$$\sum_{\mu \geq 0} \rho_T(p^\mu) X^{-\mu} := \prod_{p|f_p} ((1 - X^2) H_{n,p}(T; X)).$$

Let

$$\mathcal{D}(T) := GL_{2n}(\mathbb{Z}) \setminus \{G \in M_{2n}(\mathbb{Z}) \cap GL_{2n}(\mathbb{Q}) : T[G^{-1}] \text{ half-integral}\},$$

where $GL_{2n}(\mathbb{Z})$ operates by left-multiplication and $T[G^{-1}] = T'G^{-1}T$. Then $\mathcal{D}(T)$ is finite. For $a \in \mathbb{N}$ with $a|f_T$, let

$$(6.2) \quad \phi(a; T) := \sqrt{a} \sum_{d^2|a} \sum_{G \in \mathcal{D}(T), |\det(G)|=d} \rho_{T[G^{-1}]}(a/d^2).$$

Note that $\phi(a; T) \in \mathbb{Z}$ for all a . With these notations we state the following theorem:

Theorem 6.3 ([17]). *Suppose that $g \equiv 0, 1 \pmod{4}$ and let $k \in \mathbb{N}$ with $g \equiv k \pmod{2}$. A Siegel modular form F is in $S_{k+n}^{Maass}(\Gamma_{2g})$ if and only if there exists a modular form*

$$f(z) = \sum_{n=1}^{\infty} c(n) q^n \in S_{k+\frac{1}{2}}(\Gamma_0(4))$$

such that $A(T) = \sum_{a|f_T} a^{k-1} \phi(a; T) c \left(\frac{|D_T|}{a^2} \right)$ for all T . Here,

$$D_T := (-1)^g \cdot \det(2T)$$

and $D_T = D_{T,0} f_T^2$ with $D_{T,0}$ the corresponding fundamental discriminant and $f_T \in \mathbb{N}$.

Remark 6.4. A proof of Theorem 6.3 given in [17] implies that if $A(T) \in \mathbb{Z}$ for all T , then $c(m) \in \mathbb{Z}$ for all $m \in \mathbb{N}$.

Proof of Theorem 6. From Theorem 6.3 we can take

$$f(z) = \sum_{n=1}^{\infty} c(n) q^n \in S_{k+\frac{1}{2}}(\Gamma_0(4)) \cap \mathbb{Z}_p[[q]]$$

such that

$$F(Z) = \sum_{T>0} A(T) q^{\text{tr}(TZ)} = \sum_{T>0} \sum_{a|f_T} a^{k-1} \phi(a; T) c \left(\frac{|D_T|}{a^2} \right) q^{\text{tr}(TZ)}.$$

By Theorem 1, there exists a positive integer b such that, for every positive integer m ,

$$c(p^b m) \equiv 0 \pmod{p^j},$$

since $k \equiv 2$ or $3 \pmod{\frac{p-1}{2}}$. Suppose that $p^{b+2j} \mid |D_T|$. If $p^j \mid a$ and $a \mid f_T$, then

$$a^{k-1} \phi(a; T) c \left(\frac{|D_T|}{a^2} \right) \equiv 0 \pmod{p^j}.$$

If $p^j \nmid a$ and $a \mid f_T$, then $p^b \mid \frac{|D_T|}{a^2}$ and $a^{k-1} \phi(a; T) c \left(\frac{|D_T|}{a^2} \right) \equiv 0 \pmod{p^j}$. \square

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REFERENCES

- [1] S. Ahlgren and M. Boylan *Central Critical Values of Modular L-functions and Coefficients of Half Integral Weight Modular Forms Modulo ℓ* , Amer. J. Math. 129 (2007), no. 2, 429–454.
- [2] A. Balog, H. Darmon, K. Ono, *Congruences for Fourier coefficients of half-integer weight modular forms and special values of L-functions*, Analytic Number Theory, 105–128. Progr. Math. 138 Birkhauser, 1996.
- [3] B. Berndt and A. Yee, *Congruences for the coefficients of quotients of Eisenstein series*, Acta Arith. 104 (2002), no. 3, 297–308.
- [4] R. E. Borcherds, *Automorphic forms on $O_{s+2,2}(R)$ and infinite products*, Invent. Math. 120 (1995) 161–213.
- [5] J. H. Bruinier, K. Ono, *The arithmetic of Borcherds' exponents*, Math. Ann. 327 (2003), no. 2, 293–303.
- [6] L. Carlitz, *Arithmetic properties of generalized Bernoulli numbers*, J. Reine Angew. Math. 202 1959 174–182.

- [7] H. Cohen, *Sums involving the values at negative integers of L -functions of quadratic characters*, Math. Ann. 217 (1975), no. 3, 271–285.
- [8] D. Choi and Y. Choie, *Linear Relations among the Fourier Coefficients of Modular Forms on Groups $\Gamma_0(N)$ of Genus Zero and Their Applications*, to appear in J. Math. Anal. Appl. 326 (2007), no. 1, 655–666.
- [9] Y. Choie, W. Kohnen, K. Ono, *Linear relations between modular form coefficients and non-ordinary primes*, Bull. London Math. Soc. 37 (2005), no. 3, 335–341.
- [10] H. Cohen and J. Oesterle, *Dimensions des espaces de formes modulaires*, Lecture Notes in Mathematics, 627 (1977), 69–78.
- [11] S. Corteel and J. Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc. 356 (2004) 1623–1635.
- [12] B. Gordon and K. Hughes, *Multiplicative properties of eta-product*, Cont. Math. 143 (1993), 415–430.
- [13] P. Guerzhoy, *The Borcherds-Zagier isomorphism and a p -adic version of the Kohnen-Shimura map*, Int. Math. Res. Not. 2005, no. 13, 799–814.
- [14] Y. Kitaoka, *Dirichlet series in the theory of Siegel modular forms*, Nagoya Math. J. 95 (1984), 73–84.
- [15] N. Koblitz, *Introduction to elliptic curves and modular forms*, Graduate Texts in Mathematics, 97. Springer-Verlag, New York, 1993
- [16] W. Kohnen, *Lifting modular forms of half-integral weight to Siegel modular forms of even genus*, Math. Ann. 322 (2002), 787–809.
- [17] W. Kohnen and H. Kojima, *A Maass space in higher genus*, Compos. Math. 141 (2005), no. 2, 313–322.
- [18] P. Jenkins and K. Ono, *Divisibility criteria for class numbers of imaginary quadratic fields*, Acta Arith. 125 (2006), no. 3, 285–289.
- [19] T. Miyake, *Modular forms*, Translated from the Japanese by Yoshitaka Maeda, Springer-Verlag, Berlin, 1989
- [20] J.-P. Serre, *Formes modulaires et fonctions zeta p -adiques*, Lecture Notes in Math. 350, Modular Functions of One Variable III. Springer, Berlin Heidelberg, 1973, pp. 191–268.
- [21] J. Sturm, *On the congruence of modular forms*, Number theory (New York, 1984–1985), 275–280, Lecture Notes in Math., 1240, Springer, Berlin, 1987.
- [22] S. Treneer, *Congruences for the Coefficients of Weakly Holomorphic Modular Forms*, to appear in the Proceedings of the London Mathematical Society.
- [23] D. Zagier, *Traces of singular moduli*, Motives, polylogarithms and Hodge theory, Part I, Int. Press Lect. Ser., 3, I, Int. Press, Somerville, MA, 2002, pp.211–244.

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