FLAT MODULES OVER VALUATION RINGS

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ABSTRACT. Let R be a valuation ring and let Q be its total quotient ring. It is proved that any singly projective (respectively flat) module is finitely projective if and only if Q is maximal (respectively artinian). It is shown that each singly projective module is a content module if and only if any non-unit of R is a zero-divisor and that each singly projective module is locally projective if and only if R is self injective. Moreover, R is maximal if and only if each singly projective module is separable, if and only if any flat content module is locally projective. Necessary and sufficient conditions are given for a valuation ring with non-zero zero-divisors to be strongly coherent or π-coherent.

A complete characterization of semihereditary commutative rings which are π-coherent is given. When R is a commutative ring with a self FP-injective quotient ring Q, it is proved that each flat R-module is finitely projective if and only if Q is perfect.

In this paper, we consider the following properties of modules: P-flatness, flatness, content flatness, local projectivity, finite projectivity and single projectivity. We investigate the relations between these properties when R is a PP-ring or a valuation ring. Garfinkel ([11]), Zimmermann-Huisgen ([22]), and Gruson and Raynaud ([13]) introduced the concepts of locally projective modules and strongly coherent rings and developed important theories on these. The notions of finitely projective modules and π-coherent rings are due to Jones ([15]). An interesting study of finitely projective modules and singly projective modules is also done by Azumaya in [1]. For a module M over a ring R, the following implications always hold:

$$\begin{array}{ccccc} M \text{ is projective} & \Rightarrow & M \text{ is locally projective} & \Rightarrow & M \text{ is flat content} \\ & & \Downarrow & & \Downarrow \\ & & M \text{ is finitely projective} & \Rightarrow & M \text{ is flat} \\ & & \Downarrow & & \Downarrow \\ & & M \text{ is singly projective} & \Rightarrow & M \text{ is P}-\text{flat}, \end{array}$$

but there are not generally reversible. However, if R satisfies an additional condition, we get some equivalences. For instance, in [2], Bass defined a ring R to be right perfect if each flat right module is projective. In [23] it is proved that a ring R is right perfect if and only if each flat right module is locally projective, and if and only if each locally projective right module is projective. If R is a commutative arithmetic ring, i.e. a ring whose lattice of ideals is distributive, then any P-flat module is flat. By [1, Proposition 16], if R is a commutative domain, each P-flat module is singly projective, and by [1, Proposition 18 and 15] any flat left module is finitely projective if R is a commutative arithmetic domain or a left noetherian ring. Consequently, if R is a valuation domain each P-flat module is finitely projective.

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When R is a valuation ring, we prove that this result holds if and only if the ring Q of quotients of R is artinian. Moreover, we show that R is maximal if and only if any singly projective module is separable or any flat content module is locally projective, and that Q is maximal if and only if each singly projective module is finitely projective.

In Section 2, necessary and sufficient conditions are given for a commutative semihereditary ring to be π-coherent. Moreover we characterize commutative PP-rings for which each product of singly projective modules is singly projective.

In the last section we study the valuation rings R for which each product of content (respectively singly, finitely, locally projective) modules is content (respectively singly, finitely, locally projective). The results are similar to those obtained by Zimmermann-Huisgen and Franzen in [8], and by Kemper in [16], when R is a domain. However, each valuation domain is π-coherent but not necessarily strongly coherent. We prove that a valuation ring with non-zero zero-divisors is π-coherent if and only if it is strongly coherent.

1. Definitions and preliminaries

If A is a subset of a ring R, we denote respectively by $\ell(A)$ and $\mathrm{r}(A)$ its left annihilator and its right annihilator. Given a ring R and a left R-module M, we say that M is **P-flat** if, for any $(s,x) \in R \times M$ such that $sx = 0$, $x \in \mathrm{r}(s)M$. When R is a domain, M is P-flat if and only if it is torsion-free. As in [1], we say that M is **finitely projective** (respectively **singly projective**) if, for any finitely generated (respectively cyclic) submodule N, the inclusion map $N \to M$ factors through a free module F. A finitely projective module is called f-projective in [15]. As in [22] we say that M is **locally projective** if, for any finitely generated submodule N, there exist a free module F, an homomorphism $\phi : M \to F$ and an homomorphism $\pi : F \to M$ such that $\pi(\phi(x)) = x,\ \forall x \in N$. A locally projective module is said to be either a trace module or a universally torsionless module in [11]. Given a ring R, a left R-module M and $x \in M$, the **content ideal** $\mathrm{c}(x)$ of x in M, is the intersection of all right ideals A for which $x \in AM$. We say that M is a **content module** if $x \in \mathrm{c}(x)M,\ \forall x \in M$.

It is obvious that each locally projective module is finitely projective but the converse doesn't generally hold. For instance, if R is a commutative domain with quotient field $Q \neq R$, then Q is a finitely projective R-module: if N is a finitely generated submodule of Q, there exists $0 \neq s \in R$ such that $sN \subseteq R$, whence the inclusion map $N \to Q$ factors through R by using the multiplications by s and s^{-1}; but Q is not locally projective because the only homormorphism from Q into a free R-module is zero.

Proposition 1.1. *Let R be a ring. Then:*

1. *Each singly projective left R-module M is P-flat. The converse holds if R is a domain.*
2. *Any P-flat cyclic left module is flat.*
3. *Each P-flat content left module M is singly projective.*

Proof. (1). Let $0 \neq x \in M$ and $r \in R$ such that $rx = 0$. There exist a free module F and two homomorphisms $\phi : Rx \to F$ and $\pi : F \to M$ such that $\pi \circ \phi$ is the inclusion map $Rx \to M$. Since $r\phi(x) = 0$ and F is free, there exist

$s_1, \dots, s_n \in \mathrm{r}(r)$ and $y_1, \dots, y_n \in F$ such that $\phi(x) = s_1y_1 + \cdots + s_ny_n$. Then $x = s_1\pi(y_1) + \cdots + s_n\pi(y_n)$. The last assertion is obvious.

(2). Let C be a cyclic left module generated by x and let A be a right ideal. Then each element of $A \otimes_R C$ is of the form $a \otimes x$ for some $a \in A$. If $ax = 0$ then $\exists b \in \mathrm{r}(a)$ such that $x = bx$. Therefore $a \otimes x = a \otimes bx = ab \otimes x = 0$. Hence C is flat.

(3). Let $x \in M$. Then, since $x \in \mathrm{c}(x)M$ there exist $a_1, \dots, a_n \in \mathrm{c}(x)$ and $x_1, \dots, x_n \in M$ such that $x = a_1x_1 + \cdots + a_nx_n$. Let $b \in R$ such that $bx = 0$. Therefore $x \in \mathrm{r}(b)M$ because M is P-flat. It follows that $\mathrm{c}(x) \subseteq \mathrm{r}(b)$. So, if we put $\phi(rx) = (ra_1, \dots, ra_n)$, then ϕ is a well defined homomorphism which factors the inclusion map $Rx \to M$ through ${}_RR^n$. □

Theorem 1.2. *A ring R is left perfect if and only if each flat left module is a content module.*

Proof. If R is left perfect then each flat left module is projective. Conversely suppose that each flat left module is a content module. Let $(a_k)_{k\in\mathbb{N}}$ be a family of elements of R, let $(e_k)_{k\in\mathbb{N}}$ be a basis of a free left module F and let G be the submodule of F generated by $\{e_k - a_ke_{k+1} \mid k \in \mathbb{N}\}$. By [2, Lemma 1.1] F/G is flat. We put $z_k = e_k + G$, $\forall k \in \mathbb{N}$. Since F/G is content and $z_k = a_kz_{k+1}$, $\forall k \in \mathbb{N}$, there exist $c \in R$ and $n \in \mathbb{N}$ such that $z_0 = cz_n$ and $\mathrm{c}(z_0) = cR$. It follows that $cR = ca_n \dots a_pR$, $\forall p > n$. Since $z_0 = a_0 \dots a_{n-1}z_n$, there exists $k > n$ such that $ca_n \dots a_k = a_0 \dots a_k$. Consequently $a_0 \dots a_kR = a_0 \dots a_pR$, $\forall p \geq k$. So, R is left perfect because it satisfies the descending chain condition on principal right ideals by [2, Theorem P]. □

Given a ring R and a left R-module M, we say that M is **P-injective** if, for any $(s, x) \in R \times M$ such that $\ell(s)x = 0$, $x \in sM$. When R is a domain, M is P-injective if and only if it is divisible. As in [19], we say that M is **finitely injective** (respectively **FP-injective**) if, for any finitely generated submodule A of a (respectively finitely presented) left module B, each homomorphism from A to M extends to B. If M is an R-module, we put $M^* = \mathrm{Hom}_R(M, R)$.

Proposition 1.3. *Let R be a ring. Then:*

1. *If R is a P-injective left module then each singly projective left module is P-injective;*
2. *If R is a FP-injective left module then each finitely projective left module is FP-injective and a content module;*
3. *If R is an injective module then each singly projective module is finitely injective and locally projective.*

Proof. Let M be a left module, F a free left module and $\pi : F \to M$ an epimorphism.

1. Assume that M is singly projective. Let $x \in M$ and $r \in R$ such that $\ell(r)x = 0$. There exists a homomorphism $\phi : Rx \to F$ such that $\pi \circ \phi$ is the inclusion map $Rx \to M$. Since F is P-injective, $\phi(x) = ry$ for some $y \in F$. Then $x = r\pi(y)$.

2. Assume that M is finitely projective. Let L be a finitely generated free left module, let N be a finitely generated submodule of L and let $f : N \to M$ be a homomorphism. Then $f(N)$ is a finitely generated submodule of M. So, there exists a homomorphism $\phi : f(N) \to F$ such that $\pi \circ \phi$ is the inclusion map $f(N) \to M$. Since F is FP-injective, there exists a morphism $g : L \to F$ such that $\phi \circ f$ is the restriction of g to N. Now it is easy to check that $\pi \circ g$ is the restriction of f to N.

Let $x \in M$. There exists a homomorphism $\phi : Rx \to F$ such that $\pi \circ \phi$ is the inclusion map $Rx \to M$. Let $\{e_i \mid i \in I\}$ be a basis of F. There exist a finite subset J of I and a family $(a_i)_{i\in J}$ of elements of R such that $\phi(x) = \sum_{i\in J} a_i e_i$. Let A be the right ideal generated by $(a_i)_{i\in J}$. Then $(0 : x) = (0 : \phi(x)) = \ell(A)$. Let B be a right ideal such that $x \in BM$. Then $x = \sum_{k=1}^{p} b_k x_k$ where $b_k \in B$ and $x_k \in M$, $\forall k,\ 1 \le k \le p$. Let N be the submodule of M generated by $\{\pi(e_i) \mid i \in J\} \cup \{x_k \mid 1 \le k \le p\}$. Thus there exists a homomorphism $\varphi : N \to F$ such that $\pi \circ \varphi$ is the inclusion map $N \to M$. Therefore there exist a finite subset K of I and two families $\{d_{k,j} \mid 1 \le k \le p,\ j \in K\}$ and $\{c_{i,j} \mid (i,j) \in J \times K\}$ of elements of R such that $\varphi(\pi(e_i)) = \sum_{j\in K} c_{i,j} e_j$, $\forall i \in J$ and $\varphi(x_k) = \sum_{j\in K} d_{k,j} e_j$, $\forall k,\ 1 \le k \le p$. It follows that $\varphi(x) = \sum_{j\in K}(\sum_{i\in J} a_i c_{i,j}) e_j = \sum_{j\in K}(\sum_{k=1}^{p} b_k d_{k,j}) e_j$. So, $\sum_{i\in J} a_i c_{i,j} = \sum_{k=1}^{p} b_k d_{k,j}$, $\forall j \in K$. Let A' be the right ideal generated by $\{\sum_{i\in J} a_i c_{i,j} \mid j \in K\}$. Then $A' \subseteq A$ and $A' \subseteq B$. Moreover, $\ell(A) = (0 : x) = (0 : \varphi(x)) = \ell(A')$. By [14, Corollary 2.5] $A = A'$. So, $A \subseteq B$. We conclude that $\mathrm{c}(x) = A$ and M is a content module.

3. Let M be a singly projective module and $x \in M$. So, there exists a homomorphism $\phi : Rx \to F$ such that $\pi \circ \phi$ is the inclusion map $Rx \to M$. Since F is finitely injective, we can extend ϕ to M. By using a basis of F we deduce that $x = \sum_{k=1}^{n} \phi_k(x) x_k$ where $\phi_k \in M^*$ and $x_k \in M$, $\forall k,\ 1 \le k \le n$. Hence M is locally projective by [11, Theorem 3.2] or [22, Theorem 2.4]. By a similar proof as in (2), we show that M is finitely injective, except that L is not necessarily a finitely generated free module. □

A short exact sequence of left R-modules $0 \to N \to M \to L \to 0$ is **pure** if it remains exact when tensoring it with any right R-module. We say that N is a pure submodule of M. This property holds if L is flat.

Lemma 1.4. *Let R be a local ring, let P be its maximal ideal and let N be a flat left R-module. Assume that N is generated by a family $(x_i)_{i\in I}$ of elements of N such that $(x_i + PN)_{i\in I}$ is a basis of N/PN. Then N is free.*

Proof. Let $(e_i)_{i\in I}$ be a basis of a free left module F, let $\alpha : F \to N$ be the homomorphism defined by $\alpha(e_i) = x_i,\ \forall i \in I$ and let L be the kernel of α. It is easy to check that $L \subseteq PF$. Let $y \in L$. We have $y = \sum_{i\in J} a_i e_i$ where J is a finite subset of I and $a_i \in P,\ \forall i \in J$. Since L is a pure submodule of F, $\forall i \in J$ there exists $y_i \in L$ such that $\sum_{i\in J} a_i e_i = \sum_{i\in J} a_i y_i$. We have $y_i = \sum_{j\in J_i} b_{i,j} e_j$ where J_i is a finite subset of I, $b_{i,j} \in P,\ \forall (i,j) \in J \times J_i$. Let $K = J \cup (\cup_{i\in J} J_i)$. If $i \in K \setminus J$ we put $a_i = 0$ and $a_{i,j} = 0,\ \forall j \in K$, and if $j \in K \setminus J_i$ we put $a_{i,j} = 0$ too. We get $\sum_{i\in K} a_i e_i = \sum_{j\in K}(\sum_{i\in K} a_i b_{i,j}) e_j$. It follows that $a_j = \sum_{i\in K} a_i b_{i,j}$. So, if A is the right ideal generated by $\{a_i \mid i \in K\}$, then $A = AP$. By Nakayama lemma $A = 0$, whence $F \cong N$. □

A left R-module is said to be a **Mittag-Leffler** module if, for each index set Λ, the natural homomorphism $R^{\Lambda} \otimes_R M \to M^{\Lambda}$ is injective. The following lemma is a slight generalization of [6, Proposition 2.3].

Lemma 1.5. *Let R be a subring of a ring S and let M be a flat left R-module. Assume that $S \otimes_R M$ is finitely projective over S. Then M is finitely projective.*

Proof. By [15, Proposition 2.7] a module is finitely projective if and only if it is a flat Mittag-Leffler module. So we do as in the proof of [6, Proposition 2.3]. □

From this lemma and [15, Proposition 2.7] we deduce the following proposition. We can also see .

Proposition 1.6. *Let R be a subring of a left perfect ring S. Then each flat left R-module is finitely projective.*

Proposition 1.7. *Let R be a commutative ring and let S be a multiplicative subset of R. Then:*

1. *For each singly (respectively finitely, locally) projective R-module M, $S^{-1}M$ is singly (respectively finitely, locally) projective over $S^{-1}R$;*
2. *Let M be a singly (respectively finitely) projective $S^{-1}R$-module. If S contains no zero-divisors then M is singly (respectively finitely) projective over R.*

Proof. (1). We assume that $M \neq 0$. Let N be a cyclic (respectively finitely generated) submodule of $S^{-1}M$. Then there exists a cyclic (respectively finitely generated) submodule N' of M such that $S^{-1}N' = N$. There exists a free R-module F, a morphism $\phi : N' \to F$ and a morphism $\pi : F \to M$ such that $(\pi \circ \phi)(x) = x$ for each $x \in N'$. It follows that $(S^{-1}\pi \circ S^{-1}\phi)(x) = x$ for each $x \in N$. We get that $S^{-1}M$ is singly (respectively finitely) projective over R. We do a similar proof to show that $S^{-1}M$ is locally projective if M is locally projective.

(2) By Lemma 1.5 M is finitely projective over R if it is finitely projective over $S^{-1}R$. It is easy to check that M is singly projective over R if it is singly projective over $S^{-1}R$. □

If R is a subring of a ring Q which is either left perfect or left noetherian, then then each flat left R-module is finitely projective by [20, Corollary 7]. We don't know if the converse holds. However we have the following results:

Theorem 1.8. *Let R be a commutative ring with a self FP-injective quotient ring Q. Then each flat R-module is finitely projective if and only if Q is perfect.*

Proof. "Only if" requires a proof. Let M be a flat Q-module. Then M is flat over R and it follows that M is finitely projective over R. By Proposition 1.7(1) $M \cong Q \otimes_R M$ is finitely projective over Q. From Proposition 1.3 we deduce that each flat Q-module is content. We conclude by Theorem 1.2 □

Theorem 1.9. *Let R be a commutative ring with a Von Neumann regular quotient ring Q. Then the following conditions are equivalent:*

1. *Q is semi-simple;*
2. *each flat R-module is finitely projective;*
3. *each flat R-module is singly projective.*

Proof. (1) $\Rightarrow$ (2) is an immediate consequence of [20, Corollary 7] and (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). First we show that each Q-module M is singly projective. Every Q-module M is flat over Q and R. So, M is singly projective over R. It follows that $M \cong Q \otimes_R M$ is singly projective over Q by Proposition 1.7(1). Now let A be an ideal of Q. Since Q/A is singly projective, it is projective. So, Q/A is finitely presented over Q and A is a finitely generated ideal of Q. Hence Q is semi-simple. □

2. π-coherence and PP-rings

As in [22] we say that a ring R is left **strongly coherent** if each product of locally projective right modules is locally projective and as in [3] R is said to be right **π-coherent** if, for each index set Λ, every finitely generated submodule of R_R^Λ is finitely presented.

Theorem 2.1. *Let R be a commutative ring. Then the following conditions are equivalent:*

1. *R is π-coherent;*
2. *for each index set Λ, R^Λ is finitely projective;*
3. *each product of finitely projective modules is finitely projective.*

Proof. (1) $\Rightarrow$ (2). Let N be a finitely generated submodule of R^Λ. There exist a free module F and an epimorphism π from F into R^Λ. It is obvious that R is coherent. Consequently R^Λ is flat. So $\ker\ \pi$ is a pure submodule of F. Since N is finitely presented it follows that there exists $\phi : N \to F$ such that $\pi \circ \phi$ is the inclusion map from N into R^Λ.

(2) $\Rightarrow$ (1). Since R^Λ is flat for each index set Λ, R is coherent. Let Λ be an index set and let N be a finitely generated submodule of R^Λ. The finite projectivity of R^Λ implies that N is isomorphic to a submodule of a free module of finite rank. Hence N is finitely presented.

It is obvious that (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Let Λ be an index set, let $(M_\lambda)_{\lambda\in\Lambda}$ be a family of finitely projective modules and let N be a finitely generated submodule of $M = \prod_{\lambda\in\Lambda} M_\lambda$. For each $\lambda \in \Lambda$, let N_λ be the image of N by the canonical map $M \to M_\lambda$. We put $N' = \prod_{\lambda\in\Lambda} N_\lambda$. So, $N \subseteq N' \subseteq M$. For each $\lambda \in \Lambda$ there exists a free module F_λ of finite rank such that the inclusion map $N_\lambda \to M_\lambda$ factors through F_λ. It follows that the inclusion map $N \to M$ factors through $\prod_{\lambda\in\Lambda} F_\lambda$ which is isomorphic to $R^{\Lambda'}$ for some index set Λ'. Now the monomorphism $N \to R^{\Lambda'}$ factors through a free module F. It is easy to conclude that the inclusion map $N \to M$ factors through F and that M is finitely projective. □

By using [22, Theorem 4.2] and Proposition 1.3, we deduce the following corollary:

Corollary 2.2. *Every strongly coherent commutative ring R is π-coherent and the converse holds if R is self injective.*

Proposition 2.3. *Let R be a π-coherent commutative ring and let S be a multiplicative subset of R. Assume that S contains no zero-divisors. Then $S^{-1}R$ is π-coherent.*

Proof. Let M be a finitely generated $S^{-1}R$-module. By [3, Theorem 1] we must prove that $\mathrm{Hom}_{S^{-1}R}(M, S^{-1}R)$ is finitely generated on $S^{-1}R$. There exists a finitely generated R-submodule N of M such that $S^{-1}N \cong M$. The following sequence

$$0 \to N^* \to \mathrm{Hom}_R(N, S^{-1}R) \to \mathrm{Hom}_R(N, S^{-1}R/R)$$

is exact. Since N is finitely generated and $S^{-1}R/R$ is S-torsion, $\mathrm{Hom}_R(N, S^{-1}R/R)$ is S-torsion too. So, $\mathrm{Hom}_{S^{-1}R}(M, S^{-1}R) \cong \mathrm{Hom}_R(N, S^{-1}R) \cong S^{-1}N^*$. By [3, Theorem 1] N^* is finitely generated. Hence $\mathrm{Hom}_{S^{-1}R}(M, S^{-1}R)$ is finitely generated over $S^{-1}R$. □

Theorem 2.4. *Let R be a commutative semihereditary ring and let Q be its quotient ring. Then the following conditions are equivalent:*

1. *R is π-coherent;*
2. *Q is self injective;*

Moreover, when these conditions are satisfied, each singly projective R-module is finitely projective.

Proof. (1) $\Rightarrow$ (2). By Proposition 2.3 Q is π-coherent. We know that Q is Von Neumann regular. It follows from [18, Theorem 2] that Q is self injective.

(2) $\Rightarrow$ (1). Let $(M_i)_{i\in I}$ be a family of finitely projective R-modules, where I is an index set, and let N be a finitely generated submodule of $\prod_{i\in I} M_i$. Then N is flat. Since N is a submodule of $\prod_{i\in I} Q\otimes_R M_i$, $Q\otimes_R N$ is isomorphic to a finitely generated Q-submodule of $\prod_{i\in I} Q\otimes_R M_i$. It follows that $Q\otimes_R N$ is a projective Q-module. Hence N is projective by [6, Proposition 2.3]. We conclude by Theorem 2.1.

Let M be a singly projective R-module and let N be a finitely generated submodule of M. Then $Q\otimes_R M$ is finitely projective over Q by Propositions 1.7(1) and 1.3. It follows that $Q\otimes_R N$ is projective over Q. Hence N is projective by [6, Proposition 2.3]. □

Proposition 2.5. *Let R be a Von Neumann regular ring. Then a right R-module is content if and only if it is singly projective.*

Proof. By Proposition 1.1(3) it remains to show that each singly projective right module M is content. Let $m\in M$. Then mR is projective because it is isomorphic to a finitely generated submodule of a free module. So, mR is content. For each left ideal A, $mR\cap MA = mA$ because mR is a pure submodule of M. Hence M is content. □

A topological space X is said to be **extremally disconnected** if every open set has an open closure. Let R be a ring. We say that R is a right **Baer ring** if for any subset A of R, $\mathrm{r}(A)$ is generated by an idempotent. The ring R defined in [22, Example 4.4] is not self injective and satisfies the conditions of the following theorem.

Theorem 2.6. *Let R be a Von Neumann regular ring. Then the following conditions are equivalent:*

1. *Each product of singly projective right modules is singly projective;*
2. *Each product of content right modules is content;*
3. *R_R^R is singly projective;*
4. *R_R^R is a content module;*
5. *R is a right Baer ring;*
6. *The intersection of each family of finitely generated left ideals is finitely generated too;*
7. *For each cyclic left module C, C^* is finitely generated.*

Moreover, when R is commutative, these conditions are equivalent to the following: Spec *R is extremally disconnected.*

Proof. The conditions (2), (4), (6) are equivalent by [11, Theorem 5.15]. By Proposition 2.5 (4) $\Leftrightarrow$ (3) and (1) $\Leftrightarrow$ (2). It is easy to check that (5) $\Leftrightarrow$ (7).

(3) ⇒ (5). Let $A \subseteq R$ and let $x = (a)_{a\in A} \in R^A_R$. So, $\mathrm{r}(A) = (0 : x)$. Then xR is projective because it is isomorphic to a submodule of a free module. Thus $\mathrm{r}(A) = eR$, where e is an idempotent.

(5) ⇒ (1). Let $(M_i)_{i\in I}$ be a family of singly projective right modules and $m = (m_i)_{i\in I}$ be an element of $M = \prod_{i\in I} M_i$. For each $i \in I$, there exists an idempotent e_i such that $(0 : m_i) = e_iR$. Let e be the idempotent which satisfies $eR = \mathrm{r}(\{1 - e_i \mid i \in I\})$. Then $eR = (0 : m)$, whence mR is projective.

If R is commutative and reduced, then the closure of $D(A)$, where A is an ideal of R, is $V((0 : A))$. So, Spec R is extremally disconnected if and only if, for each ideal A there exists an idempotent e such that $V((0 : A)) = V(e)$. This last equality holds if and only if $(0 : A) = Re$ because $(0 : A)$ and Re are semiprime since R is reduced. Consequently Spec R is extremally disconnected if and only if R is Baer. The proof is now complete. □

Let R be a ring. We say that R is a right **PP-ring** if any principal right ideal is projective.

Lemma 2.7. *Let R be a right PP-ring. Then each cyclic submodule of a free right module is projective.*

Proof. Let C be a cyclic submodule of a free right module F. We may assume that F is finitely generated by the basis $\{e_1, \dots, e_n\}$. Let $p : F \to R$ be the homomorphism defined by $p(e_1r_1 + \cdots + e_nr_n) = r_n$ where $r_1, \dots, r_n \in R$. Then $p(C)$ is a principal right ideal. Since $p(C)$ is projective, $C \cong C' \oplus p(C)$ where $C' = C \cap \ker p$. So C' is a cyclic submodule of the free right module generated by $\{e_1, \dots, e_{n-1}\}$. We complete the proof by induction on n. □

Theorem 2.8. *Let R be a commutative PP-ring and let Q be its quotient ring. Then the following conditions are equivalent:*

1. *Each product of singly projective modules is singly projective;*
2. *R^R is singly projective;*
3. *R is a Baer ring;*
4. *Q satisfies the equivalent conditions of Theorem 2.6;*
5. *For each cyclic module C, C^* is finitely generated;*
6. Spec *R is extremally disconnected;*
7. Min *R is extremally disconnected.*

Proof. It is obvious that (1) ⇒ (2). It is easy to check that (3) ⇔ (5). We show that (2) ⇒ (3) as we proved (3) ⇒ (5) in Theorem 2.6, by using Lemma 2.7.

(5) ⇒ (4). Let C be a cyclic Q-module. We do as in proof of Proposition 2.3 to show that $\mathrm{Hom}_Q(C, Q)$ is finitely generated over Q.

(4) ⇒ (1). Let $(M_i)_{i\in I}$ be a family of singly projective right modules and let N be a cyclic submodule of $M = \prod_{i\in I} M_i$. Since R is PP, N is a P-flat module. By Proposition 1.1 N is flat. We do as in the proof of (2) ⇒ (1) of Theorem 2.4 to show that N is projective.

(3) ⇔ (6) is shown in the proof of Theorem 2.6.

(4) ⇔ (7) holds because Spec Q is homeomorphic to Min R. □

3. Flat modules

Let M be a non-zero module over a commutative ring R. As in [10, p.338] we set:

$$M_\sharp = \{s \in R \mid \exists 0 \neq x \in M \text{ such that } sx = 0\} \quad \text{and} \quad M^\sharp = \{s \in R \mid sM \subset M\}.$$

Then $R \setminus M_\sharp$ and $R \setminus M^\sharp$ are multiplicative subsets of R.

Lemma 3.1. *Let M be a non-zero P-flat R-module over a commutative ring R. Then $M_\sharp \subseteq R_\sharp \cap M^\sharp$.*

Proof. Let $0 \neq s \in M_\sharp$. Then there exists $0 \neq x \in M$ such that $sx = 0$. Since M is P-flat, we have $x \in (0 : s)M$. Hence $(0 : s) \neq 0$ and $s \in R_\sharp$.

Suppose that $M_\sharp \nsubseteq M^\sharp$ and let $s \in M_\sharp \setminus M^\sharp$. Then $\exists 0 \neq x \in M$ such that $sx = 0$. It follows that $x = t_1y_1 + \cdots + t_py_p$ for some $y_1, \dots, y_p \in M$ and $t_1, \dots, t_p \in (0 : s)$. Since $s \notin M^\sharp$ we have $M = sM$. So $y_k = sz_k$ for some $z_k \in M$, $\forall k,\ 1 \leq k \leq p$. We get $x = t_1sz_1 + \cdots + t_psz_p = 0$. Whence a contradiction. □

Now we assume that R is a commutative ring. An R-module M is said to be **uniserial** if its set of submodules is totally ordered by inclusion and R is a **valuation ring** if it is uniserial as R-module. If M is a module over a valuation ring R then $M_\sharp$ and $M^\sharp$ are prime ideals of R. In the sequel, if R is a valuation ring, we denote by P its maximal ideal and we put $Z = R_\sharp$ and $Q = R_Z$. Since each finitely generated ideal of a valuation ring R is principal, it follows that any P-flat R-module is flat.

Proposition 3.2. *Let R be a valuation ring, let M be a flat R-module and let E be its injective hull. Then E is flat.*

Proof. Let $x \in E \setminus M$ and $r \in R$ such that $rx = 0$. There exists $a \in R$ such that $0 \neq ax \in M$. From $ax \neq 0$ and $rx = 0$ we deduce that $r = ac$ for some $c \in R$. Since $cax = 0$ and M is flat we have $ax = by$ for some $y \in M$ and $b \in (0 : c)$. From $bc = 0$ and $ac = r \neq 0$ we get $b = at$ for some $t \in R$. We have $a(x - ty) = 0$. Since $at = b \neq 0$, $(0 : t) \subset Ra$. So $(0 : t) \subseteq (0 : x - ty)$. The injectivity of E implies that there exists $z \in E$ such that $x = t(y + z)$. On the other hand $tr = tac = bc = 0$, so $t \in (0 : r)$. □

In the sequel, if J is a prime ideal of R we denote by 0_J the kernel of the natural map: $R \to R_J$.

Proposition 3.3. *Let R be a valuation ring and let M be a non-zero flat R-module. Then:*

1. *If $M_\sharp \subset Z$ we have $\mathrm{ann}(M) = 0_{M_\sharp}$ and M is an $R_{M_\sharp}$-module;*
2. *If $M_\sharp = Z$, $\mathrm{ann}(M) = 0$ if $M_Z \neq ZM_Z$ and $\mathrm{ann}(M) = (0 : Z)$ if $M_Z = ZM_Z$. In this last case, M is a Q-module.*

Proof. Observe that the natural map $M \to M_{M_\sharp}$ is a monomorphism. First we assume that R is self FP-injective and $P = M_\sharp$. So $M^\sharp = P$ by Lemma 3.1. If $M \neq PM$ let $x \in M \setminus PM$. Then $(0 : x) = 0$ else $\exists r \in R,\ r \neq 0$ such that $x \in (0 : r)M \subseteq PM$. If $M = PM$ then P is not finitely generated else $M = pM$, where $P = pR$, and $p \notin M^\sharp = P$. If P is not faithful then $(0 : P) \subseteq \mathrm{ann}(M)$. Thus M is flat over $R/(0 : P)$. So we can replace R with $R/(0 : P)$ and assume that P is faithful. Suppose $\exists 0 \neq r \in P$ such that $rM = 0$. Then $M = (0 : r)M$.

Since $(0:r)\neq P$, let $t\in P\setminus(0:r)$. Thus $M=tM$ and $t\notin M^{\sharp}=P$. Whence a contradiction. So M is faithful or $\mathrm{ann}(M)=(0:P)$.

Return to the general case. We put $J=M_{\sharp}$.

If $J\subset Z$ then R_J is coherent and self FP-injective by [4, Theorem 11]. In this case JR_J is principal or faithful. So M_J is faithful over R_J, whence $\mathrm{ann}(M)=0_J$. Let $s\in R\setminus J$. There exists $t\in Zs\setminus J$. It is easy to check that $\forall a\in R,\ (0:a)$ is also an ideal of Q. On the other hand, $\forall a\in Q,\ Qa=(0:(0:a))$ because Q is self FP-injective. It follows that $(0:s)\subset(0:t)$. Let $r\in(0:t)\setminus(0:s)$. Then $r\in 0_J$. So, $rM=0$. Hence $M=(0:r)M=sM$. Therefore the multiplication by s in M is bijective for each $s\in R\setminus J$.

Now suppose that $J=Z$. Since Q is self FP-injective then M is faithful or $\mathrm{ann}(M)=(0:Z)$. Let $s\in R\setminus Z$. Thus $Z\subset Rs$ and $sZ=Z$. It follows that $ZM_Z=ZM$. So, M is a Q-module if $ZM_Z=M_Z$. □

When R is a valuation ring, N is a pure submodule of M if $rN=rM\cap N,\ \forall r\in R$.

Proposition 3.4. *Let R be a valuation ring and let M be a non-zero flat R-module such that $M_{M_{\sharp}}\neq M_{\sharp}M_{M_{\sharp}}$. Then M contains a non-zero pure uniserial submodule.*

Proof. Let $J=M_{\sharp}$ and $x\in M_J\setminus JM_J$. If $J\subset Z$ then M is a module over $R/0_J$ and $J/0_J$ is the subset of zero-divisors of $R/0_J$. So, after replacing R with $R/0_J$ we may assume that $Z=J$. If $rx=0$ then $x\in(0:r)M_Z\subseteq ZM_Z$ if $r\neq 0$. Hence Qx is faithful over Q which is FP-injective. So $V=Qx$ is a pure submodule of M_Z. We put $U=M\cap V$. Thus U is uniserial and $U_Z=V$. Then M/U is a submodule of M_Z/V, and this last module is flat. Let $z\in M/U$ and $0\neq r\in R$ such that $rz=0$. Then $z=as^{-1}y$ where $s\notin Z$, $a\in(0:r)\subseteq Z$ and $y\in M/U$. It follows that $a=bs$ for some $b\in R$ and $sbr=0$. So $b\in(0:r)$ and $z=by$. Since M/U is flat, U is a pure submodule of M. □

Proposition 3.5. *Let R be a valuation ring and let M be a flat R-module. Then M contains a pure free submodule N such that $M/PM\cong N/PN$.*

Proof. Let $(x_i)_{i\in I}$ be a family of elements of M such that $(x_i+PM)_{i\in I}$ is a basis of M/PM over R/P, and let N be the submodule of M generated by this family. If we show that N is a pure submodule of M, we deduce that N is flat. It follows that N is free by Lemma 1.4. Let $x\in M$ and $r\in R$ such that $rx\in N$. Then $rx=\sum_{i\in J}a_ix_i$ where J is a finite subset of I and $a_i\in R,\ \forall i\in J$. Let $a\in R$ such that $Ra=\sum_{i\in J}Ra_i$. It follows that, $\forall i\in J$, there exists $u_i\in R$ such that $a_i=au_i$ and there is at least one $i\in J$ such that u_i is a unit. Suppose that $a\notin Rr$. Thus there exists $c\in P$ such that $r=ac$. We get that $a(\sum_{i\in J}u_ix_i-cx)=0$. Since M is flat, we deduce that $\sum_{i\in J}u_ix_i\in PM$. This contradicts that $(x_i+PM)_{i\in I}$ is a basis of M/PM over R/P. So, $a\in Rr$. Hence N is a pure submodule. □

4. Singly projective modules

Lemma 4.1. *Let R be a valuation ring. Then a non-zero R-module M is singly projective if and only if for each $x\in M$ there exists $y\in M$ such that $(0:y)=0$ and $x\in Ry$. Moreover $M_{\sharp}=Z$ and $M_Z\neq ZM_Z$.*

Proof. Assume that M is singly projective and let $x\in M$. There exist a free module F, a morphism $\phi:Rx\to F$ and a morphism $\pi:F\to M$ such that $(\pi\circ\phi)(x)=x$. Let $(e_i)_{i\in I}$ be a basis of F. Then $\phi(x)=\sum_{i\in J}a_ie_i$ where

J is a finite subset of I and $a_i \in R$, $\forall i \in J$. There exists $a \in R$ such that $\sum_{i\in J} Ra_i = Ra$. Thus, $\forall i \in J$ there exists $u_i \in R$ such $a_i = au_i$. We put $z = \sum_{i\in J} u_i e_i$. Then $\phi(x) = az$. Since there is at least one index $i \in J$ such that u_i is a unit, then $(0 : z) = 0$. It follows that $(0 : \phi(x)) = (0 : a)$. But $(0 : x) = (0 : \phi(x))$ because ϕ is a monomorphism. We have $x = a\pi(z)$. So, by [4, Lemma 2] $(0 : \pi(z)) = a(0 : x) = a(0 : a) = 0$. The converse and the last assertion are obvious. □

Let R be a valuation ring and let M be a non-zero R-module. A submodule N of M is said to be **pure-essential** if it is a pure submodule and if 0 is the only submodule K satisfying $N \cap K = 0$ and $(N + K)/K$ is a pure submodule of M/K. An R-module E is said to be **pure-injective** if for any pure-exact sequence $0 \to N \to M \to L \to 0$, the following sequence is exact:

$$0 \to \mathrm{Hom}_R(L, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(N, E) \to 0.$$

We say that E is a **pure-injective hull** of K if E is pure-injective and K is a pure-essential submodule of E. We say that R is **maximal** if every family of cosets $\{a_i + L_i \mid i \in I\}$ with the finite intersection property has a non-empty intersection (here $a_i \in R$, L_i denote ideals of R, and I is an arbitrary index set).

Proposition 4.2. *Let R be a valuation ring and let M be a non-zero R-module. Then the following conditions are equivalent:*

1. *M is a flat content module;*
2. *M is flat and contains a pure-essential free submodule.*

Moreover, if these conditions are satisfied, then any element of M is contained in a pure cyclic free submodule L of M. If R is maximal then M is locally projective.

Proof. (1) ⇒ (2). Let $0 \neq x \in M$. Then $x = \sum_{1\leq i\leq n} a_i x_i$ where $a_i \in \mathrm{c}(x)$ and $x_i \in M$, $\forall i$, $1 \leq i \leq n$. Since R is a valuation ring $\exists a \in R$ such that $Ra = Ra_1 + \cdots + Ra_n$. So, we get that $\mathrm{c}(x) = Ra$ and $x = ay$ for some $y \in M$. Thus $y \notin PM$ else $\mathrm{c}(x) \subset Ra$. So $PM \neq M$ and we can apply Proposition 3.5. It remains to show that N is a pure-essential submodule of M. Let $x \in M$ such that $Rx \cap N = 0$ and N is a pure submodule of M/Rx. There exist $b \in R$ and $y \in M \setminus PM$ such that $x = by$. Since $M = N + PM$, we have $y = n + pm$ where $n \in N$, $m \in M$ and $p \in P$. Then $n \notin PN$ and $bpm = -bn + x$. Since N is pure in M/Rx there exist $n' \in N$ and $t \in R$ such that $bpn' = -bn + tx$. We get that $b(n + pn') \in N \cap Rx = 0$. So $b = 0$ because $n + pn' \notin PN$. Hence $x = 0$.

(2) ⇒ (1). First we show that M is a content module if each element x of M is of the form $s(y + cz)$, where $s \in R$, $y \in N \setminus PN$, $c \in P$ and $z \in M$. Since N is a pure submodule, $PM \cap N = PN$ whence $y \notin PM$. If $x = stw$ with $t \in P$ and $w \in M$ we get that $s(y + cz - tw) = 0$ whence $y \in PM$ because M is flat. This is a contradiction. Consequently $\mathrm{c}(x) = Rs$ and M is content. Now we prove that each element x of M is of the form $s(y + cz)$, where $s \in R$, $y \in N \setminus PN$, $c \in P$ and $z \in M$. If $x \in N$, then we check this property by using a basis of N. Suppose $x \notin N$ and $Rx \cap N \neq 0$. There exists $a \in P$ such that $0 \neq ax \in N$. Since N is pure, there exists $y' \in N$ such that $ax = ay'$. We get $x = y' + bz$ for some $b \in (0 : a)$ and $z \in M$, because M is flat. We have $y' = sy$ with $s \in R$ and $y \in N \setminus PN$. Since $as \neq 0$, $b = sc$ for some $c \in P$. Hence $x = s(y + cz)$. Now suppose that $Rx \cap N = 0$. Since N is pure-essential in M, there exist $r \in R$ and $m \in M$ such that $rm \in N + Rx$ and $rm \notin rN + Rx$. Hence $rm = n + tx$ where $n \in N$ and $t \in R$.

Thus $n = by'$ where $b \in R$ and $y' \in N \setminus PN$. Then $b \notin rR$. So, $r = bc$ for some $c \in P$. We get $bcm = by' + tx$. If $t = bd$ for some $d \in P$ then $b(cm - y' - dx) = 0$. Since M is flat, it follows that $y' \in PM \cap N = PN$. But this is false. So $b = st$ for some $s \in R$. We obtain $t(x + sy' - scm) = 0$. Since M is flat and $tsc \neq 0$ there exists $z \in M$ such that $x = s(-y' + cz)$.

Let $y \in M$. There exists $x \in M \setminus PM$ such that $y \in Rx$. We may assume that $x + PM$ is an element of a basis $(x_i + PM)_{i \in I}$ of M/PM. Then Rx is a summand of the free pure submodule N generated by the family $(x_i)_{i \in I}$.

Assume that R is maximal. Let the notations be as above. By [9, Theorem XI.4.2] each uniserial R-module is pure-injective. So, Rx is a summand of M. Let u be the composition of a projection from M onto Rx with the isomorphism between Rx and R. Thus $u \in M^*$ and $u(x) = 1$. It follows that $y = u(y)x$. Hence M is locally projective by [11, Theorem 3.2] or [22, Theorem 2.1]. □

Proposition 4.3. *Let R be a valuation ring such that $Z = P \neq 0$ and let M be a non-zero R-module. Then the following conditions are equivalent:*

1. *M is singly projective;*
2. *M is a flat content module;*
3. *M is flat and contains an essential free submodule.*

Proof. (2) ⇒ (1) by Proposition 1.1.

(1) ⇒ (2). It remains to show that M is a content module. Let $x \in M$. There exists $y \in M$ and $a \in R$ such that $x = ay$ and $(0 : y) = 0$. Since $Z = P$ then $y \notin PM$. We deduce that $\mathrm{c}(x) = Ra$.

(2) ⇔ (3). By Proposition 4.2 it remains to show that (2) ⇒ (3). Let N be a pure-essential free submodule of M. Since R is self FP-injective by [12, Lemma], it follows that N is a pure submodule of each overmodule. So, if K is a submodule of M such that $K \cap N = 0$, then N is a pure submodule of M/K, whence $K = 0$. □

Corollary 4.4. *Let R be a valuation ring. The following conditions are equivalent:*

1. *$Z = P$;*
2. *Each singly projective module is a content module.*

Proof. It remains to show that (2) ⇒ (1). By Proposition 1.7 Q is finitely projective over R. If $R \neq Q$, then Q is not content on R because, $\forall x \in Q \setminus Z$, $\mathrm{c}(x) = Z$. So $Z = P$. □

Corollary 4.5. *Let R be a valuation ring. Then the injective hull of any singly projective module is singly projective too.*

Proof. Let N be a non-zero singly projective module. We denote by E its injective hull. For each $s \in R \setminus Z$ the multiplication by s in N is injective, so the multiplication by s in E is bijective. Hence E is a Q-module which is flat by Proposition 3.2. It is an essential extension of N_Z. From Propositions 4.3 and 1.7(2) we deduce that E is singly projective. □

Let R be a valuation ring and let M be a non-zero R-module. We say that M is **separable** if any finite subset is contained in a summand which is a finite direct sum of uniserial submodules.

Corollary 4.6. *Let R be a valuation ring. Then any element of a singly projective module M is contained in a pure uniserial submodule L. Moreover, if R is maximal, each singly projective module is separable.*

Proof. By Proposition 4.2 any element of M is contained in a pure cyclic free Q-submodule G of M_Z. We put $L = M \cap G$. As in proof of Proposition 3.4 we show that L is a pure uniserial submodule of M. The first assertion is proved.

Since R is maximal L is pure-injective by [9, Theorem XI.4.2]. So, L is a summand of M. Each summand of M is singly projective. It follows that we can complete the proof by induction on the cardinal of the chosen finite subset of M. □

Corollary 4.7. *Let R be a valuation ring. Then the following conditions are equivalent:*

(1) *R is self injective;*
(2) *Each singly projective module is locally projective;*
(3) *$Z = P$ and each singly projective module is finitely projective.*

Proof. (1) ⇒ (2) by Proposition 1.3.

(2) ⇒ (3) follows from [11, Proposition 5.14(4)] and Corollary 4.4.

(3) ⇒ (1). By way of contradiction suppose that R is not self injective. Let E be the injective hull of R. By Corollary 4.5 E is singly projective. Let $x \in E \setminus R$ and $M = R + Rx$. Since E is finitely projective, then there exist a finitely generated free module F, a morphism $\phi : M \to F$ and a morphism $\pi : F \to E$ such that $(\pi \circ \phi)(y) = y$ for each $y \in M$. Let $\tilde{\phi} : M/R \to F/\phi(R)$ and $\tilde{\pi} : F/\phi(R) \to E/R$ be the morphisms induced by ϕ and π. Then $(\tilde{\pi} \circ \tilde{\phi})(y + R) = y + R$ for each $y \in M$. Since $\phi(R)$ is a pure submodule of F, then $F/\phi(R)$ is a finitely generated flat module. Hence $F/\phi(R)$ is free and E/R is singly projective. But $E/R = P(E/R)$ by [4, Lemma 12]. This contradicts that E/R is a flat content module. By Proposition 4.3 we conclude that $E = R$. □

Corollary 4.8. *Let R be a valuation ring. Then Q is self injective if and only if each singly projective module is finitely projective.*

Proof. By [17, Theorem 2.3] Q is maximal if and only if it is self injective. Suppose that Q is self injective and let M be a singly projective R-module. Then M_Z is locally projective over Q by Proposition 1.7(1) and corollary 4.7. Consequently M is finitely projective by Lemma 1.5.

Conversely let M be a singly projective Q-module. Then M is singly projective over R, whence M is finitely projective over R. If follows that M is finitely projective over Q. From Corollary 4.7 we deduce that Q is self injective. □

Theorem 4.9. *Let R be a valuation ring. Then the following conditions are equivalent:*

(1) *R is maximal;*
(2) *each singly projective R-module is separable;*
(3) *each flat content module is locally projective.*

Proof. (1) ⇒ (2) by Corollary 4.6 and (1) ⇒ (3) by Proposition 4.2.

(2) ⇒ (1) : let $\widehat{R}$ be the pure-injective hull of R. By [5, Proposition 1 and 2] $\widehat{R}$ is a flat content module. Consequently 1 belongs to a summand L of $\widehat{R}$ which is a finite direct sum of uniserial modules. But, by [7, Proposition 5.3] $\widehat{R}$ is indecomposable. Hence $\widehat{R}$ is uniserial. Suppose that $R \neq \widehat{R}$. Let $x \in \widehat{R} \setminus R$. Then there exists $c \in P$ such that $1 = cx$. Since R is pure in $\widehat{R}$ we get that $1 \in P$ which is absurd. Consequently, R is a pure-injective module. So, R is maximal by [21, Proposition 9].

(3) ⇒ (1) : since $\widehat{R}$ is locally projective then R is a summand of $\widehat{R}$ which is indecomposable. So R is maximal. □

A submodule N of a module M is said to be **strongly pure** if, $\forall x \in N$ there exists an homomorphism $u : M \to N$ such that $u(x) = x$. Moreover, if N is pure-essential, we say that M is a **strongly pure-essential extension** of N.

Proposition 4.10. *Let R be a valuation ring and let M be a flat R-module. Then M is locally projective if and only if it is a strongly pure-essential extension of a free module.*

Proof. Let M be a non-zero locally projective R-module. Then M is a flat content module. So M contains a pure-essential free submodule N. Let $x \in N$. There exist $u_1, \dots, u_n \in M^*$ and $y_1, \dots, y_n \in M$ such that $x = \sum_{i=1}^n u_i(x)y_i$. Since N is a pure submodule, $y_1, \dots, y_n$ can be chosen in N. Let $\phi : M \to N$ be the homomorphism defined by $\phi(z) = \sum_{i=1}^n u_i(z)y_i$. Then $\phi(x) = x$. So, N is a strongly pure submodule of M.

Conversely, assume that M is a strongly pure-essential extension of a free submodule N. Let $x \in M$. As in proof of Proposition 4.2, $x = s(y + cz)$, where $s \in R,\ y \in N \setminus PN,\ c \in P$ and $z \in M$. Since N is strongly pure, there exists a morphism $\phi : M \to N$ such that $\phi(y) = y$. Let $\{e_i \mid i \in I\}$ be a basis of N. Then $y = \sum_{i\in J} a_i e_i$ where J is a finite subset of I and $a_i \in R, \forall i \in J$. Since $y \in N \setminus PN$ there exists $j \in J$ such that $a_j \notin P$. We easily check that $\{y, e_i \mid i \in I,\ i \neq j\}$ is a basis of N too. Hence Ry is a summand of N. Let u be the composition of ϕ with a projection of N onto Ry and with the isomorphism between Ry and R. Then $u \in M^*,\ u(y) = 1$ and $u(y+cz) = 1+cu(z) = v$ is a unit. We put $m = v^{-1}(y+cz)$. It follows that $x = u(x)m$. Hence M is locally projective by [11, Theorem 3.2] or [22, Theorem 2.1]. □

Corollary 4.11. *Let R be a valuation ring and let M be a locally projective R-module. If M/PM is finitely generated then M is free.*

Theorem 4.12. *Let R be a valuation ring. The following conditions are equivalent:*

1. *Z is nilpotent;*
2. *Q is an artinian ring;*
3. *Each flat R-module is finitely projective;*
4. *Each flat R-module is singly projective.*

Proof. (1) ⇔ (2). If Z is nilpotent then $Z^2 \neq Z$. It follows that Z is finitely generated over Q and it is the only prime ideal of Q. So, Q is artinian. The converse is well known.

(2) ⇒ (3) is a consequence of [20, Corollary 7] and it is obvious that (3) ⇒ (4).

(4) ⇒ (2). First we prove that each flat Q-module is singly projective. By Proposition 4.3 it follows that each flat Q-module is content. We deduce that Q is perfect by Theorem 1.2. We conclude that Q is artinian since Q is a valuation ring. □

5. Strongly coherence or π-coherence of valuation rings.

In this section we study the valuation rings, with non-zero zero-divisors, for which any product of content (respectively singly, finitely, locally projective) modules is content (respectively singly, finitely, locally projective) too.

Theorem 5.1. *Let R be a valuation ring such that $Z \neq 0$. Then the following conditions are equivalent:*

1. *Each product of content modules is content;*
2. *R^R is a content module;*
3. *For each ideal A there exists $a \in R$ such that either $A = Ra$ or $A = Pa$;*
4. *Each ideal is countably generated and $R^{\mathbb{N}}$ is a content module;*
5. *The intersection of any non-empty family of principal ideals is finitely generated.*

Proof. The conditions (1), (2) and (5) are equivalent by [11, Theorem 5.15]. By [4, Lemma 29] (3) $\Leftrightarrow$ (5).

(2) $\Rightarrow$ (4). It is obvious that $R^{\mathbb{N}}$ is a content module. Since (2) $\Leftrightarrow$ (3) then P is the only prime ideal. We conclude by [4, Corollary 36].

(4) $\Rightarrow$ (3). Let A be a non-finitely generated ideal. Let $\{a_n \mid n \in \mathbb{N}\}$ be a spanning set of A. Then $x = (a_n)_{n\in\mathbb{N}} \in R^{\mathbb{N}}$. It follows that $x = ay$ for some $a \in \mathrm{c}(x)$ and $y \in R^{\mathbb{N}}$, and $\mathrm{c}(x) = Ra$. So, if $y = (b_n)_{n\in\mathbb{N}}$, we easily check that P is generated by $\{b_n \mid n \in \mathbb{N}\}$. Hence $A = aP$. $\square$

By Proposition 1.1 each valuation domain R verifies the first two conditions of the next theorem.

Theorem 5.2. *Let R be a valuation ring such that $Z \neq 0$. Then the following conditions are equivalent:*

1. *Each product of singly projective modules is singly projective.*
2. *R^R is singly projective;*
3. *C^* is a finitely generated module for each cyclic module C;*
4. *$(0 : A)$ is finitely generated for each proper ideal A;*
5. *P is principal or faithful and for each ideal A there exists $a \in R$ such that either $A = Ra$ or $A = Pa$;*
6. *Each ideal is countably generated and $R^{\mathbb{N}}$ is singly projective;*
7. *Each product of flat content modules is flat content;*
8. *R^R is a flat content module;*
9. *Each ideal is countably generated and $R^{\mathbb{N}}$ is a flat content module;*
10. *P is principal or faithful and the intersection of any non-empty family of principal ideals is finitely generated.*

Proof. It is obvious that (1) $\Rightarrow$ (2) and (7) $\Rightarrow$ (8).

(3) $\Leftrightarrow$ (4) because $(0 : A) \cong (R/A)^*$.

(2) $\Rightarrow$ (4). Let A be a proper ideal. Then R^A is singly projective too and $x = (a)_{a\in A}$ is an element of R^A. Therefore x belongs to a cyclic free submodule of R^A by Lemma 4.1. Since R^R is flat, R is coherent by [10, Theorem IV.2.8]. Consequently $(0 : A) = (0 : x)$ is finitely generated.

(4) $\Rightarrow$ (5). Then R is coherent because R is a valuation ring. Since $Z \neq 0$, $Z = P$ by [4, Theorem 10]. If P is not finitely generated then P cannot be an annihilator. So P is faithful. By [12, Lemma 3] and [17, Proposition 1.3], if A is a proper ideal then either $A = (0 : (0 : A))$ or $A = P(0 : (0 : A))$. By (4), $(0 : (0 : A)) = Ra$ for some $a \in P$.

(5) $\Rightarrow$ (1). Let $(M_i)_{i\in I}$ be a family of singly projective modules. Let $x = (x_i)_{i\in I}$ be an element of $\Pi_{i\in I} M_i$. Since M_i is singly projective for each $i \in I$ there exist $a_i \in R$ and $y_i \in M_i$ such that $x_i = a_i y_i$ and $(0 : y_i) = 0$. We have either

$\sum_{i\in I} Ra_i = Ra$ or $\sum_{i\in I} Ra_i = Pa$ for some $a \in R$. Then, $\forall i \in I$, $\exists b_i \in R$ such that $a_i = ab_i$. Therefore either $\exists i \in I$ such that b_i is a unit, or $P = \sum_{i\in I} Rb_i$. It follows that $x = ay$ where $y = (b_i y_i)_{i\in I}$. Now it is easy to check that $(0 : y) = 0$.

(6) $\Rightarrow$ (4). Since each ideal is countably generated then so is each submodule of a finitely generated free module. So, the flatness of $R^{\mathbb{N}}$ implies that R is coherent. Let A be a proper ideal generated by $\{a_n \mid n \in \mathbb{N}\}$. Then $x = (a_n)_{n\in\mathbb{N}}$ is an element of $R^{\mathbb{N}}$. Therefore x belongs to a cyclic free submodule of $R^{\mathbb{N}}$ by Lemma 4.1. Consequently $(0 : A) = (0 : x)$ is finitely generated because R is coherent.

(5) $\Rightarrow$ (9). By Theorem 5.1((3) $\Leftrightarrow$ (4)) it remains to show that $R^{\mathbb{N}}$ is flat. This is true because (5) $\Rightarrow$ (1).

(1) $\Leftrightarrow$ (7). Since (1) $\Rightarrow$ (2) or (7) $\Rightarrow$ (8), R is coherent. From $Z \neq 0$ and [4, Theorem 10] it follows that $Z = P$. Now we use Proposition 4.3 to conclude.

(2) $\Leftrightarrow$ (8). Since R^R is flat, R is coherent. We do as above to conclude.

(6) $\Leftrightarrow$ (9). Since each submodule of a free module of finite rank is countably generated, then the flatness of $R^{\mathbb{N}}$ implies that R is coherent. So we conclude as above.

(5) $\Leftrightarrow$ (10) by Theorem 5.1((3) $\Leftrightarrow$ (5)).

The last assertion is already shown. So, the proof is complete. □

Remark 5.3. When R is a valuation domain, the conditions (5), (7), (8), (9) and (10) are equivalent by [16, Theorem 4] and [4, Corollary 36].

Remark 5.4. If R satisfies the conditions of Theorem 5.1 and if P is not faithful and not finitely generated then R is not coherent and doesn't satisfy the conditions of Theorem 5.2.

By [8, Corollary 3.5] or [16, Theorem 3], a valuation domain R is strongly coherent if and only if either its order group is $\mathbb{Z}$ or if R is maximal and its order group is $\mathbb{R}$. It is easy to check that each Prüfer domain is π-coherent because it satisfies the fourth condition of the next theorem. When R is a valuation ring with non-zero zero-divisors we get:

Theorem 5.5. *Let R be a valuation ring such that $Z \neq 0$. Then the following conditions are equivalent:*

1. *R is strongly coherent;*
2. *R is π-coherent;*
3. *R^R is singly projective and separable;*
4. *C^* is a finitely generated module for each finitely generated module C;*
5. *$(0 : A)$ is finitely generated for each proper ideal A and R is self injective;*
6. *R is maximal, P is principal or faithful and for each ideal A there exists $a \in R$ such that either $A = Ra$ or $A = Pa$;*
7. *Each ideal is countably generated and $R^{\mathbb{N}}$ is singly projective and separable;*
8. *R^R is a separable flat content module;*
9. *Each ideal is countably generated and $R^{\mathbb{N}}$ is a separable flat content module;*
10. *Each product of separable flat content modules is a separable flat content module;*
11. *R is maximal, P is principal or faithful and the intersection of any non-empty family of principal ideals is finitely generated.*

Proof. By Theorem 2.1 (1) $\Rightarrow$ (2). It is obvious that (10) $\Rightarrow$ (8). By [3, Theorem 1] (2) $\Leftrightarrow$ (4). By Theorem 5.2, Theorem 4.9 and [17, Theorem 2.3] (5) $\Leftrightarrow$ (6) and (6) $\Rightarrow$ (7) . By Theorem 5.2 (6) $\Leftrightarrow$ (11), (7) $\Leftrightarrow$ (9) and (3) $\Leftrightarrow$ (8).

(4) $\Rightarrow$ (6). By Theorem 5.2 R is coherent and self FP-injective and it remains to prove that R is maximal if P is not principal. Let E be the injective hull of R. If $R \neq E$ let $x \in E \setminus R$. Since R is an essential submodule of E, $(R : x) = rP$ for some $r \in R$. Then $(R : rx) = P$. Let M be the submodule of E generated by 1 and rx. We put $N = M/R$. Then $N \cong R/P$. We get that $N^* = 0$ and M^* is isomorphic to a principal ideal of R. Moreover, since $(R : rx) = P$, for each $t \in P$ the multiplication by t in M is a non-zero element of M^*. Since P is faithful we get that $M^* \cong R$. Let $g \in M^*$ such that the restriction of g to R is the identity. For each $p \in P$ we have $pg(rx) = prx$. So $(0 : g(rx)-rx) = P$. Since P is faithful, there is no simple submodule in E. Hence $g(rx) = rx$ but this is not possible because $g(rx) \in R$ and $rx \notin R$. Consequently R is self-injective and maximal.

(2) $\Rightarrow$ (1). Since (2) $\Rightarrow$ (6) R is self injective. We conclude by proposition 1.3.

(3) $\Rightarrow$ (1). Since R^R is singly projective, by Theorem 5.2 R is coherent and self FP-injective. So, if U is a uniserial summand of R^R, then U is singly projective and consequently $U \neq PU$. Let $x \in U \setminus PU$. It is easy to check that $U = Rx$ and that $(0 : x) = 0$. Hence R^R is locally projective and R is strongly coherent.

(7) $\Rightarrow$ (4). Let $F_1 \to F_0 \to C \to 0$ be a free presentation of a finitely generated module C, where F_0 is finitely generated. It follows that F_1 is countably generated. As above we prove that $R^{\mathbb{N}}$ is locally projective. By Theorem 5.2 R is coherent and consequently each finitely generated submodule of $R^{\mathbb{N}}$ is finitely presented. Since $F_1^* \cong R^{\mathbb{N}}$ we easily deduce that C^* is finitely generated.

(1) $\Rightarrow$ (10). Since (1) $\Rightarrow$ (6), R is maximal. We use Theorem 4.9 to conclude. The proof is now complete. □

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