

# PHASE SPACES IN SPECIAL RELATIVITY: TOWARDS ELIMINATING GRAVITATIONAL SINGULARITIES

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ABSTRACT. This paper shows one way to construct phase spaces in special relativity by expanding Minkowski Space. These spaces appear to indicate that we can dispense with gravitational singularities. The key mathematical ideas in the present approach are to include a complex phase factor, such as,  $e^{i\phi}$  in the Lorentz transformation and to use both the proper time and the proper mass as parameters. To develop the most general case, a complex parameter  $\sigma = s + im$ , is introduced, where  $s$  is the proper time, and  $m$  is the proper mass, and  $\sigma$  and  $\frac{\sigma}{|\sigma|}$  are used to parameterize the position of a particle (or reference frame) in space-time-matter phase space. A new reference variable,  $u = \frac{m}{r}$ , is needed (in addition to velocity), and assumed to be bounded by 0 and  $\frac{c^2}{G} = 1$ , in geometrized units. Several results are derived: The equation  $E = mc^2$  apparently needs to be modified to  $E^2 = \frac{s^2 c^{10}}{G^2} + m^2 c^4$ , but a simpler (invariant) parameter is the “energy to length” ratio, which is  $\frac{c^4}{G}$  for any spherical region of space-time-matter. The generalized “momentum vector” becomes completely “masslike” for  $u \approx 0.79$ , which we think indicates the existence of a maximal gravity field. Thus, gravitational singularities do not occur. Instead, as  $u \rightarrow 1$  matter is apparently simply crushed into free space. In the last section of this paper we attempt some further generalizations of the phase space ideas developed in this paper.

## 1. INTRODUCTION

Phase spaces are a common and important way to model physical systems. For example, a harmonic oscillator has the energy equation  $E = \frac{p^2}{2m} + \frac{kx^2}{2}$ , which expresses the energy of the oscillator (pendulum, mass on a string, etc) as a function of the position and momentum. More generally phase spaces, usually called symplectic structures or symplectic spaces, have been extensively studied. The most basic symplectic structure is a smooth Manifold with a closed, non-degenerate 2-form, such as,  $\omega = dp_i \wedge dq^i$ . Equivalently, given a manifold of all possible configurations of a system, the phase space for the system is the cotangent bundle. See [13] for an introduction to symplectic structures.

Phase spaces within General Relativity have also been studied, for examples see [2, 3]. Unfortunately, while interesting this work does not seem to lead to any resolution of important open questions in general relativity, such as cosmic censorship or the validity of gravitational singularities. Even expanding the phase spaces to complex number phase spaces does not seem to be sufficient to answer these questions, but for an example of the complex treatment, see [6].

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Although, general relativity and especially special relativity, have survived intense experimental and observational testing [7, 15, 16, 17, 9, 4, 1, 8], the question of the existence of gravitational singularities is still unresolved. My sense is that most astro-physicists and general relativist would prefer that gravitational singularities were not predicted by general relativity, but are willing to accept them if necessary, because general relativity works so well otherwise. Thus, we have assumed in this paper that attempting to eliminate the singularities predicted by general relativity is a desirable result (see for example, [7], chapter 44).

The present paper is motivated by the desire to try to eliminate gravitational singularities (without changing anything else in a measurable way). It is somewhat surprising that we seem to be able to make progress within the framework of a complex phase space set in special relativity, especially as the present construction is not intended to incorporate gravity. The main difference between the present work and previous efforts to build relativistic phase spaces is the inclusion of the proper mass as a dynamic variable, as opposed to a constant scale factor. The consequences of using the proper mass as an additional parameter (along with proper time) seem to be significant enough to warrant beginning this study within the frame work of special relativity.

The present work begins with Minkowski space and changes this to a complex phase space, a space-time-matter configuration space, by introducing a complex “phase” factor into the Lorentz transformation. The special relativistic “four” momentum and “four” position are combined into a single vector (1-form) in  $C^4$ . Of course, the proper mass is constant, so we have to work out the transformation equations (complex Lorentz transformation) for an arbitrary mass, then substitute the actual mass of interest. Actually, we can show that there is a canonical value of the proper time as well, so that proper time and proper mass are not that different after all.

To make the phase factor dynamically significant requires another parameter to play a role similar to the velocity. This is a another key difference between the present work and previous phase space developments in general relativity. The parameter needs to have dimensions of mass over length (or coordinate time). Fortunately, this ratio is already a parameter of some interest in general relativity (although only for extremely massive objects), so we have assumed the mass to length ratio of general relativity to be the parameter we need. Then gravitational effects become significant in an extreme relativistic limit.

We have tried to introduce the phase factor into the Lorentz transformation in the most general possible way. However, it turns out to be very simple, i.e., of the form  $e^{i\phi}$ . Nevertheless, we have proceeded with the general derivation given in this paper, because some of the results of the derivation are used in the further developments in the last section. The outline of this paper is as follows:

- (1) State the postulates of the Phase Space.
- (2) Decide what forms need to be invariant and derive the conditions on the complex Lorentz group.
- (3) Derive the Phase Space Lorentz transformation.
- (4) Consider some of the consequences of these equations.
- (5) Consider some generalizations.

## 2. POSTULATES OF THE SPECIAL RELATIVISTIC PHASE SPACE

Since the phase space constructed in this paper is built within special relativity, we certainly need the postulates of special relativity. Briefly, we can recall these to be: 1. The laws of physics should have the same form (in Cartesian coordinates) for all Lorentz observers, and 2. The speed of light  $c$ , is constant for all Lorentz observers.

We need another assumption to make further progress, namely the gravity constant,  $G$  is also invariant for all Lorentz observers. I assume that most readers will find this to be a reasonable assumption that is pretty much already assumed in general relativity. One can imagine that different Lorentz observers could verify this assumption by doing extremely weak field experiments, so that the Lorentz character of the reference frames was not disturbed.

In addition, for the following calculations we have used geometric units chosen so that  $c = 1$  and  $G = 1$ . We have denoted the mass to length ratio with the parameter,  $u = \frac{m}{r}$ . Then, in geometrized units,  $0 \leq u \leq 1$ . The transformation equations derived below require this restriction on  $u$ , similar to the restrictions on  $v$ . The particular choice,  $\frac{c^2}{G} = 1$  is made to normalize  $u$  as simply as possible. This is not quite the Schwarzschild radius predicted by general relativity, which is  $\frac{c^2}{2G} = 1$ . Whether or not there is a two in denominator (or some other scalar factor is a matter to be decided by experiment, so for the purposes of this paper I have omitted the two for simplicity.

I am well aware that the assumption that  $0 \leq u \leq 1$  is likely to be disputed or rejected by some readers, but I think this requirement is simply a reflection of the cosmic censorship conjecture: what happens inside the Schwarzschild radius isn't observable from the outside, so  $u$  will be observed to be bounded above by one (the development given here does not forbid  $u = 1$ ). In any case, there is no question that for ordinary objects  $u$  is extremely small, mostly not even measurable. For example, for the earth,  $u \approx 10^{-10}$ , and for a 1 kg ball of radius 1 m,  $u \approx 10^{-27}$ .  $u$  is still smaller for elementary particles.

I have found little discussion in the literature of the parameter,  $u$ . The internet motion mountain physics text does discuss  $u$ , but treats this parameter as force. I don't think this is a good idea, because force is such a difficult concept in special relativity.

## 3. INVARIANT FORMS

To develop the phase space idea in this paper, we have assumed that the volume form,  $dV = dy^0 \wedge dy^1 \wedge dy^2$  is invariant with respect to the allowed coordinate transformations. For the time being we are considering a 3 dimensional space consisting of two space dimensions and one time dimension, with the usual index conventions (0 for time, 1 and 2 for space dimensions). To keep the problem as simple as possible the direction of relative motion is assumed to be along one of the space coordinates. The second space dimension may not be necessary, but we have included it for reasons that will become clear as the derivation proceeds.

We begin with the "position" vector,  $\mathbf{y}$ , and the two tangent vectors,  $\mathbf{y}_{,s}$  and  $\mathbf{y}_{,m}$ . Here "position" vector means position in the complex phase space, not the physical position. In these expressions,  $s$  is the proper time, and  $m$  is the proper mass of the particle. In addition, I have assumed that we can extract the magnitude of the

two form representing the cross product of tangent vectors (roughly the symplectic 2-form discussed above), using Hodge star duality. Since we don't have an explicit metric defined, the definition of the Hodge star dual is quite significant, but I have defined this in the natural way, i.e., so that we get the answer we would expect if we had a metric. Apparently the definition that is used below induces a metric on the space, but the significance of this isn't entirely clear to me.

In any case, we have assumed the surface area form is invariant with respect to the allowed coordinate transformations. The following derivation is probably much more complicated than it needs to be. I have done the derivation this way to set the stage for generalized equations of motion. However, to obtain just the complex Lorentz transformations in a simple way, we can just multiply the usual components of the Lorentz transformation by a complex "phase" factor of modulus 1, and assume a complex conjugated, Lorentz metric.

Continuing with the derivation, from the tangent vectors, we can construct the area 2-form (symplectic 2-form),

$$(3.1) \quad dA = \delta_{kn}^{01} y_s^k y_m^n dy^0 dy^1 + \delta_{kn}^{02} y_s^k y_m^n dy^0 dy^2 + \delta_{kn}^{12} y_s^k y_m^n dy^1 dy^2$$

In this expression I have used the generalized Kronecker delta (anti-symmetric in  $k$  and  $n$ , so  $\delta_{01}^{01} = 1$ ,  $\delta_{10}^{01} = -1$  and  $\delta_{kn}^{01} = 0$  for any other choice of  $k$  and  $n$ ) and there is, of course, no sum on the fixed numerical indices. I have omitted the wedge product symbol for brevity. For the rest of this section I have assumed the wedge product unless otherwise stated.

Next we need to define the Hodge star duals of the basis 2-forms, so we can find  $*dA$ .

**Definition 3.1.** The Hodge star duals of the basis two forms are defined as follows:  $*(dy^1 \wedge dy^2) = -dy^0$ ,  $*(dy^2 \wedge dy^0) = dy^1$ ,  $*(dy^0 \wedge dy^1) = dy^2$

Notice that I have defined  $*(dy^1 \wedge dy^2)$  to have the opposite sign of what might be expected. This is to avoid having to use  $i$  to keep the contributions to the volume formally positive. In addition, to this definition, I have also assumed throughout this paper that to find the Hodge star dual of a 1 or 2 form we have to use the conjugated transpose form of the components (this will be clear below). Thus, we have the following expression for  $*dA$ :

$$(3.2) \quad *dA = \delta_{kn}^{01} (y_s^k y_m^n)^\dagger dy^2 + \delta_{kn}^{20} (y_s^k y_m^n)^\dagger dy^1 - \delta_{kn}^{12} (y_s^k y_m^n)^\dagger dy^0$$

In this expression the  $\dagger$ s indicate complex conjugated transpose. Finally, we can extract the magnitude of the "symplectic 2-form" as the component of  $*dA \wedge dA$ . I have called this  $L^2$  for reasons that will be explained later.

$$(3.3) \quad L^2 = \delta_{kl}^{01} \delta_{np}^{01} (y_s^k y_m^l)^\dagger y_s^n y_m^p + \delta_{kl}^{20} \delta_{np}^{20} (y_s^k y_m^l)^\dagger y_s^n y_m^p - \delta_{kl}^{12} \delta_{np}^{12} (y_s^k y_m^l)^\dagger y_s^n y_m^p$$

To derive the complex Lorentz transformation we want to find the sub-group of  $SL(3)$  (special linear group in 3 dimensions) that keeps the righthand side of equation 3.3 invariant. We start with the transformation equations,  $y^k = B_j^k \hat{y}^j$ , where  $B_j^k$  is a  $3 \times 3$  matrix (with constant complex entries), such that  $|\det(B_j^k)| = 1$ . Substituting the transformation equation into the righthand side of equation 3.3 and subtracting the righthand side of equation 3.3, we get:

$$(3.4) \quad (\delta_{kl}^{01} \delta_{np}^{01} - \delta_{kl}^{12} \delta_{np}^{12} + \delta_{kl}^{20} \delta_{np}^{20}) \times [(B_\alpha^k \hat{y}_s^\alpha B_\beta^l \hat{y}_m^\beta)^\dagger B_\rho^n \hat{y}_s^\rho B_\sigma^p \hat{y}_m^\sigma - (\hat{y}_s^k \hat{y}_m^l)^\dagger \hat{y}_s^n \hat{y}_m^p] = 0$$

In equation 3.4, both Greek and Latin indices are summed over 0,1,2. I am using Greek indices simply because Latin indices are in short supply and also for editing purposes. The multiplication operations are all just complex numbers, so we can rearrange equation 3.4 to get:

$$(3.5) \quad (\delta_{kl}^{01} \delta_{np}^{01} - \delta_{kl}^{12} \delta_{np}^{12} + \delta_{kl}^{20} \delta_{np}^{20}) \times [(B_\alpha^k B_\beta^l)^\dagger B_\rho^n B_\sigma^p (\hat{y}_{,s}^\alpha \hat{y}_{,m}^\beta)^\dagger (\hat{y}_{,s}^\rho \hat{y}_{,m}^\sigma) - (\hat{y}_{,s}^k \hat{y}_{,m}^l)^\dagger \hat{y}_{,s}^n \hat{y}_{,m}^p] = 0$$

In this equation the †s indicate the entries in the conjugated transpose,  $B^\dagger$ . So for example,  $(B_l^k)^\dagger$  is the entry in the  $k^{th}$  row and  $l^{th}$  column of  $B^\dagger$ . There is, of course, no matrix multiplication in this equation, so we can freely move the transpose operation inside the brackets if we wish.

We seek the minimal conditions on  $B_l^k$  that will satisfy this equation without restricting the values of  $y_{,s}^k$  or  $y_{,m}^k$ . By making a fixed choice of  $k, l, n$  and  $p$  in the last term (in the square brackets) and then making the same choice for  $\alpha, \beta, \rho$  and  $\sigma$ , we can then sum over  $k, l, n$  and  $p$  in the first term to get the following equations that constrain  $B_l^k$ .

$$(3.6) \quad \Delta(B_2^2)^\dagger \Delta B_2^2 + \Delta(B_2^1)^\dagger \Delta B_2^1 - \Delta(B_2^0)^\dagger \Delta B_2^0 - 1 = 0,$$

$$(3.7) \quad \Delta(B_1^2)^\dagger \Delta B_1^2 + \Delta(B_1^1)^\dagger \Delta B_1^1 - \Delta(B_1^0)^\dagger \Delta B_1^0 - 1 = 0,$$

$$(3.8) \quad \Delta(B_0^2)^\dagger \Delta B_0^2 + \Delta(B_0^1)^\dagger \Delta B_0^1 - \Delta(B_0^0)^\dagger \Delta B_0^0 + 1 = 0.$$

In these equations,  $\Delta B_l^k$  indicates the cofactor of the entry in B in the  $k^{th}$  row and  $l^{th}$  column. Of course, we also require that  $\det B = e^{i\theta}$ , for some phase angle,  $\theta$ .

If we set  $B = \begin{bmatrix} b_{00} & b_{01} & 0 \\ b_{10} & b_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , then we can substitute into equations 3.6 - 3.8, and  $\det B = e^{i\theta}$ , to get:

$$(3.9) \quad \overline{(b_{00}b_{11} - b_{01}b_{10})}(b_{00}b_{11} - b_{01}b_{10}) - 1 = 0,$$

$$(3.10) \quad -\overline{b_{01}}b_{10} + \overline{b_{00}}b_{00} - 1 = 0,$$

$$(3.11) \quad -\overline{b_{11}}b_{11} + \overline{b_{10}}b_{01} + 1 = 0,$$

$$(3.12) \quad b_{00}b_{11} - b_{01}b_{10} = e^{i\theta}.$$

The choice of  $b_{22} = 1$  and other entries involving an index of 2 equal to zero is certainly a special choice (for example, certainly  $b_{22} = e^{i\phi}$  would be more general), but I am calculating essentially a two dimensional case. Notice that equations 3.9 and 3.12 are essentially the same statement:  $|\det B| = 1$ . Our next task is to try to make the matrix B physically reasonable by deciding (guessing) how the usual Lorentz transformation appears in the matrix B. The simplest is to attach a complex number factor to each entry in the usual Lorentz transformation:

$$(3.13) \quad B = \begin{bmatrix} (e + if)\gamma & -v\gamma(c + id) & 0 \\ -v\gamma(a + ib) & \gamma(a + ib) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting, this B into equations 3.9 - 3.12, we get the following equations:

$$(3.14) \quad -(a^2 + b^2)\gamma^2 + v^2\gamma^2(a - ib)(c + id) + 1 = 0$$

$$(3.15) \quad -v^2\gamma^2(c - id)(a + ib) + (e^2 + f^2)\gamma^2 - 1 = 0$$

$$(3.16) \quad (a^2 + b^2)\gamma^2[(e + if) - v^2(c + id)][(e - if) - v^2(c - id)] - 1 = 0$$

A straight forward calculation yields,

$$(3.17) \quad e = a, f = b, c = \frac{((a^2 + b^2)\gamma^2 - 1)a}{v^2\gamma^2(a^2 + b^2)}, \text{ and } d = \frac{((a^2 + b^2)\gamma^2 - 1)b}{v^2\gamma^2(a^2 + b^2)}.$$

We are left to decide how to eliminate the last variable. Substituting equations 3.17, using the matrix 3.13, into 3.12 gives the basic conditions on  $a$  and  $b$ , which simplifies to:

$$(3.18) \quad \frac{(a + ib)^2}{a^2 + b^2} = e^{i\theta}$$

Unfortunately, this equation doesn't seem to say much. It is tempting to suppose that  $a^2 + b^2 = 1$ , because this choice makes  $B$  symmetric, since in that case  $c = a$  and  $d = b$ , but it is also possible to find a stronger argument: The magnitude of the tangent vectors,  $y_{,s}^k$  and  $y_{,m}^k$  (using the Hodge star duality in definition one) should not be explicitly dependent on  $v$ . Thus,  $|y_{,s}^k|^2 = -(a^2 + b^2)\gamma^2 + \frac{((a^2 + b^2)\gamma^2 - 1)^2}{v^2\gamma^2(a^2 + b^2)}$ , and a short calculation (using  $\gamma^2 = \frac{1}{1-v^2}$ , of course) shows that the only solution to  $\frac{\partial}{\partial v}(|y_{,s}^k|^2) = 0$  is  $(a^2 + b^2) = 1$ . Thus, the whole exercise boils down to multiplying the usual Lorentz transformation by a complex phase factor,  $\exp(i\phi)$ .

In any case, to finish the calculation and interpret  $\exp(i\phi)$  in terms of  $u$ , the next decision we have to make is whether to solve for  $a$  or for  $b$ . If we let  $u = \frac{m}{r}$  be  $a$  or  $b$ , as the case may be, one way we get  $(a + ib) = u + i\sqrt{1 - u^2}$  and the other we get  $(a + ib) = \sqrt{1 - u^2} + iu$ . In the first case the usual (non-phase space) relativistic limit is recovered when  $u = 1$  (since the imaginary part of the factor,  $u + i\sqrt{1 - u^2}$  needs to be zero in this limit). In the second case, the usual relativistic limit occurs when  $u = 0$ . Of course, there is really no reason why the relativistic limit could not occur when both factors are pure imaginary, so that these considerations would be reversed. Thus, the simplest choice (I think it does not matter, in fact) is to let  $(a + ib) = \sqrt{1 - u^2} + iu$ .

Thus, to summarize our final result:

$$(3.19) \quad B = \begin{bmatrix} \gamma(u + i\sqrt{1 - u^2}) & -v\gamma(u + i\sqrt{1 - u^2}) & 0 \\ -v\gamma(u + i\sqrt{1 - u^2}) & \gamma(u + i\sqrt{1 - u^2}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#### 4. DISCUSSION AND CONSEQUENCES OF THE PHASE SPACE LORENTZ TRANSFORMATION

**4.1. Effective Metric.** At this point it is worth taking some time to try to interpret the transformation physically. This isn't easy, since the complex phase space we have constructed appears to be much more of a configuration space than even the usual Minkowski space. In addition, the corrections appear to be virtually unmeasurable excepting, in the relativistic phase space limit, which entails enormous gravity fields. Apart from this environment not normally being considered within

the context of special relativity, the high gravity environment is so beyond our ordinary experience that it is difficult to be sure exactly what the equations are telling us.

In the first place it is useful to ask if we effectively have a metric. Using the Hodge star dual in definition 3.1, the matrix  $B$  in equation 3.19 is clearly Lorentz orthogonal. So apparently definition 1 together with the condition,  $\frac{\partial}{\partial v}(|y_{,s}^k|^2) = 0$ , are sufficient conditions to induce a complex conjugated metric. So for the rest of the paper we will mostly just use the induced metric to simplify the considerations.

#### 4.2. Complex Phase Space, Energy Equation and Space-Time-Matter.

Following the usual procedure in special relativity, we can define a “four-velocity” (actually only a “two-velocity” here), by

**Definition 4.1.**

$$(4.1) \quad \mathbf{U} = (\sqrt{1 - u^2} + iu)\gamma < 1, v >$$

It is not immediately obvious how to construct the phase space “four momentum” from the four velocity. It is tempting to set  $\mathbf{P} = \mathbf{U}(s + im)$ , and in fact, we set several drafts of this paper with this definition. However, this definition will not do for reasons that will be explained later. Instead we need to use a scaled parameter,  $\frac{s+im}{\sqrt{s^2+m^2}}$  and construct the “momentum per unit length” vector:

**Definition 4.2.**

$$(4.2) \quad \hat{\mathbf{P}} = \langle y^0, y^1 \rangle = (\sqrt{1 - u^2} + iu)\gamma < 1, v \rangle \sigma \\ = (\sqrt{1 - u^2} + iu)\gamma < 1, v \rangle \frac{s + im}{\sqrt{s^2 + m^2}}$$

Recall that in this expression  $s$  is the proper time and  $m$  is the proper mass of the particle or object under consideration.  $\gamma < 1, v \rangle$  is the usual “2 dimensional” tangent vector. For the rest of the paper I have adopted the convention of designating “expressions per unit length” by hatting the usual notation. “Expression” means the momentum, energy, etc. In addition, we have not concerned ourselves too much with the distinction between tangent vectors (four velocity) and cotangent vectors (four momentum), since in special relativity the distinction is not too important.

Clearly, with this definition  $\hat{\mathbf{P}}$  is the position vector in a complex phase space. This is a central aspect of the current phase space construction: an important feature of the usual theory of special relativity is the merging of space and time. For example, the separation between events has space and time components, which are different for different observers. In the present development we are trying to accomplish a further merging, namely, of mass and space-time to form space-time-matter.

There is some precedent for doing this already from general relativity and general relativistic phase space, but I don’t think the merging in general relativity is complete. We represent gravity with curvature, but in my opinion the curved space is still a representation of the physical space (with masses and forces - we can talk about curved space-time around the Sun, but the Sun is obviously still there), and not an actual merging of space-time and matter. The Einstein field equations serve as a “code” for converting matter and energy into a curved space-time model, and back again, but there isn’t an actual merging.

In any case, it is of some interest to multiply-out the two complex factors in equation 4.2 to get:

$$(4.3) \quad \hat{\mathbf{P}} = \frac{[(s\sqrt{1-u^2} - um) + i(m\sqrt{1-u^2} + us)]}{\sqrt{s^2 + m^2}} \gamma < 1, v >$$

In this expression, recall that for ordinary terrestrial physics or fundamental particle physics,  $u$  and especially  $u^2$  are vanishingly small, so that to great accuracy for terrestrial experiments,  $\hat{\mathbf{P}} = \frac{s+im}{\sqrt{s^2+m^2}} \gamma < 1, v >$ . In addition, in this limiting case,  $\sqrt{s^2 + m^2} \approx s$ , so we can recover the usual momentum vector as  $\mathbf{P} = (s + im)\gamma < 1, v >$ . In this limit the space-time and mass aspects of  $\mathbf{P}$  have been separated into the usual position vector and momentum vectors of special relativity. So in this limit we have only accomplished the merging of space-time and mass transformation equations into a single expression, essentially a one parameter special relativistic phase space. This phase space is the special relativistic analogue of classical phase space.

Before discussing this further it is worth calculating the energy per unit length of the particle. Using the special relativistic equation,  $E^2 = -\mathbf{P} \cdot \mathbf{P}$ , we have immediately,

$$(4.4) \quad E^2 = s^2 + m^2$$

Recall that we are using a complex conjugated, Lorentz metric with  $\eta_{00} = -1$ , so that the “four velocity” has constant magnitude equal to -1. It is interesting to insert the clusters of constants needed to make this equation have metric units (kilograms, metres and seconds). In this case,

$$(4.5) \quad E^2 = \frac{s^2 c^{10}}{G^2} + m^2 c^4$$

The second term in equation 4.5 is just the usual relativistic mass energy term, but the first term is new. Apparently this means that as time passes, we must include the energy of free space inside a spherical ball of radius  $r = sc$ , i.e., keep  $r$  on the null cone. If we ignore the mass energy term (which is minuscule by comparison for terrestrial experiments), then we have  $E = \frac{sc^5}{G}$ , so the energy to radius ratio of a ball of free space with radius  $r = sc$  has the colossal value of,

$$(4.6) \quad \hat{E} = \frac{c^4}{G} \approx 1.21 \times 10^{44} \text{joules per metre}$$

Thus, in the present phase space construction, free space has an energy to radius ratio equal to that of matter that has been compressed to the limiting ratio of  $\frac{c^2}{G}$ . The obvious conjecture is that in this limit matter has been crushed into empty space. I don't think this is so far fetched, even from the point of view of general relativity, since in the latter theory matter is predicted to be crushed into a space-time singularity. The current approach dispenses with the need for a singularity, by assumption, to keep  $u$  bounded. The concept of energy of space-time gives the mass energy some where to go, so that we do not need a singularity. Nevertheless, there are clearly problems with energy conservation, discussed presently.

At this point we can understand what is wrong with using  $E$  instead of  $\hat{E}$ . Equation 4.5 seems to say that the energy of a region of space-time-matter, with a value of  $u \approx 1$ , has a total energy of  $E = \frac{\sqrt{2}sc^5}{G}$ , so that  $\hat{E} = \frac{\sqrt{2}c^4}{G}$ . This won't



do, because equation 4.6 says that  $\hat{E}$  for free space does not have the factor of  $\sqrt{2}$ . Thus, the conclusion I have drawn from this is simply that energy is not a relativistic phase space invariant, and instead the important (invariant) quantity is  $\hat{E}$ . If we compute  $\hat{E}$  directly using  $\hat{\mathbf{P}}$ , we get  $\hat{E} = 1$ , as expected. This says the total energy per unit length (of radius) of any ball of space-time-matter is  $\frac{c^4}{G}$ . Evidently energy is not conserved in this relativistic limiting case. We will returned to this issue.

A less obvious consequence of equation 4.6, and no doubt much more controversial, is that the external gravity field of a “black hole” is apparently zero. This follows from the direct experiential fact that the gravity field of free space is zero. Since matter at the extreme limit of  $\frac{c^2}{G}$  appears to have been crushed into free space, there is no external gravity field. The immediate objection is that there appears to be ample evidence that extremely strong gravity fields (too strong to be neutron stars) are common in the universe. However, below we have shown that equation 4.3 seems to suggests that there is a maximal gravity field with  $u$  several times larger than a neutron star, but still well below the critical value of  $u = 1$ .

**4.3. Addition formula for  $u$ .** At this point, it would be helpful to try to understand the parameter  $u$  a little better. Apparently,  $u$  is quite different from  $v$ , since each observer can measure  $u$  (for their own reference frame or another observer) within their own frame, i.e.,  $u$  appears to be absolute, whereas,  $v$  is relative between observers. Nevertheless, great caution is needed, since we have so little direct experimental data that illuminates the property being measured by  $u$ . In any case, since there is an upper bound for  $u$ , we do have to assume there is a “relativistic addition” rule. We can calculate the exact expression for addition of  $u$  (using a combination of the polar and algebraic forms for the complex factor of modulus one):

$$\begin{aligned}
 \exp(i\phi) &= \exp i(\theta_1 + \theta_2) = \exp(i\theta_1) \times \exp(i\theta_2) = \\
 &(\sqrt{1 - u_1^2} + iu_1)(\sqrt{1 - u_2^2} + iu_2) = \\
 (4.7) \quad &\sqrt{1 - u_1^2}\sqrt{1 - u_2^2} - u_1u_2 + i(u_2\sqrt{1 - u_1^2} + u_1\sqrt{1 - u_2^2})
 \end{aligned}$$

If we compare the real and imaginary parts of this last expression with the form,  $\sqrt{1 - w^2} + iw$ , it is straight forward to verify that

$$(4.8) \quad w = u_2\sqrt{1 - u_1^2} + u_1\sqrt{1 - u_2^2}$$

satisfies both the real and imaginary parts. It can also be readily verified that  $0 \leq w \leq 1$ . Thus, equation 4.8 is the phase space addition formula for  $u$ . Notice that for terrestrial values of  $u_1$  and  $u_2$  (minuscule),  $w \approx u_1 + u_2$  to a high degree of accuracy. We need to be careful how we interpret this equation: apparently this equation would apply to say an observer at the center of the earth who made a further measurement of an object at the surface of the earth. Then  $u_1$  would be for the earth and  $u_2$  would be for the object. In particular, we would apparently not use equation 4.8 to measure the results of say, a collision of two neutron stars (assuming the stars coalesced into one star).

The phase space development gives us a way to make sense of  $u$  for a small irregularly shaped object for which the value of  $r$  to be used in computing the ratio  $\frac{m}{r}$  is apparently unclear. The value of  $r$  always makes sense in the phase

space approach as  $r = sc$ , where  $s$  is the minimum possible proper time needed to measure  $u$ . So  $r$  is on the light cone. Incidentally, this definition makes  $s$  have an inherent value for each object, in the same way that  $m$  apparently seems to have an inherent value. So for example, a steel rod of length  $L$  would be understood to have  $r = \frac{L}{2}$  (observed from the center in the rest frame, which makes  $s$  as small as possible), because according to the phase space development, the rod consists of its mass and shape as a rod, but also includes a ball of space-time with proper time  $s = \frac{L}{2c}$ .

**4.4. Possibility of a Maximal Gravity Field.** Before discussing the possibility of a maximal gravity field, I would like to suggest that the result of general relativity that assumes that a black hole has an enormous gravity field depends on a principle that I think is not relativistic: the usual explanation is that the space-time around the black hole is permanently curved as the matter collapses. A principle such as this is needed because the source of the gravity field inside the event horizon has no way of communicating with the external field (at least within the context of general relativity alone and assuming either weak or strong cosmic censorship). In my opinion this is not a general relativistic principle because it is not local: according to the usual explanation the space around the event horizon stays curved for all time, i.e. for an extended separation of space-time (at least in absence of encounters with other gravity fields or quantum effects). I think the explanation that is true to relativity is that space-time must constantly be informed from the (local) source of the gravity field. Since the source is not available (it is inside the event horizon), the field vanishes.

I am well aware that gravitons, assumed to be the “messenger” for gravity, are not predicted to radiate unless the field changes. I think this just means we will have to devise a more sophisticated model of how gravity fields are sustained, and electrostatic fields for that matter. The issue here also raises the old debate of “gravity is curvature of space time” versus “curvature of space time is a representation of gravity”. The present approach is consistent only with the representation point of view, because the physics is gone behind the event horizon, so there should be no curvature as well. I believe the “gravity is curvature” view is too extreme for the simple reason that we can still go into the lab and do all sorts of non-gravity physics, say measure a current or a temperature, so it would be very peculiar if gravity was not also physical. Thus, general relativity is surely a configuration space, a way to represent gravity geometrically: we can do either physics or geometry. Thus, if the physics is gone (behind the event horizon - or crushed into free space according to the current development) the geometry must be flat.

Finally, I would like to point out that the phase phase approach considered in this paper requires one to think of space-time and matter as unified, so I am not sure that it even makes any sense within the present work to talk about a source of a gravity field that is separated from the external field. The phase space development keeps the source and the space-time around the source united by replacing the singularity with the space-time-matter energy per unit length calculated in equation 4.6. According to equation 4.6 the state of the matter in a ball of space-time-matter, if any, is irrelevant, since the energy per unit length of radius is constant.

Returning now to equation 4.3, since  $r = sc = s$  if  $c = 1$ , we can write  $m = ur = us$  to get,

$$(4.9) \quad \hat{\mathbf{P}} = \frac{[(\sqrt{1-u^2} - u^2) + i(u\sqrt{1-u^2} + u)]}{\sqrt{1+u^2}} \gamma < 1, v >$$

In this form the real and imaginary part of  $\hat{\mathbf{P}}$  have a very interesting property, namely, if

$$(4.10) \quad u = \frac{\sqrt{2\sqrt{5}-2}}{2} \approx 0.78615,$$

then the real part of  $\hat{\mathbf{P}}$  is zero, and the imaginary part takes its maximum value (= 1). It is unclear to me exactly how gravity manifests itself in the phase space approach, but I think it makes sense to argue that when the real part of  $\hat{\mathbf{P}} = 0$ ,  $\hat{\mathbf{P}}$  is entirely “mass like”, which we could understand to be representative of the state of space-time-matter for which the maximal gravity field occurs. In this picture gravity is understood to be the propensity of space-time-matter to become completely mass like. The more mass-like a region of space-time-matter is, then the stronger the external gravity field. Thus, within the discussion of this paper, I think the only reasonable interpretation of the existence of the special value of  $u$  given in equation 4.10 is that there is a maximal gravity field at this value of  $u$ .

The unique value of  $u$  in equation 4.10 is only valid for the hatted variables: The real part  $\mathbf{P}$  has a zero at this value of  $u$ , but the imaginary part of  $\mathbf{P}$  has a maximum at a larger value of  $u \approx 0.87$ . Yet another argument why the hatted variables are a more natural choice.

Of course, a solid conceptual challenge of the present development is that the parameter  $u$  is not very intuitive, whereas the similar parameter, density, is quite intuitive and constantly interferes with thinking about  $u$ . For example, for a very large region of space-time-matter, say most of the observable universe,  $u$  might be greater than 0.1, even though the density of matter is near 0. In the phase space approach a large region of space-time-matter and a compact region of space-time-matter have an immediate difference in the expression for  $\hat{\mathbf{P}}$ , namely the value of  $s$ . So, if  $s$  is large then the region of space-time-matter is large, etc. In any case, in the statement “how far a region of space-time-matter is from being completely space-time like” we have to be very careful not to confuse this with density.

It is important to observe that the value of  $u$  considered above, substantially exceeds the value of  $u$  for a typical neutron star ( $\approx 0.1 - 0.2$ ). Thus, I think the maximal gravity field concept can be used to explain all of the experimental evidence for enormous gravity fields. For example, the best evidence for super black holes asserts that  $\sim 3.7$  million solar masses reside at the center of our galaxy, inside of a ball the size of the inner solar system, approximately extending out to Jupiter [12, 11]. This is certainly an unimaginable region of space, but is far from a black hole:  $u \approx 0.02$ , so we are not even in the neutron star range. For 3-4 million solar masses to form a black hole they would need to be confined to a region with a radius of about 9 - 12 million kilometres. Thus, the latest evidence only directly supports the black hole singularity idea if we invoke a theory (general relativity) that says such a massive compact object must collapse to a singularity.

Although the phase space approach discussed in this paper is not intended to be a theory of gravity, the concept of a maximal gravity field seems to force a certain structure on a region of space-time-matter with  $u = \frac{m}{r}$  equal to the maximal value. Apparently, for such a region to be stable,  $u$  needs to have the maximal value for

any internal value of  $r$ , i.e., for any ball of space-time-matter interior to the region we need to have  $u$  equal to the maximal value for the whole region to be stable. Then the density function with respect to the observer's rest frame at the center, should be structured to keep  $u$  constant for any internal value of  $r$ . If the region has radius  $R$  and mass  $M$ , then it is straight forward to show that the density function needs to be  $\rho(r) = \frac{3M}{4\pi Rr^2}$ . With this density function for any  $r$  inside the region of space  $\frac{m}{r} = \frac{\rho(r)V(r)}{r} = \frac{M}{R}$ , where  $V(r)$  is the volume of a ball of space of radius  $r$ .

Therefore, as the observer sights along a radial line from the origin, the density of matter inside the region is observed to drop as a function of the inverse square. Of course, it is not clear exactly what this  $r$  means, since our experience with general relativity is that the gravity field affects our observation of  $r$ . However, it seems reasonable to suppose that at least asymptotically,  $r$  has some resemblance to ordinary radial distance.

However that may be, the above density function suggests that a region of space with the maximal  $u$  need not have a hard surface. This is important, because accretion disk theory suggests that the hard surfaces of neutron stars and the event horizons of black holes should be observationally distinguishable [14, 5].

In addition, it is hard not to wonder if there is some sort of duality relationship between the parameter  $u$  and the inverse square law for gravity (in the Newtonian limit). Recall that we are imagining that gravity is the propensity of a region of space-time-matter to be mass-like, so if the density of matter is reduced as the inverse square of  $r$ , this might explain why the gravity field decreases as the inverse square of  $r$ .

Another point of interest is that the energy in free space is not "mass like", because if it were, then free space would have a gravity field so large that it would be opaque. I think the phase space approach offers a way-out of the problem posed by the prediction of enormous amounts of background energy (both the present approach and quantum field theory): space-time energy simply does not generate a gravity field. Only, mass like energy generates gravity fields. Specifically, only the mass part (imaginary part) of  $\mathbf{P}$  generates a gravity field.

**4.5. Violation of Conservation of Energy.** As already mentioned, one of the most startling predictions of the phase space development given here is that in extremely high gravity fields conservation of energy apparently does not hold. What is conserved for all values of  $u$  and  $v$  is the energy to length ratio of any region of space-time-matter,  $\frac{c^4}{G}$ . Of course, if  $u$  is much less than 0.1 or so, the general phase space reduces to the special case of Minkowski phase space, where conservation of energy is correct to a high degree of approximation (I think deviations are not experimentally verifiable). The energy equation, 4.5 reduces to  $\frac{c^4}{G}$  for  $u \ll 1$ . For example, for a 1 kilogram mass with assigned proper time,  $s$ , such that  $sc = 1$  metre, the energy calculated using equation 4.5 differs from  $\frac{c^4}{G}$  by about  $3 \times 10^{-53}$  percent. Since the extreme high gravity environment is so unfamiliar experimentally, it is hard to know what this conclusion means, but apparently we experience conservation of energy in the low gravity environment, because the space-time energy is so enormous and so far undetectable. Hence, we only detect masslike energy, unmixed with space-time energy, so energy appears to be conserved, i.e.,  $E = m$  for all Lorentz observers.

**4.6. Symplectic Relativity Revised.** In the usual theory of special relativistic, Hamiltonian Mechanics (non-quantum) the Hamiltonian is constant, i.e.,  $H = m$ , where  $m$  is the rest mass of the particle. In the current development,  $H = \sqrt{s^2 + m^2}$ , so we can continue the development with a time dependent Hamiltonian, even within special relativity. Actually, we can treat the rest mass as a variable too, deriving a two parameter special relativistic symplectic mechanics. Apparently, a better choice would be to use the hatted Hamiltonian,  $\hat{H} = \frac{c^4}{G}$ , but I am not sure how hatted symplectic mechanics works. A complex Hamiltonian, such as,  $\hat{H} = \frac{s+im}{\sqrt{s^2+m^2}}$  also seems possible. Obviously, if the special relativistic phase space approach is viable, it would be a good plan to work-out some of these ideas before exploring the impact on quantum mechanics. Evidently, we need to use hatted energy and momentum operators and I don't know what these would be. However, the last section of this paper may help.

## 5. SOME FURTHER POSSIBLE DEVELOPMENTS OF THE SPECIAL RELATIVISTIC PHASE SPACE APPROACH

In this section we will consider two possible further developments of the special relativistic phase space, namely adding more parameters and deriving equations of motions from the Lagrangian given in equation 3.3. In both these discussions,  $v$  remains constant. Another possibility that I haven't discussed in this paper is extending the phase space to general relativity, i.e., a symplectic space in which both proper mass and proper time are included as parameters. I just don't understand the present phase space approach well enough at this time, to know how to extend it to general relativity.

**5.1. More Parameters.** Having taken the plunge of forming a two parameter complex phase space, merging space-time with matter, it is reasonable to wonder if that is all we can do. For example, an obvious and appealing choice is to add the charge to mass ratio,  $w = \frac{q}{m}$ , to our list of parameters,  $v$  and  $u$  (adding the current would be another possibility). There is even a convenient cluster of constants with dimension of charge to mass, namely,  $\sqrt{\epsilon_o G}$ , where  $\epsilon_o$  is the permittivity of free space. At first glance this cluster of constants will not do, because in metric units, the value of  $\sqrt{\epsilon_o G} \approx 10^{-11}$ , whereas the the charge to mass ratio of an electron is  $1.76 \times 10^{11}$  coul/kg. Of course, the charge to mass ratio of many particles (and large bodies) is zero, so apparently  $\sqrt{\epsilon_o G}$  cannot serve as a boundary value (lower or upper) in the sense of  $c$  and  $\frac{c^2}{G}$ . However, on further reflection, in the phase space development,  $w = \frac{q}{m}$  or the current,  $\frac{q}{s}$  are not really appropriate parameters, since we have ignored the whole point of the phase space approach, namely, that space-time and matter have been unified. Thus, the parameter that we want in the phase space approach of this paper is a combined parameter, such as,  $w = \frac{q}{\sqrt{s^2+m^2}}$  or possibly  $w = \frac{q}{s+im}$ . Maybe an even better choice is a hatted parameter,  $\hat{w} = \frac{qG}{rc^2}$ , where  $r$  is the relativistic Compton radius for the phase space. We have calculated  $r$  below.

For any of these parameters,  $w$  (or  $\hat{w}$ ) can be reconciled with the "boundary" cluster of constants if we include the solid angle factor in the cluster to get,  $\sqrt{4\pi\epsilon_o G}$ . To make the reconciliation we first need to recalculate the standard relativistic

expression for the Compton radius of an electron,

$$(5.1) \quad r = \frac{e^2}{4\pi\epsilon_0 mc^2} = 2.817940325 \times 10^{-15}.$$

using the phase space approach of this paper. The factor of  $mc^2$  in the denominator needs to be replaced with the phase space expression for the total energy per unit length of a region of space filled by the electron, given in equation 4.6. So we want to replace  $mc^2$  with  $\frac{rc^4}{G}$ . This gives,

$$(5.2) \quad r = \frac{e^2 G}{4\pi\epsilon_0 rc^4}$$

Solving for  $r$ , we get,

$$(5.3) \quad r = \frac{e}{c^2} \sqrt{\frac{G}{4\pi\epsilon_0}} \approx 0.13767776994 \times 10^{-35} \text{metres}$$

This seems like an interesting result: the phase space approach appears to predict that the electron Compton radius is only about 9% of the Planck Length. Of course, it would be quite significant if  $r$  worked-out to exactly the Planck length. I have searched for a simple way to adjust the above calculation, but can't find a suitable method. Obviously, we could start by normalizing both lengths over the same angle (the Compton radius is averaged over a solid angle, whereas the Planck length is averaged over a polar angle), but doing this doesn't make the two lengths equal. Evidently charge quantization is independent of spacial-energy quantization measured by Planck's constant. Why this is so when we have the energy equation,  $E = h\nu$ , clearly also an electromagnetic equation, is baffling. For the record, the ratio of the Planck Length  $L$  to the phase space Compton radius is  $\frac{L}{r_c} = \frac{\sqrt{2h\epsilon_0 c}}{e} \approx 11.4675833875$ .

In any case, the extended value of  $r$  is presumably a great success of the phase space approach, since quantum field theory works best if  $r$  is just assumed to be zero. There is a problem with this calculation that perhaps makes it not particularly useful, namely, the vast majority of the energy in the expression  $E = \frac{rc^4}{G}$  is not accessible by any known means (at least, short of applying immense gravity fields), so it will not really contribute in a practical experiment. So perhaps equation 5.1 will remain the practical Compton radius. This is also likely, simply because the phase space version of the Compton radius does not depend on the mass of the particle. It is perhaps worth mentioning that the utility of the relativistic Compton radius calculated in equation 5.1 must mostly be a coincidence, since the energy factor used in the denominator,  $E = mc^2$  is not accessible except for the highest energy experiments, e.g., an electron-positron annihilation.

Continuing with the calculation of the relativistic phase space "charge to mass ratio" of an electron: using  $r$  from equation 5.3 in  $w = \frac{q}{\sqrt{s^2 + m^2}}$ , and inserting the appropriate constants ( $q = e$ ),  $w$  works out to  $w = .8613270258 \times 10^{-10}$ . Comparing, this is exactly equal to,  $\sqrt{4\pi\epsilon_0 G}$ , at least to ten decimal places. The electron "charge to mass" ratio is actually smaller than  $\sqrt{4\pi\epsilon_0 G}$ , but according to Maple the difference is of order  $10^{-80}$ .

Alternatively, if we define  $w = \frac{q(s-im)}{s^2 + m^2}$  then the real part works out to  $.8613270258 \times 10^{-10}$  and the imaginary part works out to  $-.4223547307 \times 10^{-31}$ . Then  $|w| =$

.8613270258  $\times 10^{-10}$ . This time  $|w|$  differs from  $\sqrt{4\pi\epsilon_o G}$  in approximately the fortieth decimal place. The hatted version works-out to exactly,  $\hat{w} = \sqrt{4\pi\epsilon_o G}$ , which isn't surprising.

Unfortunately, it does not look like  $w$  (any version) is going to be a very useful parameter, because the relativistic phase space Compton radius does not depend on the mass. So  $\hat{w} = \sqrt{4\pi\epsilon_o G}$  for any particle with non-zero charge. Evidently, in the space-time-matter-charge configuration space, in geometrized units ( $4\pi\epsilon_o = 1$ ), the "charge to mass ratio" of any particle with non-zero charge is 1. This is apparently similar to the fact that in the usual theory of special relativity, the four velocity has magnitude -1 for any particle. Probably the complex version of  $w$  will be the most useful, because we can study the components.

More generally, the difference in magnitude between the standard value of the charge to mass ratio of an electron,  $1.76 \times 10^{11}$  coul/kg, and cluster of constants,  $\sqrt{4\pi\epsilon_o G}$ , seems to refute the idea that clusters of constants serve as boundary values for relativistic parameters. In particular, without the possible reconciliation available with the phase space approach, even the idea of  $u$  and its boundary of  $\frac{c^2}{G}$  would be suspect, since we would have found a simple counterexample in  $w$ . So the reconciliation explained above is of theoretical interest, since it saves the day for the idea that certain relativistic parameters are bounded by universal constants (or clusters of constants) that I have assumed have the same values for all observers. In any case, we can continue with a sketch of the phase space development with more parameters.

The next important step would be to decide on a number system with which to express the additional parameters. The quaternions seem to be the best bet, but I don't know what to do with the fourth generator (only three generators seem to be needed: one for each of  $v$ ,  $u$ , and  $w$ ). The fourth generator needs to be a physical quantity commonly parameterized by all of the proper time, proper mass and the charge. Also, the quaternions have anti-commutative multiplication between imaginary generators, and I don't know what that means (why would  $ui \times wj = uwk = -wj \times ui$ ?). Perhaps this is just what we need, since the metric we have been using for section 4 and 5 of this paper is actually a symplectic two form, which also anticommutes. The numbers that we use need to have the complex numbers as a subfield, so for example, the Dirac or Pauli matrices are out. In any case, we can construct the velocity vector in an extended phase space in  $Q^2$ , ( $Q$  for quaternions):

$$(5.4) \quad \mathbf{U} = (\sqrt{1 - u^2 - w^2 - z^2} + iu + jw + kz)\gamma < 1, v >$$

In this equation  $i$ ,  $j$ , and  $k$  are the generators of the quaternions, and  $z$  is the unknown fourth parameter. This expression is going to have to be interpreted very carefully, because charge is invariant in special relativity. In any case, we can conjecture that the momentum form is given by:

$$(5.5) \quad \hat{\mathbf{P}} = \langle y^0, y^1 \rangle = (\sqrt{1 - u^2 - w^2 - z^2} + iu + jw + kz)\gamma < 1, v > \frac{s + im + jq + kn}{\sqrt{s^2 + m^2 + q^2 + n^2}}$$

In this equation  $n$  is the proper parameter that is associated with  $z$ . Then it is tempting to conjecture that the revised energy equation will just work-out to  $E^2 = s^2 + m^2 + q^2$  (assuming  $n = 0$ ), using units in which  $c$ ,  $G$  and  $4\pi\epsilon_o$  are all

1. The justification for this conjecture is that whatever the actual formulation is, the resulting tangent vector (form) will have magnitude of -1. Thus, there will be some extended momentum vector, and  $E^2$  will still be equal to  $-\mathbf{P} \cdot \mathbf{P}$ . Inserting the appropriate constants,

$$(5.6) \quad E^2 = \frac{s^2 c^4}{G^2} + m^2 c^4 + \frac{q^2 c^4}{4\pi\epsilon_0 G}$$

Presumably this expression will need to be re-scaled in some way to give equation 4.6. But for now we can see that the contribution to the total energy from the charge term in equation 5.4 is  $E = \frac{qc^2}{\sqrt{4\pi\epsilon_0 G}}$ . For  $q = 1.6 \times 10^{-19}$  coul,  $E \approx 1.3 \times 10^3$  joules, which is enormous (approximately 16 orders of magnitude larger than the mass energy of an electron). However, the phase space approach gives no clues as to how this energy can be accessed. We can interpret this result as simply the stored energy due to the charge, in the same way that the phase space approach developed in this paper predicts that there is a huge amount of energy stored in free space. So far we do not know how to access the energy stored in free space, either. It is straight forward to show that the ratio between the charge energy and the space-time energy per unit length is just the phase space Compton radius that we found before (equation 5.3).

Notice that we will most likely need to use a parameter with unit modulus, so equation 4.6 still holds. We then have space-time energy, charge energy, and mass energy all intermixed in any region of space-time-matter-charge.

We haven't discovered how to include charge in such a way that the charge remains invariant (with respect to changes in  $v$ ) in the appropriate special cases, so the suggestions here can't be correct, but gives some idea of how a further development in this direction might go.

**5.2. Derivation of Equations of Motion.** As previously observed the phase space Lorentz transformation (equation 3.19) can be derived simply using a complex conjugated metric, but one reason to use the Lagrangian in equation 3.3 is so we can apply an action principle to derive a set of equations which we can then solve to find a general expression for  $\mathbf{P}$ . First recall equation 3.3:

$$(5.7) \quad L^2 = \delta_{kl}^{01} \delta_{np}^{01} (y_s^k y_m^l)^\dagger y_s^n y_m^p + \delta_{kl}^{20} \delta_{np}^{20} (y_s^k y_m^l)^\dagger y_s^n y_m^p - \delta_{kl}^{12} \delta_{np}^{12} (y_s^k y_m^l)^\dagger y_s^n y_m^p$$

In this expression the  $y^k$ 's are understood to be the components of the generalized phase space (simplectic) momentum vector  $\mathbf{P}$  or  $\hat{\mathbf{P}}$ . I have already given one special case for  $\hat{\mathbf{P}}$  in definition 4.2. Then the  $y_s^k$  and  $-iy_m^k$  partials are tangent vectors. The factor of  $-i$  in front of the  $m$  partial arises because  $m$  is the imaginary part of the complex parameter,  $\sigma = s + im$ . To keep the equations as simple as possible, we will continue in accordance with equation 3.19, and assume that  $y^2 = 1$ . For the following derivation I am assuming that the particle follows a stationary path defined by the Euler-Lagrange equations for  $L^2$ . Recall [10] that these equations are as follows:

$$(5.8) \quad \frac{\partial L^2}{\partial y^k} - \frac{D}{Dm} \left( \frac{\partial L^2}{\partial y_m^k} \right) - \frac{D}{Ds} \left( \frac{\partial L^2}{\partial y_s^k} \right) = 0$$

In these equations  $\frac{D}{Dm}$  and  $\frac{D}{Ds}$  are "total" derivatives, i.e., the chain rule should be applied when differentiating  $\frac{\partial L^2}{\partial y_m^k}$  and  $\frac{\partial L^2}{\partial y_s^k}$ . We have used the special notation,



because for example, the expression  $\frac{\partial L^2}{\partial y_m^k}$  doesn't have explicit dependence on  $m$ , so the ordinary partial of this expression with respect to  $m$  would be zero. This is not what we want. Also, in the second term on the left hand side of equation 5.6, factors of  $i$  that arise from differentiating with respect to  $m$ , have canceled. For simplicity, I am assuming that  $k = 0, 1$ . In any case, substitution of equation 5.5 into equation 5.6 and simplifications yields the system of partial equations that govern the motion of the particle in the special relativistic phase space discussed in this paper.

$$(5.9) \quad (y_s^1)^2 y_{mm}^0 - y_s^0 y_s^1 y_{mm}^1 - 2y_m^1 y_s^1 y_{sm}^0 + (y_m^0 y_s^1 + y_m^1 y_s^0) y_{sm}^1 + (y_m^1)^2 y_{ss}^0 - y_m^0 y_m^1 y_{ss}^1 = 0$$

$$(5.10) \quad (y_s^0)^2 y_{mm}^1 - y_s^1 y_s^0 y_{mm}^0 - 2y_m^0 y_s^0 y_{sm}^1 + (y_m^1 y_s^0 + y_m^0 y_s^1) y_{sm}^0 + (y_m^0)^2 y_{ss}^1 - y_m^1 y_m^0 y_{ss}^0 = 0$$

Since equations 5.7 and 5.8 both involve second order partials in every term, the momentum vector defined in 4.2 obviously satisfies both equations.

The system of equations in 5.7 and 5.8 has many solutions. For example,  $\langle f(s+im), f(s-im) \rangle$  satisfies both equations for any twice differentiable  $f$  (differentiable in the sense of multi-dimensional calculus- $f(s-im)$  is obviously not analytic, unless  $f$  is constant). We would like a solution that linearizes to the momentum vector defined in 4.2. One possibility is,

$$(5.11) \quad \langle y^0, y^1 \rangle = \langle \gamma(\exp((\sqrt{1-u^2}+iu)(\frac{s+im}{\sqrt{1+u^2}}))-1), v\gamma(\exp((\sqrt{1-u^2}+iu)(\frac{s+im}{\sqrt{1+u^2}}))-1) \rangle$$

It can be readily verified that any vector of the form,

$$(5.12) \quad \langle y^0, y^1 \rangle = \langle A \exp(a(s+im)), B \exp(b(s+im)) \rangle$$

satisfies 5.7 and 5.8, where  $A, B, a$  and  $b$  are arbitrary (complex) constants. Hence, the solution in equation 5.9.

Frankly, I do not understand what the solution in equation 5.9 means. Certainly,  $v$  is constant, so this equation is not about actual motion, just configuration space motion. If  $\hat{y} = \langle y^0, y^1 \rangle$ , then certainly both the tangent vectors,  $\hat{y}_s$  and  $\hat{y}_m$  have magnitudes of  $-1$ , using the complex conjugated Lorentz metric (explained section 4). It is entirely unclear to me what kind of boundary conditions to impose on the system in 5.9.

Since the constants,  $a$  and  $b$  in equation 5.10 do not need to be real, we must have wave solutions as well. In fact, the solution in 5.9 is already a wave with an increasing amplitude.

## 6. CONCLUSION

In this paper we have developed a complex phase space within the context of special relativity. Minkowski space (space-time) is changed to a space-time-matter configuration space in  $C^2$  or  $(C^4)$ , by including a complex phase factor in the Lorentz metric and allowing the proper mass to be treated as a parameter. Unfortunately, the limiting case for which  $u \approx 1$  is the domain of enormous gravity fields, so full interpretation of the results will probably require generalizations of the techniques discussed in this paper to general relativity. Nevertheless, the phase space approach seems to provide an alternative to gravitational singularities: matter is

crushed into free space as  $u$  increases past  $u \approx .79$  and there is a stable maximum gravity field with a soft surface.

According to the development in this paper, the special relativistic phase space expression for the energy is  $E^2 = s^2 + m^2$ . Among other things, this equation says that ordinary space-time contains an unfathomable amount of energy. Apparently, the quantity that is invariant in the phase space is the energy to length ratio,  $E = \frac{c^4}{G}$ . I have discussed a number of other consequences as well, such as the baffling similarity (but not equality) of the phase space Compton radius and the Planck length.

Finally, I have considered some further developments of the approach. It certainly seems likely that more parameters can be added, that an interesting two parameter Hamiltonian mechanics can be constructed (using either energy or energy per unit length), which should have consequences for quantum mechanics, and that the general equations of “motion” in space-time-matter have many possible solutions.

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