LATE-TIME BEHAVIOUR OF THE TILTED BIANCHI TYPE $VI_{-1/9}$ MODELS

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ABSTRACT. We study tilted perfect fluid cosmological models with a constant equation of state parameter in spatially homogeneous models of Bianchi type ${\rm VI}_{-1/9}$ using dynamical systems methods and numerical simulations. We study models with and without vorticity, with an emphasis on their future asymptotic evolution. We show that for models with vorticity there exists, in a small region of parameter space, a closed curve acting as the attractor.

1. Introduction

In recent papers the tilted Bianchi models [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] have been studied using dynamical systems methods [12, 13]. In this paper we will study the general tilted¹ perfect fluid Bianchi model of type $VI_{-1/9}$, with a constant linear equation of state parameter γ satisfying the causality conditions (i.e., no superluminal speed of sound), which was not studied in detail in [9]. The tilted models have shown a wide variety of phenomena including, for example, the existence of closed curves [8] and incompleteness of the fluid congruence [15, 16].

The general irrotational perfect fluid type $VI_{-1/9}$ model is the most general of the irrotational Bianchi models. In fact, the irrotational type $VI_{-1/9}$ model has a 6-dimensional state space; just one dimension less than that of the most general tilted type $VI_{-1/9}$ model. We will show that, regardless of whether we consider the general type $VI_{-1/9}$ model with or without vorticity, the important late time asymptotes are the non-tilted Collins type $VI_{-1/9}$ solution (for $2/3 < \gamma < 10/9$),

$$ds^{2} = -dt^{2} + t^{2}dx^{2} + t^{\frac{2(4-3\gamma)}{3\gamma}}e^{-\frac{2sx}{3\gamma}}dy^{2} + t^{\frac{4}{3\gamma}}e^{\frac{4sx}{3\gamma}}dz^{2},$$

where $s = \sqrt{(2-\gamma)(3\gamma-2)}$, or the Collinson-French (or Robinson-Trautman) vacuum solution (for $10/9 < \gamma < 2$),

$$ds^{2} = -dt^{2} + t^{2}dx^{2} + t^{\frac{2}{5}} \left(e^{-\frac{\sqrt{6}}{5}x} dy + \frac{\sqrt{5}}{2} t^{\frac{4}{5}} dx \right)^{2} + t^{\frac{6}{5}} e^{\frac{4\sqrt{6}}{5}x} dz^{2}.$$

The tilt of the fluid can be asymptotically non-tilted, intermediate or extreme depending on the value of γ . The Collinson-French solution is a Petrov type III vacuum solution, and hence it has the peculiar feature that all the curvature invariants of zeroth order vanish².

The spatial hypersurfaces in a Bianchi type $VI_{-1/9}$ cosmology are defined as the orbit of the action of the Bianchi type $VI_{-1/9}$ Lie group. This Lie group can be equipped with the left-invariant one-forms:

$$\widetilde{\omega}^1 = dx, \quad \widetilde{\omega}^2 = e^{-bx}dy, \quad \widetilde{\omega}^3 = e^{2bx}dz,$$

where b is a constant. The exceptional feature of this model does not relate to the group itself (for which there is nothing extraordinary) but arises from the Einstein field equations. For this particular model one of the Einstein constraint equations vanishes exactly and hence the vacuum case allows for an extra shear degree of freedom. This extra shear is present in the Collinson-French vacuum solution and therefore this metric has no analogue in the other Bianchi vacuum cosmologies.

The aim of this paper is to fill one of the gaps in the analysis of the general tilted type VI_h models presented in [9]. The h = -1/9 case is special (as explained above) and requires a separate analysis, which has only been done in part previously. In particular, we will show that closed periodic orbits exist for the general vortic model. However, for the irrotational models, no such closed curve exists. Moreover, we will

Date: October 28, 2018.

¹The non-tilted Bianchi model of type $VI_{-1/9}$ was studied in [14].

²However, considering differential invariants they will not vanish; for example, $C_{\alpha\beta\gamma\delta;\mu}C^{\alpha\beta\gamma\delta;\mu} \neq 0$.

also complete a centre manifold analysis appearing in one 6-dimensional invariant subspace describing vortic Bianchi type $VI_{-1/9}$ models. All other attractors will be included for completeness.

2. Equations of motion

2.1. The orthonormal frame approach. The line-element of a Bianchi cosmology can be written

(2.1)
$$ds^2 = -dt^2 + \delta_{ab}\omega^a\omega^b,$$

where t is the co-moving cosmological time. The one-forms ω^a are left-invariant one-forms on the hypersurfaces spanned by the group orbits and can be related to the left-invariant one-forms, $\widetilde{\omega}^i$, given above by $\omega^a = \mathrm{e}^a_{\ i}(t)\widetilde{\omega}^i$.

The geometric (or normal) congruence, n^{μ} , is given by $\mathbf{n} = \partial/\partial t$. It is also useful to define and the Hubble scalar and the shear associated with the congruence n^{μ} :

(2.2)
$$H \equiv \frac{1}{3} n^{\mu}_{;\mu}, \quad \sigma_{\mu\nu} \equiv n_{\mu;\nu} - H h_{\mu\nu}.$$

The matter variables are chosen to be the energy density, μ , and the tilt-velocity, v^a , which is defined as the 3-velocity of the fluid with respect to the geometric (or normal) congruence, n^{μ} . The equations of motion can now be written down in terms of the Hubble scalar, H; the shear, σ_{ab} ; the curvature variables n^{ab} and a_c ; and the matter variables μ and v^a .

Expansion-normalised variables are introduced (by dividing the variables with the appropriate powers of H). The papers [7, 11] contain all the details regarding the determination of the evolution equations for the tilted cosmological models under consideration. Here, we shall adopt the so-called N-gauge in which the function \mathbf{N}_{\times} is purely imaginary; this is realised by the choice $\phi' = \sqrt{3}\lambda\Sigma_{-}$, where λ is defined by $\bar{N} = \lambda \mathrm{Im}(\mathbf{N}_{\times})$. The evolution equation for \bar{N} can then be replaced by an evolution equation for λ , which ensures a closed system of equations. We will also adopt the dimensionless time parameter τ , which is related to the cosmological time t via $\mathrm{d}t/\mathrm{d}\tau = (1/H)$, where H is the Hubble scalar.

Using expansion-normalised variables, the equations of motion are (see [7, 11] for the complete derivation of the equations):

$$(2.3) \Sigma'_{+} = (q-2)\Sigma_{+} + 3(\Sigma_{12}^{2} + \Sigma_{13}^{2}) - 2N^{2} + \frac{\gamma\Omega}{2G_{+}} \left(-2v_{1}^{2} + v_{2}^{2} + v_{3}^{2}\right)$$

$$(2.4) \Sigma'_{-} = (q - 2 - 2\sqrt{3}\Sigma_{23}\lambda)\Sigma_{-} + \sqrt{3}(\Sigma_{12}^{2} - \Sigma_{13}^{2}) + 2AN + \frac{\sqrt{3}\gamma\Omega}{2G_{+}}(v_{2}^{2} - v_{3}^{2})$$

(2.5)
$$\Sigma'_{12} = \left(q - 2 - 3\Sigma_{+} - \sqrt{3}\Sigma_{-}\right)\Sigma_{12} - \sqrt{3}\left(\Sigma_{23} + \Sigma_{-}\lambda\right)\Sigma_{13} + \frac{\sqrt{3}\gamma\Omega}{G_{+}}v_{1}v_{2}$$

(2.6)
$$\Sigma'_{13} = \left(q - 2 - 3\Sigma_{+} + \sqrt{3}\Sigma_{-}\right)\Sigma_{13} - \sqrt{3}\left(\Sigma_{23} - \Sigma_{-}\lambda\right)\Sigma_{12} + \frac{\sqrt{3}\gamma\Omega}{G_{+}}v_{1}v_{3}$$

$$(2.7) \Sigma'_{23} = (q-2)\Sigma_{23} - 2\sqrt{3}N^2\lambda + 2\sqrt{3}\lambda\Sigma_{-}^2 + 2\sqrt{3}\Sigma_{12}\Sigma_{13} + \frac{\sqrt{3}\gamma\Omega}{G_{+}}v_2v_3$$

$$(2.8) N' = \left(q + 2\Sigma_+ + 2\sqrt{3}\Sigma_{23}\lambda\right)N$$

$$(2.9) \lambda' = 2\sqrt{3}\Sigma_{23} \left(1 - \lambda^2\right)$$

(2.10)
$$A' = (q + 2\Sigma_+)A.$$

The equations for the fluid are:

(2.11)
$$\Omega' = \frac{\Omega}{G_{+}} \left\{ 2q - (3\gamma - 2) + 2\gamma A v_{1} + \left[2q(\gamma - 1) - (2 - \gamma) - \gamma \mathcal{S} \right] V^{2} \right\}$$

$$(2.12) v_1' = (T + 2\Sigma_+) v_1 - 2\sqrt{3}\Sigma_{13}v_3 - 2\sqrt{3}\Sigma_{12}v_2 - A(v_2^2 + v_3^2) - \sqrt{3}N(v_2^2 - v_3^2)$$

$$(2.13) v_2' = \left(T - \Sigma_+ - \sqrt{3}\Sigma_-\right)v_2 - \sqrt{3}\left(\Sigma_{23} + \Sigma_-\lambda\right)v_3 + \sqrt{3}\lambda N v_1 v_3 + \left(A + \sqrt{3}N\right)v_1 v_2$$

$$(2.14) v_3' = \left(T - \Sigma_+ + \sqrt{3}\Sigma_-\right)v_3 - \sqrt{3}\left(\Sigma_{23} - \Sigma_-\lambda\right)v_2 - \sqrt{3}\lambda Nv_1v_2 + \left(A - \sqrt{3}N\right)v_1v_3$$

(2.15)
$$V' = \frac{V(1-V^2)}{1-(\gamma-1)V^2} [(3\gamma-4)-2(\gamma-1)Av_1 - \mathcal{S}],$$

where

$$q = 2\Sigma^{2} + \frac{1}{2} \frac{(3\gamma - 2) + (2 - \gamma)V^{2}}{1 + (\gamma - 1)V^{2}} \Omega$$

$$\Sigma^{2} = \Sigma_{+}^{2} + \Sigma_{-}^{2} + \Sigma_{12}^{2} + \Sigma_{13}^{2} + \Sigma_{23}^{2}$$

$$S = \Sigma_{ab}c^{a}c^{b}, \quad c^{a}c_{a} = 1, \quad v^{a} = Vc^{a},$$

$$V^{2} = v_{1}^{2} + v_{2}^{2} + v_{3}^{2},$$

$$T = \frac{[(3\gamma - 4) - 2(\gamma - 1)Av_{1}](1 - V^{2}) + (2 - \gamma)V^{2}S}{1 - (\gamma - 1)V^{2}}$$

$$G_{+} = 1 + (\gamma - 1)V^{2}.$$

These variables are subject to the constraints

$$(2.16) 1 = \Sigma^2 + A^2 + N^2 + \Omega$$

$$0 = 2\Sigma_{+}A + 2\Sigma_{-}N + \frac{\gamma\Omega v_{1}}{G_{+}}$$

(2.18)
$$0 = -\left[\Sigma_{12}(N + \sqrt{3}A) + \Sigma_{13}\lambda N\right] + \frac{\gamma\Omega v_2}{G_+}$$

(2.19)
$$0 = \left[\Sigma_{13} (N - \sqrt{3}A) + \Sigma_{12} \lambda N \right] + \frac{\gamma \Omega v_3}{G_+}$$

$$(2.20) 0 = 3A^2 - (1 - \lambda^2) N^2.$$

The parameter γ will be assumed to satisfy $\gamma \in (0,2)$. The generalized Friedmann equation (2.16) yields an expression which effectively defines the energy density Ω . We will assume that this energy density is non-negative: $\Omega \geq 0$. Therefore, the state vector can be considered $X = [\Sigma_+, \Sigma_-, \Sigma_{12}, \Sigma_{13}, \Sigma_{23}, N, \lambda, A, v_1, v_2, v_3]$ modulo the constraint equations (2.17)-(2.20). Thus the dimension of the physical state space is seven (and hence of equal generality to the other models of type VI_h for a given value of h). Additional details are presented in [7].

The dynamical system is invariant under the following discrete symmetries:

$$\begin{array}{l} \phi_1: \ [\Sigma_+, \Sigma_-, \Sigma_{12}, \Sigma_{13}, \Sigma_{23}, N, \lambda, A, v_1, v_2, v_3] \mapsto [\Sigma_+, \Sigma_-, \Sigma_{12}, \Sigma_{13}, \Sigma_{23}, -N, \lambda, -A, -v_1, -v_2, -v_3] \\ \phi_2: \ [\Sigma_+, \Sigma_-, \Sigma_{12}, \Sigma_{13}, \Sigma_{23}, N, \lambda, A, v_1, v_2, v_3] \mapsto [\Sigma_+, -\Sigma_-, \Sigma_{13}, \Sigma_{12}, \Sigma_{23}, -N, \lambda, A, v_1, v_3, v_2] \\ \phi_3^{\pm}: \ [\Sigma_+, \Sigma_-, \Sigma_{12}, \Sigma_{13}, \Sigma_{23}, N, \lambda, A, v_1, v_2, v_3] \mapsto [\Sigma_+, \Sigma_-, \pm \Sigma_{12}, \mp \Sigma_{13}, -\Sigma_{23}, N, -\lambda, A, v_1, \pm v_2, \mp v_3] \\ \phi_4: \ [\Sigma_+, \Sigma_-, \Sigma_{12}, \Sigma_{13}, \Sigma_{23}, N, \lambda, A, v_1, v_2, v_3] \mapsto [\Sigma_+, \Sigma_-, -\Sigma_{12}, -\Sigma_{13}, \Sigma_{23}, N, \lambda, A, v_1, -v_2, -v_3] \end{array}$$

These discrete symmetries imply that without loss of generality we can restrict the variables $A \geq 0$ and $N \geq 0$, since the dynamics in the other regions can be obtained by simply applying one or more of the maps above. The third and fourth symmetries listed imply that one can add additional constraints on the variables $\Sigma_{12}, \Sigma_{13}, v_2$ or v_3 . For the type $VI_{-1/9}$ we can use ϕ_4 to restrict v_2 to be non-negative (since $v_2 = 0$ implies $v_3 = 0$, see below); hence, we will assume $v_2 \geq 0$. There is no natural way to restrict any of the remaining variables using the symmetry ϕ_3^+ , and so we will not do so here.

2.2. Invariant sets. For the case h = -1/9 we define $(N_{ab}v^av^b)$ is identically zero for h = -1/9

(2.21)
$$\widehat{D} = \left[\lambda (\Sigma_{12}^2 + \Sigma_{13}^2) + 2\Sigma_{12}\Sigma_{13} \right].$$

In this analysis we will be concerned with the following invariant sets:

- (1) $T(VI_{-1/9})$: The general tilted type $VI_{-1/9}$ model: $|\lambda| < 1$.
- (2) $T_1(VI_{-1/9})$: A one-tilted type $VI_{-1/9}$ model: $|\lambda| < 1, v_2 = v_3 = \Sigma_{12} = \Sigma_{13} = 0$.
- (3) $T_{1,0}(VI_{-1/9})$: A one-tilted diagonal type $VI_{-1/9}$ model: $v_2 = v_3 = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = \lambda = 0$.
- (4) $N^{\pm}(VI_{-1/9})$: A class of tilted type $VI_{-1/9}$ models: $|\lambda| < 1$, $\widehat{D} = 0$.
- (5) $T_2^+(VI_{-1/9})$: A two-tilted type $VI_{-1/9}$ model. This is the fixed-point-set of ϕ_3^+ and is given by $v_3 = \Sigma_{13} = \Sigma_{23} = \lambda = 0$.
- (6) $T_2^-(VI_{-1/9})$: A one-tilted irrotational type $VI_{-1/9}$ model. This is the fixed-point-set of ϕ_3^- and is given by $v_2 = v_3 = \Sigma_{12} = \Sigma_{23} = \lambda = 0$.
- (7) $B(VI_{-1/9})$: Non-tilted type $VI_{-1/9}$: $|\lambda| < 1$, $v_1 = v_2 = v_3 = \Sigma_{12} = \Sigma_{13} = 0$.
- (8) $B_0(VI_{-1/9})$: A class of diagonal non-tilted type $VI_{-1/9}$ models $(n^{\alpha}_{\alpha} = 0)$: $V = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = \lambda = 0$
- (9) T(II): The general type II model: $\lambda = \pm 1, A = 0$.
- (10) B(I): Type I: N = A = V = 0.
- (11) $\partial T(I)$: "Tilted" vacuum type I: $\Omega = N = A = 0$.

Regarding $N^{\pm}(VI_{-1/9})$, to verify that this is indeed an invariant subspace, we calculate \widehat{D}' :

$$\widehat{D}' = 2\left(q - 2 - 3\Sigma_{+} - \sqrt{3}\lambda\Sigma_{23} + 3Av_{1}\right)\widehat{D};$$

hence, $\widehat{D} = 0$ defines an invariant subspace. Note also that this invariant set is not a manifold; it is similar to the light-cone in 2-dimensional Minkowski space. Therefore, $N^{\pm}(VI_{-1/9}) - T_1(VI_{-1/9})$ consists of 4 disconnected pieces. By the symmetry ϕ_4 , these are actually only two inequivalent pieces. Here, we choose $N^{\pm}(VI_{-1/9})$ such that

$$T_2^+(VI_{-1/9}) \subset N^+(VI_{-1/9}), \quad T_2^-(VI_{-1/9}) \subset N^-(VI_{-1/9})$$

Since $N^+(VI_{-1/9}) \cap N^-(VI_{-1/9}) = T_1(VI_{-1/9})$, both $N^+(VI_{-1/9})$ and $N^-(VI_{-1/9})$ are invariant sets. We note that the closure of the set $T(VI_{-1/9})$ is given by

$$(2.22) \overline{T(VI_{-1/9})} = T(VI_{-1/9}) \cup T(II) \cup B(I) \cup \partial T(I).$$

Since the boundaries may play an important role in the evolution of the dynamical system we must consider all of the sets in the decomposition (2.22).

Let us consider the constraint equations (2.18) and (2.19) as a linear map

$$L: (\Sigma_{12}, \Sigma_{13}) \mapsto (v_2, v_3)/G_+,$$

where L is considered as given in terms of A, N, λ and Ω . For $h \neq -1/9$, $\det(\mathsf{L}) \neq 0$ and the image of L is 2-dimensional. However, for h = -1/9, $\det(\mathsf{L}) = 0$ and the image of L is 1-dimensional; hence, in this sense the constraint equations are degenerate. This implies that (v_2, v_3) has to be restricted to a 1-dimensional submanifold. We will therefore say that the general type $\mathrm{VI}_{-1/9}$ model only allows for 2 tilt degrees of freedom. In particular, we can solve for v_3 and obtain

$$v_3 = -\frac{\lambda v_2}{1 + \sqrt{1 - \lambda^2}}.$$

It is illustrative to consider the eigenvectors of the map L:

$$\begin{aligned} \mathbf{x}_0 &= \left(-\frac{\lambda}{1 + \sqrt{1 - \lambda^2}}, \ 1 \right), & \mathsf{L}\mathbf{x}_0 &= 0, \\ \mathbf{x}_a &= \left(1, \ -\frac{\lambda}{1 + \sqrt{1 - \lambda^2}} \right), & \mathsf{L}\mathbf{x}_a &= a\mathbf{x}_a, \quad a > 0. \end{aligned}$$

For each of these eigenvectors, $\widehat{D}=0$ and hence, alternatively, we can define $N^{\pm}(VI_{-1/9})$ when $(\Sigma_{12}, \Sigma_{13})$ is proportional to one of these eigenvectors. More specifically, for $N^{+}(VI_{-1/9})$, $(\Sigma_{12}, \Sigma_{13}) \propto \mathbf{x}_{a}$, while for $N^{-}(VI_{-1/9})$, $(\Sigma_{12}, \Sigma_{13}) \propto \mathbf{x}_{0}$.

Table 1. The dimensions of the invariant sets for the Bianchi type $VI_{-1/9}$ model. The right-most column indicates the specialization in terms of the G_2 cosmologies. Here, Diag means diagonal, HO means hypersurface orthogonal, and OT means orthogonally transitive. The stars indicate the exceptional cases where the models aquire an addition shear degree of freedom.

Dim	Invariant set	No of Tilt	G_2 action
2	$B_0(VI_{-1/9})$	0	Diag
3	$T_{1,0}(VI_{-1/9})$	1	Diag
	$B_0(VI^*_{-1/9})$ *	0	НО
4	$B(VI_{-1/9})$	0	ТО
	$T_2^+(VI_{-1/9})$	2	НО
	$T_2^-(VI_{-1/9})$ *	1	НО
5	$T_1(VI_{-1/9})$	1	ТО
	$B(VI^*_{-1/9})$ *	0	Gen G_2
6	$N^+(VI_{-1/9})$	2	Gen G_2
	$N^{-}(VI_{-1/9})$ *	1	Gen G_2
7	$T(VI_{-1/9})$	2	Gen G_2

Note also that, due to the presence of the eigenvector with zero eigenvalue \mathbf{x}_0 , $(v_2, v_3) = 0$ does not necessarily mean that $(\Sigma_{12}, \Sigma_{13})$ is zero. In particular, for the non-tilted models this implies that we may have an additional shear degree of freedom; these models have usually been called the exceptional case and are denoted by an asterisk; e.g., $B_0(VI_{-1/9}^*)$ and $B(VI_{-1/9}^*)$. We also note that for the tilted models, there is an exceptional case of the one-tilted models $T_1(VI_{-1/9})$ which could be denoted $T_1(VI_{-1/9}^*)$. However, $T_1(VI_{-1/9}^*) = N^-(VI_{-1/9})$ as explained above. Therefore, we keep the notation $N^-(VI_{-1/9})$. Similarly, we have $T_{1,0}(VI_{-1/9}^*) = T_2^-(VI_{-1/9})$.

2.3. Fluid Vorticity. The various invariant subspaces can also be categorised in terms of the $(H_{\text{fluid}} \text{ normalised}, \text{ where } H_{\text{fluid}} \equiv (1/3)u^{\mu}_{;\mu})$ fluid vorticity, W^{α} . The vorticity of the fluid for the type $\text{VI}_{-1/9}$ models is given by:

(2.23)
$$W_{a} = \frac{1}{2B} (N_{ab} + \varepsilon_{abc} A^{c}) v^{b}, \quad W_{0} = 0,$$

where

$$B \equiv \frac{1 - \frac{1}{3}(V^2 + V^2 \mathcal{S} + 2A_a v^a)}{[1 - (\gamma - 1)V^2]\sqrt{1 - V^2}}.$$

For the invariant sets:

- (1) $T(VI_{-1/9})$: $W^0 = W^1 = 0$, most general vortic type $VI_{-1/9}$.
- (2) $N^+(VI_{-1/9})$: $W^0 = W^1 = 0$.
- (3) $N^-(VI_{-1/9})$: $W^0 = W^a = 0$, non-vortic.
- (4) $T_2^+(VI_{-1/9})$: $W^0 = W^1 = W^2 = 0$.
- (5) $T_2^-(VI_{-1/9})$: $W^0 = W^a = 0$, non-vortic.
- (6) $T_1(VI_{-1/9})$: $W^0 = W^a = 0$, non-vortic.
- (7) $B(VI_{-1/9})$: $W^0 = W^a = 0$, non-tilted and non-vortic.

We note that the most general non-vortic model is of dimension 6. Hence, since the non-vortic type VI_h and VII_h models – regarding h as fixed – are of dimension 5, the type $VI_{-1/9}$ model is the most general non-vortic model of all Bianchi models.

In general we can also solve for the vorticity component W^2 :

$$W^2 = \frac{\lambda}{1 + \sqrt{1 - \lambda^2}} W^3.$$

This follows from the constraint equations and eq.(2.23).

3. Qualitative behaviour

3.1. Monotone functions. There are a number of monotone functions in the state space of interest. For $0 < \gamma \le 6/7$, there exists a monotonically increasing function Z_1 defined by

(3.1)
$$Z_{1} \equiv \alpha \Omega^{1-\gamma}, \quad \alpha = \frac{(1-V^{2})^{\frac{1}{2}(2-\gamma)}}{G_{+}^{1-\gamma}V^{\gamma}},$$
$$Z'_{1} = [2(1-\gamma)q + (2-\gamma) + \gamma S] Z_{1}.$$

This function can be used to show [7]:

Theorem 3.1. For $0 < \gamma \le 6/7$, all tilted Bianchi models (with $\Omega > 0$, V < 1) of solvable type are asymptotically non-tilted at late times.

Corollary 3.2 (Cosmic no-hair). For $\Omega > 0$, V < 1, and $0 < \gamma < 2/3$ we have that

$$\lim_{\tau \to \infty} \Omega = 1, \quad \lim_{\tau \to \infty} V = 0.$$

Moreover, the following function is a monotone function in $T(VI_{-1/9})$:

(3.2)
$$Z_2 = \frac{A^4 N^2 G_+^5 \widehat{D}^2}{(1 - V^2)^{\frac{5}{2}(2 - \gamma)} \Omega^5},$$

$$(3.3) Z_2' = (5\gamma - 6)(3 - 2Av_1)Z_2,$$

where \widehat{D} is defined in eq.(2.21). This function is monotonically decreasing for $\gamma < 6/5$ and monotonically increasing for $6/5 < \gamma$.

We note that in the subspace $N^-(VI_{-1/9})$ we have the monotone function:

(3.4)
$$Z_3 = \frac{v_1^2 \Omega}{A^2 G_{\perp} (1 - V^2)^{\frac{1}{2}(2 - \gamma)}},$$

$$(3.5) Z_3' = -(2-\gamma)(3-2Av_1)Z_3.$$

This function is monotonically decreasing in $N^-(VI_{-1/9})$.

The monotonic function Z_3 immediately implies:

Theorem 3.3 (Future behaviour in $N^-(VI_{-1/9})$). For $2/3 < \gamma < 2$, $\Omega > 0$, A > 0, $v_1^2 < 1$, $v_2 = v_3 = 0$ we have that:

either
$$\lim_{\tau \to \infty} \Omega = 0$$
, or $\lim_{\tau \to \infty} V = 0$.

This implies that all irrotational type $VI_{-1/9}$ universes are either asymptotically vacuum or non-tilted at late times.

3.2. Equilibrium points.

- 3.2.1. B(I): Equilibrium points of Bianchi type I.
 - (1) $\mathcal{I}(I)$: $\Sigma_{+} = \Sigma_{-} = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = A = N = V = 0$ and $\Omega = 1$. Here, $|\lambda| < 1$ and is an unphysical parameter. This represents the flat Friedman-Lemaître model.

The remaining equilibrium points are all in $\partial T(I)$.

3.2.2. T(II): Equilibrium points of Bianchi type II. All of the tilted equilibrium points come in pairs. These represent identical solutions (they differ by a frame rotation); however, since their embeddings in the full state space are inequivalent, two of their eigenvalues are different. All of these equilibrium points have an unstable direction with eigenvalue $-2\sqrt{3}\Sigma_{23}$ corresponding to the variable A. These equilibrium points are given in [8].

3.2.3. $T(VI_{-1/9})$: Equilibrium points of Bianchi type $VI_{-1/9}$.

- (1) $\mathcal{C}(-1/9)$: Collins perfect fluid solution, $2/3 < \gamma < 5/3$ $\Sigma_{12} = \Sigma_{13} = \Sigma_{23} = \lambda = V = 0, \; \Sigma_{+} = -\frac{1}{4}(3\gamma - 2), \; \Sigma_{-} = \frac{\sqrt{3}}{12}(3\gamma - 2), \; N^{2} = \frac{3}{16}(3\gamma - 2)(2 - \gamma), \; N^{3} = \frac{3}{16}(3\gamma$ $A = N/\sqrt{3}$, $\Omega = \frac{1}{3}(5-3\gamma)$. This equilibrium point is in $B(VI_{-1/9})$.
- (2) $\mathcal{R}^+(-1/9)$: The Apostolopoulos h = -1/9 solution [17], $4/3 < \gamma < 3/2$ This is a vortic solution lying in the invariant subspace $T^+(VI_{-1/9})$. The solution is given in terms of the expansion-normalised variables in [11] with h = -1/9, k = 1/3.
- (3) Bianchi type $VI_{-1/9}$ vacuum plane waves. All of these solutions have

$$\Omega = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0, \ \Sigma_{-} = N = \sqrt{-\Sigma_{+}(1 + \Sigma_{+})}, \ A = (1 + \Sigma_{+}), \ -1 < \Sigma_{+} < -3/4, \ |\lambda| < 1.$$

It is avantageous to introduce $r \equiv \sqrt{1-\lambda^2}$, which implies that we can write

$$\Sigma_{+} = -\frac{3}{3+r^2}, \quad 0 < r \le 1.$$

We will also define ρ by

$$\rho = v_2^2 + v_3^2.$$

The equilibrium points are then determined by the tilt velocities:

- (a) $\mathcal{L}(-1/9)$: $v_1 = v_2 = v_3 = 0$. These represent 'non-tilted' plane waves and lie in $B(VI_{-1/9})$.
- (b) $\widetilde{\mathcal{L}}(-1/9)$: $v_1 = \frac{3\gamma(3+r^2)-2(9+2r^2)}{2r^2(\gamma-1)}$, $v_2 = v_3 = 0$, $\frac{6(3+r^2)}{9+5r^2} < \gamma < 2$. These represent 'intermediately' tilted' plane waves and lie in $T_1(VI_{-1/9})$.
- (c) $\mathcal{L}_{\pm}(-1/9)$: $v_1 = \pm 1$, $v_2 = v_3 = 0$. These represent 'extremely tilted' plane waves and lie in $T_1(VI_{-1/9}).$
- (d) $\widetilde{\mathcal{F}}^+(h)$: Here, $(9+7r^2)/[3(3+r^2)] \le \gamma \le 3(3+r^2)/(9-r^2)$ and

$$v_1 = -\frac{3\gamma(3+r^2) - (9+7r^2)}{2r^2(3-\gamma)}, \quad v_2^2 - v_3^2 = \rho r$$

$$\rho = \frac{(9+r^2)\left[5 - 3\gamma\right]\left[3\gamma(3+r^2) - (9+7r^2)\right]}{8r^4(3-\gamma)^2}$$

These represent 'intermediately tilted' plane waves and lie in $N^+(VI_{-1/9})$ (for $\lambda = 0$ they lie in $T_2^+(VI_{-1/9})$.

(e) $\widetilde{\mathcal{E}}_{p}^{+}(-1/9)$, $0 < \gamma < 2$:

$$v_1 = -\frac{4r^2}{3(3-r^2)}, \quad v_2^2 - v_3^2 = \rho r$$

$$\rho = 1 - v_1^2 = \frac{(9+r^2)(9-7r^2)}{9(3-r^2)^2}$$

These represent 'extremely tilted' plane waves and lie in $N^+(VI_{-1/9})$. For $\lambda=0$ these equilibrium points lie in $T_2^+(VI_{-1/9})$, and due to the special importance for the late-time behaviour we will denote $\widetilde{\mathcal{E}}_{p0}^+(-1/9) \equiv \widetilde{\mathcal{E}}_p^+(-1/9)\Big|_{\lambda=0}$. Therefore,

$$\widetilde{\mathcal{E}}_{p0}^+(-1/9): v_1 = -\frac{2}{3}, \quad v_2 = \frac{\sqrt{5}}{3}$$

(4) \mathcal{CF} : The Collinson-French (Robinson-Trautmann) solution is given by:

$$\Sigma_{+} = -\frac{1}{3}, \quad \Sigma_{-} = \frac{1}{3\sqrt{3}}, \quad \Sigma_{13} = \frac{\sqrt{15}}{9}, \quad N = \frac{1}{\sqrt{2}}, \quad A = \frac{1}{\sqrt{6}},$$

$$\Sigma_{12} = \Sigma_{23} = \Omega = \lambda = 0.$$

There are the following equilibrium points associated with the Collinson-French solution:

- (a) \mathcal{CF}_0 : $v_1 = v_2 = 0$, $0 < \gamma < 2$.
- (b) \widetilde{CF}_{1+} : $v_1 = -\frac{\sqrt{6}(3\gamma 4)}{2(3-\gamma)}$, $v_2 = \frac{\sqrt{5(3\gamma 4)(3-2\gamma)}}{\sqrt{2}(3-\gamma)}$, $\frac{4}{3} < \gamma < \frac{3}{2}$. (c) \widetilde{CF}_2 : $v_1 = \frac{\sqrt{6}(9\gamma 14)}{6(\gamma 1)}$, $v_2 = 0$, $\frac{24-\sqrt{6}}{15} < \gamma < \frac{24+\sqrt{6}}{15}$.

(d)
$$\widetilde{\mathcal{ECF}}_{\pm}$$
: $v_1 = \pm 1, v_2 = 0, 0 < \gamma < 2$.

These equilibrium points, and their stability, were studied in [7].

(5) W: Wainwright $\gamma = 10/9$ solution:

$$\Sigma_{+} = -\frac{1}{3}, \quad \Sigma_{-} = \frac{1}{3\sqrt{3}}, \quad 0 < \Sigma_{13} < \frac{\sqrt{15}}{9}, \quad N = \frac{1}{6}\sqrt{8 + 54\Sigma_{13}^2}, \quad A = \frac{1}{\sqrt{3}}N, \quad \Omega = \frac{5}{9} - 3\Sigma_{13}^2,$$
 $\Sigma_{12} = \Sigma_{23} = \lambda = V = 0.$

4. Late-time behaviour

The late-time behaviour of models with $0 < \gamma < 2/3$ is determined by (the Cosmic no-hair) Corollary 3.2.

- 4.1. The invariant subspace $N^+(VI_{-1/9})$. Here, we have the following late-time attractors:
 - $2/3 < \gamma \le 4/3$: The Collins solution, $\mathcal{C}(-1/9)$.
 - $4/3 < \gamma < 3/2$: The Apostolopoulos solution, $\mathcal{R}^+(-1/9)$.
 - $3/2 \le \gamma < 2$: "Extremely tilted" vacuum plane waves, $\widetilde{\mathcal{E}}_{p0}^+(-1/9)$

The stability of these points for $\gamma < 3/2$ follows from the eigenvalues of the linearised matrix. For $3/2 \le \gamma < 2$ several zero-eigenvalues occur and a centre manifold analysis is needed to determine the late-time behaviour.

4.1.1. The case $3/2 < \gamma < 2$: the centre manifold. Let us present the centre manifold analysis of the equilibrium point $\widetilde{\mathcal{E}}_{p0}^+(-1/9)$ in some detail. The centre manifold in this case is a 2-dimensional submanifold of the 5-dimensional extremely tilted invariant subspace $N^+(VI_{-1/9})|_{V=1}$. To find the centre manifold, we will therefore set V=1. Let us choose variables

(4.1)
$$(\Sigma_{+}, \Sigma_{23}, N, \lambda, v_{2}) = \left(-\frac{3}{4} + x_{1}, x_{2}, \frac{\sqrt{3}}{4} + x_{3}, x_{4}, \frac{\sqrt{5}}{3} + x_{5} \right),$$

let Σ_- , Σ_{12} , Σ_{13} and Ω be determined from the constraint equations, and $v_1^2 = 1 - v_2^2 - v_3^2$. Let us define the column vector $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]^T$. We can now expand the equations of motion to 2nd order in \mathbf{x} :

$$(4.2) x' = Ax + C(x,x) + \mathcal{O}(x^3),$$

where C(-,-) is a bilinear vector-valued function. It is convenient to align the vector x with the Jordan canonical form of A. This can be accomplished by defining

(4.3)
$$\mathsf{P} = \begin{bmatrix} \frac{7}{5} & -\frac{2}{5} & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1\\ \frac{2\sqrt{3}}{5} & -\frac{2\sqrt{3}}{5} & 0 & 0 & 0\\ 4\sqrt{3} & -4\sqrt{3} & 0 & 4\sqrt{3} & -4\sqrt{3}\\ -\frac{48\sqrt{5}}{25} & \frac{64\sqrt{5}}{45} & \frac{112\sqrt{5}}{225} & 0 & 0 \end{bmatrix}.$$

Now, defining $y = P^{-1}x$, the equation for y becomes

$$\mathbf{v}' = \mathsf{J}\mathbf{v} + \tilde{\mathsf{C}}(\mathbf{v}, \mathbf{v}) + \mathcal{O}(\mathbf{v}^3),$$

where J is the Jordan block matrix

$$\mathsf{J} = \mathrm{diag}\left(0, -\frac{1}{2}, -\frac{5}{6}, 0, -\frac{1}{2}\right).$$

The centre manifold correspond to the zero-eigenvalues of J. We can therefore parameterise the centre manifold using the variables (y_1, y_4) . The next step is to expand the variables y_i , i = 2, 3, 5, in terms of (y_1, y_4) on the centre manifold, to second order. We therefore define the quadratic forms $Y_i(y_1, y_4)$, i = 2, 3, 5, such that $y_i - Y_i(y_1, y_4) = 0$ is (to second order) an invariant subspace. On the centre manifold we then have:

$$(4.4) y_i = Y_i(y_1, y_4) + \mathcal{O}(y^3).$$

By substituting these into the y_1 and y_4 evolution equations, we finally get, on the centre manifold:

$$y'_1 = -16y_1^2 + \mathcal{O}(y^3),$$

(4.5)
$$y'_4 = -16y_1y_4 + \mathcal{O}(y^3)$$

To lowest order, these equations can be solved to give $y_1 \approx 1/(16\tau)$, $y_4 \approx C/\tau$, where a constant of integration has been eliminated by a translation of time. Note that, in principle, there are essentially two kinds of behaviour: for $\tau < 0$ the variables diverge, while for $\tau > 0$ the variables decay. However, requiring $\Omega > 0$ leaves only the decaying mode as physically acceptable. These decay rates will therefore be the dominant ones since the centre manifold will dominate the behaviour at late times.

In terms of the original variables, we therefore get the decay rates in $N^+(VI_{-1/9})$ (only dominant decay rates are included):

$$\Sigma_{+} \approx -\frac{3}{4} \left(1 - \frac{7}{60\tau} \right),$$

$$\Sigma_{-} \approx \frac{\sqrt{3}}{4} \left(1 + \frac{1}{60\tau} \right),$$

$$\Sigma_{12} \approx \frac{\sqrt{15}}{60\tau},$$

$$\Sigma_{13} \approx -\frac{(1 + 16C)\sqrt{5}}{160\tau^{2}},$$

$$\Sigma_{23} \approx -\frac{(1 + 16C)}{8\tau^{2}},$$

$$N \approx \frac{\sqrt{3}}{4} \left(1 + \frac{1}{10\tau} \right),$$

$$\lambda \approx \frac{(1 + 16C)\sqrt{3}}{4\tau},$$

$$\Omega \approx \frac{3}{40\tau},$$

$$v_{1} \approx -\frac{2}{3} \left(1 + \frac{9}{20\tau} \right),$$

$$v_{2} \approx \frac{\sqrt{5}}{3} \left(1 - \frac{9}{25\tau} \right),$$

$$(4.6)$$

This confirms that the 'extremely tilted' vacuum plane wave $\widetilde{\mathcal{E}}_{p0}^+(-1/9)$ is the attractor for $N^+(VI_{-1/9})$, $3/2 < \gamma < 2$. Note, however, that this solution is unstable in the fully tilted space $T(VI_{-1/9})$, which can be verified by calculating, for example, $\widehat{D}' \approx (3/4)\widehat{D}$ close to $\widetilde{\mathcal{E}}_{p0}^+(-1/9)$.

- 4.2. The invariant subspace $N^-(VI_{-1/9})$: non-vortic type $VI_{-1/9}$ models.
 - $2/3 < \gamma < 10/9$: The Collins solution, $\mathcal{C}(-1/9)$.
 - $\gamma = 10/9$: The Wainwright solution, W.
 - $10/9 < \gamma < 2$: The Collinson-French solution, \widetilde{CF} . We have the following refinement for the asymptotic tilt (which follows from an analysis of the eigenvalues):
 - * $10/9 < \gamma \le (24 \sqrt{6})/15$: The tilt is asymptotically zero (\mathcal{CF}_0) .
 - * $(24 \sqrt{6})/15 < \gamma \le 14/9$: The tilt is either asymptotically zero (\mathcal{CF}_0) or extreme $(\widetilde{\mathcal{ECF}}_-)$.
 - * $14/9 < \gamma < (24 + \sqrt{6})/15$: The tilt is either asymptotically intermediate $(\widetilde{\mathcal{CF}}_2)$ or extreme $(\widetilde{\mathcal{ECF}}_2)$.
 - * $(24 + \sqrt{6})/15 \le \gamma < 2$: The tilt is asymptotically extreme $(\widetilde{\mathcal{ECF}}_{-})$.
- 4.3. $T(VI_{-1/9})$: The fully tilted type $VI_{-1/9}$ model. For the fully tilted model, the late-time behaviour can be summarised as follows:
 - $2/3 < \gamma < 10/9$: The Collins solution, $\mathcal{C}(-1/9)$.
 - $\gamma = 10/9$: The Wainwright solution, W.
 - $10/9 < \gamma < 2$: The Collinson-French solution, $\widetilde{\mathcal{CF}}$. We have the following refinement for the asymptotic tilt:

- * $10/9 < \gamma \le (24 \sqrt{6})/15$: The tilt is asymptotically zero (\mathcal{CF}_0) .
- * $(24 \sqrt{6})/15 < \gamma \le 4/3$: The tilt is either asymptotically zero (\mathcal{CF}_0) or extreme $(\widetilde{\mathcal{ECF}}_-)$.
- * $4/3 < \gamma \le 5/6 + \sqrt{721}/42$: The tilt is either asymptotically intermediate $(\widetilde{\mathcal{CF}}_{1+})$ or extreme $(\widetilde{\mathcal{ECF}}_{-})$.
- * $5/6 + \sqrt{721}/42 < \gamma < \gamma_H \approx 1.47392$: The tilt is either asymptotically oscillatory (closed curve) or extreme $(\widetilde{\mathcal{ECF}}_{-})$.
- * $\gamma_H < \gamma < 2$: The tilt is asymptotically extreme $(\widetilde{\mathcal{ECF}}_-)$.

Most of these results comes from an analysis of the eigenvalues of the various equilibrium points. However, for $\gamma = 5/6 + \sqrt{721}/42$ the equilibrium point \widetilde{CF}_{1+} undergoes a Hopf-bifurcation and a stable closed curve results. The analysis of this Hopf-bifurcation is given below. Interestingly, this closed curve co-exists with an extremely tilted attractor.

4.3.1. The Hopf-bifurcation. Our main aim for this section is to show the following.

Theorem 4.1. There exists a γ_0 such that for $\frac{5}{6} + \frac{\sqrt{721}}{42} < \gamma < \gamma_0$ there exists a closed orbit, $c(\tau)$, acting as the future attractor for a set of non-zero measure of tilted Bianchi type $VI_{-1/9}$ models.

To prove this theorem we will first show the existence of a closed period orbit acting as an attractor in a particular subset. We consider the invariant subset given by the Collinson-French solution with 2 tilts. We introduce $(X,Y) = (v_1, v_2^2)$:

$$X' = \left(T - \frac{2}{3}\right)X - \frac{2\sqrt{6}}{3}Y,$$

$$Y' = 2\left(T + \frac{2\sqrt{6}}{3}X\right)Y.$$
(4.7)

We set $X_0 = -\frac{\sqrt{6}(3\gamma - 4)}{2(3 - \gamma)}$, $Y_0 = \frac{5(3\gamma - 4)(3 - 2\gamma)}{2(3 - \gamma)^2}$, and perform the transformation $x = X - X_0$, $y = Y - Y_0$ with respect to the equilibrium point \widehat{CF}_{1+} .

The normal form of a Hopf-bifurcation can be written

$$(4.8) Z' = (\lambda + b|Z|^2)Z,$$

where b is some complex number and $\lambda = \alpha + i\beta$ is a parameter. If Re(b) < 0 for $\alpha = 0$, there exists a stable closed orbit for $0 < \alpha$ sufficiently small.

We will therefore set $\gamma = \frac{5}{6} + \frac{\sqrt{721}}{42}$ and expand to cubic terms in x and y. It is also convenient to introduce a complex variable z chosen such that it aligns with the Jordan form of the linearised matrix. This can be achieved by setting:

$$z = x + iay$$

where a is a real number chosen such that the linear term of $z' = f(z, \bar{z})$ is

(4.9)
$$\partial_z f(0,0) = \frac{i}{6} \sqrt{-1205 + 45\sqrt{721}}, \quad \partial_{\bar{z}} f(0,0) = 0.$$

By a transformation

$$Z = z + a_{11}z^2 + a_{12}z\bar{z} + a_{22}\bar{z}^2 + a_{111}z^3 + a_{112}z^2\bar{z} + a_{122}z\bar{z}^2 + a_{222}\bar{z}^3,$$

we can choose the coefficients a_{ij} and a_{ijk} so that the equation for Z takes the form

(4.10)
$$Z' = (\lambda + b|Z|^2)Z + \mathcal{O}(|Z|^4),$$

where

$$\lambda = \frac{i}{6}\sqrt{-1205 + 45\sqrt{721}}, \quad \text{Re}(b) = \frac{1}{291600} \left(-15990233 + 595193\sqrt{721}\right) \approx -0.029.$$

Hence, there exists a γ_0 such that there exists a closed stable orbit for $\frac{5}{6} + \frac{\sqrt{721}}{42} < \gamma < \gamma_0$.

The next step is to show that this orbit is also stable in the fully tilted type $VI_{-1/9}$ models. We can show this as follows: for a function B, we introduce the average, $\langle B \rangle$, with respect to the closed orbit $c(\tau)$, defined by

$$\langle B \rangle = \frac{1}{T} \oint_{c(\tau)} B d\tau, \quad T = \oint_{c(\tau)} d\tau.$$

We can use this average, using similar manipulations as for the closed curves in the type IV, VI_h and VII_h models [8, 11] to show that:

Theorem 4.2. Assume that there exists a closed periodic orbit $c(\tau)$ for the dynamical system (4.7). Then

$$\langle X \rangle = -\frac{\sqrt{6}(3\gamma - 4)}{2(3 - \gamma)}, \quad \langle \lambda_{\Omega} \rangle = -\frac{5(5\gamma - 6)}{3(3 - \gamma)}.$$

Proof. From the Y equation, we get $\langle T \rangle = -2\sqrt{6} \langle X \rangle /3$. A manipulation of the V equation yields $\langle T \rangle = \langle \mathcal{S} \rangle$, and $\langle \mathcal{S} \rangle = (3\gamma - 4) - 2(\gamma - 1)A \langle X \rangle$. These can now be solved to yield the desired value for $\langle X \rangle$. A similar manipulation of the Ω equation yields $\langle \lambda_{\Omega} \rangle$.

This implies that the vacuum solution is stable when the closed curve is perturbed by Ω . Hence, since the Collinson-French solution is stable with respect to vacuum perturbations, this closed curve is stable for the fully tilted type $VI_{-1/9}$ models whenever $\frac{5}{6} + \frac{\sqrt{721}}{42} < \gamma < \gamma_0$.

It remains to determine the maximal value for γ_0 ? It seems that this limiting value is related to the

It remains to determine the maximal value for γ_0 ? It seems that this limiting value is related to the existence of a heteroclinic orbit originating and ending at the saddle points \mathcal{CF}_0 and $\widetilde{\mathcal{CF}}_2$, respectively. For the limiting value of γ , which we will call γ_H , there exists such a heteroclinic orbit, while for values $\gamma \neq \gamma_H$ no such heteroclinic orbit exists connecting these two equilibrium points. We can use this to numerically estimate the value for γ_H . Our estimate gives:

$$1.473920 < \gamma_H < 1.473921.$$

Since $5/6 + \sqrt{721}/42 = 1.472653409...$ this means that the region in which the closed orbit exists is extremely small.³

Note also that this implies that these closed orbits co-exist with the extremely tilted attractor \mathcal{ECF}_{-} , which makes this closed orbit even more difficult to detect. In addition, it appears as if the curves asymptote to the closed curve relatively slowly, which implies that the numerics have to run for a relatively long time in order to see the late-time asymptote. A numerical plot of some generally tilted type $VI_{-1/9}$ models approaching this closed orbit is shown in Fig.1.

5. Conclusion

In this paper we have examined the tilted Bianchi type $VI_{-1/9}$ model in some detail. This model is a special case (due to the vanishing of one of the constraint equations) and necessitates a separate analysis from the general type VI_h models. We showed that in these models there exists a tiny region of parameter space where there exists a closed curve acting as an attractor. This closed curve co-exists with an extremely tilted attractor. In the case of the most general irrotational models this closed curve, which appears in terms of the tilt velocities, is absent. We have also confirmed the analytical results with an extensive numerical investigation.

 $^{^{3}}$ However, this seems to be typical for these models, the loophole for the type IV and VII_h models also appears to be extremely small.

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FIGURE 1. point $\widetilde{\mathcal{ECF}}_-$ Bianchi type VI_{1/9} universes approaching a closed curve and the equilibrium – ($\gamma=1.4735).$









