

# Infinite loop superalgebras of the Dirac theory on the Euclidean Taub-NUT space

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## Abstract

The Dirac theory in the Euclidean Taub-NUT space gives rise to a large collection of conserved operators associated to genuine or hidden symmetries. They are involved in interesting algebraic structures as dynamical algebras or even infinite-dimensional algebras or superalgebras. One presents here the infinite-dimensional superalgebra specific to the Dirac theory in manifolds carrying the Gross-Perry-Sorkin monopole. It is shown that there exists an infinite-dimensional superalgebra that can be seen as a twisted loop superalgebra.

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## 1 Introduction

In the quantum physics on curved space-times an interesting problem is to find the algebras of operators that commute with the field equation. In general these operators have to be the generators of isometries [1, 2] or special operators associated with more subtle hidden symmetries that can occur in association with some supersymmetries [3].

We mention that there are two generalization of the Killing vectors which become of interest in physics, namely the Stäckel-Killing (S-K) tensors and the Killing-Yano (K-Y) tensors. A symmetric tensor field  $K_{\mu_1 \dots \mu_r}$  is called a S-K tensor of valence  $r$  if  $K_{(\mu_1 \dots \mu_r; \lambda)} = 0$ . The usual Killing (K) vectors correspond

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to valence  $r = 1$  while the hidden symmetries are encapsulated in S-K tensors of valence  $r > 1$ . A tensor  $f_{\mu_1 \dots \mu_r}$  is called a K-Y tensor of valence  $r$  if it is totally antisymmetric and it satisfies the equation [4]  $f_{\mu_1 \dots (\mu_r; \lambda)} = 0$ .

An example of a background presenting all these types of symmetries and supesymmetries is the space-time of the Gross-Perry-Sorkin (GPS) monopole defined as the Euclidean Taub-NUT space with the time trivially added [5]. The space part is known to be a hyper-Kähler manifold possessing three covariantly constant Killing-Yano (K-Y) tensors with real-valued components which constitute a hypercomplex structure. This generates a  $\mathcal{N} = 4$  superalgebra of Dirac-type operators [6], in a similar way as in pseudo-classical spinning models [7, 8]. The Euclidean Taub-NUT space has, in addition, a non-covariantly constant K-Y tensor related to its specific hidden symmetry [9, 10, 11, 7, 8] giving a conserved Runge-Lenz type operator [12, 13, 9, 14, 15]. In the Dirac theory the corresponding operator can be constructed with the help of the Dirac-type operators produced by the four K-Y tensors of this space [16, 17]. Thus one obtains a rich algebra of conserved observables [18] that offers many possibilities of choosing sets of commuting operators for defining quantum modes [19, 6]. On the other hand, hereby one can select dynamical algebras typical for the Kepler problems [17, 18], or even interesting infinite-dimensional algebras or superalgebras [20]. Similar problems appear in the study of chiral supersymmetry for spin fields in self-dual backgrounds [21].

Our main objective here is to present the content of the operator algebra of the Dirac theory on Euclidean Taub-NUT space showing that this can be seen as a twisted loop superalgebra in the sense of [22]. To this end we review our previous results paying more attention to the Casimir operators involved in our construction. Thus we define a new infinite-dimensional superalgebra, different from that given in [20], that leads to a twisted loop superalgebra in a natural way.

We start in section 2 presenting the main features of the Euclidean Taub-NUT geometry and the operators of the scalar quantum theory pointing out some useful algebraic properties. The next section is devoted to the relationships among the Pauli and Dirac conserved operators that are given in section 4. In section 5 we construct our improved version of infinite-dimensional superalgebra showing in section 6 that this is a twisted loop superalgebra.

## 2 Euclidean Taub-NUT space

The manifold of the GPS monopole, denoted from now by  $\mathfrak{M}$ , is a 5-dimensional Kaluza-Klein space-time whose space part is the Euclidean Taub-NUT space. There are static charts with Cartesian coordinates  $x^\mu$  ( $\mu, \nu, \dots = 0, 1, 2, 3, 4$ ) where the time is  $t = x^0$ ,  $x^i$  ( $i, j, \dots = 1, 2, 3$ ) are the *physical* Cartesian space coordinates while  $x^4$  is the Cartesian extra-coordinate. Taking the metric of the flat model  $\eta = (-1, 1, 1, 1, 1)$  we can use the three-dimensional vector notations,

$\vec{x} = (x^1, x^2, x^3)$ ,  $r = |\vec{x}|$  and  $dl^2 = d\vec{x} \cdot d\vec{x}$ , for writing the GPS line element

$$ds^2 = -dt^2 + \frac{1}{V(r)}dl^2 + V(r)[dx^4 + A_i^{em}(\vec{x})dx^i]^2, \quad (1)$$

defined by the specific functions

$$\frac{1}{V} = 1 + \frac{\mu}{r}, \quad A_1^{em} = -\frac{\mu}{r} \frac{x^2}{r + x^3}, \quad A_2^{em} = \frac{\mu}{r} \frac{x^1}{r + x^3}, \quad A_3^{em} = 0. \quad (2)$$

The real number  $\mu$  is a parameter of the theory. If one interprets  $\vec{A}^{em}$  as the vector potential (or gauge field) it results the magnetic field with central symmetry

$$\vec{B}^{em} = \mu \frac{\vec{x}}{r^3}. \quad (3)$$

The Taub-NUT geometry possesses a special type of isometries which form the isometry group  $I(\mathfrak{M}) = T(1)_t \otimes SO(3) \otimes U(1)_4$  constituted by time translations, space rotations and  $U(1)$  transformations of extra-coordinate. The universal covering group of  $I(\mathfrak{M})$  is the external symmetry group (in the sense of [2])  $S(\mathfrak{M}) = T(1)_t \otimes SU(2) \otimes U(1)_4$ . In Ref. [23] we pointed out that these isometries combine space transformations with gauge ones in a non-trivial manner generating non-linear transformations of the coordinate  $x^4$  under rotations. Fortunately, the complications due to this phenomenon can be avoided in a special gauge where the symmetry under rotations becomes *global*. In the Cartesian charts this gauge is given by the gauge fields  $\hat{e}^{\hat{\alpha}}$  and  $e_{\hat{\alpha}}$  having the non-vanishing components [24]

$$\begin{aligned} \hat{e}_0^0 = 1, \quad \hat{e}_j^i = \frac{1}{\sqrt{V}} \delta_{ij}, \quad \hat{e}_i^4 = \sqrt{V} A_i^{em}, \quad \hat{e}_4^4 = \sqrt{V}, \\ e_0^0 = 1, \quad e_j^i = \sqrt{V} \delta_{ij}, \quad e_i^4 = -\sqrt{V} A_i^{em}, \quad e_4^4 = \frac{1}{\sqrt{V}}. \end{aligned} \quad (4)$$

In this context one can correctly define  $P_4 = -i\partial_4$  and the three-dimensional physical momentum  $\vec{P}$  whose components (in the mentioned local frames) are  $P_i = -i(\partial_i - A_i^{em}\partial_4)$ . Moreover, the angular momentum can be written in covariant form as

$$\vec{L} = \vec{x} \times \vec{P} - \mu \frac{\vec{x}}{r} P_4. \quad (5)$$

These operators obey  $[P_i, P_j] = i\varepsilon_{ijk} B_k^{em} P_4$ ,  $[P_i, P_4] = 0$  and  $[L_i, P_j] = i\varepsilon_{ijk} P_k$  which indicate that  $\vec{P}$  behaves as a vector under rotations. The scalar quantum mechanics in GPS geometry [15] is based on the Schrödinger or Klein-Gordon equations involving the static operator

$$\Delta = V\vec{P}^2 + \frac{1}{V}P_4^2, \quad (6)$$

which is either proportional with the Hamiltonian operator of the Schrödinger theory or represents the static part of the Klein-Gordon operator [18].

The space part of the manifold with GPS monopole is the Euclidean Taub-NUT space which is a hyper-Kähler manifold possessing a triplet of hypercomplex structures,  $\mathbf{f} = \{f^{(1)}, f^{(2)}, f^{(3)}\}$ , defined as

$$f^{(i)} = f_{\hat{\alpha}\hat{\beta}}^{(i)} \hat{e}^{\hat{\alpha}} \wedge \hat{e}^{\hat{\beta}} = 2\hat{e}^i \wedge \hat{e}^4 - \varepsilon_{ijk} \hat{e}^j \wedge \hat{e}^k, \quad (7)$$

where the 1-forms  $\hat{e}^{\hat{\alpha}} = \hat{e}_{\mu}^{\hat{\alpha}} dx^{\mu}$  are defined by the gauge fields (4). In addition, there exists a fourth K-Y tensor,

$$f^Y = f_{\hat{\alpha}\hat{\beta}}^Y \hat{e}^{\hat{\alpha}} \wedge \hat{e}^{\hat{\beta}} = \frac{x^i}{r} f^{(i)} + \frac{2x^i}{\mu V} \varepsilon_{ijk} \hat{e}^j \wedge \hat{e}^k, \quad (8)$$

which is not covariantly constant. The presence of  $f^Y$  is related to the existence of the hidden symmetries of the Euclidean Taub-NUT geometry, encapsulated in three non-trivial S-K tensors,  $k_i^{\mu\nu}$ . These are interpreted as the components of the so-called Runge-Lenz vector of the Euclidean Taub-NUT geodesics and can be expressed as symmetrized products of K-Y tensors [8, 7]. The corresponding conserved vector operator,

$$\vec{K} = -\frac{1}{2} \nabla_{\mu} \vec{k}^{\mu\nu} \nabla_{\nu} = \frac{1}{2} (\vec{P} \times \vec{L} - \vec{L} \times \vec{P}) - \mu \frac{\vec{x}}{r} \left( \frac{1}{2} \Delta - P_4^2 \right), \quad (9)$$

play the same role as the Runge-Lenz vector operator in the usual quantum mechanical Kepler problem [15].

This operator transforms as a vector under rotations such that one can write the following complete system of commutation relations

$$\begin{aligned} [L_i, L_j] &= i\varepsilon_{ijk} L_k, \\ [L_i, K_j] &= i\varepsilon_{ijk} K_k, \\ [K_i, K_j] &= i\varepsilon_{ijk} L_k B^2, \end{aligned} \quad (10)$$

where  $B^2 = P_4^2 - \Delta$ . The operators  $L_i$  and  $K_i$  commute with  $B$  since they commute with  $\Delta$  and  $P_4$ . Moreover, it is known [14] that the operators

$$C_1 = \vec{L}^2 B^2 + \vec{K}^2 = \mu^2 P_4^2 B^2 + \frac{\mu^2}{4} (B^2 + P_4^2)^2 - B^2 \quad (11)$$

$$C_2 = \vec{L} \cdot \vec{K} = -\frac{\mu^2}{2} P_4 (B^2 + P_4^2), \quad (12)$$

play the role of Casimir operators for the open algebra (10). With their help we can define the new Casimir operators,

$$C^{\pm} = C_1 \pm 2BC_2 + B^2 = \frac{\mu^2}{4} (P_4 \mp B)^4 = B^2 (N \pm \mu P_4)^2. \quad (13)$$

where  $N$  is the operator whose eigenvalues are just the values of the principal quantum number of the discrete energy spectra [20].

We note that the algebra (10) does not close to a finite Lie algebra because of the factor  $B^2$  that affects the last commutation relation. Nevertheless, one can obtain Lie algebras replacing the operators  $P_4$  and  $\Delta$  by their eigenvalues  $\hat{q}$  and respectively  $E^2$ . Then one can replace  $B^2$  by  $\hat{q}^2 - E^2$  and rescale the generators  $K_i$ . In this manner one obtains three different dynamical algebras: the  $o(4)$  algebra for the discrete energy spectrum in the domain  $E < |\hat{q}|$ , the  $o(3,1)$  algebra for continuous spectrum in the domain  $E > |\hat{q}|$  and the  $e(3)$  algebra corresponding only to the ground energy of the continuous spectrum,  $E = |\hat{q}|$  [9, 10, 14, 15].

### 3 Conserved Dirac and Pauli operators

For building the Dirac theory we consider the Cartesian chart, the usual four-dimensional space of the Dirac spinors,  $\Psi$ , and the Dirac matrices  $\gamma^{\hat{\alpha}}$ , that satisfy  $\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2\eta_{\hat{\alpha}\hat{\beta}}$ , in the following representation

$$\gamma^i = -i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \quad (14)$$

where  $\sigma_i$  are the Pauli matrices. In addition, we take  $\gamma^0 = i\gamma^1\gamma^2\gamma^3\gamma^4 = i \text{diag}(\mathbf{1}_2, -\mathbf{1}_2)$ . With these notations the *standard* Dirac operator without explicit mass term reads  $D = \gamma^\alpha \nabla_\alpha$  [6, 16] giving the corresponding *massless* Hamiltonian operator [6, 17]

$$H = -i\gamma^0 D = \begin{pmatrix} 0 & \boldsymbol{\alpha}^* \\ \boldsymbol{\alpha} & 0 \end{pmatrix} = \begin{pmatrix} 0 & V\pi^* \frac{1}{\sqrt{V}} \\ \sqrt{V}\pi & 0 \end{pmatrix}, \quad (15)$$

where  $\pi = \sigma_P - iV^{-1}P_4$  and  $\pi^* = \sigma_P + iV^{-1}P_4$  depending on  $\sigma_P = \vec{\sigma} \cdot \vec{P}$ . These operators obey

$$\Delta = \boldsymbol{\alpha}^* \boldsymbol{\alpha} = V\pi^* \pi. \quad (16)$$

We specify that here the star superscript is a mere notation that does not coincide with the Hermitian conjugation of the Pauli operators. The operator  $H$  is the central piece of the Dirac theory and has the remarkable property to produce the *same* energy spectrum as those given by the static Klein-Gordon equation,  $\Delta\phi = E^2\phi$ .

Here we focus on the *conserved* operators of the Dirac theory which *commute* with  $H$ . We denote by  $\mathbf{D} = \{X \mid [X, H] = 0\}$  the algebra of the conserved Dirac operators observing that they can be related to Pauli operators commuting with  $\Delta$  which form the algebra  $\mathbf{P} = \{\hat{X} \mid [\hat{X}, \Delta] = 0\}$  where we include the orbital operators having this property. All these operators are considered as conserved operators in the sense of the Klein-Gordon theory. Notice that the Pauli operators are interesting here since they are involved in different versions of the dyon theory [25] (see also [21]) which may be compared to our approach.

In Ref. [18] we have demonstrated that for any conserved Pauli operator  $\hat{X} \in \mathbf{P}$  we can construct the diagonal Dirac operator

$$\mathcal{D}(\hat{X}) = \begin{pmatrix} \hat{X} & 0 \\ 0 & \boldsymbol{\alpha}\hat{X}\Delta^{-1}\boldsymbol{\alpha}^* \end{pmatrix}, \quad (17)$$

which is also conserved. Particularly, for  $\hat{X} = \mathbf{1}_2$  we obtain the projection operator

$$I = \mathcal{D}(\mathbf{1}_2) = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & \boldsymbol{\alpha}\Delta^{-1}\boldsymbol{\alpha}^* \end{pmatrix}, \quad (18)$$

on the space  $\Psi_D = I\Psi$  in which the eigenspinors  $\psi_E$  of  $H$  form a (generalized) basis. This projection operator splits the algebra  $\mathbf{D} = \mathbf{D}_0 \oplus \mathbf{D}_1$  in two subspaces of the projections  $XI \in \mathbf{D}_0$  and  $X(\mathbf{1} - I) \in \mathbf{D}_1$  of all  $X \in \mathbf{D}$ . One can demonstrate that the subalgebra  $\mathbf{D}_1$  is an ideal in  $\mathbf{D}$  [18].

Another type of conserved Dirac operators are the  $\mathcal{Q}$ -operators defined in [6] as

$$\mathcal{Q}(\hat{X}) = \left\{ H, \begin{pmatrix} \hat{X} & 0 \\ 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 & \hat{X}\boldsymbol{\alpha}^* \\ \boldsymbol{\alpha}\hat{X} & 0 \end{pmatrix}, \quad (19)$$

where  $\hat{X}$  may be any Pauli operator. However, if  $\hat{X} \in \mathbf{P}$  then  $\mathcal{Q}(\hat{X}) \in \mathbf{D}_0$  since  $[\mathcal{Q}(\hat{X}), H] = 0$  and  $\mathcal{Q}(\hat{X})I = \mathcal{Q}(\hat{X})$ . If  $\hat{X} = \mathbf{1}_2$  we obtain just the Hamiltonian operator  $H = \mathcal{Q}(\mathbf{1}_2) \in \mathbf{D}_0$ . Consequently, the inverse of  $H$  with respect to  $I$  can be represented as  $H^{-1} = \mathcal{Q}(\Delta^{-1})$ . The mappings  $\mathcal{D} : \mathbf{P} \rightarrow \mathbf{D}_0$  and  $\mathcal{Q} : \mathbf{P} \rightarrow \mathbf{D}_0$  are linear and have the following algebraic properties

$$\mathcal{D}(\hat{X})\mathcal{D}(\hat{Y}) = \mathcal{D}(\hat{X}\hat{Y}), \quad (20)$$

$$\mathcal{Q}(\hat{X})\mathcal{Q}(\hat{Y}) = \mathcal{D}(\hat{X}\hat{Y}\Delta), \quad (21)$$

$$\mathcal{D}(\hat{X})\mathcal{Q}(\hat{Y}) = \mathcal{Q}(\hat{X})\mathcal{D}(\hat{Y}) = \mathcal{Q}(\hat{X}\hat{Y}), \quad (22)$$

for any  $\hat{X}, \hat{Y} \in \mathbf{P}$ . Moreover, the relations  $[\gamma^0, \mathcal{D}(\hat{X})] = 0$  and  $\{\gamma^0, \mathcal{Q}(\hat{X})\} = 0$  show us that, according to the usual terminology [26],  $\mathcal{D}$  and  $\gamma^0\mathcal{D}$  are *even* Dirac operators while  $\mathcal{Q}$  and  $\gamma^0\mathcal{Q}$  are *odd* ones. We note that there are many other odd or even operators which do not have such forms.

Since  $I$  is the projection operator on the space of the Dirac spinors  $\Psi_D$  we say that the projection  $IXI \in \mathbf{D}_0$  of any Dirac operator  $X$  represents the *physical part* of  $X$ . The physical part of any Dirac operator is conserved and can be written in terms of  $\mathcal{D}$  or  $\mathcal{Q}$ -operators [17, 18]. The action of  $X$  reduces thus to that of Pauli operators allowing us to rewrite the problems of the Dirac theory in terms of Pauli operators [16, 17].

Notice that the off-diagonal operators can be transformed at any time in diagonal ones using the multiplication with  $H$  or  $H^{-1}$ . For example,  $H$  itself which is off-diagonal is related to the diagonal operators  $H^2 = \mathcal{D}(\Delta)$  or  $I$ . Thus each Dirac operator from  $\mathbf{D}$  can be brought in a diagonal form associated with an operator from  $\mathbf{P}$ .

## 4 The operators of the Dirac theory

Carter and McLenaghan showed that in the theory of Dirac fermions for any isometry with K vector  $R_\mu$  there is an appropriate operator [1]:

$$X_k = -i(R^\mu \hat{\nabla}_\mu - \frac{1}{4}\gamma^\mu \gamma^\nu R_{\mu;\nu}) \quad (23)$$

which commutes with the standard Dirac operator  $D$ . In this geometry among the K vectors corresponding to the  $S(\mathfrak{M})$  generators only one is time-like,  $i\partial_t$ , generating time-translations. The other K vectors are time-independent, giving rise to conserved operators.

In other respects, each K-Y tensor  $f_{\mu\nu}$  produces a *non-standard* Dirac operator of the form

$$D_f = -i\gamma^\mu (f_\mu{}^\nu \hat{\nabla}_\nu - \frac{1}{6}\gamma^\nu \gamma^\rho f_{\mu\nu;\rho}) \quad (24)$$

which *anticommutes* with the standard one,  $D$ . In the case of the Euclidean Taub-NUT space Dirac-type operators are constructed from the K-Y tensors of this metric.

The simplest operators of  $\mathbf{D}$  which commute with  $H$ ,  $D$ , and  $\gamma^0$  are the generators of the spinor representation of  $S(\mathfrak{M})$  carried by the space  $\Psi$ . The expressions of these operators are strongly dependent on the gauge fixing. For this reason one prefers the gauge (4) where the spinor fields transform *manifestly covariant* under isometries. In this gauge, the rotation generators of the spinor representation of  $S(\mathfrak{M})$  are just the standard components of the *total* angular momentum

$$J_i = L_i + S_i, \quad S_i = \frac{1}{2}\varepsilon_{ijk} S^{jk} = \frac{1}{2}\text{diag}(\sigma_i, \sigma_i), \quad (25)$$

with point-independent spin operators [6]. In the same way one can show that the  $U(1)_4$  generator,  $P_4$ , does not get a spin term. Hence it results that the spinor representation of  $S(\mathfrak{M})$  is *reducible* being a sum of two irreducible representations carried by spaces of two-dimensional Pauli spinors where the components of the total angular momentum are  $\hat{J}_i = L_i + \frac{1}{2}\sigma_i$ . Moreover, the physical part of the total angular momentum reads

$$\mathcal{J}_i = J_i I = \mathcal{D}(\hat{J}_i) = \mathcal{D}(L_i) + \frac{1}{2}\mathcal{D}(\sigma_i), \quad (26)$$

where both the orbital and the spin terms *separately* commute with  $H$  since  $L_i$  and  $\sigma_i$  commute with  $\Delta$ .

The triplet  $\mathbf{f}$  defined by Eq. (7) gives rise to the spin-like operators

$$\Sigma^{(i)} = \frac{i}{4} f_{\hat{\alpha}\hat{\beta}}^{(i)} \gamma^{\hat{\alpha}} \gamma^{\hat{\beta}} = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad (27)$$

and produce the Dirac-type operators [6]

$$D^{(i)} = -f_{\mu,\nu}^{(i)} \gamma^\nu \nabla^\mu = i[D, \Sigma^{(i)}] = -i \begin{pmatrix} 0 & \sigma_i \boldsymbol{\alpha}^* \\ \boldsymbol{\alpha} \sigma_i & 0 \end{pmatrix} = -i\mathcal{Q}(\sigma_i), \quad (28)$$

which anticommute with  $D$  and  $\gamma^0$ . The operators  $D$  and  $D^{(i)}$ ,  $i = 1, 2, 3$ , form the basis of the  $\mathcal{N} = 4$  superalgebra [17]. In current calculations, when one is not interested to exploit the  $\mathcal{N} = 4$  superalgebra, it is indicated to use the simpler operators

$$Q_i = iH^{-1}D^{(i)} = H^{-1}\mathcal{Q}(\sigma_i) = \mathcal{D}(\sigma_i), \quad (29)$$

instead of  $D^{(i)}$ . However, in this case the fourth partner of the operators  $Q^i$  is rather trivial since this is just  $I$ . Therefore, these operators form a representation of the quaternion units (or of the algebra of Pauli matrices) with values in  $\mathbf{D}_0$ ,

$$Q_i Q_j = \delta_{ij}I + i\varepsilon_{ijk}Q_k, \quad (30)$$

producing an evident  $\mathcal{N} = 3$  superalgebra.

The corresponding Dirac-type operator of the last K-Y tensor,  $f^Y$  was obtained in [16]. This has the form

$$D^Y = -\mathcal{Q}(\sigma_r) + \frac{2i}{\mu\sqrt{V}} \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}, \quad (31)$$

where the Pauli operators  $\sigma_r = \vec{\sigma} \cdot \vec{x}/r$  and  $\lambda = \sigma_L + \mathbf{1}_2 + \mu\sigma_r P_4$  have suitable properties that help one to find the equivalent forms reported in [16] and verify that  $D^Y$  commutes with  $H$  and  $P_4$  and anticommutes with  $D$  and  $\gamma^0$ . Moreover, we observe that the physical part of  $D^Y$  can be put in the form

$$D^Y I = \mathcal{Q}(\sigma^Y \Delta^{-1}), \quad \sigma^Y = \frac{2}{\mu} [\sigma_K + (\sigma_L + \mathbf{1}_2)P_4], \quad (32)$$

where  $\sigma^Y$  is a new conserved Pauli operator associated to

$$Q^Y = HD^Y = HD^Y I = \mathcal{D}(\sigma^Y) \in \mathbf{D}_0. \quad (33)$$

We note that the Pauli operators  $\sigma_L = \vec{\sigma} \cdot \vec{L}$  and  $\sigma_K = \vec{\sigma} \cdot \vec{K}$  are conserved and satisfy  $\{\sigma_K, \sigma_L + \mathbf{1}_2\} = 2\vec{L} \cdot \vec{K}$  and  $\{\sigma_r, \sigma_L + \mathbf{1}_2\} = -2\mu P_4$ .

As in the case of the Klein-Gordon theory, we can define the components of the conserved Runge-Lenz operator of the Dirac theory [16, 17] giving directly their physical parts,

$$\mathcal{K}_i = \frac{\mu}{4} \{Q^Y, Q_i\} + \frac{1}{2}(\mathcal{B} - P_4)Q_i - \mathcal{J}_i P_4 \in \mathbf{D}_0, \quad (34)$$

where  $\mathcal{B}^2 = P_4^2 I - H^2 = \mathcal{D}(B^2)$ . Consequently, we can express

$$\mathcal{K}_i = \mathcal{D}(\hat{K}_i), \quad \hat{K}_i = K_i + \frac{\sigma_i}{2} B \in \mathbf{P}. \quad (35)$$

The operators  $\mathcal{J}_i$  and  $\mathcal{K}_i$  are involved in the following system of commutation relations

$$\begin{aligned} [\mathcal{J}_i, \mathcal{J}_j] &= i\varepsilon_{ijk}\mathcal{J}_k, \\ [\mathcal{J}_i, \mathcal{K}_j] &= i\varepsilon_{ijk}\mathcal{K}_k, \\ [\mathcal{K}_i, \mathcal{K}_j] &= i\varepsilon_{ijk}\mathcal{J}_k B^2, \end{aligned} \quad (36)$$



and commute with the operators  $Q_i$  as [18],

$$[\mathcal{J}_i, Q_j] = i\varepsilon_{ijk}Q_k, \quad [\mathcal{K}_i, Q_j] = i\varepsilon_{ijk}Q_k\mathcal{B}. \quad (37)$$

The algebra (36) does not close as a Lie algebra because of the factor  $\mathcal{B}^2$ . The dynamical algebras of the Dirac theory have to be obtained as in the scalar case by replacing this operator with its eigenvalue  $\hat{q}^2 - E^2$  and rescaling the operators  $\mathcal{K}_i$ . One obtains thus the same dynamical algebras as those governing the scalar modes but in different representations [17].

The Casimir operators of the open algebra (36) are

$$\mathcal{C}_1 = \vec{\mathcal{J}}^2\mathcal{B}^2 + \vec{\mathcal{K}}^2, \quad \mathcal{C}_2 = \vec{\mathcal{J}} \cdot \vec{\mathcal{K}}. \quad (38)$$

In addition, we can define a new Casimir-type operator

$$Q = \frac{\mu}{2}Q^Y + (\mathcal{B} - P_4)\mathcal{D}(\sigma_L + \mathbf{1}_2) = \mathcal{D}[\sigma_K + (\sigma_L + \mathbf{1}_2)B], \quad (39)$$

which is an operator from  $\mathbf{D}_0$  related to  $Q^Y$ . This satisfies the simple algebraic relations,

$$[Q, \mathcal{J}_i] = 0, \quad [Q, \mathcal{K}_i] = 0, \quad \{Q, Q_i\} = 2(\mathcal{K}_i + \mathcal{J}_i\mathcal{B}), \quad (40)$$

and the identity

$$Q^2 = \frac{\mu^2}{4}(P_4I - \mathcal{B})^4, \quad (41)$$

resulting from Eqs. (11), (12) and (39). Moreover, using Eqs. (13) we find two new operators that can be put in a closed form,

$$\mathcal{C}^+ = \mathcal{C}_1 + 2\mathcal{B}\mathcal{C}_2 + \mathcal{B}^2 = (Q + \mathcal{B})^2, \quad (42)$$

$$\mathcal{C}^- = \mathcal{C}_1 - 2\mathcal{B}\mathcal{C}_2 + \mathcal{B}^2 = \frac{\mu^2}{4}(P_4I + \mathcal{B})^4. \quad (43)$$

The operators  $Q$  and  $\mathcal{C}^+$  are Casimir operators only for the algebra (36) but  $Q^2$  and  $\mathcal{C}^-$  are general Casimir operators since they commute with any other conserved Dirac operator.

Finally we observe that we can take over the operator  $N$  of the scalar theory since the Dirac and the Klein-Gordon particles have the same energy spectrum. This offers us the opportunity to introduce the new Casimir operator

$$M = (N + \mu P_4)^2 I \in \mathbf{D}_0 \quad (44)$$

that allows us to write  $Q^2 = \mathcal{B}^2 M$ , as it results from Eqs. (13) and (41).

## 5 Infinite-dimensional superalgebra

Now we may ask how could be organized this very rich set of conserved Dirac operators. There are many commutation and anticommutation relations that

can not be ignored such that it seems that the suitable structure may be a superalgebra. Thus we start with the open superalgebra  $\mathcal{S}_0$  generated by the operators  $\{I, M, \mathcal{J}_i, \mathcal{K}_i, Q, Q_i\} \subset \mathbf{D}_0$  which satisfy Eqs. (36), (37) and (40) completed with the obvious anticommutation rule

$$\{Q, Q\} = 2\mathcal{B}^2 M. \quad (45)$$

We observe that, as in the non-relativistic quantum Kepler problem, there are algebraic relations which remain open because of the factors  $\mathcal{B}$ . Therefore, we are forced to embed all the above ingredients in an *infinite-dimensional* superalgebra constructed in the same manner as the infinite algebra of Ref. [22]. The difference is that here we have a superalgebra with generators of bosonic or fermionic type.

Let us define of the *bosonic* operators

$$I_n = I\mathcal{B}^n, \quad M_n = M\mathcal{B}^n, \quad J_n^i = \mathcal{J}_i\mathcal{B}^n, \quad K_n^i = \mathcal{K}_i\mathcal{B}^n, \quad (46)$$

and the supercharges of the *fermionic* sector

$$Q_n = Q\mathcal{B}^n, \quad Q_n^i = Q_i\mathcal{B}^n, \quad (47)$$

for any  $n = 0, 1, 2, \dots$ . The operators  $I_n$  and  $M_n$  are Casimir-type operators commuting between themselves and with all the operators of the bosonic or fermionic sectors. Then, according to Eqs. (36) and (46), we obtain the following non-trivial commutators of the bosonic sector

$$[J_n^i, J_m^j] = i\varepsilon_{ijk} J_{n+m}^k, \quad (48)$$

$$[J_n^i, K_m^j] = i\varepsilon_{ijk} K_{n+m}^k, \quad (49)$$

$$[K_n^i, K_m^j] = i\varepsilon_{ijk} J_{n+m+2}^k, \quad (50)$$

while from Eqs. (30),(40) and (45) we deduce the anticommutators of the fermionic sector,

$$\{Q_n^i, Q_m^j\} = 2\delta_{ij} I_{n+m}, \quad (51)$$

$$\{Q_n, Q_m^i\} = 2(K_{n+m}^i + J_{n+m+1}^i), \quad (52)$$

$$\{Q_n, Q_m\} = 2M_{m+n+2}. \quad (53)$$

The commutations relations between the bosonic and fermionic operators are

$$[Q_n, J_m^j] = 0, \quad [Q_n^i, J_m^j] = i\varepsilon_{ijk} Q_{n+m}^k, \quad (54)$$

$$[Q_n, K_m^j] = 0, \quad [Q_n^i, K_m^j] = i\varepsilon_{ijk} Q_{n+m+1}^k. \quad (55)$$

Thus we constructed an infinite-dimensional superalgebra  $\mathcal{S}$  generated by the countable set of operators  $\{I_n, M_n, J_n^i, K_n^i, Q_n, Q_n^i\}$ ,  $n \geq 0$ . We observe that the typical algebraic structure related to the Kepler problem is the infinite-dimensional algebra  $\mathcal{A}$  generated by  $\{J_n^i, K_n^i\}$  which is a subalgebra in  $\mathcal{S}$ .

## 6 Twisted loop superalgebras

Now we intend to show that the superalgebra  $\mathcal{S}$  can be seen as a *twisted* Kac-Moody superalgebra such that its subalgebra  $\mathcal{A}$  should be a twisted loop algebra of the usual  $so(4)$  algebra, in the sense of Ref. [22].

First we define the finite-dimensional superalgebra  $\mathcal{W}_0$  generated by the operators  $\{E, F, A^i, B^i, G, G^i\}$ . We assume that  $E$  and  $F$  commute with any other generator and that the generators  $\{A^i, B^i\}$  satisfy the  $so(4)$  algebra,

$$[A^i, A^j] = i\varepsilon_{ijk}A^k, \quad [A^i, B^j] = i\varepsilon_{ijk}B^k, \quad [B^i, B^j] = i\varepsilon_{ijk}A^k. \quad (56)$$

The operators  $G$  and  $G^i$  are fermionic supercharges obeying

$$\{G^i, G^j\} = 2\delta_{ij}E, \quad \{G, G^i\} = 2(A^i + B^i), \quad \{G, G\} = 2F \quad (57)$$

and the commutation relations

$$\begin{aligned} [A^i, G] &= 0, & [A^i, G^j] &= i\varepsilon_{ijk}G^k, \\ [B^i, G] &= 0, & [B^i, G^j] &= i\varepsilon_{ijk}G^k. \end{aligned} \quad (58)$$

This superalgebra has simple finite-dimensional representations as we briefly present in Appendix.

Furthermore, we consider the corresponding Kac-Moody infinite loop superalgebra  $\mathcal{W}$  generated by the operators  $\{E_n, F_n, A_n^i, B_n^i, G_n, G_n^i\}$ ,  $n \in \mathbb{Z}$ , with the following properties

$$\begin{aligned} [A_n^i, A_m^j] &= i\varepsilon_{ijk}A_{n+m}^k, & \{G_n^i, G_m^j\} &= 2\delta_{ij}E_{n+m}, \\ [A_n^i, B_m^j] &= i\varepsilon_{ijk}B_{n+m}^k, & \{G_n, G_m^j\} &= 2(A_{n+m}^i + B_{n+m}^i), \\ [B_n^i, B_m^j] &= i\varepsilon_{ijk}A_{n+m}^k, & \{G_n, G_m\} &= 2F_{n+m}, \end{aligned} \quad (59)$$

and

$$\begin{aligned} [A_n^i, G_m] &= 0, & [A_n^i, G_m^j] &= i\varepsilon_{ijk}G_{n+m}^k, \\ [B_n^i, G_m] &= 0, & [B_n^i, G_m^j] &= i\varepsilon_{ijk}G_{n+m}^k, \end{aligned} \quad (60)$$

understanding that the generators  $E_n$  and  $F_n$ ,  $n \in \mathbb{Z}$ , commute with any other generator of  $\mathcal{W}$ .

The next step is to define the involution automorphism  $\tau : \mathcal{W} \rightarrow \mathcal{W}^\tau$  selecting the countable subset of operators

$$\{E_{2n}, F_{2n}, A_{2n}^i, B_{2n+2}^i, G_{2n+2}, G_{2n}^i\}, \quad n \in \mathbb{Z}, \quad (61)$$

which generates the superalgebra  $\mathcal{W}^\tau \subset \mathcal{W}$ . The algebraic properties of this superalgebra are given by the commutation relations of the bosonic sector,

$$\begin{aligned} [A_{2n}^i, A_{2m}^j] &= i\varepsilon_{ijk}A_{2(n+m)}^k, \\ [A_{2n}^i, B_{2m+2}^j] &= i\varepsilon_{ijk}B_{2(n+m)+2}^k, \\ [B_{2n+2}^i, B_{2m+2}^j] &= i\varepsilon_{ijk}A_{2(n+m+2)}^k, \end{aligned} \quad (62)$$

the anticommutation relations of the fermionic sector

$$\begin{aligned}\{G_{2n}^i, G_{2m}^j\} &= 2\delta_{ij}E_{2(n+m)}, \\ \{G_{2n+2}, G_{2m}^j\} &= 2(A_{2(n+m+1)} + B_{2(n+m)+2}), \\ \{G_{2n+2}, G_{2m+2}\} &= 2F_{2(n+m+2)},\end{aligned}\quad (63)$$

and the commutation relations among both sectors,

$$\begin{aligned}[A_{2n}^i, G_{2m+2}] &= 0, & [A_{2n}^i, G_{2m}^j] &= i\varepsilon_{ijk}C_{2(n+m)}^k, \\ [B_{2n+2}^i, G_{2m+2}] &= 0, & [B_{2n+2}^i, G_{2m}^j] &= i\varepsilon_{ijk}C_{2(n+m+1)}^k.\end{aligned}\quad (64)$$

In this way we have constructed the *twisted* loop superalgebra  $\mathcal{W}^\tau$  the positive part of which (with  $n \geq 0$ ) will be denoted by  $\mathcal{W}_+^\tau$ .

Now we can show that the mapping  $\phi : \mathcal{W}_+^\tau \rightarrow \mathcal{S}$  defined by

$$I_n = \phi(E_{2n}), \quad M_n = \phi(F_{2n}), \quad J_n^i = \phi(A_{2n}^i), \quad K_n^i = \phi(B_{2n+2}^i), \quad (65)$$

and

$$Q_n = \phi(G_{2n+2}), \quad Q_n^i = \phi(G_{2n}^i), \quad n = 0, 1, 2, \dots \quad (66)$$

is an *homomorphism*. Indeed, if we consider, for example, the last of Eqs. (50) we can write

$$\begin{aligned}[\phi(B_{2n+2}^i), \phi(B_{2m+2}^j)] &= [K_n^i, K_m^j] = i\varepsilon_{ijk}J_{n+m+2}^k \\ &= i\varepsilon_{ijk}\phi(A_{2(n+m+2)}^k) = \phi([B_{2n+2}^i, B_{2m+2}^j]).\end{aligned}\quad (67)$$

In this manner one can demonstrate step by step that for any pair of generators,  $X$  and  $Y$ , of  $\mathcal{W}_+^\tau$  we have either  $[\phi(X), \phi(Y)] = \phi([X, Y])$  or  $\{\phi(X), \phi(Y)\} = \phi(\{X, Y\})$ . Reversely, if we start with Eqs. (65) and (66) supposing that  $\phi$  is an homomorphism, then we recover the superalgebra  $\mathcal{S}$ . For example, the last of Eqs. (55) results from

$$\begin{aligned}[Q_n^i, K_m^j] &= [\phi(G_{2n}^i), \phi(B_{2m+2}^j)] = \phi([G_{2n}^i, B_{2m+2}^j]) \\ &= i\varepsilon_{ijk}\phi(G_{2(n+m+1)}^k) = i\varepsilon_{ijk}Q_{n+m+1}^k.\end{aligned}\quad (68)$$

In this way, we bring arguments that our superalgebra  $\mathcal{S}$  can be seen as a twisted loop superalgebra.

It finally should be mentioned that the above construction of the twisted loop superalgebras could be regarded differently. The connection between the set of operators (46), (47) and (61) can be realized directly assigning grades to each operator [27] as follows:

$$\begin{aligned}E_{2n} &:= I\mathcal{B}^n, & F_{2n} &:= M\mathcal{B}^n, & A_{2n}^i &:= J_i\mathcal{B}^n, \\ B_{2n+2}^i &:= \mathcal{K}_i\mathcal{B}^n, & G_{2n+2} &:= Q\mathcal{B}^n, & G_{2n}^i &:= Q_i\mathcal{B}^n.\end{aligned}\quad (69)$$

Thus we achieve a graded loop superalgebra of the Kac-Moody type and the sum of the grades is conserved under (anti)commutations.

## 7 Concluding remarks

Here we constructed the infinite-dimensional superalgebra  $\mathcal{S}$  starting with the finite-dimensional open superalgebra  $\mathcal{S}_0$  formed only by conserved operators commuting with  $I, H, P_4$  and the whole set of Casimir operators freely generated by these three operators.

In  $\mathcal{S}_0$  we explicitly used two Casimir operators, namely  $I$  and  $M$ . As mentioned,  $I$  is the projector on the physical spinor subspace playing the role of identity operator. More interesting is the operator  $M$  since this depends on  $N$  which is in some sense similar with the operator  $(-2H_K)^{-1/2}$  of the  $so(4, 2)$  dynamical algebra of the quantum Kepler problem governed by the non-relativistic Hamiltonian operator  $H_K = -\frac{1}{2}\Delta - r^{-1}$  [28]. We remind the reader that the  $so(4, 2)$  dynamical algebra of the Kepler problem contains not only conserved operators but even operators that do not commute with  $H_K$ . In this case, the conserved operators, i. e. the angular momentum and the Runge-Lenz vector operator, are *orthogonal* and generate an *open* algebra that can be rescaled obtaining thus the dynamical algebra  $o(4) \subset so(4, 2)$  of the discrete energy spectrum. The first Casimir operator of  $o(4)$ , that reads  $C_K^1 = (-2H_K)^{-1} - I$ , has a similar form with our operator  $M$ . However, the second Casimir operator of  $o(4)$  vanishes while our operator  $\mathcal{C}_2$ , given by Eq. (38), is different from zero since the vector operators  $\vec{\mathcal{J}}$  and  $\vec{\mathcal{K}}$  are not orthogonal.

In these circumstances we can say that the subalgebra (36) of  $\mathcal{S}_0$  corresponds to the open algebra that gives the  $o(4)$  dynamical algebra of the Kepler problem. This explains why our twisted loop superalgebra was constructed in a similar way as that of the Kepler case [22]. The main difference between these two theories is that  $\mathcal{S}_0$  is an open superalgebra containing the supercharges  $Q$  and  $Q_i$  that naturally arise from the very special geometry of the Euclidean Taub-NUT space. For this reason we were forced to include, in addition, the bosonic Casimir operator  $M$  for writing down Eq. (45). We specify that this is more than a simple artifice since the resulting infinite superalgebra  $\mathcal{S}$  is a twisted loop superalgebra arising from a coherent algebraic structure, namely the superalgebra  $\mathcal{W}_0$  the representations of which are presented in Appendix.

In other respects, it is clear that the operators  $Q$  and  $Q_i$  appear only in the Dirac theory on  $\mathfrak{M}$  since they are in fact Dirac-type operators. Therefore, it is interesting to compare our results with relativistic systems with spin half whose non-relativistic limit is the quantum Kepler problem. Thus, in the case of the Dirac electron in external Coulomb field there exists a hidden symmetry even if one does not have a conserved Runge-Lenz operator. This symmetry is related to another operator, called the Johnson-Lippmann operator [29], that is a scalar conserved operator. In the non-relativistic limit this becomes the projection of the usual Runge-Lenz vector operator of the Kepler problem on the electron spin direction [30]. In our approach, we can say that the Johnson-Lippmann operator is just the supercharge  $Q$  whose first term given by Eq. (39) is  $\mathcal{D}(\vec{\sigma} \cdot \vec{K})$ .

Finally we note that our open superalgebra  $\mathcal{S}_0$  could be enlarged adding non-conserved operators that can be either leader operators or operators related to

the manifest supersymmetry [6] of our Hamiltonian  $H$ . However, this problem will be considered elsewhere.

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## Appendix

### The superalgebra $\mathcal{W}_0$

Here we would like to show that a fundamental representation of the superalgebra  $\mathcal{W}_0$  arises from a particular representation of the  $so(4)$  Lie algebra.

We start with a finite-dimensional representation,  $\rho$ , of this algebra generated by the linear operators  $\{A_\rho^i, B_\rho^i\}$  defined on the space  $\mathfrak{M}_\rho$  and obeying Eqs. (56). The identity operator of on  $\mathfrak{M}_\rho$  is denoted by  $1_\rho$ . The  $su(2) \times su(2)$  content of the  $so(4)$  algebra can be pointed out in the new basis  $\{J_{\rho+}^i, J_{\rho-}^i\}$  given by the operators  $J_{\rho\pm}^i = \frac{1}{2}(A_\rho^i \pm B_\rho^i)$  that satisfy the  $su(2)$  commutation relations,

$$[J_{\rho+}^i, J_{\rho+}^j] = i\varepsilon_{ijk}J_{\rho+}^k, \quad [J_{\rho-}^i, J_{\rho-}^j] = i\varepsilon_{ijk}J_{\rho-}^k, \quad [J_{\rho+}^i, J_{\rho-}^j] = 0. \quad (\text{A.1})$$

The representation  $\rho = (j^+, j^-)$  is completely determined by the  $su(2)$  weights defined by the Casimir operators  $\vec{J}_{\rho\pm}^2 = j^\pm(j^\pm + 1)1_\rho$ . However, here we have to consider, in addition, the usual Casimir operators  $C_{\rho 1} = \vec{A}_\rho^2 + \vec{B}_\rho^2$  and  $C_{\rho 2} = \vec{A}_\rho \cdot \vec{B}_\rho$  or the new ones

$$C_{\rho\pm} = C_{\rho 1} \pm 2C_{\rho 2} + 1_\rho = 4\vec{J}_{\rho\pm}^2 + 1_\rho = (2j^\pm + 1)^2 1_\rho. \quad (\text{A.2})$$

We note that when  $j^+ = j^-$  then  $\vec{A}_\rho$  and  $\vec{B}_\rho$  are orthogonal, as in the case of the dynamical algebra of the quantum Kepler problem.

Our purpose is to construct the superalgebra  $\mathcal{W}_0$  in the carrier space  $\mathfrak{M} = \mathfrak{M}_\rho \otimes \mathfrak{M}_{(\frac{1}{2}, 0)}$  of the reducible representation  $(j^+, j^-) \otimes (\frac{1}{2}, 0)$  given by arbitrary weights  $j^\pm$  taking positive real values. The representation,  $(\frac{1}{2}, 0)$ , is generated by the operators  $\hat{A}^i = \frac{1}{2}\sigma_i$  and  $\hat{B}^i = \frac{1}{2}\sigma_i$  acting in the two-dimensional space  $\mathfrak{M}_{(\frac{1}{2}, 0)}$  where the identity operator is  $1_2$ . In these circumstances we define first the identity operator on  $\mathfrak{M}$ ,  $E = 1_\rho \otimes 1_2$ , and the Casimir-type operator  $F = C_{\rho+} \otimes 1_2$ . The  $so(4)$  generators of this representation are

$$A^i = A_\rho^i \otimes 1_2 + \frac{1}{2}1_\rho \otimes \sigma_i, \quad B^i = B_\rho^i \otimes 1_2 + \frac{1}{2}1_\rho \otimes \sigma_i. \quad (\text{A.3})$$

Moreover, we introduce the supercharges

$$G_i = 1_\rho \otimes \sigma_i, \quad G = \vec{A}_\rho \otimes \vec{\sigma} + \vec{B}_\rho \otimes \vec{\sigma} + E, \quad (\text{A.4})$$

so that  $G^2 = F$ . Now it is a simple exercise to show that the operators  $\{E, F, A^i, B^i, G, G^i\}$  satisfy Eqs. (56), (57) and (58).

The conclusion is that the superalgebra  $\mathcal{W}_0$  can be realized in the carrier space of any reducible representation  $\rho \otimes (\frac{1}{2}, 0)$  of the  $so(4)$  algebra.

## References

- [1] B. Carter and R. G. McLenaghan, *Phys. Rev. D* **19** (1979) 1093.
- [2] I. I. Cotăescu, *J. Phys. A: Math. Gen.* **33** (2000) 9177.
- [3] M. Cariglia, *Class. Quantum Grav.* **21** (2004) 1051.
- [4] K. Yano, *Ann. Math.* **55** (1952) 328.
- [5] D. J. Gross and M. J. Perry, *Nucl. Phys.* **B226** (1983) 29; R. D. Sorkin, *Phys. Rev. Lett.* **51** (1983) 87;
- [6] I. I. Cotăescu and M. Visinescu, *Int. J. Mod. Phys. A* **16** (2001) 1743.
- [7] D. Vaman and M. Visinescu, *Phys. Rev. D* **57** (1998) 3790.
- [8] D. Vaman and M. Visinescu, *Fortschr. Phys* **47** (1999) 493.
- [9] G. W. Gibbons and P. J. Ruback, *Phys. Lett.* **B188** (1987) 226.
- [10] G. W. Gibbons and P. J. Ruback, *Commun. Math. Phys.* **115** (1988) 267.
- [11] J. W. van Holten, *Phys. Lett.* **B342** (1995) 47.
- [12] G. W. Gibbons and C. A. R. Herdeiro, *Class. Quant. Grav.* **16** (1999) 3619.
- [13] G. W. Gibbons and N. S. Manton, *Nucl. Phys.* **B274** (1986) 183.
- [14] L. Gy. Feher and P. A. Horvathy, *Phys. Lett.* **B183** (1987) 182; id. (E) **B188** (1987) 512.
- [15] B. Cordani, L. Gy. Feher and P. A. Horvathy, *Phys. Lett.* **B201** (1988) 481.
- [16] I. I. Cotăescu and M. Visinescu, *Phys. Lett.* **B502** (2001) 229.
- [17] I. I. Cotăescu and M. Visinescu, *Class. Quantum Grav.* **18** (2001) 3383.
- [18] I. I. Cotăescu and M. Visinescu, *J. Math. Phys.* **43** (2002) 2987.
- [19] I. I. Cotăescu and M. Visinescu, *Mod. Phys. Lett. A* **15** (2000) 145.
- [20] I. I. Cotăescu and M. Visinescu, *Symmetries and supersymmetries of the Dirac operators in curved spacetimes in Progress in General Relativity and Quantum Cosmology Research*, (Nova Science N.Y. 2006)

- [21] L. Gy. Feher, P. A. Horvathy and L. O’Raifeartaigh, *Int. J. Mod. Phys. A* **4** (1989) 5277.
- [22] J. Daboul, P. Slodowy and C. Daboul, *Phys. Lett.* **B317** 321 (1993); C. Daboul, J. Daboul and P. Slodowy, [hep-th/9408080](#).
- [23] I. I. Cotăescu and M. Visinescu, *Mod. Phys. Lett. A* **43** (2004) 2987.
- [24] H. Boutaleb - Joutei and A. Chakrabarti, *Phys. Rev. D* **21** (1979) 2280.
- [25] E. D’Hoker and L. Vinet, *Phys. Lett.* **B137** (1984) 72; F. Bloore and P. A. Horváthy, *J. Math. Phys.* **33** (1992) 1869; F. De Jonge, A. J. Macfarlane, K. Peters, and J. W. van Holten, *Phys. Lett.* **B359** (1995) 114.
- [26] B. Thaller, *The Dirac Equation* (Springer Verlag, Berlin Heidelberg 1992).
- [27] J. Daboul, [math-ph/0608008](#)
- [28] A. O. Barut and H. Kleinert, *Phys. Rev.* **156** (1967) 1541; *ibid.* **157** (1967) 1180; C. Fronsdal, *Phys. Rev.* **156** (1967) 1665; Y. Nambu, *Phys. Rev.* **160** (1967) 1171; A. O. Barut and G. L. Bronzin, *J. Math. Phys.* **12** (1971) 841.
- [29] M. H. Johnson and B. A. Lippmann, *Phys. Rev.* **78** (1950) 329.
- [30] T. T. Khachidze and A. A. Khelashvili, *Mod. Phys. Lett. A* **20** (2005) 2277.