COUPLED PAINLEVÉ III SYSTEMS WITH AFFINE WEYL GROUP SYMMETRY OF TYPES $B_4^{(1)}$, $D_4^{(1)}$ AND $D_5^{(2)}$

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ABSTRACT. We find and study four kinds of a 4-parameter family of four-dimensional coupled Painlevé III systems with affine Weyl group symmetry of types $B_4^{(1)}$, $D_4^{(1)}$ and $D_5^{(2)}$. We also show that these systems are equivalent by an explicit birational and symplectic transformation, respectively.

1. Introduction

In [5, 6], we presented some types of coupled Painlevé systems with various affine Weyl group symmetries. In this paper, we present a 4-parameter family of 2-coupled Painlevé III systems with affine Weyl group symmetry of type $D_4^{(1)}$ explicitly given by

$$\frac{dx}{dt}=\frac{\partial H_{D_4^{(1)}}}{\partial y},\quad \frac{dy}{dt}=-\frac{\partial H_{D_4^{(1)}}}{\partial x},\quad \frac{dz}{dt}=\frac{\partial H_{D_4^{(1)}}}{\partial w},\quad \frac{dw}{dt}=-\frac{\partial H_{D_4^{(1)}}}{\partial z} \tag{1}$$

with the Hamiltonian

$$\begin{aligned} H_{D_4^{(1)}} &= H_{III}(x,y,t;\alpha_1,\frac{2\alpha_2+\alpha_3+\alpha_4}{2},\alpha_0) \\ &\quad + \tilde{H}_{III}(z,w,t;\alpha_3,\frac{\alpha_4-\alpha_3}{2},1-\alpha_4)-\frac{2yw}{t}. \end{aligned} \tag{2}$$

Here x, y, z and w denote unknown complex variables and $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and α_4 are complex parameters satisfying the relation $\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4=1$. The symbols $H_{III}, \tilde{H}_{III}$ are given by

$$H_{III}(q,p,t;\gamma_0,\gamma_1,\gamma_2)=\frac{q^2p(p-1)+q\{(\gamma_0+\gamma_2)p-\gamma_0\}+tp}{t} \quad (\gamma_0+2\gamma_1+\gamma_2=1), \tag{3}$$

$$\tilde{H}_{III}(q,p,t;\gamma_0,\gamma_1,\gamma_2)=\frac{q^2p(p-t)-q\{(-\gamma_0+\gamma_2)p+\gamma_0 t\}+p}{t} \tag{4}$$

with the relation

$$\begin{aligned} &dp\wedge dq-dH_{III}(q,p,t;\gamma_0,\gamma_1,\gamma_2)\wedge dt \\ &=dP\wedge dQ-d\tilde{H}_{III}(Q,P,t;\gamma_0,\gamma_1,\gamma_2)\wedge dt. \end{aligned} \tag{5}$$

Here the relation between (q,p) and (Q,P) is given by

$$(Q,P)=(1/q,-q(qp+\gamma_0)). \tag{6}$$

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We remark that for this system we tried to seek its first integrals of polynomial type with respect to x, y, z, w. However, we can not find. Of course, the Hamiltonian $H_{D_4^{(1)}}$ is not the first integral.

The Bäcklund transformations of this system satisfy Noumi-Yamada's universal description for $D_4^{(1)}$ root system (see [3]). Since these universal Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry. The aim of this paper is to introduce the system of type $D_4^{(1)}$ and show the relationship between this system and the system of type $B_4^{(1)}$ (see [6]) by an explicit birational and symplectic transformation. We remark that the Bäcklund transformations of that system of type $B_4^{(1)}$ do not have Noumi-Yamada's universal description for $B_4^{(1)}$ root system. In this vein, it had been an open question whether our system of type $B_4^{(1)}$ can be obtained by similarity reduction of a Drinfeld-Sokolov hierarchy. After our discovery of this system, they were studied from the viewpoint of Drinfeld-Sokolov hierarchy by K. Fuji independently (cf. [1]), and he succeeded to obtain our system by similarity reduction of the Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$. His paper will appear soon.

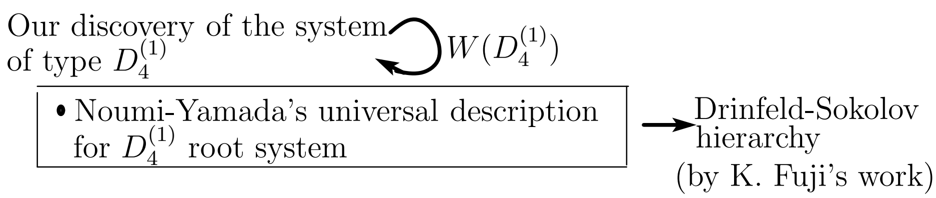

FIGURE 1.

Moreover, we presented three kinds of a 4-parameter family of 2-coupled Painlevé III systems with extended affine Weyl group symmetry of types $B_4^{(1)}$ and $D_5^{(2)}$ (see [6]), whose Hamiltonians $H_{B_4^{(1)}}, \tilde{H}_{B_4^{(1)}}$ and $H_{D_5^{(2)}}$ are given by

$$\begin{aligned} H_{B_4^{(1)}} &= \tilde{H}_{III}(x,y,t;\alpha_1,\alpha_2+\frac{\alpha_3+\alpha_4}{2},2\alpha_0+\alpha_1) \\ &+ \tilde{H}_{III}(z,w,t;\alpha_3,\frac{\alpha_4-\alpha_3}{2},1-\alpha_4)+\frac{2xw(xy+\alpha_1)}{t}, \end{aligned} \tag{7}$$

$$\begin{aligned} \tilde{H}_{B_4^{(1)}} &= H_{III}(x,y,t;\alpha_1,\alpha_2+\alpha_3+\alpha_4,\alpha_0) \\ &+ H_{III}(z,w,t;\alpha_3,\alpha_4,1-\alpha_3-2\alpha_4)+\frac{2yz(zw+\alpha_3)}{t}, \end{aligned} \tag{8}$$

$$\begin{aligned} H_{D_5^{(2)}} &= \tilde{H}_{III}(x,y,t;\alpha_1,\alpha_2+\alpha_3+\alpha_4,2\alpha_0+\alpha_1) \\ &+ H_{III}(z,w,t;\alpha_3,\alpha_4,1-\alpha_3-2\alpha_4)-\frac{2xz(xy+\alpha_1)(zw+\alpha_3)}{t}. \end{aligned} \tag{9}$$

These systems coincide with the system of type $D_4^{(1)}$ by an explicit birational and symplectic transformation, respectively. In each chart of the phase space, there appear different coupled systems with symmetries of various types.

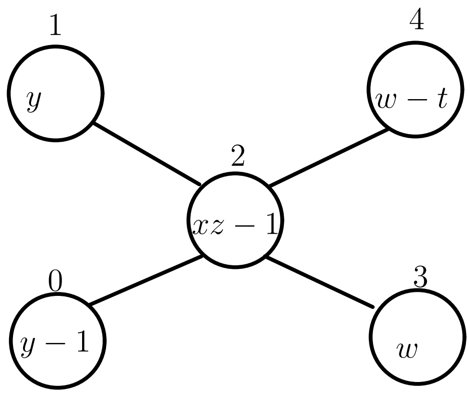

FIGURE 2. Dynkin diagram of type $D_4^{(1)}$

This paper is organized as follows. In Section 2, we introduce the system of type $D_4^{(1)}$ and its Bäcklund transformations. In Section 3, we introduce two kinds of a 4-parameter family of 2-coupled Painlevé III systems with extended affine Weyl group symmetry of type $B_4^{(1)}$ and its Bäcklund transformations. Moreover, these systems coincide with the system of type $D_4^{(1)}$ by an explicit birational and symplectic transformation, respectively. In Section 4, we introduce a 4-parameter family of 2-coupled Painlevé III systems with extended affine Weyl group symmetry of type $D_5^{(2)}$ and its Bäcklund transformations. Moreover, this system coincides with the system of type $D_4^{(1)}$ by an explicit birational and symplectic transformation.

2. The system of type $D_4^{(1)}$

In this section, we present a 4-parameter family of polynomial Hamiltonian systems that can be considered as 2-coupled Painlevé III systems in dimension four given by

$$
(10) \quad \begin{cases}
\dfrac{dx}{dt} = \dfrac{\partial H_{D_4^{(1)}}}{\partial y} = \dfrac{2x^2y - x^2 + (\alpha_0+\alpha_1)x - 2w}{t} + 1, \\
\dfrac{dy}{dt} = -\dfrac{\partial H_{D_4^{(1)}}}{\partial x} = \dfrac{-2xy^2 + 2xy - (\alpha_0+\alpha_1)y + \alpha_1}{t}, \\
\dfrac{dz}{dt} = \dfrac{\partial H_{D_4^{(1)}}}{\partial w} = \dfrac{2z^2w - tz^2 - (1-\alpha_3-\alpha_4)z + 1 - 2y}{t}, \\
\dfrac{dw}{dt} = -\dfrac{\partial H_{D_4^{(1)}}}{\partial z} = \dfrac{-2zw^2 + 2tzw + (1-\alpha_3-\alpha_4)w + \alpha_3 t}{t}
\end{cases}
$$

with the Hamiltonian (2).

THEOREM 2.1. *The system* (10) *admits affine Weyl group symmetry of type* $D_4^{(1)}$ *as the group of its Bäcklund transformations* (cf. [4]), *whose generators are explicitly given as follows*: *with the notation* $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_4)$,

$$
\begin{aligned}
&s_0 : (*) \rightarrow (x + \frac{\alpha_0}{y-1}, y, z, w, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4),\\
&s_1 : (*) \rightarrow (x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4),\\
&s_2 : (*) \rightarrow (x, y - \frac{\alpha_2 z}{xz-1}, z, w - \frac{\alpha_2 x}{xz-1}, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4 + \alpha_2),\\
&s_3 : (*) \rightarrow (x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4),\\
&s_4 : (*) \rightarrow (x, y, z + \frac{\alpha_4}{w-t}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_4, \alpha_3, -\alpha_4),\\
&\pi_1 : (*) \rightarrow (-x, 1-y, -z, -w, -t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4),\\
&\pi_2 : (*) \rightarrow (x, y, z, w-t, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_4, \alpha_3),\\
&\pi_3 : (*) \rightarrow (tz, \frac{w}{t}, \frac{x}{t}, ty, t; \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0),\\
&\pi_4 : (*) \rightarrow (-tz, \frac{t-w}{t}, -\frac{x}{t}, t - ty, t; \alpha_3, \alpha_4, \alpha_2, \alpha_0, \alpha_1).
\end{aligned}
$$

REMARK 2.2. The transformations π_2, π_3 and π_4 satisfy the following relation:

$$
\pi_4 = \pi_2 \pi_3 \pi_2. \tag{11}
$$

PROPOSITION 2.3. *Let us define the following translation operators* (see [2])

$$
\begin{aligned}
&T_1 := s_3 s_0 s_2 s_4 s_1 s_2 \pi_4, \quad T_2 := s_4 s_1 s_2 s_3 s_0 s_2 \pi_4,\\
&T_3 := s_3 s_2 s_0 s_1 s_2 s_3 \pi_1 \pi_2, \quad T_4 := s_4 s_3 s_2 s_1 s_0 s_2 \pi_1 \pi_2.
\end{aligned} \tag{12}
$$

These translation operators act on parameters α_i as follows:

$$
\begin{aligned}
T_1(\alpha_0, \alpha_1, \dots, \alpha_4) =&(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (1, 0, -1, 1, 0),\\
T_2(\alpha_0, \alpha_1, \dots, \alpha_4) =&(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 1, -1, 0, 1),\\
T_3(\alpha_0, \alpha_1, \dots, \alpha_4) =&(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 0, 0, 1, -1),\\
T_4(\alpha_0, \alpha_1, \dots, \alpha_4) =&(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 0, -1, 1, 1).
\end{aligned} \tag{13}
$$

THEOREM 2.4. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y, z, w]$. We assume that*

(A1) $deg(H) = 5$ with respect to x, y, z, w.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate r_i $(i = 0, 1, 3, 4)$:

$$
\begin{aligned}
&r_0 : x_0 = 1/x,\ y_0 = -((y-1)x + \alpha_0)x,\ z_0 = z,\ w_0 = w,\\
&r_1 : x_1 = 1/x,\ y_1 = -(yx + \alpha_1)x,\ z_1 = z,\ w_1 = w,\\
&r_3 : x_3 = x,\ y_3 = y,\ z_3 = 1/z,\ w_3 = -z(wz + \alpha_3),\\
&r_4 : x_4 = x,\ y_4 = y,\ z_4 = 1/z,\ w_3 = -z((w-t)z + \alpha_4).
\end{aligned}
$$

$(A3)$ *In addition to the assumption* $(A2)$*, the Hamiltonian system in the coordinate* r_1 *becomes again a polynomial Hamiltonian system in the coordinate* r_2:

$$r_2 : x_2 = -((x_1 - z_1)y_1 - \alpha_2)y_1,\ y_2 = 1/y_1,\ z_2 = z_1,\ w_2 = w_1 + y_1.$$

Then such a system coincides with the system (10).

Each coordinate r_i $(i = 0, 1, 3, 4)$ contains a three-parameter family of meromorphic solutions of (10).

Theorems 2.1 and 2.4 can be checked by a direct calculation, respectively.

We note that the following transformations

$$\begin{aligned}
&w_0 : (*) \to (x + \frac{\alpha_0}{y-1}, y, z, w, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4),\\
&w_1 : (*) \to (x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4),\\
&w_2 : (*) \to (x, y - \frac{\alpha_2}{x-z}, z, w + \frac{\alpha_2}{x-z}, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4 + \alpha_2),\\
&w_3 : (*) \to (x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4),\\
&w_4 : (*) \to (x, y, z + \frac{\alpha_4}{w-t}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_4, \alpha_3, -\alpha_4)
\end{aligned}$$

define a representation of the affine Weyl group of type $D_4^{(1)}$. However, we can not find polynomial Hamiltonian systems with affine Weyl group symmetry of type $D_4^{(1)}$ described above.

Moreover, from the viewpoint of holomorphy conditions let us consider a polynomial Hamiltonian system with $H \in \mathbb{C}(t)[x, y, z, w]$. We assume that

(A) This system becomes again a polynomial Hamiltonian system in each coordinate r_i $(i = 0, 1, \ldots, 4)$:

$$\begin{aligned}
&r_0 : x_0 = 1/x,\ y_0 = -((y-1)x + \alpha_0)x,\ z_0 = z,\ w_0 = w,\\
&r_1 : x_1 = 1/x,\ y_1 = -(yx + \alpha_1)x,\ z_1 = z,\ w_1 = w,\\
&r_2 : x_2 = -((x-z)y - \alpha_2)y,\ y_2 = 1/y,\ z_2 = z,\ w_2 = w + y,\\
&r_3 : x_3 = x,\ y_3 = y,\ z_3 = 1/z,\ w_3 = -z(wz + \alpha_3),\\
&r_4 : x_4 = x,\ y_4 = y,\ z_4 = 1/z,\ w_3 = -z((w-t)z + \alpha_4).
\end{aligned}$$

It is still an open question whether we can find a system satisfying the assumption (A).

We also give an explicit description of a confluence from 2-coupled Painlevé V system with $W(D_5^{(1)})$-symmetry to the system of type $D_4^{(1)}$. At first, we recall 5-parameter family of 2-coupled Painlevé V systems with $W(D_5^{(1)})$-symmetry (see [6]) explicitly given by

$$\frac{dx}{dt} = \frac{\partial H_{D_5^{(1)}}}{\partial y},\quad \frac{dy}{dt} = -\frac{\partial H_{D_5^{(1)}}}{\partial x},\quad \frac{dz}{dt} = \frac{\partial H_{D_5^{(1)}}}{\partial w},\quad \frac{dw}{dt} = -\frac{\partial H_{D_5^{(1)}}}{\partial z} \tag{14}$$

with the Hamiltonian

$$H_{D_5^{(1)}} = H_V(x,y,t;\beta_2+\beta_5,\beta_1,\beta_2+2\beta_3+\beta_4) + H_V(z,w,t;\beta_5,\beta_3,\beta_4) + \frac{2yz\{(z-1)w+\beta_3\}}{t}, \tag{15}$$

where the symbol $H_V(q,p,t;\gamma_1,\gamma_2,\gamma_3)$ denotes the Hamiltonian of the second-order Painlevé V systems given by

$$H_V(q,p,t;\gamma_1,\gamma_2,\gamma_3) = \frac{q(q-1)p(p+t)-(\gamma_1+\gamma_3)qp+\gamma_1 p+\gamma_2 tq}{t}.$$

Here $\beta_0,\beta_1,\dots,\beta_5$ are complex parameters normalized as $\beta_0+\beta_1+2\beta_2+2\beta_3+\beta_4+\beta_5=1$.

The system (14) admits affine Weyl group symmetry of type $D_5^{(1)}$ as the group of its Bäcklund transformations, whose generators $w_0,w_1,\dots,w_5$ defined as follows: with *the notation* $(*) := (x,y,z,w,t;\beta_0,\beta_1,\dots,\beta_5)$,

$$\begin{aligned}
w_0 : (*) \rightarrow & (x+\frac{\beta_0}{y+t},y,z,w,t;-\beta_0,\beta_1,\beta_2+\beta_0,\beta_3,\beta_4,\beta_5),\\
w_1 : (*) \rightarrow & (x+\frac{\beta_1}{y},y,z,w,t;\beta_0,-\beta_1,\beta_2+\beta_1,\beta_3,\beta_4,\beta_5),\\
w_2 : (*) \rightarrow & (x,y-\frac{\beta_2}{x-z},z,w+\frac{\beta_2}{x-z},t;\beta_0+\beta_2,\beta_1+\beta_2,-\beta_2,\beta_3+\beta_2,\beta_4,\beta_5),\\
w_3 : (*) \rightarrow & (x,y,z+\frac{\beta_3}{w},w,t;\beta_0,\beta_1,\beta_2+\beta_3,-\beta_3,\beta_4+\beta_3,\beta_5+\beta_3),\\
w_4 : (*) \rightarrow & (x,y,z,w-\frac{\beta_4}{(z-1)},t;\beta_0,\beta_1,\beta_2,\beta_3+\beta_4,-\beta_4,\beta_5),\\
w_5 : (*) \rightarrow & (x,y,z,w-\frac{\beta_5}{z},t;\beta_0,\beta_1,\beta_2,\beta_3+\beta_5,\beta_4,-\beta_5).
\end{aligned} \tag{16}$$

PROPOSITION 2.5. *For the system of type $D_5^{(1)}$, we make the change of parameters and variables*

$$\beta_0=\alpha_0,\ \beta_1=\alpha_1,\ \beta_2=\alpha_2,\ \beta_3=\alpha_3,\ \beta_4=\alpha_4-\alpha_3-\frac{1}{\varepsilon},\ \beta_5=\frac{1}{\varepsilon}, \tag{17}$$

$$t=-\varepsilon T,\ x=1+\frac{X}{\varepsilon T},\ y=\varepsilon TY,\ z=1+\frac{1}{\varepsilon TZ},\ w=-\varepsilon T(ZW+A_3)Z \tag{18}$$

from $\beta_0,\beta_1,\dots,\beta_5,t,x,y,z,w$ to $\alpha_0,\alpha_1,\dots,\alpha_4,\varepsilon,T,X,Y,Z,W$. Then the system can also be written in the new variables T,X,Y,Z,W and parameters $\alpha_0,\alpha_1,\dots,\alpha_4,\varepsilon$ as a Hamiltonian system. This new system tends to the system (10) *of type $D_4^{(1)}$ as $\varepsilon\to 0$.*

By proving the following theorem, we see how the degeneration process in Proposition 2.5 works on the Bäcklund transformation group $W(D_5^{(1)}) =< w_0,w_1,\dots,w_5 >$ described above.

PROPOSITION 2.6. *For the degeneration process in Proposition 2.5, we can choose a subgroup*

$$W_{D_5^{(1)} \rightarrow D_4^{(1)}} := \{< s_0, \dots, s_4 > | s_i := w_i \ (i = 0, 1, 2, 3), \ s_4 := w_4 w_5 w_3 w_4 w_5\}$$

of the Bäcklund transformation group $W(D_5^{(1)})$ *so that* $W_{D_5^{(1)} \rightarrow D_4^{(1)}}$ *converges to* $W(D_4^{(1)})$ *as* $\varepsilon \rightarrow 0$.

3. The system of type $B_4^{(1)}$

In this section, we propose two types of a 4-parameter family of 2-coupled Painlevé III systems in dimension four with affine Weyl group symmetry of type $B_4^{(1)}$. Each of them is equivalent to a polynomial Hamiltonian system, however, each has a different representaion of type $B_4^{(1)}$. We also show that each of them is equivalent to the system (10) by a birational and symplectic transformation.

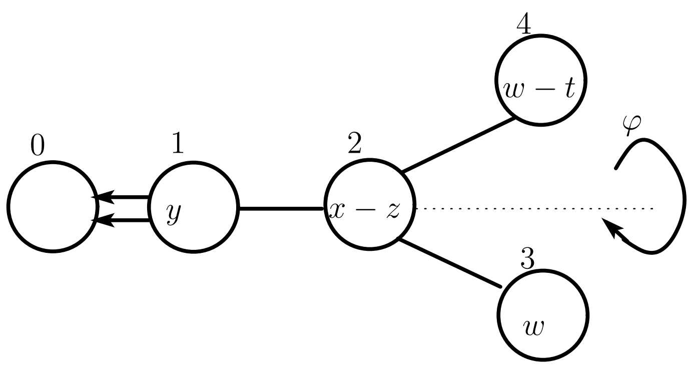

FIGURE 3. Dynkin diagram of type $B_4^{(1)}$

The first member is given by

$$\text{(19)} \quad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H_{B_4^{(1)}}}{\partial y} = \dfrac{2x^2y - tx^2 - 2\alpha_0 x + 1}{t} + \dfrac{2x^2 w}{t}, \\ \dfrac{dy}{dt} = -\dfrac{\partial H_{B_4^{(1)}}}{\partial x} = \dfrac{-2xy^2 + 2txy + 2\alpha_0 y + \alpha_1 t}{t} - \dfrac{2w(2xy + \alpha_1)}{t}, \\ \dfrac{dz}{dt} = \dfrac{\partial H_{B_4^{(1)}}}{\partial w} = \dfrac{2z^2 w - tz^2 - (1 - \alpha_3 - \alpha_4)z + 1}{t} + \dfrac{2x(xy + \alpha_1)}{t}, \\ \dfrac{dw}{dt} = -\dfrac{\partial H_{B_4^{(1)}}}{\partial z} = \dfrac{-2zw^2 + 2tzw + (1 - \alpha_3 - \alpha_4)w + \alpha_3 t}{t} \end{cases}$$

with the Hamiltonian (7). Here x, y, z and w denote unknown complex variables and $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and α_4 are complex parameters satisfying the relation $2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$.

THEOREM 3.1. *The system* (19) *admits extended affine Weyl group symmetry of type* $B_4^{(1)}$ *as the group of its Bäcklund transformations* (cf. [4]), *whose generators are explicitly given as follows*: *with the notation* $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_4)$,

$$
\begin{aligned}
&s_0 : (*) \to (-x, -y + \frac{2\alpha_0}{x} - \frac{1}{x^2}, -z, -w, -t; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2, \alpha_3, \alpha_4),\\
&s_1 : (*) \to (x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4),\\
&s_2 : (*) \to (x, y - \frac{\alpha_2}{x - z}, z, w + \frac{\alpha_2}{x - z}, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4 + \alpha_2),\\
&s_3 : (*) \to (x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4),\\
&s_4 : (*) \to (x, y, z + \frac{\alpha_4}{w - t}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_4, \alpha_3, -\alpha_4),\\
&\varphi : (*) \to (x, y, z, w - t, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_4, \alpha_3).
\end{aligned}
$$

THEOREM 3.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $H \in \mathbb{C}(t)[x, y, z, w]$. *We assume that*

(A1) $deg(H) = 5$ *with respect to* x, y, z, w.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate r_i $(i = 0, 1, \dots, 4)$:

$$
\begin{aligned}
&r_0 : x_0 = x,\ y_0 = y - \frac{2\alpha_0}{x} + \frac{1}{x^2},\ z_0 = z,\ w_0 = w,\\
&r_1 : x_1 = 1/x,\ y_1 = -(yx + \alpha_1)x,\ z_1 = z,\ w_1 = w,\\
&r_2 : x_2 = -((x - z)y - \alpha_2)y,\ y_2 = 1/y,\ z_2 = z,\ w_2 = w + y,\\
&r_3 : x_3 = x,\ y_3 = y,\ z_3 = 1/z,\ w_3 = -z(wz + \alpha_3),\\
&r_4 : x_4 = x,\ y_4 = y,\ z_4 = 1/z,\ w_4 = -z((w - t)z + \alpha_4).
\end{aligned}
$$

Then such a system coincides with the system (19).

Theorems 3.1 and 3.2 can be checked by a direct calculation, respectively.

THEOREM 3.3. *For the system* (10) *of type* $D_4^{(1)}$, *we make the change of parameters and variables*

$$A_0 = \frac{\alpha_0 - \alpha_1}{2}, \quad A_1 = \alpha_1, \quad A_2 = \alpha_2, \quad A_3 = \alpha_3, \quad A_4 = \alpha_4, \tag{20}$$

$$X = \frac{1}{x}, \quad Y = -(xy + \alpha_1)x, \quad Z = z, \quad W = w \tag{21}$$

from $\alpha_0, \alpha_1, \dots, \alpha_4, x, y, z, w$ *to* $A_0, A_1, \dots, A_4, X, Y, Z, W$. *Then the system* (10) *can also be written in the new variables* X, Y, Z, W *and parameters* $A_0, A_1, \dots, A_4$ *as a Hamiltonian system. This new system tends to the system* (19) *with the Hamiltonian* (7).

PROOF. Notice that

$$2A_0 + 2A_1 + 2A_2 + A_3 + A_4 = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$$

and the change of variables from (x, y, z, w) to (X, Y, Z, W) is symplectic. Choose S_i $(i = 0, 1, \ldots, 4)$ and φ as

$$S_0 := \pi_1,\ S_1 := s_1,\ S_2 := s_2,\ S_3 := s_3,\ S_4 := s_4,\ \varphi := \pi_2.$$

Then the transformations S_i are reflections of the parameters $A_0, A_1, \ldots, A_4$. The transformation group $\tilde{W}(B_4^{(1)}) =< S_0, S_1, \ldots, S_4, \varphi >$ coincides with the transformations given in Theorem 3.1. □

The second member (see (4),(5)) is given by

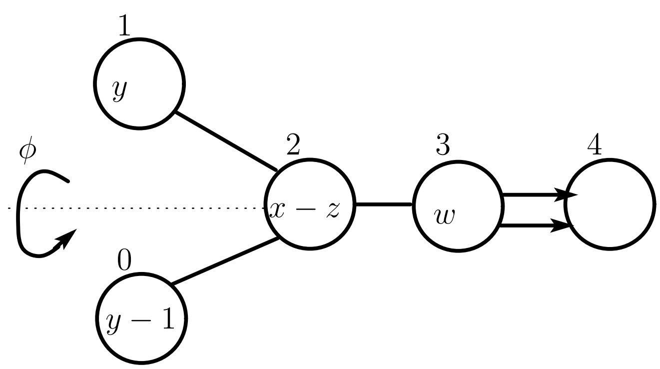

FIGURE 4. Dynkin diagram of type $B_4^{(1)}$

$$(22)\quad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial \tilde{H}_{B_4^{(1)}}}{\partial y} = \dfrac{2x^2y - x^2 + (\alpha_0 + \alpha_1)x + t}{t} + \dfrac{2z(zw + \alpha_3)}{t}, \\ \dfrac{dy}{dt} = -\dfrac{\partial \tilde{H}_{B_4^{(1)}}}{\partial x} = \dfrac{-2xy^2 + 2xy - (\alpha_0 + \alpha_1)y + \alpha_1}{t}, \\ \dfrac{dz}{dt} = \dfrac{\partial \tilde{H}_{B_4^{(1)}}}{\partial w} = \dfrac{2z^2w - z^2 + (1 - 2\alpha_4)z + t}{t} + \dfrac{2yz^2}{t}, \\ \dfrac{dw}{dt} = -\dfrac{\partial \tilde{H}_{B_4^{(1)}}}{\partial z} = \dfrac{-2zw^2 + 2zw - (1 - 2\alpha_4)w + \alpha_3}{t} - \dfrac{2y(2zw + \alpha_3)}{t} \end{cases}$$

with the Hamiltonian (8). Here x, y, z and w denote unknown complex variables and $\alpha_0, \alpha_1, \ldots, \alpha_4$ are complex parameters satisfying the relation $\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4 = 1$.

THEOREM 3.4. *The system* (22) *admits extended affine Weyl group symmetry of type* $B_4^{(1)}$ *as the group of its Bäcklund transformations* (cf. [4]), *whose generators are explicitly* *given as follows: with the notation* $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_4)$,

$$\begin{aligned}
s_0 &: (*) \to (x + \frac{\alpha_0}{y-1}, y, z, w, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4),\\
s_1 &: (*) \to (x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4),\\
s_2 &: (*) \to (x, y - \frac{\alpha_2}{x-z}, z, w + \frac{\alpha_2}{x-z}, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4),\\
s_3 &: (*) \to (x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3),\\
s_4 &: (*) \to (x, y, z, w - \frac{2\alpha_4}{z} + \frac{t}{z^2}, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4, -\alpha_4),\\
\phi &: (*) \to (-x, 1-y, -z, -w, -t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4).
\end{aligned}$$

Theorem 3.5. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $H \in \mathbb{C}(t)[x, y, z, w]$*. We assume that*

(A1) $deg(H) = 5$ *with respect to* x, y, z, w.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate r_i $(i = 0, 1, \dots, 4)$:

$$\begin{aligned}
&r_0 : x_0 = 1/x, \ y_0 = -((y-1)x + \alpha_0)x, \ z_0 = z, \ w_0 = w,\\
&r_1 : x_1 = 1/x, \ y_1 = -(yx + \alpha_1)x, \ z_1 = z, \ w_1 = w,\\
&r_2 : x_2 = -((x-z)y - \alpha_2)y, \ y_2 = 1/y, \ z_2 = z, \ w_2 = w + y,\\
&r_3 : x_3 = x, \ y_3 = y, \ z_3 = 1/z, \ w_3 = -z(wz + \alpha_3),\\
&r_4 : x_4 = x, \ y_4 = y, \ z_4 = z, \ w_4 = w - \frac{2\alpha_4}{z} + \frac{t}{z^2}.
\end{aligned}$$

Then such a system coincides with the system (22).

Theorems 3.4 and 3.5 can be checked by a direct calculation, respectively.

Theorem 3.6. *For the system* (10) *of type* $D_4^{(1)}$*, we make the change of parameters and variables*

$$A_0 = \alpha_0, \quad A_1 = \alpha_1, \quad A_2 = \alpha_2, \quad A_3 = \alpha_3, \quad A_4 = \frac{\alpha_4 - \alpha_3}{2}, \tag{23}$$

$$X = x, \quad Y = y, \quad Z = \frac{1}{z}, \quad W = -(zw + \alpha_3)z \tag{24}$$

from $\alpha_0, \alpha_1, \dots, \alpha_4, x, y, z, w$ *to* $A_0, A_1, \dots, A_4, X, Y, Z, W$*. Then the system* (10) *can also be written in the new variables* X, Y, Z, W *and parameters* $A_0, A_1, \dots, A_4$ *as a Hamiltonian system. This new system tends to the system* (22) *with the Hamiltonian* (8).

Proof. Notice that

$$A_0 + A_1 + 2A_2 + 2A_3 + 2A_4 = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$$

and the change of variables from (x,y,z,w) to (X,Y,Z,W) is symplectic. Choose S_i $(i=0,1,\dots,4)$ and ϕ as

$$S_0:=s_0,\ S_1:=s_1,\ S_2:=s_2,\ S_3:=s_3,\ S_4:=\pi_1,\ \phi:=\pi_2.$$

Then the transformations S_i are reflections of the parameters $A_0,A_1,\dots,A_4$. The transformation group $\tilde{W}(B_4^{(1)})=<S_0,S_1,\dots,S_4,\phi>$ coincides with the transformations given in Theorem 3.4. □

By using Theorems 3.3 and 3.6, it is easy to see that the system (19) coincides with the system (22) by an explicit birational and symplectic transformation.

Proposition 3.7. *For the system* (19) *of type $B_4^{(1)}$, we make the change of parameters and variables*

$$A_0=2\alpha_0+\alpha_1,\quad A_1=\alpha_1,\quad A_2=\alpha_2,\quad A_3=\alpha_3,\quad A_4=\frac{\alpha_4-\alpha_3}{2},$$
$$X=\frac{1}{x},\quad Y=-(xy+\alpha_1)x,\quad Z=\frac{1}{z},\quad W=-(zw+\alpha_3)z$$

from $\alpha_0,\alpha_1,\dots,\alpha_4,x,y,z,w$ to $A_0,A_1,\dots,A_4,X,Y,Z,W$. Then the system (19) *can also be written in the new variables X,Y,Z,W and parameters $A_0,A_1,\dots,A_4$ as a Hamiltonian system. This new system tends to the system* (22) *with the Hamiltonian* (8).

4. The system of type $D_5^{(2)}$

In this section, we propose a 4-parameter family of 2-coupled Painlevé III systems in dimension four with affine Weyl group symmetry of type $D_5^{(2)}$ given by

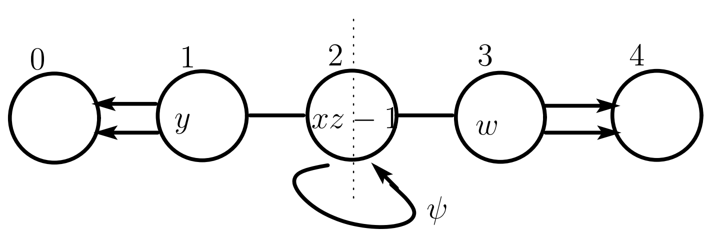

Figure 5. Dynkin diagram of type $D_5^{(2)}$

$$(25)\quad \begin{cases} \dfrac{dx}{dt}=\dfrac{\partial H_{D_5^{(2)}}}{\partial y}=\dfrac{2x^2y-tx^2-2\alpha_0x+1}{t}-\dfrac{2x^2z(zw+\alpha_3)}{t},\\ \dfrac{dy}{dt}=-\dfrac{\partial H_{D_5^{(2)}}}{\partial x}=\dfrac{-2xy^2+2txy+2\alpha_0y+\alpha_1t}{t}+\dfrac{2z(zw+\alpha_3)(2xy+\alpha_1)}{t},\\ \dfrac{dz}{dt}=\dfrac{\partial H_{D_5^{(2)}}}{\partial w}=\dfrac{2z^2w-z^2+(1-2\alpha_4)z+t}{t}-\dfrac{2xz^2(xy+\alpha_1)}{t},\\ \dfrac{dw}{dt}=-\dfrac{\partial H_{D_5^{(2)}}}{\partial z}=\dfrac{-2zw^2+2zw-(1-2\alpha_4)w+\alpha_3}{t}+\dfrac{2x(xy+\alpha_1)(2zw+\alpha_3)}{t} \end{cases}$$

with the Hamiltonian (9). Here x, y, z and w denote unknown complex variables and $\alpha_0, \alpha_1, ldots, \alpha_4$ are complex parameters satisfying the relation $\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4=\frac{1}{2}$.

THEOREM 4.1. *The system* (25) *admits extended affine Weyl group symmetry of type* $D_5^{(2)}$ *as the group of its Bäcklund transformations* (cf. [4]), *whose generators are explicitly given as follows*: *with the notation* $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_4)$,

$$\begin{aligned}
&s_0 : (*) \rightarrow (-x, -y + \frac{2\alpha_0}{x} - \frac{1}{x^2}, -z, -w, -t; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2, \alpha_3, \alpha_4),\\
&s_1 : (*) \rightarrow (x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4),\\
&s_2 : (*) \rightarrow (x, y - \frac{\alpha_2 z}{xz-1}, z, w - \frac{\alpha_2 x}{xz-1}, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4),\\
&s_3 : (*) \rightarrow (x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3),\\
&s_4 : (*) \rightarrow (x, y, z, w - \frac{2\alpha_4}{w} + \frac{t}{z^2}, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4, -\alpha_4),\\
&\psi : (*) \rightarrow (\frac{z}{t}, tw, tx, \frac{y}{t}, t; \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0).
\end{aligned}$$

THEOREM 4.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $H \in \mathbb{C}(t)[x, y, z, w]$. *We assume that*

(A1) $deg(H) = 5$ *with respect to* x, y, z, w.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate r_i $(i = 0, 1, 3, 4)$:

$$\begin{aligned}
&r_0 : x_0 = x,\ y_0 = y - \frac{2\alpha_0}{x} + \frac{1}{x^2},\ z_0 = z,\ w_0 = w,\\
&r_1 : x_1 = 1/x,\ y_1 = -(yx + \alpha_1)x,\ z_1 = z,\ w_1 = w,\\
&r_3 : x_3 = x,\ y_3 = y,\ z_3 = 1/z,\ w_3 = -z(wz + \alpha_3),\\
&r_4 : x_4 = x,\ y_4 = y,\ z_4 = z,\ w_4 = w - \frac{2\alpha_4}{w} + \frac{t}{z^2}.
\end{aligned}$$

(A3) In addition to the assumption (A2), the Hamiltonian system in the coordinate r_1 *becomes again a polynomial Hamiltonian system in the coordinate* r_2:

$$r_2 : x_2 = -((x_1 - z_1)y_1 - \alpha_2)y_1,\ y_2 = 1/y_1,\ z_2 = z_1,\ w_2 = w_1 + y_1.$$

Then such a system coincides with the system (25).

Theorems 4.1 and 4.2 can be checked by a direct calculation, respectively.

THEOREM 4.3. *For the system* (10) *of type* $D_4^{(1)}$, *we make the change of parameters and variables*

$$A_0 = \frac{\alpha_0 - \alpha_1}{2}, \quad A_1 = \alpha_1, \quad A_2 = \alpha_2, \quad A_3 = \alpha_3, \quad A_4 = \frac{\alpha_4 - \alpha_3}{2}, \tag{26}$$

$$X = \frac{1}{x}, \quad Y = -(xy + \alpha_1)x, \quad Z = \frac{1}{z}, \quad W = -(zw + \alpha_3)z \tag{27}$$

from $\alpha_0, \alpha_1, \dots, \alpha_4, x, y, z, w$ *to* $A_0, A_1, \dots, A_4, X, Y, Z, W$. *Then the system* (10) *can also be written in the new variables* X, Y, Z, W *and parameters* $A_0, A_1, \dots, A_4$ *as a Hamiltonian system. This new system tends to the system* (25) *with the Hamiltonian* (9).

Proof. Notice that

$$2(A_0 + A_1 + A_2 + A_3 + A_4) = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$$

and the change of variables from (x, y, z, w) to (X, Y, Z, W) is symplectic. Choose S_i $(i = 0, 1, \dots, 4)$ and ψ as

$$S_0 := \pi_1, \; S_1 := s_1, \; S_2 := s_2, \; S_3 := s_3, \; S_4 := \pi_2, \; \psi := \pi_3.$$

Then the transformations S_i are reflections of the parameters $A_0, A_1, \dots, A_4$. The transformation group $\tilde{W}(D_5^{(2)}) =< S_0, S_1, \dots, S_4, \psi >$ coincides with the transformations given in Theorem 4.1. □

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