Differential rotation of stretched and twisted thick magnetic flux tube dynamos in Riemannian spaces

by

L.C. Garcia de Andrade

Departamento de Física Teórica − IF − Universidade do Estado do Rio de Janeiro-UERJ
Rua São Francisco Xavier, 524
Cep 20550-003, Maracanã, Rio de Janeiro, RJ, Brasil
Electronic mail address: garcia@dft.if.uerj.br

Abstract

The topological mapping between a torus of big radius and a sphere is applied to the Riemannian geometry of a stretched and twisted very thick magnetic flux tube, to obtain spherical dynamos solving the magnetohydrodynamics (MHD) self-induction equation for the magnetic flux tubes undergoing differential (non-uniform) rotation along the tube magnetic axis. Constraints on the shear is also computed. It is shown that when the hypothesis of the convective cyclonic dynamo is used the rotation is constant and a solid rotational body is obtained. As usual toroidal fields are obtained from poloidal magnetic field and these fields may be expressed in terms of the differential rotation $\Omega(r, \theta(s))$. In the case of non-cyclonic dynamos the torsion in the Frenet frame is compute in terms of the dynamo constant. The flux tube shear $\frac{\partial}{\partial r}\Omega$ is also computed. The untwisted tube case is shown to be trivial in the sense that does not support any dynamo action. This case is in agreement with Cowlings antidynamo theorem, since in the untwisted case the tube becomes axially symmetric which the refereed theorem rules out. We also show that it is consistent with the Zeldovich antidynamo theorem which rules out planar dynamos. Knowledge of the differential rotation of the Earth, for example, allows one to place limits on the curvature and torsion of the flux tube axis and vice-versa , knowledge of the topology permit us to infer differential rotation and other physical parameters of the stars and planets. **PACS numbers:**

02.40.Hw-Riemannian geometries

I Introduction

Earlier Parker [1] investigated the so-called convective cell cyclonic dynamos in terms of the differential rotation of celestial and astrophysical objects and solutions of self-induction equation equations coupled with the self-induction equation of the magnetic fields. Non-homogeneous rotation of celestial and astrophysical [2] also called differential rotation, happens due to the fact that these bodies are not solid, but undergo distinct rotations between the poles and equator. This physical phenomena is supposed to produce and amplify magnetic fields in the so-called dynamo mechanics. On the other hand, Arnold, Zeldovich, Ruzmaikin and Sokoloff [3] showed dynamos could considered as stationary solutions of self-induction equations in Riemannian three-dimensional spaces. Here we consider the Riemann metric of a very thick, stretched and twisted magnetic flux tube recently Garcia de Andrade [4, 5] to investigate magnetic flux tubes in superconducting plasmas, and use the map between spheres and very thich tori to obtain spherical dynamo solutions of MHD self-induction equation. Thiffault and Boozer [6] following the same reasoning applied the methods of Riemann geometry in the context of chaotic flows and fast dynamos. Yet more recently Thiffeault [7] investigated the stretching and Riemannian curvature of material lines in chaotic flows as possible dynamos models. In this paper he argued that filaments with torsion can also be constructed. Also Boozer [8] has obtained a geomagnetic dynamo from conservation of magnetic helicity. This can also be shown here in the generalization to non-holonomic Frenet frame [9]. Since in the case of kinematical dynamos we address here, the flow does not depend on the magnetic field (nonlinear MHD), we consider that the differential rotation $\Omega(r, \theta(s))$ depends on the radial and angular coordinates. Since we know that the Zeldovich [10] stretch, twist and fold method allows us to obtain kinematical dynamos, the stretch and twist of the magnetic flux tube seems providential to be consider as the germ of a spherical dynamo which has been so usually employed to explain the origin and physical nature of the geomagnetic and solar magnetic fields. The paper is organised as follows: In section II a brief outline of the explanation of the Riemannian geometry of magnetic flux tubes and how we may transform it into the Riemann metric [11] of a sphere is presented. Section III presents anti-dynamo tests theorems of Zeldovich and Cowlings to show that both of them are fulfilled by the stretched, and twisted

thick flux tube. In section IV general thick flux tube solutions of the self-induction equation on the Riemannian magnetic flux tube background metric as is usually done in handling Maxwells equations in Einsteins general theory of relativity, are presented. In section V conclusions are presented.

II   Spherical dynamos from closed flux tubes

According to Ricca [12] the Riemann metric of a twisted magnetic flux tube may be written as

$$dl^2 = dr^2 + r^2 d\theta_R{}^2 + K^2(s) ds^2 \tag{II.1}$$

where the tube coordinates are (r, θ_R, s) [12] where $\theta(s) = \theta_R - \int \tau ds$ where τ is the Frenet torsion and κ is the curvature of the tube axis and $K(s)$ is given by

$$K^2(s) = [1 - r\kappa(s)cos\theta(s)]^2 \tag{II.2}$$

Note that the limit of a very thin tube is $K := 1$, since the radial coordinate r tends to zero by shrinkink the tube. But if we substitute this value of K into the flux tube Riemann metric (II.1) one obtains

$$dl^2 = dr^2 + r^2 d\theta_R{}^2 + ds^2 \tag{II.3}$$

which by substituting the coordinate-s along the magnetic flux tube axis by the straight cylindrical coordinate-z one obtains the Riemann flat metric of a very thin cylindrical tube, which implies that when we compute the self-induced equations in this metric the presence of the tube will not be fully felt by this equation. To remedy this situation, we shall address here the other extreme of a very thick tube where the internal radius can even surpass the value of the radius of the tube. In this case expression (II.2) becomes

$$K^2(s) = [r\kappa(s)cos\theta(s)]^2 \tag{II.4}$$

by taking into account that the Frenet curvature $\kappa = \frac{1}{R}$ where R is the local radius of the tube, substitution of (II.4) into (II.1) yields

$$dl^2 = dr^2 + r^2[d\theta_R{}^2 + cos^2\theta(s)d\phi^2] \tag{II.5}$$

4

where the coordinate ϕ given by $d\phi := \frac{ds}{R}$ is the angular coordinate in the plane of the torus. Expression (II.5) is clearly the Riemann metric describing the 3D sphere. Since most of the celestial bodies including planets and stars can be described by spherical or spheroidal symmetries, solving the self-induction equation for non-turbulent fluids inside the twisted and stretched thick magnetic flux tubes, seems to be the a useful approach to dynamo theories. Computing the Riemannian gradient operator ∇ in terms of the thick flux tube curvilinear coordinates [13] reads

$$\nabla = [\partial_r, \frac{1}{r}\partial_\theta, \frac{1}{K}\partial_s] \tag{II.6}$$

where $\partial_j := \frac{\partial}{\partial x^j}$. The magnetic field here can be expressed as

$$\vec{B} = e^{pt}\vec{B_0} := e^{pt}[B_\theta(r,\theta)\vec{e_\theta} + B_s(r)\vec{t}] \tag{II.7}$$

where p is the dynamo constant and real parameter here, the dynamo condition is $p \geq 0$. These solutions will be tested as dynamos in the next section.

III   Testing Cowling and Zeldovich anti-dynamo theorems in stretched and twisted thick flux tubes

To test Cowlings antidynamo theorem which states that axially symmetric magnetic devices cannot support dynamo action, in this section we solve the MHD equations

$$\nabla.\vec{B} = 0 \tag{III.8}$$

$$\frac{\partial}{\partial t}\vec{B} - \nabla\times[\vec{v}\times\vec{B}] - \epsilon\nabla^2\vec{B} = 0 \tag{III.9}$$

$$\nabla.\vec{v} = 0 \tag{III.10}$$

where $\vec{u}$ is a solenoidal field while ϵ is the resistivity coefficient. Equation (III.8) represents the induction equation. The expression $\vec{v} = [0, \Omega r, v_0]$ where v_0 is the constant speed of the flow along the magnetic axis. Since here we shall only consider nondissipative flows , ϵ vanishes and we do not need to compute the Riemannian Laplacian ∇^2. Here

$$\vec{e_\theta} = -\vec{n}cos\theta + \vec{b}sin\theta \tag{III.11}$$

which by using the Frenet frame relations

$$\vec{t}' = \kappa\vec{n} \tag{III.12}$$

$$\vec{n}' = -\kappa\vec{t} + \tau\vec{b} \tag{III.13}$$

$$\vec{b}' = -\tau\vec{n} \tag{III.14}$$

where the dash represents the ordinary derivation with respect to coordinate-s , yields

$$\frac{\partial}{\partial s}\vec{e_\theta} = \kappa sin\theta\vec{t} \tag{III.15}$$

Substitution of these expressions into the MHD equations yields

$$p\vec{B_0} + [\frac{v_\theta}{r}\partial_\theta + \frac{v_0}{K}\partial_s]\vec{B_0} - [\frac{B_\theta}{r}\partial_\theta + \frac{B_s}{K}\partial_s]\vec{v} = 0 \tag{III.16}$$

Substitution of expressions (II.7), (III.11) and (III.15) into (III.16) yields the following three scalar equations along the Frenet basis $(\vec{t}, \vec{n}, \vec{b})$

$$p = \tau tg\theta[v_0 + \frac{\Omega r}{Tw}] \tag{III.17}$$

$$p = \frac{\tau}{Tw}[-v_0 + \Omega r] \tag{III.18}$$

$$p = \frac{v_0 tg\theta}{r}[1 + Tw] \tag{III.19}$$

Here we consider the twist definition as $Tw = \frac{B_\theta}{B_s}$. We also consider in this derivation the other MHD equations

$$\frac{\partial B_\theta}{\partial s} = \frac{B_\theta\kappa\tau r sin\theta}{K} \tag{III.20}$$

which is valid also for v_θ. An immediate astrophysical consequence of these equations is that the twisted flux tube does not support dynamo action when the tube is planar. By planar here, we mean that the torsion of the magnetic axis is planar which geometrically means that the Frenet torsion vanishes. This is exactly the Zeldovich anti-dynamo theorem [10]. To prove this result here we make the substitution $\tau = 0$ into equations (III.17),(III.18), and (III.19), which yields

$$p = 0 \tag{III.21}$$

and

$$p = \frac{v_0 tg\theta}{r}[1 + Tw] \tag{III.22}$$

these two last equations together imply that either $v_0 = 0$ or coordinate r at infinity. The first and more realistic hypothesis yield a planar circular flow. In other words, the helical flux inside the magnetic flux tube reduces to a planar circular flow. Of course our main result is in equation (III.21) which shows the dynamo action is not supported. Let us now turn our attention to show that the Cowling theorem is fulfilled, or that the untwisted, axially symmetric flux tubes do not support dynamo actions. To accomplished this task we simply substitute the expression $Tw = 0$ for the unwisted tube, into the same equations were used to test Zeldovich theorem, which in turn yields

$$\tau tg\theta \Omega r = 0 \tag{III.23}$$

$$v_0 = \Omega r \tag{III.24}$$

$$p = \frac{v_0 tg\theta}{r} \tag{III.25}$$

Assuming that the torsion does not vanish in the equation (III.23), and since the tube being thick, coordinate r cannot vanish, we conclude that the angular velocity Ω of the flow also vanish which from equation (III.24) that v_0 and from the last equation (III.25) we obtain $p = 0$. In the next section we shall analyse the general dynamo solution of the stretched , twisted tube.

IV   Differential rotation of stretched, twisted thick flux tubes dynamos

In this section we shall consider the general solution of self-induced equations and also consider the constraints of the other divergence-free equations on the non-uniform motion (differential rotation) of the flux tube. Expressions (III.17),(III.18) and (III.19) altogether yields an algebraic equation to the twist of the tube, given by

$$Tw^2 - [\tau r - v_0]Tw - \frac{\tau \Omega r^2}{v_0} = 0 \tag{IV.26}$$

Solutions of this algebraic equation for the tube twist are

$$Tw = \frac{1}{2}[\tau r \pm ((\tau r)^2[1 + 4\Omega])^{\frac{1}{2}}] \tag{IV.27}$$

to simplify this first solution let us assume the cyclonic hypothesis [1], where $\Omega >> 1$. Substitution of this value into the last expression yields

$$Tw = \frac{1}{2}\tau r[\pm(2\Omega)^{\frac{1}{2}}] \tag{IV.28}$$

where we have consider the strong torsion bound $\tau r >> v_0$. This allows us to determine the differential rotation in terms of the twist as

$$\frac{1}{2}\Omega(r,\theta(s)) = \frac{Tw^2}{\tau^2 r^2} \tag{IV.29}$$

Since the flux tube twist is given by the ratio between the poloidal and toroidal components of the magnetic field, we obtain a relation between these components and the differential rotation as

$$B_s{}^2 = \frac{\sqrt{2}}{2}\frac{\Omega}{[\tau r]^2}B_\theta{}^2 \tag{IV.30}$$

Assuming that the cyclonic hypothesis also implies that $\Omega >> v_0$ equation (III.18) becomes

$$p = \frac{\tau}{Tw}[\Omega r] \tag{IV.31}$$

Substitution of expression (IV.29) into last expression yields

$$p = \sqrt{2\Omega} > 0 \tag{IV.32}$$

Since $p > 0$ a dynamo action is supported , however since p by hypothesis is constant, the differential rotation degenerates in a solid homogeneous rotation. Let us now investigate the case of non-cyclonic rotation where $\Omega << 1$ and $\Omega << v_0$. Under these bounds the twist algebraic solutions reduces to $Tw = \tau r$. Substitution of this result into the expression (III.18) yields

$$p = \Omega \tag{IV.33}$$

which supports also anti-dynamos or non-dynamos (p¡0) for anti-cyclonic rotations ($\Omega < 0$). To further investigate the diferential rotation let us consider the equation

$$\frac{\partial v_\theta}{\partial s} = \frac{v_\theta \kappa \tau r sin\theta}{K} \tag{IV.34}$$

which by substitution of the thickness condition on K and $v_\theta = \Omega r$ yields the shear relation

$$\frac{\partial \Omega}{\partial s} = \Omega \tau tg\theta \tag{IV.35}$$

Since in this our case Ω is constant either Ω or τ vanishes, which does not represent dynamos as we have just seen, or yet $tg\theta$ vanishes which yields $\theta = 0$ region. The only dynamo condition which finally survives is to consider that the product between torsion and $\Omega = p$ product is very weak. This is however a not very efficient dynamo since though $p > 0$, it is close to zero, which also gives a very weak rotation which also yields an anti-dynamo as for example, in the case of the Venus planet which does not support a magnetic field provenient from dynamo actions since its rotation is 243 lower that the Earths. As a final attempt to obtain our dynamo solution let us drop the strong torsion of the flux tube dynamo and $\tau r = v_0$, which from expression (IV.26) yields

$$Tw^2 = \Omega r \qquad (IV.36)$$

which upon substitution into expression (III.18) yields the differential rotation as

$$\Omega(r,s) = \frac{p^2}{\tau^2 r} \qquad (IV.37)$$

which is now is not constant and a true differential rotation and besides it is also a dynamo since $p = \tau\sqrt{\Omega r} > 0$ if $\tau > 0$. The shear also does not vanish and is written as $\frac{\partial \Omega}{\partial s} = -\frac{2p^2}{r}\tau^{-3}tg\theta$ since $\tau > 0$. Substitution of this last result into equation (IV.35), allows us to determine the torsion in terms of the dynamo constant p as $\tau = \frac{p}{\sqrt{rtg\theta}}$ which to be real only on certain branches of the flux tube unless the constant p or this purely imaginary, which gives us the general dynamo condition as $Re(p) > 0$ [14].

V   Conclusions

In conclusion, we have tested Cowling and Zeldovich anti-dynamo theorems in thick stretched and twisted flux tubes, as solution of MHD cyclonic flows. Two dynamo solutions are obtained , one which is a very inefficient dynamo and the other which is a better efficient dynamo where the differential rotation is also computed along the shear along the magnetic axis of the tube. Since the Riemann metric of the very thick tube coincides with the sphere metric in three dimensions we may argue that spherical dynamos are also obtained by this technique. Other interesting test for Cowling antidynamo theorem can be obtained by using other metrics besides the four-dimensional black hole spacetime considering recently numerically by Brandenburg

[15]. Future prospects included the investigation of general relativistic MHD dynamos on the background of Lewis metric.

Acknowledgements

References

[1] E.N. Parker, Cosmical Magnetic Fields (1980), Oxford University Press.

[2] G. Rudiger and R. Hollerbach, The Magnetic Universe (2004), Wiley.

[3] V. Arnold, Ya B. Zeldovich, R. Ruzmaikin, D. Sokoloff, Soviet Physics J. JETP 54 (6) , (1981) 1083.

[4] L. C. Garcia de Andrade, Physics of Plasmas 13, 022309 (2006).

[5] L.C. Garcia de Andrade, Twist transport in strongly torsioned astrophysical flux tubes,Astrophysics and Space Science (2007) in press.

[6] J. Thiffeault and A.H.Boozer,Chaos 11,(2001) 16.

[7] J. Thiffeault, Stretching and Curvature of Material Lines in Chaotic Flows,(2004) Los Alamos arXiv:nlin. CD/0204069.

[8] A.H. Boozer, Phys. of Fluids B 5 (7), (1993) 2271.

[9] C. Thakur, Astrophysics and space science 149 (1988) 83.

[10] Ya B. Zeldovich, A.A. Ruzmaikin and D.D. Sokoloff, The Almighty Chance (1990) World sci. Press. S. Childress and A. Gilbert, Stretch, Twist and Fold: The Fast Dynamo (1996)(Springer).

[11] V. Arnold and B. Khesin, Topological Methods in Hydrodynamics, Applied Mathematics Sciences 125 (1991) Springer.

[12] R. Ricca, Solar Physics 172,241 (1997).

[13] W.D. D'haesseleer, W. Hitchon, J. Callen and J.L. Shohet, Flux Coordinates and Magnetic field Structure (1991) Spinger.

[14] S. Childress and A. Gilbert, Stretch, Twist and Fold: The Fast Dynamo (1996)(Springer).

[15] A. Brandenburg, Ap. J 465:L115 (1996).