

# Superluminal motion and closed signal curves

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**ABSTRACT:** We discuss some properties of Lorentz invariant theories which allow for superluminal motion. We show that, if signals are always sent forward in time, closed curves along which signals propagate can be formed. This leads to problems with causality and with the second law of thermodynamics. Only if one singles out one frame with respect to which all signals travel forward in time, the formation of 'closed signal curves' can be prevented. However, the price to pay is that in some reference frames perturbations propagate towards the past or towards the future, depending on the direction of emission.

**KEYWORDS:** Cosmology of theories beyond the SM.

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## 1. Introduction

It is a matter of debate in the literature whether a theory that admits superluminal propagation is acceptable [1, 2, 3, 4, 5, 6, 7, 8, 9]. It has been argued that superluminal motion needs not lead to closed 'timelike' curves, and is therefore not problematic. Furthermore, it has been put forward that perturbations on a background which is not Lorentz invariant (i.e., around a solution of the equations of motion which breaks Lorentz invariance) can very well propagate faster than the speed of light, without leading to serious problems with causality.

In this paper we first show that whenever the Lagrangian for a field is such that field modes can propagate at superluminal speeds, closed curves along which a signal propagates can be constructed. We call them 'closed signal curves' or short CSC's. In a next step we show that, for a fixed cosmological background solution, the same result holds if one requires that observers can send signals only forward in time, i.e., a forward time direction exists unambiguously in each reference frame. Only if we require that all signals, independently from the frame with respect to which they have been emitted, travel forward with respect to the time of the cosmological reference frame, we can avoid the possibility of CSC's. However, this goes at the cost that observers traveling at high (but sub-luminal) speed with respect to the cosmological frame must send signals backwards in their time for some specific directions. In other words, fluctuations in these frames propagate sometimes with the advanced and sometimes with the retarded Green function.

It seems clear to us, that in a universe with closed signal curves, physics, as we know it, is no longer possible. For example, the second law of thermodynamics is violated, since after one turn in a closed loop, the original state of the system must be re-established hence entropy cannot have grown. If these loops are of Planckian size or much larger than the age of the universe, there may be a way out of contradiction with every day experience, but if the loops can be of mm or cm size, this becomes very difficult. Especially, it is not clear to us whether thermodynamics in general, and the concept of entropy in particular, can still make sense in a universe with closed signal curves.

The point of the present note is to show that theories which do admit superluminal motion, either admit closed signal curves or force some observers to send signals backwards in their time. This finding is independent of the fact whether or not the background breaks Lorentz invariance.

In the next section we construct closed signal curves in a field theory which allows for superluminal motion. We discuss our result and show that it can be avoided by additional assumptions if we have a preferred reference frame, like in cosmology. We also formulate the conditions under which scalar field Lagrangians allow superluminal motion. In Section III we discuss in more detail the cosmological situation concentrating especially on the example of k-essence [10, 11, 12, 13] and in Section IV we conclude. The speed of light is  $c = 1$  and we use the metric signature  $(+, -, -, -)$ .

## 2. Closed signal curves from superluminal velocities

It is well known that covariant Lagrangians can lead to superluminal motion. To be specific and to simplify matters, let us consider the Lagrangian of a scalar field  $\phi$ , leading to a covariant equation of motion of the form

$$G^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \text{lower order terms} , \quad (2.1)$$

where  $G^{\mu\nu}$  is a symmetric tensor field given by  $\phi$  and other degrees of freedom. It need not be the spacetime metric. If  $G^{\mu\nu}$  is non-degenerate and has Lorentzian signature, Eq. (2.1) is a hyperbolic equation of motion. We assume this to be the case (see [12] for a discussion about this issue). The null-cone of the co-metric  $G^{\mu\nu}$  is the characteristic cone of this equation. The rays are defined by the 'metric'  $(G^{-1})_{\mu\nu}$  such that  $G^{\mu\nu}(G^{-1})_{\nu\lambda} = \delta^\mu_\lambda$ . The characteristic cone limits the propagation of field modes in the sense that the value of the field at some event  $q$  is not affected by the values outside the past characteristic cone and, on the other hand, that the value at  $q$  cannot influence the field outside the future characteristic cone [14].

For very high frequencies, the lower order terms are subdominant and the field propagates along the characteristic cone. At lower frequencies, lower order terms act similarly to an effective mass and the field propagates inside the characteristic cone. We now show that, closed signal curves can be constructed if this cone is wider than the light cone defined by the spacetime metric  $g_{\mu\nu}$ .

If the characteristic cone of  $G_{\mu\nu}$  is wider than the light cone, the maximal propagation velocity  $v_{\max}$  of the field  $\phi$ , which satisfies  $G_{\mu\nu}v_{\max}^\mu v_{\max}^\nu = 0$ , is spacelike (with respect to  $g_{\mu\nu}$ ). Since the notion 'spacelike' is frame independent, this is true in every reference frame. Of course, the characteristic cone for  $\phi$  is not invariant under Lorentz transformations, but the fact that it is spacelike is.

We consider two reference frames  $R$  and  $R'$  with common origin  $q_0$ :  $(0, 0) = (t, x) = (t', x')$ .  $R'$  is boosted with respect to  $R$  in  $x$ -direction with velocity  $v < 1$ . For an event  $q$  with coordinates  $(t, x)$  in  $R$  and  $(t', x')$  in  $R'$  we have the usual transformation laws

$$\begin{aligned} t' &= \gamma(t - vx), & x' &= \gamma(x - vt), & \gamma &= \frac{1}{\sqrt{1 - v^2}}, \\ t &= \gamma(t' + vx'), & x &= \gamma(x' + vt'). \end{aligned} \quad (2.2)$$

We assume that  $v$  is sufficiently large such that the superluminal velocity  $1/v$  in  $x$ -direction is inside the characteristic cone of  $\phi$ . We can then send a  $\phi$ -signal from  $(0, 0)$  with speed

$v_1 > 1/v$  in  $x$ -direction to the event  $q_1$ , see Fig. 1. The signal is received in  $x_1$  at time  $t_1 = x_1/v_1$ . In  $R'$  this event has the coordinates  $t'_1 = \gamma t_1(1 - vv_1) < 0$  and  $x'_1 = \gamma t_1(v_1 - v) > 0$ . Note also that  $v'_1 = x'_1/t'_1 = (v_1 - v)/(1 - vv_1) < 0$ , and the signal is propagating into the past of  $R'$ .

We can choose  $x_1$  and correspondingly  $t_1$  very small so that curvature is negligible on these scales and we may identify the spacetime manifold with its tangent space at  $(0, 0)$ . In other words we want to choose these dimensions sufficiently small so that we may neglect the position dependence of both, the light cone and the characteristic cone for  $\phi$ . The situation is then exactly analogous to the one of special relativity.

An observer in the frame  $R'$  now receives the signal emitted at  $(0, 0)$  in  $R$  and returns it with velocity  $v'_2$  to  $x_2 = 0 = \gamma(x'_2 + vt'_2)$ . We denote the arrival event by  $q_2$ . It has the coordinates  $(t_2, 0)$  with respect to  $R$  and  $(t'_2, x'_2)$  with respect to  $R'$ . Since  $v'_2 \neq v'_1$  in order for a CSC to form, we may have to transform the signal to another frequency to allow it to travel with speed  $v'_2$ . If the returned signal arrives at some time  $t_2 < 0$ , the observer in  $R$  which has received the signal simply stores it until the time  $|t_2|$  has elapsed after which this signal has propagated along the closed curve  $q_0 \rightarrow q_1 \rightarrow q_2 \rightarrow q_0$  shown in Fig. 1, a CSC has been generated.

Let us elaborate on the requirement  $t_2 < 0$ . We assume here that an arbitrary observer can send signals only into her future, so that  $t'_2 > t'_1$ . Hence we want to choose  $v'_2$  such that even though  $\Delta t' = t'_2 - t'_1 > 0$ , we have  $\Delta t = t_2 - t_1 < 0$ . When sending a signal with speed  $v_2$  in frame  $R$ , respectively  $v'_2$  in  $R'$ , the times which elapse while the signal travels a distance  $\Delta x$  respectively  $\Delta x'$  are related by  $(\Delta x' = v'_2 \Delta t')$

$$\Delta t = t_2 - t_1 = \gamma(t'_2 + vx'_2 - t'_1 - vx'_1) = \gamma(\Delta t' + v\Delta x') = \gamma(1 + vv'_2)\Delta t' , \quad (2.3)$$

$$\Delta t' = \gamma(1 - vv_2)\Delta t . \quad (2.4)$$

In order to achieve  $\Delta t < 0$  and at the same time  $\Delta t' > 0$  we need  $vv_2 > 1$ , hence  $v_2 > 1/v > 1$ .

From Fig. 1 it is evident that  $v_2$  which is the inverse of the slope of the line from  $q_2$  to  $q_1$  is smaller than  $v_1$  which is the inverse of the slope from  $q_0$  to  $q_1$ . This is also obtained from

$$0 < v_2 = \frac{-x_1}{t_2 - t_1} = \frac{x_1}{t_1 - t_2} < \frac{x_1}{t_1} = v_1 .$$

For the inequality sign we have used  $t_2 < 0$ . Therefore, also  $v_2$  is inside the characteristic cone of  $\phi$  and is admitted as a propagation velocity. Note that since  $v_2 > 1$  the distance between the events  $q_1$  and  $q_2$  is spacelike. Also in the reference frame  $R'$ ,  $v'_1 > v'_2$ , but both these velocities are negative hence for the absolute values we have  $|v'_2| > |v'_1|$ .

## 2.1 Lagrangians which allow for superluminal motion

We now identify scalar field Lagrangians which allow for superluminal motion leading to the causal problem discussed above. Consider a Lagrangian characterized by a non-standard kinetic term, with the action

$$S = \int d^4x \sqrt{-g} \left[ -\frac{R}{6} + K(\phi)p(X) - V(\phi) \right] , \quad (2.5)$$

were  $\phi$  is the scalar field (for example, a tachyon [15], the k-essence field [10], or the k-inflaton [16]).  $X = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$  is the kinetic energy; we use units with  $\frac{8\pi G}{3} = 1$ . The

equation of motion for  $\phi$  is given by

$$Kp_{,X}G^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi = -K_{,\phi}p - V_{,\phi} - 2Xp_{,X}K_{,\phi} . \quad (2.6)$$

The potential term and first order derivatives are irrelevant for the characteristics of the field equation. These are given by the co-metric  $G^{\mu\nu}$ . If a prime denotes derivative with respect to  $X$ , the co-metric is

$$G^{\mu\nu} = g^{\mu\nu} + \frac{p''(X)}{p'(X)}\nabla^{\mu}\phi\nabla^{\nu}\phi . \quad (2.7)$$

As discussed above, for the signal not to propagate faster than the speed of light, the characteristic cone should not lie outside the metric cone. This means that the unit normal to the characteristics  $S_{\mu}$  must not be timelike with respect to  $g^{\mu\nu}$  [3, 17]. The condition

$$G^{\mu\nu}S_{\mu}S_{\nu} = 0 \quad (2.8)$$

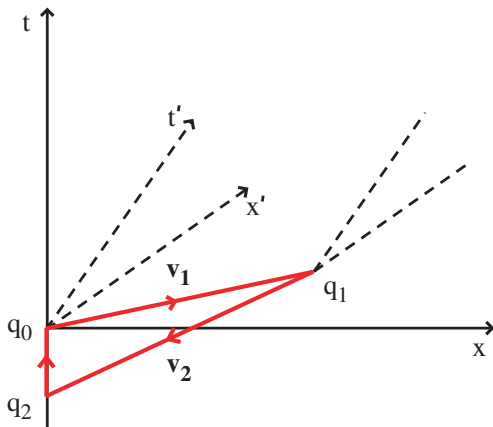
implies

$$g^{\mu\nu}S_{\mu}S_{\nu} = -\frac{p''(X)}{p'(X)}(\nabla^{\mu}\phi S_{\mu})^2 . \quad (2.9)$$

Therefore,  $S_{\mu}$  is not timelike if and only if

$$\frac{p''(X)}{p'(X)} \geq 0 . \quad (2.10)$$

Every theory that does not fulfill this conditions runs into the problem discussed above. Already in the 60ties, the appearance of superluminal motion has led to the exclusion of generic covariant higher spin  $s \geq 1$  Lagrangians [18]. Examples are the Lagrangian of a self-interacting neutral vector field, a minimally coupled spin 2 field, or the minimally coupled Rarita-Schwinger equation for a spin 3/2 particle [19].



**Figure 1:** A closed signal curve going along  $q_0 \rightarrow q_1 \rightarrow q_2 \rightarrow q_0$ .

spacetime metric), the construction leading to a CSC presented in the previous section is still possible. However, now there is in principle a way out. If we require that signals

### 3. Closed signal curves on a background

So far, we have not specified any background upon which the  $\phi$ -signal propagates. As we have seen above, the Lagrangian can be such that the presence of a non-vanishing signal is sufficient for the characteristic cone of  $\phi$  to be spacelike or, equivalently, its normal to be timelike, and hence the propagation to be superluminal.

On the other hand, one can consider the propagation of fluctuations upon a fixed background  $\phi_0$ . If  $\nabla\phi_0 \neq 0$  is timelike, this generates a preferred frame of reference, the one in which  $\nabla\phi_0$  is parallel to  $t$ . Let us call this reference frame  $R_0$ . If the null-cone of the metric  $G^{\mu\nu}(\phi_0)$  is spacelike (always with respect to the

always propagate forward in time in the frame  $R_0$ , closed signal curves become impossible. The CSC  $q_0 \rightarrow q_1 \rightarrow q_2 \rightarrow q_0$  is also closed in  $R_0$ . As it encloses a non-vanishing area it must contain both, a part where it advances in time and a part where it goes backward in time, so that it violates the requirement that the signal can only advance in time in the frame  $R_0$ .

This is the main point. In relativity, events with spacelike separations have no well defined chronology. Depending on the reference frame we are using,  $q_2$  is either before (in  $R$ ) or after (in  $R'$ )  $q_1$ . If we can send a signal from  $q_1$  to  $q_2$ , this signal travels forward in time in  $R'$  and backward in time in  $R$ . In the frame which is boosted with respect to  $R$  with velocity  $1/v_2$  in  $x$ -direction, the signal has even infinite velocity:  $q_1$  and  $q_2$  have the same time coordinate in this reference frame.

If we require every signal to travel forward with respect to the time coordinate of  $R_0$ , we shall no longer have closed curves along which a signal propagates, but we then have signals propagating into the past in the boosted reference frame  $R'$  in which they have been emitted. Moreover, the field value at some point  $q$  can now be influenced by field values in the future. This may sound very bizarre; however, as far as we can see, it is not contradictory since the events in the future which can influence  $q$  are in its spacelike future and cannot be influenced by  $q$ . On the other hand, the events in its past which  $q$  can influence are in its spacelike past and they cannot influence  $q$ , see Fig. 2. In the limit in which the maximal propagation velocity  $v_{\max}$  derived from Eq. (2.1) approaches infinity in the frame  $R_0$ , the past and future cones in the boosted reference frame will approach each other, but never overlap. The cone edge  $x' = v'_{\max} t'$  is always flatter than the one  $x' = (-v_{\max})' t'$ : one has  $(-v_{\max})' = -(v_{\max} + v)/(1 + vv_{\max})$ , and  $v'_{\max} = (v_{\max} - v)/(1 - vv_{\max})$ , which both tend to  $-1/v$  in the limit  $v_{\max} \rightarrow \infty$ . The opening angle  $\alpha$  between  $v'_{\max}$  and  $(-v_{\max})'$  is given by

$$\alpha = \frac{2v_{\max}}{v_{\max}^2 - 1} \left( \frac{1 - v^2}{1 + v^2} \right). \quad (3.1)$$

Hence  $\alpha \rightarrow 0$  if either  $v \rightarrow 1$  or  $v_{\max} \rightarrow \infty$ .

From Fig. 2 it is clear that there is no immediate contradiction since there are no points which are simultaneously in the past and future characteristic cone of  $q$ , hence no closed signal curves or CSC's are possible. The physical interpretation is however quite striking for an observer sitting at the origin of  $R'$ . When sending a signal with a velocity close to  $v_{\max}$  to the left, it naturally propagates into the observer's future; when sending it to the right, it has to propagate into her past, from where it can reach her again later at a  $t' > 0$ , when the past cone from  $(t', 0)$  intersects the future cone from  $(0, 0)$ . However, also in the boosted frame  $R'$ ,  $\phi_0$  is a solution of the equation of motion and were it not for the cosmological symmetry, there would be no reason to prefer one frame over the other.

### 3.1 k-essence

We now consider in somewhat more detail the particular example of k-essence [10, 11]. We show that k-essence signals with different wave numbers can propagate with different superluminal velocities.

The k-essence action is given by

$$S = \int d^4x \sqrt{-g} \left[ -\frac{R}{6} + \frac{p(X)}{\phi^2} \right], \quad (3.2)$$

where now  $\phi$  is the k-essence field and again  $X = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi$ .

In [13] it has been shown that, in every k-essence model which solves the coincidence problem and leads to a period of acceleration, the field has to propagate superluminally during some stage of its evolution. Therefore, k-essence can lead to the formation of CSC's. As discussed above, CSC's can be constructed using two different superluminal propagation velocities (see Fig. 1). In particular, in the frame  $R_0$  where the background is homogeneous and isotropic we need  $v_2 < v_1$ . This can be achieved because the equation of motion of k-essence perturbations contains an effective mass term which leads to dispersion. Therefore, different wave-numbers propagate with different velocities. In the following, we calculate the group velocity of the k-essence perturbations using the WKB approximation.

We split the k-essence field in the cosmic background solution and a perturbation,  $\phi = \phi_0(t) + \delta\phi(t, \mathbf{x})$ . In longitudinal gauge the metric is

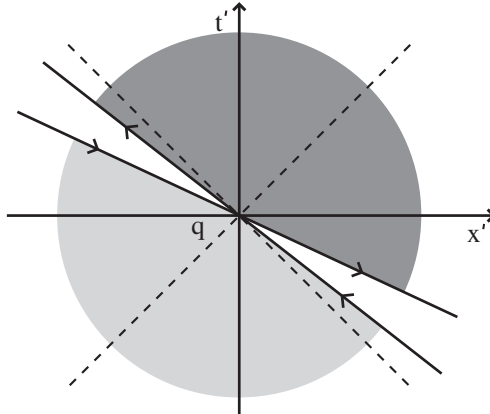
$$ds^2 = \left(1 + 2\Psi(t, \mathbf{x})\right) dt^2 - a(t)^2 \left(1 - 2\Psi(t, \mathbf{x})\right) \delta_{ij} dx^i dx^j, \quad (3.3)$$

where  $a(t)$  is the scale factor and  $\Psi(t, \mathbf{x})$  is the Bardeen potential. We restrict our calculations to the case in which k-essence is subdominant with respect to matter and radiation. This is the case when k-essence evolves from the radiation fixed point to the de Sitter fixed point (see for example Fig. 1 in [13]). As it is shown in [13], during this stage the sound velocity  $c_s^2$  has to be larger than one. The equation of motion of k-essence perturbations depends on the choice of initial conditions. One possibility is to consider standard adiabatic initial conditions, for which the ratio  $\delta\rho_i/\rho_i$  is of the same order of magnitude for matter, radiation and k-essence. Since the Bardeen potential  $\Psi$  is related to  $(\sum_i \delta\rho_i)/\rho_{\text{tot}}$ , it is sourced mainly by the perturbations in the dominant component of the universe. Therefore, when k-essence is subdominant, we can write the equation of motion for k-essence perturbations considering the Bardeen potential as an external source, which does not influence the propagation properties. This equation is of the form

$$\delta\ddot{\phi} + \alpha\dot{\delta\phi} + \beta\delta\phi + c_s^2\Delta\delta\phi = \mu\dot{\Psi} + \nu\Psi, \quad (3.4)$$

where  $\Delta\delta\phi = g^{ij}\partial_i\partial_j(\delta\phi)$ . Here the over-dot denotes derivative with respect to physical time  $t$  and  $\alpha, \beta, c_s^2, \mu$  and  $\nu$  are functions of  $t$ . Similar perturbation equations have also been derived in [20]. This equation is of the type (2.1). The wave fronts are given by the characteristics, which determine the maximal speed of signal propagation, here  $c_s$ . This sound velocity is achieved in the limit of high wave-numbers  $k \rightarrow \infty$ , and is given by

$$c_s^2 = \frac{p'}{2Xp'' + p'} \quad , \quad ' = \frac{d}{dX}. \quad (3.5)$$



**Figure 2:** The characteristic cone  $v'_{\text{max}}$  and  $(-v_{\text{max}})'$  is shown with arrows indicating the forward time direction in the preferred frame  $R_0$ . All events inside the 'backward' characteristic cone (light grey) can influence the event  $q$ , whereas  $q$  can influence all events inside the 'forward' characteristic cone (dark grey).

For the effective mass term  $\beta$ , we find

$$\beta = \frac{2\rho_k}{2Xp'' + p'} , \quad (3.6)$$

which is always positive since the energy density of k-essence,  $\rho_k = (2Xp' - p)/\phi^2$ , is positive, and  $2Xp'' + p' > 0$  in a stable theory [11]. For the damping term  $\alpha\dot{\phi}$  we find

$$\alpha = 3H \left( 1 - \frac{2Xp'(3p'' + 2Xp''')}{(p' + 2Xp'')^2} \right) - 2\frac{\dot{\phi}}{\phi} \left( 1 + \frac{(p - 2Xp')(3p'' + 2Xp''')}{(p' + 2Xp'')^2} \right) . \quad (3.7)$$

For illustrative purpose, we can now calculate the group velocity using a WKB approximation. For simplicity we neglect the source term which does not affect the propagation properties. We set

$$\delta\phi(x, t) = \int d^3k e^{ikx} \varphi(k, t) , \quad (3.8)$$

where  $t, k$  are the physical time and momentum. The Fourier transformed function satisfies the equation

$$\ddot{\varphi} + \alpha\dot{\varphi} + (\beta + c_s^2 k^2)\varphi = 0 . \quad (3.9)$$

In order to put this equation in a form suitable for the WKB approximation, we perform the substitution

$$\varphi(k, t) = e^{-\int_0^t \frac{\alpha(t')}{2} dt'} A(k)W(t) , \quad (3.10)$$

so that (3.9) reduces to

$$\ddot{W}(t) + \omega^2(k, t)W(t) = 0 , \quad (3.11)$$

where we have identified

$$\omega^2(k, t) \equiv \beta + c_s^2 k^2 - \frac{\alpha^2}{4} - \frac{\dot{\alpha}}{2} . \quad (3.12)$$

We define the effective mass term

$$m^2 \equiv \beta - \frac{\alpha^2}{4} - \frac{\dot{\alpha}}{2} . \quad (3.13)$$

Within the WKB approximation we neglect the time derivatives of  $c_s^2$  and of  $m^2$ ,  $\dot{c}_s/c_s \ll \omega$  and  $\dot{m}/m \ll \omega$ , yielding the approximate solution

$$\delta\phi(x, t) = e^{-\int_0^t \frac{\alpha}{2} dt'} \int d^3k \frac{A(k)}{\sqrt{\omega(k, t)}} e^{ikx - i \int_0^t \omega(k, t') dt'} . \quad (3.14)$$

As customary in the evaluation of the group velocity, we now suppose that  $A(k)$  is a function sharply peaked around a given wave-number  $k_0$ , and that it stays so for at least a few oscillations. We can therefore Taylor expand  $\omega(k, t)$  at first order both in  $k - k_0$  and in  $t$ , and within the WKB approximation we then find

$$\delta\phi(x, t) \simeq f(k_0, t) \delta\phi(y, 0) , \quad (3.15)$$

where  $f(k_0, t)$  is an irrelevant phase and

$$y = x - \partial_k \omega(k_0, 0)t . \quad (3.16)$$



The group velocity is therefore given by

$$v_g(k_0) = \partial_k \omega(k_0, 0) = \frac{c_s^2 k_0}{\sqrt{c_s^2 k_0^2 + m^2}} . \quad (3.17)$$

If  $m^2$  is positive, the velocity of the perturbation is always smaller than  $c_s$ , and approaches it in the limit  $k_0 \rightarrow \infty$ . If  $m^2$  is negative, low wave numbers with  $c_s^2 k_0^2 < -m^2$  are unstable. Because of the properties of hyperbolic equations of motion [14], we know that the maximal speed of the signal is again  $c_s$ . Therefore in this case, Eq. (3.17) no longer correctly describes the signal propagation speed.

In Fig. 3, we plot  $m^2$  and  $c_s^2$  for the k-essence Lagrangian given in ref. [10]

$$p(X) = -2.01 + 2\sqrt{1+X} + 3 \cdot 10^{-17} X^3 - 10^{-24} X^4 . \quad (3.18)$$

We see that the condition  $m^2 > 0$  (solid line) is verified for most of the region of interest given by  $c_s^2 > 1$  (dashed line). In the example considered,  $c_s^2 > 1$  after  $z \simeq 1.4 \times 10^6$  and stays so until today [13]. Note that the part where  $m^2 < 0$  (dotted line) corresponds to a stage where the background varies so quickly that in any case the WKB approximation breaks down, and our calculation does not apply any longer. If  $m^2 > 0$ , the group velocity is given by equation (3.17) for sufficiently large values of  $k_0$ . In order to construct the CSC of Fig. 1, we now simply need to choose  $k_1 > k_2$  in order to have  $v_1 = v_g(k_1) > v_2 = v_g(k_2)$ , and  $k_1, k_2$  large enough to have  $v_1, v_2 > 1/v > 1$ .

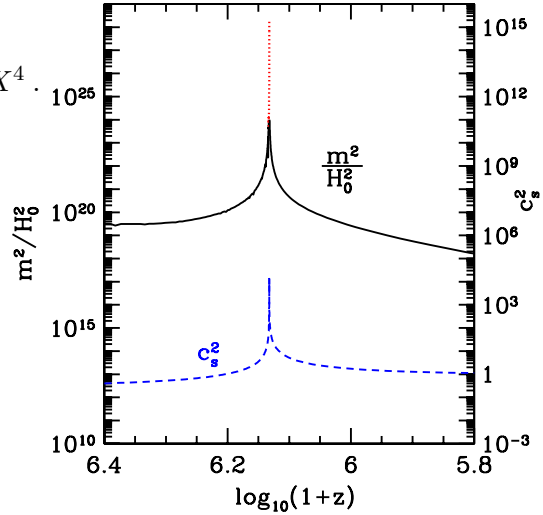
The situation is analogous if we choose non-adiabatic initial conditions where k-essence perturbations are much larger than matter and radiation perturbations. Of course the sound velocity  $c_s^2$  which only depends on the second order spatial derivatives in the equation of motion will remain the same. Therefore the fact that the theory has a speed of sound larger than the speed of light does not depend on the particular choice of the initial conditions. But the group velocity can be different in the two cases.

Combining the three Einstein equations  $00, 0i$  and  $ii$ , which relate the evolution of the k-essence perturbations  $\delta\phi$  to the Bardeen potential  $\Psi$ , we obtain a second order equation of motion for  $\Psi$  which has the form

$$\ddot{\Psi} + \tilde{\alpha}\dot{\Psi} + \tilde{\beta}\Psi + c_s^2 \Delta \Psi = 0 , \quad (3.19)$$

where now

$$\tilde{\beta} = 2\dot{H} + 2\frac{H}{p' + 2Xp''} \left( 3H(p' + Xp'') - \frac{2g(X)}{p'\phi\dot{\phi}} \right) \quad (3.20)$$



**Figure 3:** The mass term  $m^2$  and the sound velocity  $c_s^2$  as functions of redshift for the example (3.18). We have plotted the absolute value of  $m^2$  and the solid line corresponds to a positive  $m^2$  whereas the dotted line corresponds to a negative  $m^2$ .

and the damping term

$$\tilde{\alpha} = \frac{7p' + 8Xp''}{p' + 2Xp''}H - \frac{4g(X)}{(p' + 2Xp'')p'\dot{\phi}}, \quad (3.21)$$

with  $g(X) = Xp'^2 - pp' - Xpp''$ .

The above calculation of the group velocity can be straightforwardly repeated in this case. One finds for the Bardeen potential the same form of the group velocity as in (3.17), but in terms of the new effective mass  $\tilde{m}$  given by  $\tilde{m}^2 \equiv \tilde{\beta} - \frac{\tilde{\alpha}^2}{4} - \frac{\dot{\tilde{\alpha}}}{2}$ . As in the previous case, we have evaluated  $\tilde{m}^2$  for the particular Lagrangian (3.18), and we find the same qualitative behaviour as for  $m^2$  in Fig. 3.

## 4. Conclusions

We have shown that if superluminal motion is possible and if a signal emitted in some reference frame  $R'$  propagates always forward with respect to the frame time  $t'$ , closed signal curves, CSC's can be constructed. Note that these are neither closed timelike curves nor closed causal curves (timelike or lightlike) in the sense of Hawking and Ellis [21], since they contain spacelike parts. Hawking and Ellis call spacetimes which do not admit closed causal curves 'causally stable', and they show that stable causality is equivalent to the existence of a Lorentzian metric and of a function  $t$  the gradient of which is globally timelike and past-directed [21], p198ff. This condition may very well be satisfied in our case since the field  $\phi$  may be weak and the metric nearly flat.

However the relevant question is whether the existence of a global past-directed timelike gradient  $\nabla^\alpha t$  prevents also the existence of closed signal curves which are partially spacelike, as constructed in Fig. 1. As argued in [21] (see also [22]), if a past-directed timelike gradient  $\nabla^\alpha t$  exists, closed timelike or lightlike curves cannot be formed since for every future-directed timelike or lightlike curve with tangent  $v^\alpha$  the derivative of  $t$  along the curve  $g_{\alpha\beta}v^\alpha\nabla^\beta t < 0$ . This means that  $t$  can only decrease along such a curve and therefore can never return to its initial value.

The situation is different if one allows a signal to propagate along a spacelike curve, even if it remains inside a given cone defined by a Lorentzian metric  $G_{\alpha\beta}$ . Indeed the notion of 'future-directed curve' is not well defined for a spacelike curve; it depends on the reference frame. Therefore we cannot apply the same argument as before; first we have to choose a notion of 'future-directed' for spacelike curves. Let us first use the notion which has led to the CSC: We define (unambiguously) a curve to be future-directed, if a signal along this curve always propagates forward in time with respect to the reference frame in which it has been emitted. With this definition the curve from  $q_0$  to  $q_1$  as well as the one from  $q_1$  to  $q_2$  are both future-directed. But, denoting by  $v_1^\alpha$  and  $v_2^\alpha$  their tangent vectors, we clearly have  $g_{\alpha\beta}v_1^\alpha\nabla^\beta t < 0$  but  $g_{\alpha\beta}v_2^\alpha\nabla^\beta t > 0$  and  $g_{\alpha\beta}v_1^\alpha\nabla^\beta t' > 0$  but  $g_{\alpha\beta}v_2^\alpha\nabla^\beta t' < 0$ . Every timelike coordinate which grows along one part of our curve, decays along the other part. Therefore the conditions of the theorem are no longer met and one can construct closed signal curves.

On the other hand if we introduce a preferred frame and require that signals always propagate forward in time with respect to this frame, we can define the notion of future-directed curve in this frame and the theorem does apply: no closed curves can be constructed. But the price to pay is that in other reference frames emitters can send signal

which are past-directed with respect to their proper time. In the reference frame with velocity  $v = 1/v_1$  the signal even propagates with infinite velocity, which means that the 'propagation equation' is no longer hyperbolic but elliptic. In this frame the propagation of the fluctuations  $\delta\phi$  becomes non-local.

Finally, one may consider to apply the Hawking and Ellis theorem not to the spacetime metric  $g_{\alpha\beta}$  but to the metric  $G_{\alpha\beta}$ . In this case even if a curve has a superluminal velocity, it can be a timelike future-directed curve with respect to  $G_{\alpha\beta}$  and therefore  $G_{\alpha\beta}v^\alpha\nabla^\beta t < 0$ , which implies that no closed timelike (with respect to  $G_{\alpha\beta}$ ) curve can be formed. But this notion is invariant only with respect to 'Lorentz transformations' which leave  $G_{\alpha\beta}$  invariant and not the lightcone. Therefore, now the speed of light depends on the reference frame. Furthermore, local Lorentz symmetry with respect to  $G_{\alpha\beta}$  would now imply that we have to take covariant derivatives with respect to this metric. Hence it is  $G_{\alpha\beta}$  and no longer  $g_{\alpha\beta}$  which defines the structure of spacetime, and we replace general relativity by a bi-metric theory of gravity.

Hence the Hawking and Ellis theorem confirms our conclusions: if superluminal motion is possible and if a signal emitted in some reference frame  $R'$  propagates always forward in frame-time  $t'$ , closed signal curves can be constructed. These curves, even if they are not timelike or lightlike, are 'time machines'. They allow us, e.g., to influence the present with knowledge of the future. After watching the 6 numbers on TV on Saturday evening we can send this information back to Friday afternoon and enter them in our lottery bulletin.

On the other hand, if a background which defines a preferred timelike direction is present, the ruin of all lottery companies can sometimes be prevented: we just have to require that signals travel forward in time in a preferred rest frame which can be defined unambiguously if  $\nabla\phi_0$  is timelike. But this implies that in other reference frames a signal can propagate either towards the future or towards the past, depending on the direction of emission (or it can even behave non-locally).

As we have started with a Lorentz invariant Lagrangian, we would in principle expect that all solutions of the equations of motion are viable and that their perturbations have to be handled in the same way. However in order to avoid CSC's, theories that allow for superluminal motion have, in addition to the Lagrangian, to provide a rule which tells us when to take the retarded and when the advanced Green function to propagate perturbations on a background. If the background solution has no special symmetries, it is not straightforward to implement such a rule.

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