

Simple pairs potential-density for flat rings

Patricio S. Letelier*

*Departamento de Matemática Aplicada
Instituto de Matemática,
Estatística e Computação Científica
Universidade Estadual de Campinas
13083-970 Campinas, S. P., Brazil*

Pairs potential-density in terms of elementary functions that represents flat rings structures are presented. We study structures representing one or several concentric flat rings. Also disks surrounded by concentric flat rings are exhibited. The stability of concentric circular orbits of particles moving on a flat ring structure is analyzed for radial perturbations.

I. INTRODUCTION

Flat rings systems are a common feature of the four giant planets of the Solar System. Also ringy structures are present in several observed galaxies and nebulae, e.g., in the Hourglass nebula we have as an outstanding characteristic: the presence of a very net and large isolated ring. The exact gravitational potential of a ring of zero thickness and constant linear density is given in its exact form by an elliptic integral that is seldom used for practical purposes. This potential is usually approximated by truncated series of spherical harmonics, i.e., a multipolar expansion. As far as we know there not simple expressions for the exact gravitational potential of a flat ring of any surface density.

The potential of a ring enclosing a disk can be obtained by a process of complexification of the potential of a punctual mass [1], [14], [4]. This potential can be used to built a family of similar structures [8]. On the other hand simple pairs potential-density for thin disks are known. The simpler is the Plummer-Kuzmin disk [[11] and [6]] that represents a simple model of galactic disk with a concentration of mass in its center and density that decays as $1/r^3$ on the plane of the disk, see also [3]. This structure has no boundary even though for practical purposes one can put a cutoff radius wherein the main part of the mass is inside, say 98% of the mass. Another simpler models of disks are the [10] disks. These disks have a mass concentration on their centers and finite radius. The [12] inversion theorem can be used to invert the Morgan and Morgan disks to produce an infinite disk with a central hole of the same radius of the original disk. We can also put a cutoff in these inverted disks. Therefore the inverted Morgan and Morgan disks can be considered as representing a flat rings. In the context Einstein theory of gravitation one of this inverted rings was used to study the gravity of a disk with a central black hole, [7].

The purpose of this article is to use the family of Morgan and Morgan disks to construct flat ring like struc-

tures. We shall use two different approaches, the first consists in the superposition of finite disks of different densities. By using this method we can construct structures that represent one or several concentric flat rings and families of disks surrounded by flat rings. These structures have a finite outer radius. The second method is the Lord Kelvin inversion method that will be used to construct the corresponding inverse structures obtained using the first method. In this case the structures extend to infinite, but as before one can also put a cutoff.

As a very first test of stability for these structures we study the linear stability of particles moving in circular orbits inside them. In other words, we assume the very naive model that the structures are build from particles moving only in concentric circles. The stability of circular orbits in several axially symmetric systems is studied in both Newton and Einstein theories of gravitation in [9].

This article is divided as follows, in Sec. II we give a quick revision of the pair potential-density associated to the Morgan and Morgan disks. In Sec. III we study several classes of superpositions that give rise to pairs potential-density of structures that can be considered as representing one or several concentric flat rings. In the next section, Sec. IV, we construct pairs potential-density associated to structures representing a central disk surrounded by one or several concentric flat rings. In the following section, Sec. V, we study the Kelvin inversion of the previous studied structures. In the penultimate section, Sec VI, we study the stability under linear radial perturbations of four of the ring structures already presented. Finally, in Sec VII, we summarize and discuss some of the previous results.

II. MORGAN AND MORGAN DISKS

The [10] disks are obtained by solving the Laplace equation in the natural coordinates to represent the gravitational potential of a disk like structure, i.e., prolate coordinates (ν, μ, φ) that are related to the usual cylindrical coordinates (r, z, φ) by

$$a\nu = -\text{Im}\mathcal{R}, \quad a\mu = \text{Re}\mathcal{R}, \quad \mathcal{R} \equiv \sqrt{r^2 + (z - ia)^2}, \quad (1)$$

*Electronic address: letelier@ime.unicamp.br

where a is a positive constant. From the previous equations we find

$$r = a\sqrt{\nu^2 + 1}(1 - \mu^2), \quad z = a\nu\mu. \quad (2)$$

Note that on the plane $z = 0$ we have $\nu = 0$ and $\mu = \sqrt{1 - p^2}$, with $p = r/a$.

From the Laplace equation in prolate coordinates one can find, [2], the potential of a disk parallel to the plane $z = 0$ and centered on the origin of the coordinate system,

$$V_{2m} = (1 - 2m)GM_{2m}(-1)^m P_{2m}(0)P_{2m}(\mu)q_{2m}(\nu)/a, \quad (3)$$

where M_{2m} is the mass of the disk with potential V_{2m} and a its radius. $P_n(x)$ are the usual Legendre polynomials and $q_n(x)$ are related to the usual Legendre functions, $Q_n(x)$, by $q_n(x) = i^{n+1}Q_n(ix)$. We recall the identity,

$$\begin{aligned} (-1)^n P_{2n}(0) &= 1 \cdot 3 \cdot 5 \dots (2n - 1)/(2 \cdot 4 \cdot 6 \dots 2n), \\ &= (2n - 1)!!/(2n)!!, \end{aligned} \quad (4)$$

that will frequently be used.

Associated to the potential (3) we have the surface density, [2],

$$\sigma_{2m} = (2m + 1)M_{2m}P_{2m}(1 - p^2)/(2\pi a^2 \sqrt{1 - p^2}). \quad (5)$$

[10] considered the following superposition of the above mentioned solutions of Laplace equation,

$$V^{(n)} = \sum_{m=0}^{m=n} A_{2m,2n} V_{2m}, \quad (6)$$

with

$$A_{2m,2n} = \frac{(4m + 1)2^{2m}(2n)!}{(n + m)!(n - m)!(2n + 2m + 1)!}. \quad (7)$$

This superposition represents the potential of disks with the simple associated surface density,

$$\sigma^{(n)}(r) = (2n + 1)M^{(n)}(1 - p^2)^{n - \frac{1}{2}}/(2\pi a^2), \quad (8)$$

where we have taken $M_{2n} = M^{(n)}$ for all n .

The potential on the plane $z = 0$ can be obtained using the algorithm, [10],

$$V^{(n)} = -\frac{GM^{(n)}}{a}(2n + 1)(-1)^n P_{2n}(0) \int_0^{\pi/2} [1 - p^2 \cos^2 \theta]^n d\theta \quad (9)$$

The $n = 0$ disk is singular on its rim and using (1) can be interpreted as the potential associated to a complexified bar of constant density, [8]. The subsequent members of the family represents disks of finite density with their maximum on their centers and zero density on their rims.

III. FLAT RINGS

Let us consider disks of the same radius a and decreasing mass,

$$M^{(n)} = 2\pi\sigma_c a^2/(2n + 1), \quad (10)$$

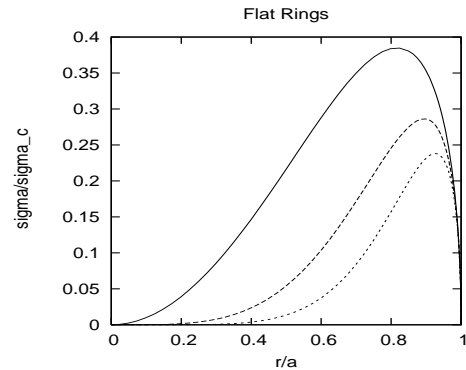


FIG. 1: The surface density of the first three members of a family disks with a central hole, $\{\sigma_{1r}^{(n)}\}$, i.e., flat rings, the size of the central hole is bigger for larger n .

where σ_c is a constant and will be taken equal for all disks of the Morgan and Morgan family, $n = 1, 2, 3, \dots$. For these family of disks we have that the corresponding surface density is

$$\sigma^{(n)} = \sigma_c(1 - p^2)^{\frac{1}{2}}(1 - p^2)^{n-1}. \quad (11)$$

Now let us consider the following superposition,

$$\sigma_{1r}^{(n)} = \sum_{k=0}^n C_k^n (-1)^{n-k} \sigma^{(n+1-k)}, \quad (12)$$

$$= \sigma_c(1 - p^2)^{\frac{1}{2}} p^{2n}, \quad (13)$$

where $C_k^n = n!/[(n-k)!k!]$. We have that all these superpositions give disks of radius a with zero density on their centers, i.e., disks with a hole in their centers, in other words flat rings. The densities associated to the three first members of this family of flat rings are presented in Fig. 1. We see a disk with a hole in the center with a residual density that is small for larger n . The mass of the three first flat rings are $\pi\sigma_c a^2/16$, $\pi\sigma_c a^2/32$, and $5\pi\sigma_c a^2/256$, respectively. The potentials associated to these flat rings can be found using a superposition with the same coefficients as the ones used for the densities, e.g., the potential associated to σ_{1r} is $V_{1r} = V^{(1)} - V^{(2)}$, etc.

Now let us consider, instead a superposition of disks, a superposition of flat rings,

$$\sigma_{2r}^{(n)} = \sigma_{1r}^{(n)} - 2b^2\sigma_{1r}^{(n)} + b^4\sigma_{1r}^{(n)}, \quad (14)$$

where b is a constant such that $b > a$. This superposition put a gap at $p = 1/b$ ($r = a/b$) on the flat one ring given us a family of two concentric flat rings. In Fig. 2 we present $\sigma_{2r}^{(1)}$ and $\sigma_{2r}^{(2)}$, with $b^2 = 2$, as before we see that the larger the n the larger is the size of the central hole. In Fig. 3 we show $\sigma_{2r}^{(1)}/\sigma_c$ for $b^2 = 2, 1.7, 1.4$.

We can construct several families of two rings using one rings in such a way that the final density be

$$\sigma_{2r}^{(n,m)} = \sigma_c(1 - p^2)^{\frac{1}{2}}(1 - b^2 p^2)^{2m} p^{2n} \quad (15)$$

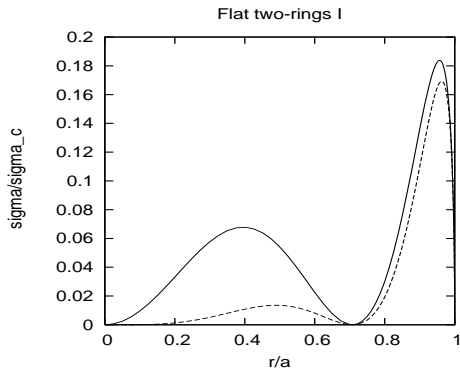


FIG. 2: The surface density of the first two members of a family disks with a center hole and a gap, $\sigma_{2r}^{(1)}$ and $\sigma_{2r}^{(2)}$, i.e., flat double rings, the size of the central hole is bigger for larger n . We take $b^2 = 2$.

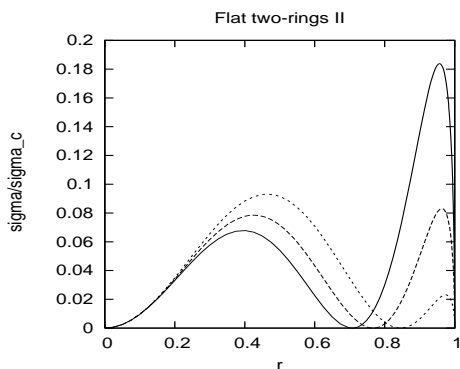


FIG. 3: The surface density of the first member of the family of two-rings, $\sigma_{2r}^{(1)}$, for $b^2 = 2, 1.7, 1.4$.

Of course this kind of superposition can be used to have families of solutions representing n concentric flat rings with density,

$$\sigma_{nr}^{(l, \vec{m})} = \sigma_c (1-p^2)^{\frac{1}{2}} (1-b_{(1)}^2 p^2)^{2m_1} (1-b_{(2)}^2 p^2)^{2m_2} \times \dots \times (1-b_{(n-1)}^2 p^2)^{2m_{n-1}} p^{2l}, \quad (16)$$

with $\vec{m} = (m_1, \dots, m_{n-1}) \in N^{n-1}$. The positions of the gaps are given by $p_i = 1/b_i$, with $i = 1 \dots n-1$, and $b_i > a$.

IV. DISKS WITH FLAT RINGS

By using the same type of superpositions presented in the previous section we can construct the pair potential-density for disks surrounded by flat rings. Let us consider models of a disk surrounded by one concentric flat ring. The density

$$\sigma_{d1r}^{(1)} = \sigma^{(1)} - 2b^2 \sigma_{(1r)}^{(1)} + b^4 \sigma_{1r}^{(2)}, \quad (17)$$

$$= \sigma_c (1-p^2)^{\frac{1}{2}} (1-b^2 p^2)^2, \quad (18)$$

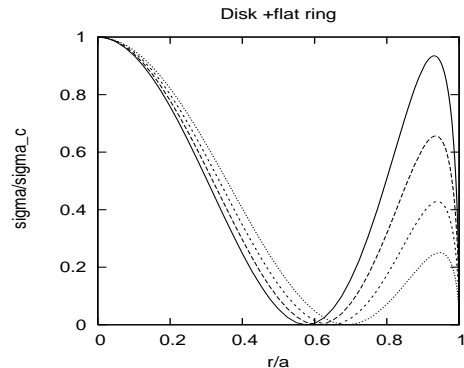


FIG. 4: The surface density of the first member of the family of disk-one-rings, $\sigma_{d1r}^{(1)}$, for $b^2 = 3, 2.7, 2, 4, 2.1$. For b small we have a larger central disk.

represents a disk of radius $r = a/b$ with a flat ring between $r = a/b$ and $r = a$. In Fig. 4 we present $\sigma_{d1r}^{(1)}/\sigma_c$ for $b^2 = 3, 2.7, 2, 4, 2.1$ we see that the position of the gap, as well as, the maximum of the flat ring density depends on b .

As before, we can build a family of disks with surrounded by one ring with surface density,

$$\sigma_{d1r}^{(n)} = \sigma_c (1-p^2)^{\frac{1}{2}} (1-b^2 p^2)^{2n}, \quad (19)$$

and the general case of a disk surrounded n concentric flat rings belonging to the above mentioned family,

$$\sigma_{dnr}^{(\vec{m})} = \sigma_c (1-p^2)^{\frac{1}{2}} (1-b_{(1)}^2 p^2)^{2m_1} (1-b_{(2)}^2 p^2)^{2m_2} \times \dots \times (1-b_{(n)}^2 p^2)^{2m_n} \quad (20)$$

where now $\vec{m} = (m_1, \dots, m_n) \in N^n$.

V. KELVIN INVERTED FLAT RINGS

To construct new family of flat rings we can use the inversion theorem of [12] that in cylindrical coordinates tells us that: if the pair potential-density $V(r, z)$, $\sigma(r, z)$ is a solution of Poisson equation the pair,

$$\hat{V}(r, z) = V \left[\frac{a^2 r}{r^2 + z^2}, \frac{a^2 z}{r^2 + z^2} \right], \quad (21)$$

$$\hat{\sigma}(r, z) = (a/r)^3 \sigma \left[\frac{a^2 r}{r^2 + z^2}, \frac{a^2 z}{r^2 + z^2} \right] \quad (22)$$

is also a solution of the same equation, see for instance, [5]. The inversion of the Morgan and Morgan family of disks with the density (8) gives us,

$$\hat{\sigma}^{(n)}(r) = (2n+1)M^{(n)}/(2\pi a^2 q^3)(1-q^2)^{n-\frac{1}{2}}, \quad (23)$$

where $q = a/r$ and $n \geq 1$. Note that that the Kelvin inverted Morgan and Morgan disks can be considered as representing flat rings, we have a disk with a hole of radius a . The density of these flat rings decays in a similar way as the Plummer-Kuzmin disks, i.e, as $1/r^3$, not too fast.

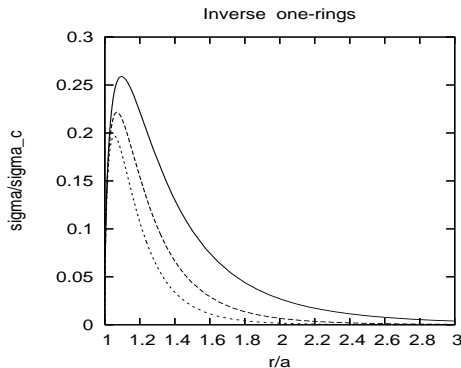


FIG. 5: The surface density of the first three member of the family of inverse one-rings, $\{\hat{\sigma}_{1r}^{(n)}\}$. We have a central hole of radius $r = a$. Even though the rings extent to infinity we see a clear cutoff.

The mass of the Kelvin inverted Morgan and Morgan disks is given by,

$$\hat{M}^{(n)} = (2n + 1)!!(2n)!!M^{(n)}/(2n)!! \quad (24)$$

The Kelvin inverted one-rings (13) gives other family of one-rings with density

$$\hat{\sigma}_{1r}^{(n)} = \sigma_c(1 - q^2)^{\frac{1}{2}}q^{2n+3}. \quad (25)$$

Note that the density decays as $\hat{\sigma}_{1r}^{(n)} \sim 1/r^{2n+3}$ for this family of flat rings. In Fig. 5 we plot the inverted density of the first three members of the one-flat-ring family. Note the central hole of radius $r = a$. Due to the fast decay rate with the distance we have that on can put a clear cutoff and to consider this flat rings as finite, Eq. (25). We can also invert the families of two and several flat rings discussed in Sec. 3, we will also have new families of rings that decay with the distance very fast.

VI. STABILITY OF CIRCULAR ORBITS

In this section we study the stability under radial perturbations of circular orbits concentric to the flat rings. By assuming that flat rings are made of particles moving along circular orbits the study of the stability of these orbits can be considered as an order zero test of stability of the flat rings. In this case the collective behavior of the particles of the ring is not taken into account. The epicyclic frequency of the perturbed orbit is, [3],

$$\kappa^2 = a^2 \left(\frac{\partial^2 V}{\partial p^2} + \frac{3}{p} \frac{\partial V}{\partial p} \right). \quad (26)$$

The epicyclic frequencies of rings with densities $\sigma_{1r}^{(1)}, \sigma_{1r}^{(2)}$ and their associated Kelvin inverted flat single

rings with densities $\hat{\sigma}_{1r}^{(1)}, \hat{\sigma}_{1r}^{(2)}$ are:

$$\kappa_{1r}^{(1)} = \pi [a^3 \sigma_c (27p^2 - 8)/8]^{\frac{1}{2}}, \quad (0 \leq p \leq 1), \quad (27)$$

$$\kappa_{1r}^{(2)} = \frac{\pi}{4} [a^3 \sigma_c (105p^4 - 81p^2 + 20)]^{\frac{1}{2}}, \quad (0 \leq p \leq 1) \quad (28)$$

$$\hat{\kappa}_{1r}^{(1)} = 3 \frac{\pi}{p^3} [a^3 \sigma_c / 8]^{\frac{1}{2}}, \quad (p \geq 1), \quad (29)$$

$$\hat{\kappa}_{1r}^{(1)} = \frac{\pi}{p^4} [3a^3 \sigma_c (-18p^2 + 35)/32]^{\frac{1}{2}}, \quad (p \geq 1). \quad (30)$$

The one-ring with density $\sigma_{1r}^{(1)}$ is not completely stable, we have stability only for $r > (2\sqrt{2}/3)a$. The flat rings with densities, $\sigma_{1r}^{(2)}$, and $\hat{\sigma}_{1r}^{(1)}$ are stable in all their extensions. We have that only the inner part, $1 < (r/a) < \sqrt{35/18}$, of the flat ring with density $\hat{\sigma}_{1r}^{(2)}$ is stable.

VII. DISCUSSION

We presented several families of pairs potential-density that can be used to represent one or several concentric rings, also a disk surrounded by flat rings. We have shown in some detail the superposition of surface densities to build the families indicating that the same superposition is valid to build the associated potential.

We have considered only the density as representing the physics inside the different structures studied along this work, in principle, one can find the other thermodynamics variables by solving the Fokker-Planck equation. In a recent work we presented a method for solving this equation for a disk, [13], that with simple modifications can be used to find a distribution function for these structures.

With our very first approach to the stability of flat rings presented in the penultimate section, we find stable configurations, as well as, partially stable ones. The structure of one-ring standing alone is not very realist. For instance in the case of planetary flat rings we have a planet located in center of the ring that will contribute to stabilize the structure. To be more specific let us consider a planet with mass M surrounded by a flat ring. Let κ_r^2 be the square of the epicyclic frequency associated to the ring alone. Now the square of the epicyclic frequency of a particle orbiting the system planet-ring is $\kappa_{pr}^2 = \kappa_r^2 + GMa^3/p^3$. Since the contribution of the planet to the epicyclic frequency of the particles of the disk is a positive quantity we have that the planet tends to stabilize the rings. To perform a study of the stability of the structures above described taken into account the collective behavior of the particles we need to know the fluid dynamics variables that in principle can be found solving the Fokker-Planck equation using the method already mentioned.

Acknowledgments

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