The higher Hilbert pairing via (ϕ, G)-modules

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Abstract

Following the strategy in [6], we prove a Tate duality for higher dimensional local fields of mixed characteristic $(0, p)$, $p \neq 2$. The main tool is the theory of higher fields of norms as developed in [1] and [7]. Assuming that p is not ramified in the basefield, we then use this construction to define the higher Hilbert pairing. In particular, we show that the Hilbert pairing is non-degenerate, and we re-discover the formulae of Brückner and Vostokov.

Contents

1 Introduction

1.1 Statement of the main result

Let p be an odd prime, and let K be a d-dimensional local field of mixed characteristic $(0,p)$ Denote by $\mathcal{G}_K$ the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$. For $n \geq 1$, denote by μ_{p^n} the group of p^nth roots of unity.

This paper consists of two parts. In the first part, we prove a higher Tate duality for the $\mathcal{G}_K$-module μ_{p^n}:-

Theorem 1.1. *Let K be a d-dimensional local field of mixed characteristic $(0,p)$, and denote by $\mathcal{G}_K$ its absolute Galois group. Let F be the maximal algebraic extension of $\mathbb{Q}_p$ contained in K, and assume that O_K/O_F is formally smooth. Then for $i \in \{0,\ldots,d+1\}$ and all $n \in \mathbb{N}$ we have a perfect pairing*

$$H^i(\mathcal{G}_K, \mu_{p^n}^{\otimes i}) \times H^{d+1-i}(\mathcal{G}_K, \mu_{p^n}^{\otimes d-i}) \to \mathbb{Q}_p/\mathbb{Z}_p.$$

Remark. The above pairing should certainly be the same as the cup product pairings, but this is not that easy to show.

To prove Theorem 1.1, we follow the strategy of Herr in [6] and express the Galois cohomology groups in terms of the (ϕ, G)-module of μ_{p^n}. We will also prove a higher dimensional Tate isomorphim:-

Theorem 1.2. *Let K be a d-dimensional local field of mixed characteristic $(0,p)$. Let F be the maximal algebraic extension of $\mathbb{Q}_p$ contained in K, and assume that O_K/O_F is formally smooth. Then there is a canonical isomorphism*

$$H^{d+1}(\mathcal{G}_K, \mu_{p^\infty}^{\otimes d}) \cong \mathbb{Q}_p/\mathbb{Z}_p.$$

In the second part of the paper, we assume that p is prime in K, and we use Theorem 1.1 to define a pairing

$$K_d(K_n) \times K_1(K_n) \to \mu_{p^n}.$$

Composing it with the natural multiplication map $K_1(K_n)^{\times d} \to K_d(K_n)$, we obtain a pairing $V_n : K_1(K_n)^{\times(d+1)} \to \mu_{p^n}$ which factors through

$$\mathfrak{V}_n : \big(K_1(K_n)/p^n\big)^{\times(d+1)} \to \mu_{p^n}.$$

In Section 4 we give a an explicit description of $\mathfrak{V}_n$:- For $1 \leq i \leq d+1$, let $\alpha_i \in O_K^\times$ such that $\alpha_i \cong 1 \mod \bar{\pi}_n$, and let $F_i(X) \in \mathbb{A}_K^+$ such that $h_n(F_i) = \alpha_i$. Let $f_i(X) = (1 - \frac{\phi}{p}) \log F(X)$.

Theorem 1.3. *The pairing $\mathfrak{V}_n$ is non-degenerate. Moreover, we have*

$$\mathfrak{V}_n(\alpha_1, \ldots, \alpha_{d+1}) = \mu_{p^n}^{\mathrm{Tr}\,\mathrm{Res}_{\pi_n, T_1, \ldots, T_d}(\Phi)},$$

where Φ *is given by the formula*

$$\Phi = \frac{1}{\pi}\sum_{i=1}^{d+1}\frac{(-1)^{d+1-i}}{p^{d+1-i}} f_i(\pi_n) d\log F_1(\pi_n)\wedge\cdots\wedge d\log F_{i-1}(\pi_n)$$
$$\wedge\, d\log F^{\phi}_{i+1}(\pi_n)\wedge\cdots\wedge d\log F^{\phi}_{d+1}(\pi_n).$$

Comparing these formulae with the explicit descriptions of the higher Hilber pairing of Brückner and Vostokov (c.f. [3] and [8]), we get the following result:-

Corollary 1.4. *The pairing* $\mathfrak{V}_n$ *is the higher Hilbert pairing.*

For proving Theorem 1.3, we follow the strategy of Benois in [2].

Remarks. (1) To keep the notation as simple as possible, we will prove the above results for local fields of dimension 2. However, the proofs generalize without problems to local fields of arbitrarily high dimension.
(2) Theorem 1.1 can certainly be generalized to an arbitrary $\mathbb{Z}_p$- of $\mathcal{G}_K$. We will deal with this in a different paper.

1.2 Notation

∗ For a 2-dimensional local field K with ring of integers O_K, let $k_K \cong \mathbb{F}_p((T))$ be the residue field.
∗ For a (ϕ, G)-module M, we sometimes denote the action of the Frobenius operator on M by ϕ_M.
∗ For a 2-dimensional local field K, let $\mathcal{G}_K = \mathrm{Gal}(\bar{K}/K)$.

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2 Higher (ϕ, G)-modules

2.1 Setup

Let K be a 2-dimensional local field of mixed characteristic $(0, p)$, and let F be the maximal algebraic extension of $\mathbb{Q}_p$ contained in K. Let k_F be the residue field of F and let ω_F be a uniformizer of F. Assume that O_K/O_F is formally smooth, i.e. ω_F is a uniformizer of K. Let X be a unit in K whose reduction $\bar{X}$ is a p-basis for the residue field k_K of K, so $k_K \cong k_b((X))$ for some finite extension k_b of k_F. Fix an algebraic closure $\bar{K}$ of K. Let $(\xi_i)_{i\geq 0}$ be compatible system of primitive p^ith roots of unity, and let $(X_i)_{i\geq 0}$ be a compatible system of p^ith roots of X. Denote by μ_{p^i} the group of p^ith roots of unity.

Let $K_i = K(\mu_{p^i}, X_i)$ and $K_\infty = \bigcup K_i$. Also, let $F_i = F(\mu_{p^i})$.

Lemma 2.1. *The extension K_∞ is a 2-dimensional p-adic Lie extension of K. More precisely, we have $\mathrm{Gal}(K_\infty/K) \cong \Gamma_1 \rtimes \Gamma_2$, where*

$$\Gamma_2 = \mathrm{Gal}(K_\infty/K(\mu_{p^\infty})) \cong \mathbb{Z}_p.$$

and $\Gamma = \mathrm{Gal}(K(\mu_{p^\infty})/K)$ is isomorphic (via the cyclotomic character χ) to an open subgroup of $\mathbb{Z}_p^\times$.

Let γ_1 and γ_2 be topological generators of Γ_1 and Γ_2, respectively. Let $a \in \mathbb{Z}_p$ such that

$$\gamma_1\gamma_2 = \gamma_2^a\gamma_1. \tag{1}$$

Note. We have $a = \chi(\gamma_1) \in \mathbb{Z}_p^\times$. It follows that in particular we have

$$\gamma_1\gamma_2^{\frac{1}{a}} = \gamma_2\gamma_1. \tag{2}$$

Let $\mathbb{E}_F$ be the field of norms of the tower $(F_i)_{i\geq 0}$, and let $k_{\mathcal{F}}$ be its residue field. Let $\bar{\pi}_F$ be a uniformizer of $\mathbb{E}_F$, so $\mathbb{E}_F \cong k_{\mathcal{F}}((\bar{\pi}_F))$. Let $\mathbb{E}_K$ be the field of norms of the tower (K_i). Let $\epsilon = (1, \xi_1, \xi_2, \dots)$ and $\mathfrak{T} = (X_i)_{i\geq 1} \in \mathbb{E}_K$. Define $\bar{\pi} = \epsilon - 1$. Let $k = k_b k_{\mathcal{F}}$.

Lemma 2.2. *The field $\mathbb{E}_K$ is given by*

$$\begin{aligned}\mathbb{E}_K &\cong k_{\mathcal{F}}((\bar{\pi}_F))\hat{\otimes}_{k_F} k_b((\mathfrak{T})) \\ &\cong k((\mathfrak{T}))((\bar{\pi}_F)).\end{aligned}$$

Proof. See the section on Kummer towers in [7]. □

2.2 Lift to characteristic 0

Let $\mathbb{A}_F$ be a lift of $\mathbb{E}_F$ to characteristic 0, so $\mathbb{A}_F \cong W(k_{\mathcal{F}})[[\pi_F]][\pi_F^{-1}]^\vee$, where π_F is a lift of $\bar{\pi}_F$. Let ϕ be a lift to $\mathbb{A}_F$ of the Frobenius operator commuting with the action of Γ_1. Let $T = [\mathfrak{T}]$. Define

$$\mathbb{A}_K = W(k)[[T]][T^{-1}]^\vee[[\pi_F]][\pi_F^{-1}]^\vee.$$

Then $\mathbb{A}_K$ is a lift of $\mathbb{E}_K$ to characteristic 0. Let $\mathbb{B}_K = \mathbb{A}_K[\frac{1}{p}]$ be its field of fractions. Note that $\mathbb{A}_F \subset \mathbb{A}_K$. Define a lift of Frobenius to $\mathbb{A}_K$ by $\phi(T) = T^p$. Note that ϕ commutes with the action of G on $\mathbb{A}_K$. Define $N \in \mathbb{Z}_p$ by

$$\begin{aligned}\gamma_2(\pi_F) &= \pi_F, \\ \gamma_2(T) &= (\pi + 1)^N T.\end{aligned}$$

One can show that $N \in \mathbb{Z}_p^\times$ since X is a p-basis of k_K.

Note that $\mathbb{A}_K$ is a free finitely generated module over $\phi(\mathbb{A}_K)$ of degree p^2. It follows that we can define a left inverse ψ of ϕ by the formula

$$\phi(\psi(x)) = \frac{1}{p^2}\mathrm{Tr}_{\mathbb{A}_K/\phi(\mathbb{A}_K)}(x).$$

Proposition 2.3. *Let $\mathbb{E}$ be an algebraic closure of $\mathbb{E}_K$. Then we have an isomorphism of Galois groups*

$$\mathrm{Gal}(\mathbb{E}/\mathbb{E}_K) \cong \mathrm{Gal}(\bar{K}/K_\infty).$$

Proof. See [7]. ☐

Let $\mathbb{A}$ be a lift of $\mathbb{E}$ containing $\mathbb{A}_K$. Then the actions of ϕ and ψ can be extended uniquely to $\mathbb{A}$ (c.f. [7]).

2.3 The ring $\mathbb{A}_{\mathrm{cris}}$

Let $\mathbb{C}_K$ be the p-adic completion of $\bar{K}$, and let $O_{\mathbb{C}_K}$ be its ring of integers. Let $\tilde{\mathbb{E}}$ be the set of sequence $x = (x^{(0)}, x^{(1)}, \dots)$ of elements in $O_{\mathbb{C}_K}$ satisfying $(x^{(i+1)})^p = x^{(i)}$. Then $\tilde{\mathbb{E}}$ has a natural structure as a ring of charactristic p. For $n \geq 1$, let $\epsilon_n = (\zeta^{(n)}, \zeta^{(n+1)}, \dots)$ be the $p^n$the root of ϵ in $\tilde{\mathbb{E}}$. Let $\mathbb{A}_{\mathrm{inf}} = W(\tilde{\mathbb{E}})$ be the ring of Witt vectors of $\tilde{\mathbb{E}}$, ϕ the Froenius of $\mathbb{A}_{\mathrm{inf}}$, and if $x \in \tilde{\mathbb{E}}$, then denote by $[x]$ its Teichmüller representative in $\mathbb{A}_{\mathrm{inf}}$. Then the homomorphism

$$\begin{aligned} \theta :& \mathbb{A}_{\mathrm{inf}} \to O_{\mathbb{C}_K} \\ & \sum p^n [x_n] \to \sum p^n x_n^{(0)} \end{aligned}$$

is surjective and its kernel is a principal ideal with generator $\omega = \frac{[\epsilon]-1}{[\epsilon_1]-1}$. Let $\mathbb{B}_{\mathrm{inf}} = \mathbb{A}_{\mathrm{inf}}(p^{-1})$. Note that θ extends to a homomorphism $\mathbb{B}_{\mathrm{inf}} \to \mathbb{C}_K$. Define $\mathbb{B}_{\mathrm{dR}}^{\nabla +} = \varprojlim \mathbb{B}_{\mathrm{inf}}^{+}/(\ker \theta)^n$, and extend θ by continutiy to a homomorphism $\mathbb{B}_{\mathrm{dR}}^{\nabla +} \to \mathbb{C}_p$. This makes $\mathbb{B}_{\mathrm{dR}}^{\nabla +}$ into a discrete valuation ring with maximal ideal $\ker(\theta)$ and residue field $\mathbb{C}_K$. The action of $\mathcal{G}_K$ on $\mathbb{B}_{\mathrm{inf}}^{+}$ extends by continuity to a continuous action on $\mathbb{B}^{\nabla}+_{\mathrm{dR}}$.

Let $\mathbb{A}_{\mathrm{cris}}$ be the subring of $(\mathbb{B}_{\mathrm{dR}}^{\nabla})^+$ consisting of the elements of the form $\sum_{n=0}^{\infty} a_n \frac{\omega^n}{p!}$, where a_n is a sequence of elements in $\mathbb{A}_{\mathrm{inf}}$ tending to 0 as $n \to +\infty$.

2.4 Differentials, residues and duality

Let $\Omega^1_{\mathbb{A}_K}$ be the module of continuous $\mathbb{Z}_p$-linear 1-differentials of $\mathbb{A}_K$. Note that we have an isomorphism of $\mathbb{A}_K$-modules

$$\Omega^1_{\mathbb{A}_K} \cong \mathbb{A}_K d\pi_F \oplus \mathbb{A}_K dT.$$

Let $\Omega^1_{\mathbb{A}_K}$ be the module of continuous $\mathbb{Z}_p$-linear 1-differentials of $\mathbb{A}_K$, and let

$$\Omega^2_{\mathbb{A}_K} = \bigwedge^2 \Omega^1_{\mathbb{A}_K}.$$

Lemma 2.4. *We have an isomorphism of $\mathbb{A}_K$-modules*

$$\Omega^2_{\mathbb{A}_K} \cong \mathbb{A}_K d\pi_F \wedge dT.$$

Corollary 2.5. *If K' is a finite separable extension of K, then the natural map*

$$\mathbb{A}_{K'} \otimes_{\mathbb{A}_K} \Omega^2_{\mathbb{A}_K} \to \Omega^2_{\mathbb{A}_{K'}}$$

is an isomorphism.

Proof. Clear. ☐

Corollary 2.6. *If $K' = K \otimes_F F'$ for some finite unramified extension F' of F, then the natural map*

$$W(k_{F'}) \otimes_{W(k_F)} \Omega^2_{\mathbb{A}_K} \to \Omega^2_{\mathbb{A}_{K'}}$$

is a $W(k_F)$-linear isomorphism.

It follows from Lemma 2.4 that if $\omega \in \Omega^2_{\mathbb{A}_K}$, then there exist $a_{i,j} \in \mathbb{Z}_p$ such that $\omega = (\sum a_{i,j} T^i \pi_F^j) d\pi_F \wedge dT$.

Definition. Define the residue map

$$\begin{aligned} \mathrm{Res} : \Omega^2_{\mathbb{A}_K} &\to \mathbb{Z}_p, \\ \mathrm{Res}(\omega) &= a_{-1,-1}. \end{aligned}$$

Since ϕ is a lift of the Frobenius operator, we have $\phi(\pi_F) = u\pi_F^p$ for some $u \in \mathbb{A}_K^\times$ satisfying $u \cong 1 \mod p$.

Lemma 2.7. *For any $\lambda \in \mathbb{A}_K$, we have*

$$\mathrm{Res}(\phi(\lambda) d\phi(\pi_F) \wedge d\phi(T)) = p^2 \phi(\mathrm{Res}(\lambda d\pi_F \wedge dT)).$$

Proof. It is sufficient to prove the formula for $\lambda = \frac{1}{\pi_F T}$. Write $u = 1 + pa$ for some $a \in \mathbb{A}_K$. We have $\phi(T) = T^p$, so

$$\phi(\lambda) d\phi(\pi_F) \wedge d\phi(T) = \frac{p^2}{\pi_F T} d\pi_F \wedge dT + \frac{p}{uT} du \wedge dT.$$

But $\frac{1}{uT} du \wedge dT = T^{-1} \sum d(\frac{(-1)^{r+1} p^r}{r} a^r) \wedge dT$. It is easy to see that the coefficient of π_F^{-1} in $\sum d(\frac{(-1)^{r+1} p^r}{r} a^r)$ is 0, which finishes the proof. ☐

Definition. Let M be a torsion (ϕ, G)-module. Define

$$\tilde{M} = \mathrm{Hom}_{\mathbb{A}_K}(M, \mathbb{B}_K/\mathbb{A}_K \otimes_{\mathbb{A}_K} \Omega^2_{\mathbb{A}_K}).$$

Lemma 2.8. *The residue map induces an isomorphism*

$$\mathrm{TR} : \tilde{M} \to M^\vee.$$

Proof. We imitate the proof of Lemma 1.3 in [6]. By continuity, the residue gives a homomorphism from $\tilde{M}$ to $M^\vee$. Now $\Omega^2_{\mathbb{A}_K}$ is a free $\mathbb{A}_K$-module of rank 1. The ring $\mathbb{A}_K$ is principal and the $\mathbb{A}_K$-module $\mathbb{B}_K/\mathbb{A}_K \otimes_{\mathbb{A}_K} \Omega^2_{\mathbb{A}_K}$ is divisible and hence injective, and the functor which associates $\tilde{M}$ to M is exact. Also, the functor which to M associates $M^\vee$ is exact, so by the snake lemma we can assume without loss of generality that M is an $\mathbb{A}_K$-module of length 1 and hence a 1-dimensional vector space over $\mathbb{E}_K$. By choosing a basis, we can therefore assume that $M = \mathbb{E}_K$. Note that we can identify $\mathbb{E}_K^\vee$ with $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{E}_K, \mathbb{F}_p)$ and $\widetilde{\mathbb{E}_K}$ with $\mathrm{Hom}_{\mathbb{E}_K}(\mathbb{E}_K, \Omega^2_{\mathbb{A}_K}/p\Omega^2_{\mathbb{A}_K})$. We need to show that the natural map

$$\widetilde{\mathbb{E}_K} = \mathrm{Hom}_{\mathbb{E}_K}(\mathbb{E}_K, \Omega^2_{\mathbb{A}_K}/p\Omega^2_{\mathbb{A}_K}) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{E}_K, \mathbb{F}_p)$$

induced by the residue is bijective. Now if $f \in \widetilde{\mathbb{E}_K}$ is non-zero, then since the image of f is an $\mathbb{E}_K$-vector space of dimension 1 and the residue map is surjective, it is clear that $\mathrm{TR}(f)$ is non-zero. Now let $a \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{E}_K, \mathbb{F}_p)$. For all $m, n \in \mathbb{Z}$, let $\alpha_{m,n} = a(T^m \bar{\pi}_F^n)$. Since a is continuous, there exists $N \in \mathbb{N}$ such that $\alpha_{m,n} = 0$ for all $n \geq N$ and all $m \in \mathbb{Z}$. Define

$$\omega = (\sum_{n \geq -N} \sum_{m \in \mathbb{Z}} \alpha_{-m,-n} T^{m-1} \pi_F^{n-1}) dT \wedge d\pi.$$

Then for all $m, n \in \mathbb{Z}$, the class $\mod p$ of $\mathrm{Res}(T^m \pi^n \omega)$ is $\alpha_{m,n}$. It follows that if we define $f \in \widetilde{\mathbb{E}_K}$ by $f(1) = \omega \mod p$, then $\mathrm{TR}(f) = a$. □

2.5 The equivalence of categories

Denote by $\mathcal{G}_K$ the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$.

Definition. A $\mathbb{Z}_p$-representation of $\mathcal{G}_K$ is a $\mathbb{Z}_p$-module V of finite type equipped with a continuous linear action of $\mathcal{G}_K$. If V is annihilated by a power of p, then it is called a p-torsion module.

Theorem 2.9. *The functor $V \to D(V) = (V \otimes_{\mathbb{Z}_p} \mathbb{A})^{H_K}$ gives an equivalence of categories*

$$(\mathbb{Z}_p\textit{-representations of } \mathcal{G}_K) \to (\textit{étale } (\phi, G)\textit{-modules over } \mathbb{A}_K),$$

and an essential inverse is given by $D \to (\mathbb{A} \otimes_{\mathbb{A}_K} D)^{\phi=1}$.

Proof. See [1] or [7]. □

2.6 The module $\mathbb{Z}_p(2)$

For an element $a \in \mathbb{E}_K$, denote by $[a]$ its Teichmüller representative. Recall that if K contains a primitive pth root of unity, then $[\epsilon] = \pi + 1$ is an element of $\mathbb{A}_K$. To simplify notation, let $\Omega(K) = \Omega^2_{\mathbb{A}_K}$.

Lemma 2.10. *Let $K' = K(\mu_p)$, and define an $\mathbb{A}$-linear map*

$$\rho_{K'} : \mathbb{A} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) \to \mathbb{A} \otimes_{\mathbb{A}_{K'}} \Omega(K'),$$
$$\lambda \otimes \epsilon^2 \to \lambda \otimes d\log[\epsilon] \wedge d\log T.$$

Then $\rho_{K'}$ is an isomorphism of $\mathbb{A}$-modules which does not depend on the choice of the generator ϵ of $\mathbb{Z}_p(1)$.

Proof. Imitate the proof of Lemma 3.6 in [6]. □

Note that we can give $\Omega(K)$ the structure of a (ϕ, G)-module by defining

$$\phi_{\Omega(K)}(xdy \wedge dz) = \frac{1}{p^2}\phi(x)d\phi(y) \wedge \phi(z),$$
$$g(xdy \wedge dz) = \chi(g)g(x)dg(y) \wedge dg(z)$$

for $g \in G$.

Proposition 2.11. *With this stucture as a (ϕ, G)-module, $\Omega(K)$ is isomorphic to $D(\mathbb{Z}_p(2))$.*

Proof. Let $K' = K(\mu_p)$. By Corollary 2.5 the natural map $\Omega^2_{\mathbb{A}_K} \to \Omega^2_{\mathbb{A}_{K'}}$ is injective. Composing it with the natural $\mathrm{Gal}(\bar{K}/K)$-equivariant injection $\Omega^2_{\mathbb{A}_{K'}} \to \mathbb{A} \otimes_{\mathbb{A}_{K'}} \Omega^2_{\mathbb{A}_{K'}}$ gives an $\mathbb{A}_K$-linear $\mathcal{G}_K$-equivariant map $\Omega^2_{\mathbb{A}_K} \to (\mathbb{A}_K \otimes_{\mathbb{A}_{K'}} \Omega^2_{\mathbb{A}_{K'}})^{H_K}$. Explicit calculation shows that this map is also surjective. Composing it with the restriction of $\rho^{-1}_{K'}$ to the points fixed by H_K gives an isomorphism of $\mathbb{A}_K$-modules between $\Omega^2_{\mathbb{A}_K}$ and $D_K(\mathbb{Z}_p(2))$ which commutes with the action of $\mathcal{G}_K$. It is easy to see that it also commutes with the action of ϕ, which finishes the proof. □

2.7 Calculation of the Galois cohomology

Let V be a $\mathbb{Z}_p$-representation of $\mathcal{G}_K$, and denote by D the corresponding (ϕ, G)-module $D_K(V)$. Define the following complex:-

$$\mathcal{C}_{\phi,\gamma_1,\gamma_2}(D) : 0 \longrightarrow D \xrightarrow{f_1} D^{\oplus 3} \xrightarrow{f_2} D^{\oplus 3} \xrightarrow{f_3} D \longrightarrow 0, \qquad (3)$$

where the maps f_i are defined as follows:-

$$f_1 : x \to [(\phi - 1)x, (\gamma_1 - 1)x, (\gamma_2 - 1)x],$$
$$f_2 : (x, y, z) \to [(\phi - 1)y - (\gamma_1 - 1)x,$$
$$(\phi - 1)z - (\gamma_2 - 1)x,$$
$$(\gamma_2 - 1)y - (\gamma_1 \frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1} - 1)z],$$
$$f_3 : (x, y, z) \to (\gamma_2 - 1)x - (\gamma_1 \frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1} - 1)y - (\phi - 1)z.$$

Similarly, define the complex

$$\mathcal{C}_{\psi,\gamma_1,\gamma_2}(D) : 0 \longrightarrow D \xrightarrow{g_1} D^{\oplus 3} \xrightarrow{g_2} D^{\oplus 3} \xrightarrow{g_3} D \longrightarrow 0, \tag{4}$$

where the maps g_i are defined as follows:-

$$\begin{aligned} g_1 : x \to & [(\psi - 1)x, (\gamma_1 - 1)x, (\gamma_2 - 1)x], \\ g_2 : (x, y, z) \to & [(\psi - 1)y - (\gamma_1 - 1)x, \\ & (\psi - 1)z - (\gamma_2 - 1)x, \\ & (\gamma_2 - 1)y - (\gamma_1 \frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1} - 1)z], \\ g_3 : (x, y, z) \to & (\gamma_2 - 1)x - (\gamma_1 \frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1} - 1)y - (\psi - 1)z. \end{aligned}$$

Definition. Denote by $H^i_{\phi,\gamma_1,\gamma_2}(D)$ (resp. $H^i_{\psi,\gamma_1,\gamma_2}(D)$) the cohomology groups of the complex $\mathcal{C}_{\phi,\gamma_1,\gamma_2}(D)$ (resp. $\mathcal{C}_{\psi,\gamma_1,\gamma_2}(D)$).

Proposition 2.12. *Let $V = \mathbb{Z}_p(1)$ or μ_{p^n}, and let D be its (ϕ, G)-module. Then for all $0 \leq i \leq 3$, we have isomorphisms*

$$H^i(\mathcal{G}_K, V) \cong H^i_{\phi,\gamma_1,\gamma_2}(D) \cong H^i_{\psi,\gamma_1,\gamma_2}(D).$$

Remark. This result can certainly also be shown to be true for a general $\mathbb{Z}_p$-representation V of $\mathcal{G}_K$. However, in the general case the argument is technically much more complicated, and since our main interest is the construction of the Hilbert pairing, we restrict ourselves to the case above. We will prove the general case in [9].

Proof. Scholl [7] and Andreatta [1] have shown that we have isomorphisms $H^i(\mathcal{G}_K, V) \cong H^i_{\phi,\gamma_1,\gamma_2}(D)$. It is therefore sufficient to explicitely give isomorphisms $\iota_i : H^i_{\psi,\gamma_1,\gamma_2}(D) \cong H^i_{\phi,\gamma_1,\gamma_2}(D)$.

$i = 3$:- Let $x \in D$. Since ψ is surjective, we can choose $u \in D$ such that $\psi(u) = x$. Define $\iota_3(x) = u$. This is well-defined:- If u' also satisfies $\psi(u') = x$, then $a = u - u' \in D^{\psi=0}$. Write $u = \sum_{i \in \mathbb{Z}} f_i(\pi_F)T^i$. Then in particular $\psi(f_0(\pi)) = 0$. In Proposition I.5.1 in [4], it is shown that $\gamma_1 - 1$ is invertible on $D_F(V)^{\psi=0}$. Let $h_0 = (\gamma_1 - 1)^{-1}(f_0)$. For $j \neq 0$, let $\alpha_j = \frac{(\gamma_2 - 1)T^j}{T^j} \in \mathbb{A}_F^*$. Let $v = \sum_{j \neq 0} \alpha_j f_j(\pi)T^j$. Then $u = (\gamma_1 - 1)h_0 + (\gamma_2 - 1)v$ and hence is zero in $H^3_{\phi_D,\gamma_1,\gamma_2}(D)$.

$i = 2$:- Let $[x, y, z] \in D^{\oplus 3}$ satisfy $(\gamma_2 - 1)x - (\gamma_1 - 1)y - (\psi - 1)z = 0$. Denote $\iota_2([x, y, z])$ by $[u, v, w]$. Write $(\phi - 1)w = \sum_{i \in \mathbb{Z}} a_i f_i(\pi)T^i$. Note that $f_0(\pi_F) \in D_f(\mathbb{Z}_p(1))^{\psi=0}$, so by Proposition I.5.1 in [4], there exists $h_0 \in D_F(\mathbb{Z}_p(1))$ such that $(\gamma_1 - 1)h_0 = f_0$. Also, when $i \neq 0$, then $\alpha_i = \frac{(\gamma_2 - 1)T^i}{T^i}$ is invertible in $\mathbb{A}_F$. Define $u = x + h_0(\pi)$, $v = y + \sum_{j \neq 0} f_j(\pi)\alpha_j^{-1}T^j$ and $w = z$.

$i = 1$:- Let $y, z \in D$ satisfy

$$(\gamma_2 - 1)y = (\gamma_1 \frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1} - 1)z \tag{5}$$

and $(\psi - 1)y = (\psi - 1)z = 0$. We need to show that there exists $x \in D$ such that $(\phi - 1)y = (\gamma_1 - 1)x$ and $(\phi - 1)z = (\gamma_2 - 1)x$. How do we construct this x? Write $(\phi - 1)z = \sum_{i \in \mathbb{Z}} f_i(\pi_F)T^i$. By Proposition I.5.1 in [4], $(\gamma_1 - 1)^{-1} f_0$ is well-defined. When $i \neq 0$, then $\alpha_i = \frac{(\gamma_2 - 1)T^i}{T^i} \in \mathbb{A}_F^\times$. Define

$$x = (\gamma_1 - 1)^{-1} f_0(\pi_F) + \sum_{i \in \mathbb{Z}} \alpha_i^{-1} f_i(\pi_F)T^i.$$

Using (5) it is not difficult to see that x has the required properties.

$i = 0$:- $\iota_0 = \mathrm{id}$. When $x \in D$ satisfies $\psi x = x$, $\gamma_1 \frac{\gamma_2^{\frac{1}{n}} - 1}{\gamma_2 - 1} x = x$ and $\gamma_2 x = x$, then it is easy to see (using again the result from [4] mentioned above) that $\phi x = x$.

It is not difficult to see that the above maps are indeed isomorphisms. □

Remark. If γ_1' and γ_2' is a different pair of topological generators of Γ_1 and Γ_2, then the complexes $\mathcal{C}_{\phi,\gamma_1,\gamma_2}$ and $\mathcal{C}_{\phi,\gamma_1',\gamma_2'}$ are quasi-isomorphic. More generally, we can replace γ_1 and γ_2 by $\omega_1\gamma_1$ and $\omega_2\gamma_2$ for any $\omega_1, \omega_2 \in \Lambda(G)^\times$.

2.8 Construction of the pairing

Definition. The Pontryagin dual $M^\vee$ of an étale (ϕ, G)-torsion module M is defined as the continuous homomorphisms

$$M^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p).$$

As shown in Lemma 2.8, the map TR induces an isomorphism $\tilde{M} \to M^\vee$. We can therefore give $M^\vee$ the structure of a (ϕ, G)-module. Denote the operation of Frobenius on it by $\phi_{M^\vee}$. We quote the following result from [6]:-

Proposition 2.13. *Let $\mathcal{C} = (M^i, d^i : M^i \to M^{i+1})$ be a cochain comlex of abelian groups which are compact and locally separated (with M^i in degree i). Suppose that the d^i are strict homomorphisms with closed images. Then $\mathcal{C}^\vee = (N_i := (M^i)^\vee, d_i :=^t d^{i-1} : N_i \to N_{i-1})$ is a chain complex of abelian groups which are compact and locally separated (with N_i in degree i). The d_i are strict homomorphisms with closed image, and for all i, we have natural isomorphisms*

$$\alpha_i : H_i(\mathcal{C}^\vee) \cong (H^i(\mathcal{C}))^\vee.$$

In order to be able to apply Proposition 2.13 to the complex $\mathcal{C}_{\phi,\gamma_1,\gamma_2}(M)$, we need the following result:-

Lemma 2.14. *If M is an étale (ϕ, G)-torsion module over $\mathbb{A}_K$, then the image of $\phi - 1$ contains a compact neighbourhood of 0 on which $\phi - 1$ induces a bijection and hence a homeomorphism by compactness.*

Proof. Reduce to the 1-dimensional case, using Proposition 2.4 in [5]. □

For the rest of this section, let $M = D(\mu_{p^n})$. Note that as a ϕ-module, we have $M \cong \mathbb{A}_K \mod p^n$. In particular, this implies that ψ_M is defined.

Proposition 2.15. *The map $\psi : M^\vee \to M^\vee : f \to f \circ \phi_M$ agrees with $\psi_{M^\vee}$.*

Proof. Imitate the argument in Section 5.5.1 in [6]. □

Let $0 \le i \le 3$, and define an isomorphism

$$\begin{aligned} v_i : (M^\vee)^{\oplus\binom{3}{i}} &\to (M^{\oplus\binom{3}{i}})^\vee, \\ (g_j)_{1\le j\le\binom{3}{i}} &\to \oplus_{1\le j\le\binom{3}{i}} g_j. \end{aligned}$$

Lemma 2.16. *For all $0 \le i \le 3$, the v_i induce isomorphisms (which we will also denote by v_i)*

$$H^i(\mathcal{C}_{\psi_{M^\vee},\gamma_1^{-1},\gamma_2^{-1}}(M^\vee)) \to H_{3-i}([\mathcal{C}_{\phi_M,\gamma_1,\gamma_2}(M)]^\vee).$$

Proof. We only have to show that the actions of $\psi_{M^\vee}$ and ϕ_M are compatible with the maps v_i. But this is shown in Proposition 2.15. □

Combining the isomorphisms of Proposition 2.12 and Lemmas 2.13 and 2.16, we therefore get for all $0 \le i \le 3$ an isomorphism

$$\begin{aligned} u_i(M) : H^i(\mathcal{C}_{\phi_{M^\vee},\gamma_1,\gamma_2}(M^\vee)) &\to H^i(\mathcal{C}_{\psi_{M^\vee},\gamma_1^{-1},\gamma_2^{-1}}(M^\vee)) \\ &\to^{v_i} H_{3-i}([\mathcal{C}_{\phi_M,\gamma_1,\gamma_2}(M)]^\vee) \\ &\to^{\alpha_i} [H^{3-i}(\mathcal{C}_{\phi_M,\gamma_1,\gamma_2}(M))]^\vee \end{aligned}$$

Using these isomorphisms, we get the following proposition:-

Proposition 2.17. *For all $0 \le i \le 3$, we have a perfect pairing*

$$\begin{aligned} H^i(\mathcal{C}_{\phi_M,\gamma_1,\gamma_2}(M)) \times H^{3-i}(\mathcal{C}_{\phi_{M^\vee},\gamma_1,\gamma_2}(M^\vee)) &\to \mathbb{Q}_p/\mathbb{Z}_p, \\ (x,y) &\to [u_{3-i}(M)(y)](x). \end{aligned}$$

Making the above isomorphisms explicit (which is very messy, so we omit the details), one can show that

$$H^1_{\phi_M,\gamma_1,\gamma_2}(M) \times H^2_{\phi_{M^\vee},\gamma_1,\gamma_2}(M^\vee) \to \mathbb{Q}_p/\mathbb{Z}_p, \tag{6}$$

$$(x,y,z) \times (f,g,h) \to h(\gamma_2\gamma_1 x) - g(\gamma_2\phi_M(y)) \tag{7}$$

$$+ f(\gamma_1 \frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1}\phi_M(z)) + g(\tilde{\omega}\phi_M(z)), \tag{8}$$

$$H^2_{\phi_M,\gamma_1,\gamma_2}(M) \times H^1_{\phi_{M^\vee},\gamma_1,\gamma_2}(M^\vee) \to \mathbb{Q}_p/\mathbb{Z}_p, \tag{9}$$

$$(x,y,z) \times (f,g,h) \to -h(\gamma_2 x) - g(\gamma_1 \frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1} y) - h(\tilde{\omega} y) \tag{10}$$

$$+ f(\phi_M(z)), \tag{11}$$

where $\tilde{\omega} \in \Lambda(G)$ is the element satisfying

$$\tilde{\omega}(\gamma_2^{-1} - 1) = n\frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1} - 1.$$

3 The Tate pairing

3.1 Proof of the Tate isomosphism

Definition. Define the map

$$\mathrm{TR}_K : \Omega^2_{\mathbb{A}_K} \to \mathbb{Z}_p$$

to be the composition $\mathrm{Tr}_{W(k)/\mathbb{Q}_p} \circ \mathrm{Res}_K$. Note that TR_K is $\mathbb{Z}_p$-linear.

Proposition 3.1. *For all $\omega \in \Omega(K)$, we have*

$$\mathrm{TR}_K((\gamma_1 \frac{\gamma_2^{\frac{1}{n}} - 1}{\gamma_2 - 1} - 1)\omega) = 0, \tag{12}$$

$$\mathrm{TR}_K((\gamma_2 - 1)\omega) = 0, \tag{13}$$

$$\mathrm{TR}_K((\phi_{\Omega(K)} - 1)\omega) = 0. \tag{14}$$

Proof. Since $\gamma_2(T) = (1+\pi)^N T$ from some $N \in \mathbb{Z}_p^\times$, it is clear that (13) holds. Write $\omega = \sum_{i \in \mathbb{Z}} a_i T^i dT \wedge d\pi_F$, where $a_i \in \mathbb{A}_F$ for all $i \in \mathbb{Z}$. As shown in [6], we have $\mathrm{TR}_K((\phi - 1)a_0 d\pi_F) = 0$, which (by the compatibilities of the actions of ϕ) implies (14). It remains to show (12). Let $x = \gamma_2 - 1$. Expanding $\frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1}$ in terms of x gives

$$\frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1} = \frac{1}{a} + \text{higher order terms.}$$

Since γ_2 is trivial on $\mathbb{A}_F$, the operator $\gamma_1 \frac{\gamma_2^{\frac{1}{a}} - 1}{\gamma_2 - 1} - 1$ acts as $\gamma_1 - 1$ on a_0. Equation (12) therefore follows from the 1-dimensional case as treated in [6]. $\square$

For $j \geq 1$, let $\Omega_j(K) = \Omega(K) \mod p^j$, which is an étale (ϕ, G)-torsion module over $\mathbb{A}_K$ isomorphic to $D_K(\mu_{p^j})$. By reduction $\mod p^j$, Res induces a canonical $\mathbb{Z}/p^j\mathbb{Z}$-linear map

$$\mathrm{TR}_{K,j} : \Omega_j(K) \to \mathbb{Z}/p^j\mathbb{Z}.$$

By Proposition 3.1, $\mathrm{TR}_{K,j}$ factorizes through the quotient of $\Omega_j(K)$ by $(\tau_1(\Omega_j(K) + \tau_2(\Omega_j(K) + (\phi - 1)(\Omega_i(K))$, where $\tau_i = \gamma_i - 1$. We therefore get a homomorphism

$$\mathrm{TR}_{K,j} : H^3_{\phi,\gamma_1,\gamma_2}(\Omega_i(K)) \to \mathbb{Z}/p^j\mathbb{Z}.$$

Passing to the direct limit gives a map

$$H^3_{\phi,\gamma_1,\gamma_2}(\mathbb{B}_K/\mathbb{A}_K \otimes_{\mathbb{A}_K} \Omega(K)) \to \mathbb{Q}_p/\mathbb{Z}_p.$$

Let v_p be the p-adic valuation of $\mathbb{Z}_p$ normalized by $v_p(p) = 1$. Let $n_p(\gamma_{1,n}) = v_p(\log\chi(\gamma_1))$ and $n_p(\gamma_{2,n}) = v_p(\eta(\gamma_{2,n}))$. Let $c = \frac{p^{n_p(\gamma_{1,n})}}{\log\chi(\gamma_{1,n})}\frac{p^{n_p(\gamma_{2,n})}}{\eta(\gamma_{2,n})}$.

Proposition 3.2. *The map $-c\,\mathrm{TR}$ gives a canonical isomorphism between the groups $H^3_{\phi,\gamma_1,\gamma_2}(\mathbb{B}_K/\mathbb{A}_K \otimes_{\mathbb{A}_K} \Omega(K))$ and $\mathbb{Q}_p/\mathbb{Z}_p$.*

Remark. The factor c may seem bizarre, but we will see its use in Section 4.

Proof. It is sufficient to show that $\mathrm{TR}_{K,j}$ gives an isomorphism for all j. Since we can expand any element in $\Omega_j(K)$ as a power series in T with coefficients in $\mathbb{A}_F \mod p^j$, it is sufficient to show that for all $k \neq 0$, $\gamma_2 - 1$ is surjective on $T^k\mathbb{A}_F$. The proposition follows from Herr's proof of the Tate isomorphism in the 1-dimensional case (c.f. Theorem 5.2 in [6]). Expanding the power series shows that $(\gamma_2 - 1)(T^k) = T^k f_k(\pi)$, where $f_k(\pi) \in \mathbb{A}_F^\times$, which finishes the proof. □

Combining this result with Proposition 2.11 and the main result of Section 2.7 proves Theorem 1.2.

3.2 Relation to Pontryagin duality

Let V be a torsion $\mathbb{Z}_p$-representation of $\mathcal{G}_K$, and let $M = D(V)$. Recall that

$$\tilde{V} = \mathrm{Hom}_{\mathbb{Z}_p}(V, \mu_{p^\infty}).$$

Lemma 3.3. *The (ϕ, G)-module $D_K(\tilde{V})$ is isomorphic to $\mathrm{Hom}_{\mathbb{A}_K}(M, \mathbb{B}_K/\mathbb{A}_K \otimes_{\mathbb{A}_K} \Omega(K))$.*

Proof. It follows from the equivalence of categories (Theorem 2.9) that $D_K(\tilde{V})$ is isomorphic to $\mathrm{Hom}_{\mathbb{A}_K}(M, D(\mu_{p^\infty}^{\otimes n}))$. It is now sufficient to observe that

$$D_K(\mu_{p^\infty}^{\otimes n}) = \varinjlim D_K(\mu_{p^j}^{\otimes n}) \cong \varinjlim \frac{\Omega(K)}{p^j\Omega(K)} \cong \mathbb{B}_K/\mathbb{A}_K \otimes_{\mathbb{A}_K} \Omega(K).$$

□

Theorem 1.1 is therefore a consequence of Lemma 2.8 and Theorem 2.17.

4 The higher Hilbert pairing

4.1 Construction of the pairing

Let F be the maximal algebraic extension of $\mathbb{Q}_p$ contained in K. Throughout this section, we assume that the extension of F over $\mathbb{Q}_p$ is unramified. As in the previous sections, let $\epsilon = (1, \xi_p, \xi_{p^2}, \dots) \in \mathbb{E}_K$ and $\pi = [\epsilon] - 1$, where $[\epsilon]$ is the Teichmüller representative of ϵ. Also, let $\mathfrak{X} = (X, X^{\frac{1}{p}}, X^{\frac{1}{p^2}}, \dots) \in \mathbb{E}_K$, and let $T = [\mathfrak{X}] \in \mathbb{A}_K$ be its Teichmüller representative.

Fix $n \geq 1$. Let $\pi_n = \phi^{-n}(\pi) = [(\xi_{p^n}, \xi_{p^{n+1}} \dots)] - 1$ and $T_n = \phi^{-n}(T) = [(X^{\frac{1}{p^n}}, X^{\frac{1}{p^{n+1}}}, \dots)]$.

Proposition 4.1. *Let $n \geq 1$. Then for all $i \geq 0$, taking cup product with ξ_{p^n} gives an isomorphism of* $\mathrm{Gal}(K_n/K)$*-modules*

$$\cup\xi_{p^n} : H^i(G_{K_n}, \mathbb{Z}/p^n\mathbb{Z}) \to H^i(G_{K_n}, \mu_{p^n}).$$

Proof. Let $D = D(\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{A}_{K_n} \mod p^n$. Let $\Gamma_1^{(n)} = \mathrm{Gal}(K(\mu_{p^\infty}, X^{\frac{1}{p^n}})/K_n)$ and $\Gamma_2^{(n)} = \mathrm{Gal}(K_\infty/K(\mu_{p^\infty}, X^{\frac{1}{p^n}}))$. Let $\gamma_{1,n}$ and $\gamma_{2,n}$ be topological generators of $\Gamma_1^{(n)}$ and $\Gamma_2^{(n)}$, respectively. Recall that the G_{K_n}-cohomology of $\mathbb{Z}/p^n\mathbb{Z}$ is given by the complex

$$\mathcal{C}_{\phi,\gamma_{1,n},\gamma_{2,n}}(D) : 0 \longrightarrow D \xrightarrow{f_1} D^{\oplus 3} \xrightarrow{f_2} D^{\oplus 3} \xrightarrow{f_3} D \longrightarrow 0,$$

where

$$\begin{aligned} f_1 : x \to & [(\phi - 1)x, (\gamma_{1,n} - 1)x, (\gamma_{2,n} - 1)x], \\ f_2 : (x, y, z) \to & [(\phi - 1)y - (\gamma_{1,n} - 1)x, \\ & (\phi - 1)z - (\gamma_{2,n} - 1)x, \\ & (\gamma_{2,n} - 1)y - (\gamma_{1,n}\frac{\gamma_{2,n}^{\frac{1}{a}} - 1}{\gamma_{2,n} - 1} - 1)z], \\ f_3 : (x, y, z) \to & (\gamma_{2,n} - 1)x - (\gamma_{1,n}\frac{\gamma_{2,n}^{\frac{1}{a}} - 1}{\gamma_{2,n} - 1} - 1)y - (\phi - 1)z. \end{aligned}$$

Since $\mu_{p^n} =< \xi_{p^n} >$, it is easy to see from this description of the cohomology groups that cup product (which is the same as multiplication) with ξ_{p^n} gives a $\mathrm{Gal}(K_n/K)$-equivariant isomorphims $H^i(G_{K_n}, \mathbb{Z}/p^n\mathbb{Z}) \to H^i(G_{K_n}, \xi_{p^n})$ for all $0 \leq i \leq 3$. □

By Theorem 1.1, we have a perfect pairing

$$H^2(G_{K_n}, \mu_{p^n}^{\otimes 2}) \times H^1(G_{K_n}, \mathbb{Z}/p^n\mathbb{Z}) \to \mathbb{Q}_p/\mathbb{Z}_p.$$

Taking cup product with ξ_{p^n} and using Proposition 4.1 gives a perfect pairing

$$H^2(G_{K_n}, \mu_{p^n}^{\otimes 2}) \times H^1(G_{K_n}, \mu_{p^n}) \to \mu_{p^n}. \tag{15}$$

Definition. The Hilbert pairing

$$K_2(K_n) \times K_1(K_n) \to \mu_{p^n}.$$

is the composition of (15) with the Galois symbol map

$$\delta^2 \times \delta : K_2(K_n) \times K_1(K_n) \to H^2(G_{K_n}, \mu_{p^n}^{\otimes 2}) \times H^1(G_{K_n}, \mu_{p^n})$$

Note that we have a natural (surjective) multiplication map $K_1(K_n) \times K_1(K_n) \to K_2(K_n)$. We therefore interprete the Hilbert pairing as a pairing

$$V_n : K_1(K_n) \times K_1(K_n) \times K_1(K_n) \to \mu_{p^n}. \tag{16}$$

From the definition of the Galois symbol it is clear that V_n factors through

$$\mathfrak{V}_n : \left(K_1(K_n)/p^n\right)^{\times 3} \to \mu_{p^n}. \tag{17}$$

Since the pairing (15) is perfect, the pairing $\mathfrak{V}_n$ in (17) is non-degenerate.

Lemma 4.2. *We have a commutative diagram*

$$\begin{array}{ccc} K_1(K_n) \times K_1(K_n) & \xrightarrow{\delta \times \delta} & H^1(K_n, \mu_{p^n}) \times H^1(K_n, \mu_{p^n}) \\ \downarrow & & \downarrow \cup \\ K_2(K_n) & \xrightarrow{\delta^2} & H^2(K_n, \mu_{p^n}) \end{array}$$

4.2 The Kummer map

Let F by the maximal extension of $\mathbb{Q}_p$ contained in K, and let $R = O_F[[T_n]][T_n^{-1}]$. Note that we can identifiy $\mathbb{A}^+_{K_n}$ with the abstract power series ring $R[[Y]]$ (where $Y = \pi_n$, but we forget this for the time being). Let $\mathfrak{m} = (p, Y)$ be the maximal ideal of $\mathbb{A}^+_{K_n}$, and let $\mathcal{A} = 1 + \mathfrak{m}$. For $F(Y) \in \mathcal{A}$, define

$$l(F(Y)) = (1 - \frac{\phi}{p}) \log F(Y).$$

To shorten notation, let $f(Y) = l(F(Y))$. Define the differential operators $D_1 = (Y+1)\frac{d}{dY}$ and $D_2 = T_n \frac{d}{dT_n}$. Fix $n \geq 1$, and let $S_n = R[[\pi_n]]$.

Note. (1) The action of γ_2 on $\mathbb{A}^+_{K_n}$ is trivial $\mod \pi$.
(2) The element ϵ is a generator of $\mathbb{Z}_p(1)$, so $\epsilon \mod p^n$ is a generator of μ_{p^n} and can be identified with ξ_{p^n}.

Let $\epsilon^{(n)} = \epsilon \mod p^n$ and $\tau = \frac{1}{\pi} - \frac{1}{2}$.

Proposition 4.3. *Let $F(Y) \in \mathcal{A}$. Then there exist unique $a_{\gamma_{1,n}}(\pi_n), b_{\gamma_{2,n}}(\pi_n) \in S_n$ such that*

$$(\phi - 1)(a_{\gamma_{1,n}}(\pi_n) \otimes \epsilon^{(n)}) = (\gamma_{1,n} - 1)(f(\pi_n)\tau \otimes \epsilon^{(n)}),$$
$$(\phi - 1)(b_{\gamma_{2,n}}(\pi_n) \otimes \epsilon^{(n)}) = (\gamma_{2,n} - 1)(f(\pi_n)\tau \otimes \epsilon^{(n)}).$$

Moreover, we have

$$a_{\gamma_{1,n}}(\pi_n) = \frac{1 - \chi(\gamma_{1,n})}{p^n} D_1 \log F(\pi_n) \mod \pi,$$
$$b_{\gamma_{2,n}}(\pi_n) = \eta_n(\gamma_{2,n}) D_2 \log F(\pi_n) \mod \pi.$$

Proof. Arguing as in the proof of Lemma 2.1.3 in [2] we have

$$(\gamma_{1,n} - 1)(f(\pi_n)\tau \otimes \epsilon^{(n)}) = -\frac{1 - \chi(\gamma_{1,n})}{p^n} D_1 \log f(\pi_n) \otimes \epsilon^{(n)} \mod \pi.$$

Using the identity $D_1\phi = p\phi D_1$, we can write

$$(\gamma_{1,n}-1)(f(\pi_n)\tau \otimes \epsilon^{(n)}) = (\phi-1)(\frac{1-\chi(\gamma_{1,n})}{p^n} D_1 \log F(\pi_n) \otimes \epsilon^{(n)}) \mod \pi.$$

Let $\tilde{a}_{\gamma_{1,n}}(\pi_n) = \frac{1-\chi(\gamma_{1,n})}{p^n} D_1 \log F(\pi_n)$, so

$$(\phi-1)(\tilde{a}_{\gamma_{1,n}}(\pi_n) \otimes \epsilon^{(n)}) = (\gamma_{1,n}-1)(f(\pi_n)\tau \otimes \epsilon^{(n)}).$$

Since $\phi-1$ is invertible on πS_n, we deduce that there exists a unique $a_{\gamma_{1,n}}(\pi_n) \in S_n$ such that $a_{\gamma_{1,n}}(\pi_n) = \tilde{a}_{\gamma_{1,n}}(\pi_n) \mod \pi$ and $(\phi-1)(a_{\gamma_{1,n}}(\pi_n) \otimes \epsilon^{(n)}) = (\gamma_{1,n}-1)(f(\pi_n)\tau \otimes \epsilon^{(n)})$.

The existence of $b_{\gamma_{2,n}}(\pi_n)$ follows from similar arguments:- Let $\eta_n(\gamma_{2,n}) = \frac{\eta(\gamma_{2,n})}{p^n}$. Note that

$$\gamma_{2,n}(T_n) = (1+\pi)^{\eta_n(\gamma_{2,n})} T_n, \tag{18}$$

so

$$\gamma_{2,n}(T_n) = T_n + \eta_n(\gamma_{2,n}) T_n \pi \mod \pi^2,$$

and hence

$$\gamma_{2,n} f(\pi_n) = f(\pi_n) + \eta_n(\gamma_{2,n}) D_2 f(\pi_n)\pi \mod \pi^2,$$
$$(\gamma_{2,n}-1)(f(\pi_n)\tau \otimes \epsilon^{(n)}) = \eta_n(\gamma_{2,n}) D_2 f(\pi_n) \otimes \epsilon^{(n)} \mod \pi.$$

By assumption we have

$$f(\pi_n) = (1-\frac{\phi}{p}) \log F(\pi_n).$$

Since $D_2\phi = p\phi D_2$, it follows that

$$(\gamma_{2,n}-1) f(\pi_n)\tau \otimes \epsilon^{(n)} = (1-\phi)\eta_n(\gamma_{2,n}) D_2 \log F(\pi_n) \otimes \epsilon^{(n)} \mod \pi.$$

Let $\tilde{b}_{\gamma_{2,n}}(\pi_n) = \eta_n(\gamma_{2,n}) D_2 \log F(\pi_n) \otimes \epsilon^{(n)}$. It follows that $(1-\phi)\tilde{b}_{\gamma_{2,n}}(\pi_n) = (\gamma_{2,n}-1) f(\pi_n)\tau \otimes \epsilon^{(n)}$. Since $\phi-1$ is invertible on πS_n, there exists a unique $b_{\gamma_{2,n}}(\pi_n) \in S_n$ such that $b_{\gamma_{2,n}} = \tilde{b}_{\gamma_{2,n}} \mod \pi$ and

$$(\phi-1) b_{\gamma_{2,n}}(\pi_n) \otimes \epsilon^{(n)} = (\gamma_{2,n}-1)(f(\pi_n)\tau \otimes \epsilon^{(n)}).$$

□

Definition. Let $\iota_n : \mathcal{A} \to H^1_{\phi,\gamma_{1,n},\gamma_{2,n}}(C_n)$ be the homomorphism

$$F(X) \to [f(\pi_n)\tau \otimes \epsilon^{(n)}, a_{\gamma_{1,n}}(\pi_n) \otimes \epsilon^{(n)}, b_{\gamma_{2,n}} \otimes \epsilon^{(n)}]. \tag{19}$$

Lemma 4.4. *The map ι_n is well-defined.*

Proof. Explicit calculation shows that

$$(\gamma_{2,n}-1)(a_{\gamma_{1,n}}(\pi_n)\otimes\epsilon^{(n)}) = (\gamma_{1,n}\frac{\gamma_2^{\frac{1}{N}}-1}{\gamma_{2,n}-1}-1)(b_{\gamma_{2,n}}(\pi_n)\otimes\epsilon^{(n)}). \tag{20}$$

It follows that ι_n is really a map into $H^1_{\gamma_{1,n},\gamma_{2,n},\phi}$. □

Proposition 4.5. *Let $\delta_n : K_n^\times \to H^1(G_{K_n}, \mathbb{Z}_p(1))$ be the Kummer map. We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\iota_n} & H^1_{\phi,\gamma_{1,n},\gamma_{2,n}}(C_n) \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle \cong} \\ K_n^\times & \xrightarrow{\delta_n} & H^1(G_{K_n}, \mathbb{Z}_p(1)) \end{array}$$

To prove the proposition, we follow the strategy of Benois in the proof of Proposition 2.1.5 in [2]. We split the proof of the proposition into a sequence of lemmas.

Note that the action of $\mathcal{G}_K$ on $\mathbb{A}_K$ factors through $G_K = \mathrm{Gal}(K_\infty/K)$. Recall that $G_K \cong \Gamma_1 \rtimes \Gamma_2$, where Γ_1 is congruent (via the cyclotomic character χ_K) to an open subgroup of $\mathbb{Z}_p^\times$ and Γ_2 is congruent (via a character η_K) to $\mathbb{Z}_p$.

Lemma 4.6. *Let $[x, y, z] \in H^1_{\gamma_{1,n},\gamma_{2,n},\phi}(C_n)$, and let $u \in \mathbb{A}$ be a solution of $(\phi - 1)u = x$. Then $h^1([x, y, z])$ is given by the cocycle $\sigma \to c(\sigma)$ which is defined as follows:- Let $\tilde{\sigma}$ be the image of σ in G_K under the projection map. Let $k = \chi(\tilde{\sigma})$ and $l = \eta(\tilde{\sigma})$, so $\tilde{\sigma} = \gamma_{1,n}^k\gamma_{2,n}^l$. Then*

$$c(\sigma) = (\sigma-1)u - \frac{\gamma_{1,n}^k-1}{\gamma_{1,n}-1}y - \gamma_{1,n}^k\frac{\gamma_{2,n}^l-1}{\gamma_{2,n}-1}z.$$

Proof. Let $N_{x,y,z} = D(\mu_{p^n}) \oplus \mathbb{A}_{K_n}e$, where the action of $\phi, \gamma_{1,n}$ and $\gamma_{2,n}$ on e is given by $\phi(e) = e + x$, $\gamma_{1,n}(e) = e + y$, $\gamma_{2,n}(e) = e + z$. Then the long exact sequence associated to the short exact sequence of G_{K_n}-modules

$$0 \to D(\mu_{p^n}) \to N_{x,y,z} \to \mathbb{A}_{K_n} \to 0 \tag{21}$$

gives the connecting homomorhism $\delta : H^0(\mathbb{A}_{K_n}) \to H^1_{\gamma_{1,n},\gamma_{2,n},\phi}(C_n)$, and an easy diagram search shows that $\delta(1) = [x, y, z]$. Applying $(\phi - 1)$ to (21) gives a short exact sequence

$$0 \to \mu_{p^n} \to T_{x,y,z} \to \mathbb{Z}_p \to 0$$

and a connecting homomorphism $\delta_{\mathrm{Gal}} : \mathbb{Z}_p \to H^1(G_K, \mu_{p^n})$. We have $u + e \in N_{x,y,z} \otimes_{\mathbb{A}_{K_n}} \mathbb{A}$ and $(\phi - 1)(u + e) = 0$, so $u + e \in T_{x,y,z}$. So $\delta_{\mathrm{Gal}}(1)$ can be represented by the cocycle

$$\begin{aligned} \sigma \to & \sigma(u+e) - (u-e) \\ = & (\sigma-1)u + (\sigma-1)e. \end{aligned}$$

Now $\mathcal{G}_K$ acts on $\mathbb{A}_{K_n}$ via the quotient G_{K_n}. Since $e \in \mathbb{A}_{K_n}$, we have

$$\begin{aligned}(\tilde{\sigma} - 1)e &= (\gamma_{1,n}^k \gamma_{2,n}^l - 1)e \\ &= \gamma_{1,n}^k(\gamma_{2,n}^l - 1)e + (\gamma_{1,n}^k - 1)e \\ &= \gamma_{1,n}^k \frac{\gamma_{2,n}^l - 1}{\gamma_{2,n} - 1} z + \frac{\gamma_{1,n}^k - 1}{\gamma_{1,n} - 1} y\end{aligned}$$

The lemma now follows from the commutativity of the diagram

$$\begin{array}{ccc} H^0(\mathbb{A}_{K_n}) & \xrightarrow{\delta} & H^1_{\gamma_{1,n},\gamma_{2,n},\phi}(C_n) \\ \Big\downarrow{\scriptstyle h^0} & & \Big\downarrow{\scriptstyle h^1} \\ \mathbb{Z}_p & \xrightarrow{\delta_{\mathrm{Gal}}} & H^1(\mathcal{G}_{K_n}, \mu_{p^n}) \end{array}$$

□

In particular, $h^1(\iota_n(F))$ is given by

$$\begin{aligned}\sigma \to & (\sigma - 1)u - \frac{\gamma_{1,n}^{\chi(\sigma)} - 1}{\gamma_{1,n} - 1} a_{\gamma_{1,n}}(\pi_n) \otimes \epsilon^{(n)} \\ & - \gamma_{1,n}^{\chi(\sigma)} \frac{\gamma_{2,n}^{\eta(\sigma)} - 1}{\gamma_{2,n} - 1} b_{\gamma_{2,n}}(\pi_n) \otimes \epsilon^{(n)},\end{aligned}$$

where $(1 - \phi)u = f(\pi_n)\tau$. Since $\gamma_{1,n}(\pi_n) = \pi_n \mod \pi$ and $\gamma_{2,n}(T_n) = T_n \mod \pi$, we have

$$\frac{\gamma_{1,n}^{\chi(\sigma)} - 1}{\gamma_{1,n} - 1} a_{\gamma_{1,n}}(\pi_n) \otimes \epsilon^{(n)} \cong \chi(\sigma) \frac{1 - \chi(\sigma)}{p^n} D_1 \log F(\pi_n) \otimes \epsilon^{(n)} \mod \pi,$$

$$\gamma_{1,n}^{\chi(\sigma)} \frac{\gamma_{2,n}^{\eta(\sigma)} - 1}{\gamma_{2,n} - 1} b_{\gamma_{2,n}}(\pi_n) \otimes \epsilon^{(n)} \cong \eta(\sigma)\eta_n(\gamma_{2,n}) D_2 \log F(\pi_n) \mod \pi.$$

These congruences imply that

$$c(\sigma) \cong (\chi(\sigma)\sigma - 1)u + \frac{1 - \chi(\sigma)}{p^n} D_1 \log F(\pi_n) + \frac{\eta(\sigma)}{\chi(\sigma)} \eta_n(\gamma_{2,n}) D_2 \log F(\pi_n) \mod \pi.$$

We now interprete $c(\sigma)$ in terms of $\mathbb{A}_{\mathrm{cris}}$. Denote by I the ideal of $\mathbb{A}_{\mathrm{cris}}$ generated by π^2 and $\frac{\pi^{p-1}}{p}$.

Lemma 4.7. *There exists a unique* $x \in \mathrm{Fil}^1 \mathbb{A}_{\mathrm{cris}}$ *such that* $x = u(\pi - \frac{\pi^2}{2}) \mod I$ *and*

$$(1 - \frac{\phi}{p})x = f(\pi_n).$$

Proof. Imitate the proof of Lemma 2.1.6.2 in [2]. □

Define the element

$$\mu(\sigma) = (\sigma - 1)x - \log(\frac{\sigma(F(\pi_n))}{F(\pi_n)}),$$

which belongs to $1 + \mathrm{Fil}^1 \mathbb{A}_{\mathrm{cris}}$ for all $\sigma \in \mathcal{G}_{K_n}$, and it is easy to check that the map $\mu : \mathcal{G}_{K_n} \to \mathrm{Fil}^1 \mathbb{A}_{\mathrm{cris}}$ is a cocycle.

Lemma 4.8. *We have* $\mu(\sigma) = c(\sigma)$.

Proof. We have $(1-\frac{\phi}{p})\mu(\sigma) = 0$, so $\mu(\sigma)$ has the form $a(\sigma)t$ for some $a(\sigma) \in \mathbb{Q}_p$. On the other hand, from the congruences

$$\begin{aligned}\tilde{\sigma}F(\pi_n) =& F(\pi_n) + \chi(\sigma)\frac{1-\chi(\sigma)}{p^n}D_1 \log F(\pi_n)\pi \\ &+ \eta(\sigma)\eta_n(\gamma_{2,n})D_2 \log F(\pi_n)\pi \mod \pi^2\end{aligned}$$

and $(\sigma - 1)x = (\chi(\sigma)\sigma - 1)u\pi \mod I$ it follows that

$$\begin{aligned}\mu(\sigma) =& (\chi(\sigma)\sigma - 1)u\pi + \frac{1-\chi(\sigma)}{p^n}D_1 \log F(\pi_n)\pi \\ &+ \frac{\eta(\sigma)}{\chi(\sigma)}\eta_n(\gamma_{2,n})D_2 \log F(\pi_n)\pi \mod I \\ =& c(\sigma)T \mod I,\end{aligned}$$

which implies that $\mu(\sigma) = tc(\sigma)$. □

Corollary 4.9. *One has*

$$[\epsilon]^{c(\sigma)} = \exp(\mu(\sigma)) = \frac{\sigma \exp(x)}{\exp(x)} \frac{F(\pi_n)}{\sigma F(\pi_n)}.$$

Proof of Proposition 4.5. Let $y = \exp(x)$. Then the equation $(1 - \frac{\phi}{p})x = f(\pi_n)$ can be written of the form

$$\frac{y^p}{\phi(y)} = \exp(pf(\pi_n)).$$

Consider the short exact sequence

$$1 \to [\epsilon]^{\mathbb{Z}_p} \to 1 + W^1(R) \to^{\nu} 1 + pW(R) \to 1,$$

where $\nu(a) = \frac{a^p}{\phi(a)}$. It shows that the inclusion $W(R) \subset \mathbb{A}_{\mathrm{cris}}$ gives a $1-1$ correspondence between solutions Y of $\frac{Y^p}{\phi(Y)} = \exp(pf(\pi_n))$ and solutions $X = \log Y$ of $(1 - \frac{\phi}{p})X = f(\pi_n)$. Hence $y \in 1 + W^1(R)$, and it is easy to see by induction that

$$\frac{y^{p^n}}{\phi^n(y)} = \frac{F(\pi_n)^{p^n}}{\phi^n(F(\pi_n))}.$$

Let $z = \phi^{-n}(yF(\pi_n)^{-1})$. Applying the map $\theta : W(R) \to O_{\mathbb{C}_p}$ to both sides of this equation, we obtain that

$$\theta(z)^{p^n} = h_n(F)^{-1}.$$

Hence the connecting map δ_n sends $h_n(F)$ to the class of the cocycle $\sigma \to \theta(z/\sigma(z))$. On the other hand, one has

$$\theta(\frac{z}{\sigma(z)}) = \theta\phi^{-n}(\frac{y\sigma F(\pi_n)}{\sigma(y)F(\pi_n)}) = \theta\phi^{-n}([\epsilon]^{-c(\sigma)}) = \xi_{p^n}^{-c(\sigma)},$$

which finishes the proof.

4.3 Vostokov's formulae

In this section we reprove Vostokov's formulae for (16). More precisely, we prove the following result:- For $1 \leq i \leq 3$, let $\alpha_i \in O_K^\times$ such that $\alpha_i \cong 1 \mod \bar{\pi}_n$, and let $F_i(X) \in \mathbb{A}_K^+$ such that $h_n(F_i) = \alpha_i$. Let $f_i(X) = (1 - \frac{\phi}{p}) \log F(X)$.

Theorem 4.10. *We have*

$$V_n(\alpha_1, \alpha_2, \alpha_3) = \mu_{p^n}^{\mathrm{Tr}\,\mathrm{Res}_{\pi_n,T}(\Phi)}, \tag{22}$$

where Φ is given by the formula

$$\begin{aligned}\Phi = &-\frac{1}{\pi}(\frac{1}{p^2} f_1(\pi_n) d\log F_2^\phi(\pi_n) \wedge d\log F_3^\phi(\pi_n) \\ &- \frac{1}{p} f_2(\pi_n) d\log F_1(\pi_n) \wedge d\log F_3^\phi(\pi_n) \\ &+ f_3(\pi_n) d\log F_1(\pi_n) \wedge d\log F_2(\pi_n)).\end{aligned}$$

We will prove the theorem in the rest of this section.

Lemma 4.11. *Let $[x, y, z], [x', y', z'] \in H^1_{\phi,\gamma_{1,n},\gamma_{2,n}}(C_n)$. If $[\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]$ represents the cohomology class of $[x, y, z] \cup [x', y', z']$, then*

$$\mathfrak{x} = y \otimes \gamma_{1,n} x' - x \otimes \phi y',$$
$$\mathfrak{y} = z \otimes \gamma_{2,n} x' - x \otimes \phi z'.$$

Moreover, if $z, z' \in S_n$, then

$$\mathfrak{z} = x \otimes \gamma_{2,n} y' - y \otimes \gamma_{1,n} x' \mod \pi.$$

Proof. Since $\Gamma_1^{(n)}$ (resp. $\Gamma_2^{(n)}$) is isomorphic to an open subgroup of $\mathbb{Z}_p^\times$ (resp. $\mathbb{Z}_p$), the formulae for $\mathfrak{x}$ and $\mathfrak{y}$ follow from [6], using that the cup product is compatible with restriction. The formula for $\mathfrak{z}$ follows from the observation that γ_1 and γ_2 commute on $S_n \mod \pi$. ☐

For $1 \leq i \leq 3$, let $\iota_n(F_i(X)) = [f_i(\pi_n)\tau \otimes \epsilon, a^{(i)}_{\gamma_{1,n}}(\pi_n) \otimes \epsilon, b^{(i)}_{\gamma_{2,n}} \otimes \epsilon]$.

Corollary 4.12. *If* $[\mathfrak{x},\mathfrak{y},\mathfrak{z}] = \iota_n(F_1(X)) \cup \iota_n(F_2(X))$, *then*

$$\begin{aligned}
\mathfrak{x} =& \frac{1-\chi(\gamma_{1,n})}{p^n}(D_1 \log F_1(\pi_n) \otimes f_2(\pi_n) \\
& - p^{-1} f_1(\pi_n) \otimes D_1 \log F_2^{\phi}(\pi_n)) \otimes \tau \otimes \epsilon^2 \mod S_n, \\
\mathfrak{y} =& \eta_n(\gamma_{2,n})(D_2 \log F_1(\pi_n) \otimes f_2(\pi_n) \\
& - p^{-1} f_1(\pi_n) \otimes D_2 \log F_2^{\phi}(\pi_n)) \otimes \tau \otimes \epsilon^2 \mod S_n, \\
\mathfrak{z} =& \frac{1-\chi(\gamma_{1,n})}{p^n}\eta_n(\gamma_{2,n})(D_1 \log F_2(\pi_n) D_2 \log F_1(\pi_n) \\
& - D_2 \log F_1(\pi_n) D_1 \log F_2(\pi_n)) \otimes \epsilon^2 \mod \pi
\end{aligned}$$

Proof. Observe that

$$\begin{aligned}
\gamma_{1,n}(f_i(\pi_n)\tau) &= \chi^{-1}(\gamma_{1,n}) f_i(\pi_n)\tau \mod S_n, \\
\gamma_{2,n}(f_i(\pi_n)\tau) &= f_i(\pi_n)\tau \mod S_n.
\end{aligned}$$

The lemma now follows from the previous lemma and Proposition 4.3. □

Proof of Theorem 1.3. We prove the Theorem using the formulae (6). Let

$$H_{\alpha_1,\alpha_2,\alpha_3} = \iota_n(F_3(X)) \cup [\mathfrak{x},\mathfrak{y},\mathfrak{z}],$$

where (to simplify the notation) we write

$$[x, y, z] = [f_3(\pi_n)\tau, a^{(3)}_{\gamma_{1,n}}(\pi_n), b^{(3)}_{\gamma_{2,n}}].$$

Recall that $\gamma_{1,n}$ and $\gamma_{2,n}$ commute on $S_n \mod \pi$. It follows that the formulae (6) simplify to

$$\iota_n(F_3(X)) \cup [\mathfrak{x},\mathfrak{y},\mathfrak{z}] = \mathfrak{z} \otimes \gamma_2\gamma_1(x) - \mathfrak{y} \otimes \gamma_2\phi_M(y) + \mathfrak{x} \otimes \gamma_1\phi_M(z) \mod S_n.$$

Using the formulae in Corollary 4.12 it follows that

$$\begin{aligned}
H_{\alpha_1,\alpha_2,\alpha_3} = & \gamma_1\phi_M(b^{(3)}_{\gamma_{2,n}}(\pi_n))(\frac{1-\chi(\gamma_{1,n})}{p^n}(D_1 \log F_1(\pi_n) \otimes f_2(\pi_n) \\
& \qquad - p^{-1} f_1(\pi_n) \otimes D_1 \log F_2^{\phi}(\pi_n)) \otimes \tau \otimes (\epsilon^{(n)})^2) \\
& -\gamma_2\phi_M(a^{(3)}_{\gamma_{1,n}}(\pi_n))(\eta_n(\gamma_{2,n})(D_2 \log F_1(\pi_n) \otimes f_2(\pi_n) \\
& \qquad - p^{-1} f_1(\pi_n) \otimes D_2 \log F_2^{\phi}(\pi_n)) \otimes \tau \otimes (\epsilon^{(n)})^2) \\
& +\gamma_2\gamma_1(f_3(\pi_n)\tau)(\frac{1-\chi(\gamma_{1,n})}{p^n}\eta_n(\gamma_{2,n})(D_1 \log F_2(\pi_n) D_2 \log F_1(\pi_n) \\
& \qquad - D_2 \log F_1(\pi_n) D_1 \log F_2(\pi_n)) \otimes (\epsilon^{(n)})^2) \mod S_n
\end{aligned}$$

As before, we have

$$\begin{aligned}
\gamma_{1,n}(f_i(\pi_n)\tau) &= \chi^{-1}(\gamma_{1,n}) f_i(\pi_n)\tau \mod S_n, \\
\gamma_{2,n}(f_i(\pi_n)\tau) &= f_i(\pi_n)\tau \mod S_n,
\end{aligned}$$

so (rearranging the terms) the above formula simplifies to

$$\begin{aligned}
H_{\alpha_1,\alpha_2,\alpha_3} =& \frac{1-\chi(\gamma_{1,n})}{p^n}\eta_n(\gamma_{2,n})\pi^{-1}\times \\
-p^{-2}&f_1(\pi_n)(D_1\log F_2^{\phi}(\pi_n)D_2\log F_3^{\phi}(\pi_n) \\
&\quad + D_2\log F_2^{\phi}(\pi_n)D_1\log F_3^{\phi}(\pi_n))) \\
+p^{-1}&f_2(\pi_n)(D_1\log F_1(\pi_n)D_2\log F_3^{\phi}(\pi_n) \\
&\quad + D_2\log F_1(\pi_n)D_1\log F_3^{\phi}(\pi_n)) \\
-&f_3(\pi_n)(D_1\log F_2(\pi_n)D_2\log F_1(\pi_n) \\
&\quad + D_2\log F_1(\pi_n)D_1\log F_2(\pi_n))\otimes(\epsilon^{(n)})^2 \mod S_n.
\end{aligned}$$

Recall that $D_1=(\pi_n+1)\frac{d}{d\pi_n}$ and $D_2=T_n\frac{d}{dT_n}$. It follows that the image in $\Omega(K)$ of the above expression (which we also denote by $H_{\alpha_1,\alpha_2\alpha_3}$) with respect to the map in Lemma 2.10 is

$$\begin{aligned}
\frac{1-\chi(\gamma_{1,n})}{p^n}\eta_n(\gamma_{2,n})\pi^{-1}\times(&-p^{-2}f_1(\pi_n)d\log F_2^{\phi}(\pi_n)\wedge d\log F_3^{\phi}(\pi_n) \\
&+p^{-1}f_2(\pi_n)d\log F_1(\pi_n)\wedge d\log F_3^{\phi}(\pi_n) \\
&-f_3(\pi_n)d\log F_2(\pi_n)\wedge d\log F_1(\pi_n)) \mod S_n.
\end{aligned}$$

Taking into account that $p^{-n}\log\chi(\gamma_{1,n})=p^{-n}(\chi(\gamma_{1,n})-1) \mod p^n$, we obtain that

$$\begin{aligned}
-c\,\mathrm{TR}(H_{\alpha_1,\alpha_2,\alpha_3}) =& -\mathrm{Tr}_{F/\mathbb{Q}_p}\mathrm{Res}_{\pi_n,T}\,\pi^{-1}(p^{-2}f_1(\pi_n)d\log F_2^{\phi}(\pi_n)\wedge d\log F_3^{\phi}(\pi_n) \\
&-p^{-1}f_2(\pi_n)d\log F_1(\pi_n)\wedge d\log F_3^{\phi}(\pi_n) \\
&+f_3(\pi_n)d\log F_2(\pi_n)\wedge d\log F_1(\pi_n)),
\end{aligned}$$

which finishes the proof.

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