No phase transition for Gaussian fields with bounded spins

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Summary Let $a < b$, $\Omega = [a, b]^{\mathbb{Z}^d}$ and H be the (formal) Hamiltonian defined on Ω by

$$H(\eta) = \frac{1}{2} \sum_{x,y\in\mathbb{Z}^d} J(x-y)\,(\eta(x) - \eta(y))^2 \tag{1}$$

where $J : \mathbb{Z}^d \to \mathbb{R}$ is any summable non-negative symmetric function ($J(x) \geq 0$ for all $x \in \mathbb{Z}^d$, $\sum_x J(x) < \infty$ and $J(x) = J(-x)$). We prove that there is a unique Gibbs measure on Ω associated to H. The result is a consequence of the fact that the corresponding Gibbs sampler is attractive and has a unique invariant measure.

Keywords truncated Gaussian fields, bounded spins, quadratic potential, no phase transition.

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1 Introduction

Let $\Omega = [a, b]^{\mathbb{Z}^d}$. Let the function $J : \mathbb{Z}^d \to \mathbb{R}^+$ be summable, non-negative and symmetric: $J(x) \geq 0$ for all $x \in \mathbb{Z}^d$ and $0 < \|J\| := \sum_x J(x) < \infty$; it is convenient to also assume $J(0) = 0$. For each finite $\Lambda \subset \mathbb{Z}^d$, consider the "ferromagnetic" Hamiltonian $H^\Lambda : \Omega \to \mathbb{R}$ given by the quadratic potential

$$H^\Lambda(\eta) := \frac{1}{2} \sum_{\{x,y\}\not\subset\Lambda^c} J(y-x)(\eta(x) - \eta(y))^2. \tag{2}$$

Let $\Gamma = \{\mu^{\Lambda,\gamma} : \Lambda \subset \mathbb{Z}^d \text{ finite}, \gamma \in \Omega\}$ be the family of local *specifications* induced by H^Λ: for finite Λ and $\gamma \in \Omega$ let $\mu^{\Lambda,\gamma}$ be the measure on $[a, b]^\Lambda$ with boundary conditions γ

defined by

$$\mu^{\Lambda,\gamma}(d\eta_\Lambda) \;:=\; \frac{1}{Z^{\Lambda,\gamma}}\exp\left(-H^\Lambda(\eta_\Lambda\gamma_{\Lambda^c})\right)d\eta_\Lambda, \tag{3}$$

where $Z^{\Lambda,\gamma}$ is the normalizing constant and $(\eta_\Lambda\gamma_{\Lambda^c}) \in \Omega$ is the *juxtaposition* of η_Λ and γ_{Λ^c}:

$$\eta_\Lambda\gamma_{\Lambda^c}(x) \;=\; \begin{cases} \eta(x) & \text{if } x \in \Lambda, \\ \gamma(x) & \text{if } x \in \Lambda^c. \end{cases}$$

A *Gibbs measure compatible* with Γ is a measure μ on Ω satisfying the "DLR" (Dobrushin, Lanford y Ruelle, [1], [2]) equations

$$\int \mu(d\gamma)\int \mu^{\Lambda,\gamma}(d\eta_\Lambda)f(\eta_\Lambda\gamma_{\Lambda^c}) = \int \mu(d\eta)f(\eta). \tag{4}$$

for continuous $f : \Omega \to \mathbb{R}$. Using the notation $\mu f = \int \mu(d\eta)f(\eta)$, the DLR equations read

$$\mu\left(\mu^{\Lambda,(\cdot)}f\right) = \mu f. \tag{5}$$

We prove that for this model there exists a unique Gibbs measure:

Theorem 1 Let $J : \mathbb{Z}^d \to \mathbb{R}^+$ be summable non-negative symmetric function such that $0 < \|J\| < \infty$. Let $\Gamma = \{\mu^{\Lambda,\gamma} : \Lambda \in \mathcal{S}, \gamma \in \Omega\}$ be the family of specifications (3) induced by the Hamiltonian (2). Then there exists a unique Gibbs measure compatible with Γ.

This theorem is proven at the end of Section 3.

Since $H^\Lambda(\eta) = \|J\|H^\Lambda(\eta/\sqrt{\|J\|})$ where $(\eta/c)(x) = \eta(x)/c$ for all x and the interval $[a, b]$ is arbitrary, we can and will assume

$$\|J\| = \sum_{x\in\mathbb{Z}^d} J(x) = 1 \tag{6}$$

without losing generality. In fact, if we choose $\|J\| = 1$ and introduce an inverse temperature β defining

$$\mu_\beta^{\Lambda,\gamma}(d\eta_\Lambda) \;:=\; \frac{1}{Z_\beta^{\Lambda,\gamma}}\exp\left(-\beta H^\Lambda(\eta_\Lambda\gamma_{\Lambda^c})\right)d\eta_\Lambda, \tag{7}$$

we have $\beta H^\Lambda(\eta) = H^\Lambda(\sqrt{\beta}\eta)$. If $\eta \in [a^*, b^*]^{\mathbb{Z}^d}$, then $\sqrt{\beta}\eta \in [\sqrt{\beta}a^*, \sqrt{\beta}b^*]^{\mathbb{Z}^d}$. Since Theorem 1 is true for any interval, substituting $[a, b]$ with $[\sqrt{\beta}a, \sqrt{\beta}b]$ we obtain that the model at inverse temperature β and spins in $[a, b]$ has a unique Gibbs measure. It is then sufficient to consider the case $\beta = 1$ because the other cases reduce to this one.

Anti ferromagnetic case in bipartite graphs The usual trick permits to extend Theorem 1 to negative J in bipartite graphs. Assume J satisfies the conditions of Theorem 1 and $J(x-y)=0$ if $x,y\in\Upsilon_1$ or $x,y\in\Upsilon_2$ for a partition Υ_1, Υ_2 of $\mathbb{Z}^d$. Define $\tilde J(x)=-J(x)$ and $\tilde\Gamma$ the specifications constructed with $\tilde J$. Define the transformation $(R\eta)(x)=\eta(x)$ for $x\in\Upsilon_1$ and $(R\eta)(x)=a+b-\eta(x)$ for $x\in\Upsilon_2$. For a measure μ on Ω, call $R\mu$ the measure induced by this transformation. Then μ is Gibbs for Γ if and only if $R\mu$ is Gibbs for $\tilde\Gamma$. This implies that Theorem 1 holds also for the specifications $\tilde\Gamma$.

2 Stochastic domination and Gibbs sampler

In this Section we collect some general known results about stochastic domination, introduce the Gibbs sampler process and discuss properties of the set of invariant measures for the Gibbs sampler related to attractiveness of the process. The particular form of the specifications is not relevant here. Most results are easy extensions to the continuous space Ω of results of Chapters 3 and 4 of Liggett [4] for the space $\{0,1\}^{\mathbb{Z}^d}$.

Stochastic domination in Ω. For $\eta,\xi\in\Omega$ say that $\eta\le\xi$ if and only if $\eta(x)\le\xi(x)$ for all $x\in\mathbb{Z}^d$. A function $f:\Omega\to\mathbb{R}$ is increasing if and only if $f(\eta)\le f(\xi)$ for $\eta\le\xi$. Let μ_1 and μ_2 probability measures on Ω. We say that μ_2 dominates stochastically μ_2, and denote $\mu_1\preceq\mu_2$, if $\mu_1 f\le\mu_2 f$ for each increasing measurable function f. $\mu_1\preceq\mu_2$ if there exists a coupling $(\hat\eta_1,\hat\eta_2)$ with marginals μ_1 and μ_2 such that $\hat\eta_1\le\hat\eta_2$ almost surely [5, 4].

Gibbs Sampler The Gibbs sampler associated to a specification Γ is a continuous time Markov process $(\eta_t:t\ge 0)$ on Ω with infinitesimal generator L defined on cylinder continuous functions $f:\Omega\to\mathbb{R}$ by:

$$Lf(\eta):=\sum_{x\in\mathbb{Z}^d}L_xf(\eta),\qquad L_xf(\eta):=\int_a^b\mu^{\{x\},\eta}(ds)[f(\eta+(s-\eta(x))\theta_x)-f(\eta)],\tag{8}$$

where $\theta_x\in\{0,1\}^{\mathbb{Z}^d}$ is defined by $\theta_x(x)=1$ and $\theta_x(z)=0$ for $z\ne x$. In words, at rate 1, at each site $x\in\mathbb{Z}^d$ the spin $\eta(x)\in[a,b]$ is updated with the law $\mu^{\{x\},\eta}$. The existence of a process η_t with generator L such that $\frac{d}{dt}\mathbb{E}(f(\eta_t)|\eta_0=\eta)=Lf(\eta)$ is standard, using a graphical construction and a percolation argument. Call $S(t)$ the corresponding semigroup defined by $S(t)f(\eta)=\mathbb{E}(f(\eta_t)|\eta_0=\eta)$. The semigroup acts on measures via the formula $(\mu S(t))f=\mu(S(t)f)$; $\mu S(t)$ is the law of the process at time t when the initial distribution is μ. We say that μ is invariant for the process if $\mu S(t)=\mu$. A measure μ is invariant if and only if $\mu Lf=0$ for all continuous cylinder f.

Proposition 2 If a measure μ is Gibbs for specifications Γ then it is invariant for the Gibbs sampler associated to Γ.

Proof. It suffices to show $\mu L_x f = 0$ for all $x \in \mathbb{Z}^d$ and continuous cylinder f.

$$\mu L_x f \;=\; \int \mu(d\eta) \int_a^b \mu^{\{x\},\eta}(ds)[f(\eta + (s - \eta(x))\theta_x) - f(\eta)] \;=\; \mu(\mu^{\{x\},(\cdot)} f) - \mu f \;=\; 0, \quad (9)$$

by (5). □

Attractiveness A process is *attractive* if $\mu_1 \preceq \mu_2$ implies $\mu_1 S(t) \preceq \mu_2 S(t)$. A sufficient condition for attractiveness of Gibbs sampler is

$$\mu^{\{x\},\eta} \preceq \mu^{\{x\},\xi} \quad \text{if} \quad \eta \leq \xi \qquad (10)$$

Let δ^a and δ^b be the measures concentrating mass on the configuration "all a" and "all b" respectively. Clearly, $\delta^a \preceq \mu \preceq \delta^b$ for any measure μ. If the process is attractive, $\delta^b S(t)$ is non increasing and $\delta^a S(t)$ is non decreasing in t. Hence both sequences have a (weak) limit when $t \to \infty$ that we call μ^b and μ^a, respectively. Both μ^b and μ^a are invariant measures called upper and lower invariant measures respectively. For any measure μ attractiveness implies $\delta^a S(t) \;\preceq\; \mu S(t) \;\preceq \delta^b S(t)$ for all t. If μ is invariant $\mu S(t) = \mu$ for all t and taking limits as $t \to \infty$,

$$\mu^a \;\preceq\; \mu \;\preceq\; \mu^b \qquad (11)$$

Proposition 3 Assume the Gibbs sampler associated to Γ is attractive and $\mu^a = \mu^b$. Then if μ is a Gibbs measure compatible with Γ, $\mu = \mu^a = \mu^b$.

Proof. By Proposition 2 Gibbs measures are invariant for the Gibbs sampler, hence any Gibbs measure μ must satisfy (11), showing uniqueness. □

3 Truncated Gaussian fields and Gibbs sampler

In this section we discuss some basic properties of truncated Gaussian variables, show that the truncated Gaussian Gibbs sampler is attractive and that the upper and lower invariant measures for the Gibbs sampler coincide, proving Theorem 1.

Truncated Normal variables. Denote X_m a random variable with *truncated normal distribution* $\mathcal{N}_{a,b}(m,1)$ whose density g is given by

$$g(u) := \phi\,(u - m)\,[\Phi\,(b - m) - \Phi\,(a - m)]^{-1}\,\mathbf{1}\{a \leq u \leq b\}, \qquad (12)$$

where $a < b$, $m \in \mathbb{R}$, $\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$ is the standard normal distribution and $\Phi(u) = \int_{-\infty}^{u} \phi(s) ds$ is the cumulative distribution. The truncated normal $\mathcal{N}_{a,b}(m,1)$ is just the normal $\mathcal{N}(m,1)$ conditioned to the interval $[a,b]$. A simple computation shows

$$\text{If } m_1 < m_2, \text{ then } X_{m_1} \preceq X_{m_2} \qquad (13)$$

Also,

$$\begin{aligned} \mathbb{E}[X_m] &= m - \varphi(m)\,, \quad (14)\\ \text{where} \quad \varphi(m) &:= \frac{\phi(b-m)-\phi(a-m)}{\Phi(b-m)-\Phi(a-m)}. \quad (15) \end{aligned}$$

(see Sec.7, Cap. 13 de [3]). The function φ is odd with respect to $m_0 = \frac{a+b}{2}$: $\varphi\,(m_0+m) = -\varphi_{a,b}\,(m_0-m)$ for all $0 \le m \le \frac{b-a}{2}$. Furthermore φ is increasing, continuous and invertible in the interval $a \le m \le b$.

Truncated Gaussian Gibbs sampler The specification $\mu^{\{x\},\eta}$ given by (3) with $\|J\| = 1$ is a truncated normal distribution $\mathcal{N}(\overline{\eta}(x), 1)$, where

$$\overline{\eta}(x) = \sum_{y \neq x} J(y-x)\eta(y).$$

Since $h(\eta) = \overline{\eta}(x)$ is an increasing function of η, (13) implies $\mu^{\{x\},\eta}$ satisfies (10). Hence the corresponding Gibbs sampler is attractive. Furthermore for $x \in \mathbb{Z}^d$ and $f(\eta) = \eta(x)$,

$$Lf(\eta) \;=\; \int_a^b \mu^{\{x\},\eta}(ds)(s-\eta(x)) \;=\; \overline{\eta}(x) - \varphi(\overline{\eta}(x)) - \eta(x).$$

using (14). We abuse notation writing $\overline{\eta}(x)$ and $\eta(x)$ instead of h and f, for $h(\eta) = \overline{\eta}(x)$ and $f(\eta) = \eta(x)$.

Lemma 4 Let μ be invariant for the Gibbs Sampler and translation invariant. Then

$$\mu\varphi(\overline{\eta}(x)) = 0. \quad (16)$$

Proof. Since μ is invariant for Gibbs Sampler,

$$0 = \mu L(\eta(x)) = \mu(\overline{\eta}(x)) - \mu(\varphi(\overline{\eta}(x))) - \mu(\eta(x)). \quad (17)$$

On the other hand, by translation invariance, $\mu(\eta(x))$ does not depend on x. Hence,

$$\mu(\overline{\eta}(x)) = \mu(\eta(x)) \sum_{y: y\neq x} J(y-x) = \mu(\eta(x)). \quad (18)$$

(recall $\sum_{y\neq 0} J(y) = 1$). □

Proof of Theorem 1. Existence of a Gibbs measure μ is proven in Chapter 4 of [1] as $[a, b]$ is a *standard Borel space* of finite measure and the potential J is absolutely summable.

Since the Gibbs sampler is attractive, the upper and lower invariant measures μ^b and μ^a are well defined. By Proposition 3 it suffices to show $\mu^a = \mu^b$. Let (η^a, η^b) be a random vector with marginals μ^a and μ^b and such that $\eta^a \leq \eta^b$. The function $\overline{\eta}(x)$ is increasing in η and $\varphi(m)$ is increasing in m. Hence $\varphi(\overline{\eta}^a(x)) \leq \varphi(\overline{\eta}^b(x))$. Since the limit defining μ^a and μ^b is translation invariant, so are μ^a and μ^b and by (16), $\varphi(\overline{\eta}^a(x))$ and $\varphi(\overline{\eta}^b(x))$ have expected value 0. Hence $\varphi(\overline{\eta}^a(x)) = \varphi(\overline{\eta}^b(x))$ a.s.. Since φ is invertible, $\overline{\eta}^a(x) = \overline{\eta}^b(x)$ a.s.. That is,

$$\sum_{y:y\neq x} J(y-x)(\eta^b(y) - \eta^a(y)) = 0 \tag{19}$$

Since $\eta^a \leq \eta^b$, (19) implies $\eta^b(y) = \eta^a(y)$ for all y such that $J(y-x) > 0$. Since x is arbitrary, this implies $\eta^a(y) = \eta^b(y)$ almost surely for all y. □

4 Specifications are truncated multivariate normal distributions

In this section (which can be read independently of the others, except for notation) we show that the specifications are truncated multivariate normal distributions.

Lemma 5 For each finite Λ and $\gamma \in \Omega$, the specification $\mu^{\Lambda,\gamma}$ is a multivariate Normal distribution $\mathcal{N}_\Lambda(\mathbf{m}_\Lambda^\gamma, \mathbf{\Sigma}_\Lambda)$ truncated to the box $[a,b]^\Lambda$, where

$$\mathbf{m}_\Lambda^\gamma = (A^\Lambda)^{-1} B^{\Lambda,\Lambda^c} \gamma_{\Lambda^c} \quad \text{and} \quad \mathbf{\Sigma}_\Lambda = (A^\Lambda)^{-1} \tag{20}$$

with

$$A^\Lambda(x,y) := \begin{cases} \sum_{y\in\Lambda\setminus\{x\}} J(y-x) + \|J\|, & \text{if } x = y \in \Lambda, \\ \\ -J(y-x), & \text{if } x \in \Lambda \text{ and } y \in \Lambda\setminus\{x\} \end{cases} \tag{21}$$

and

$$B^{\Lambda,\Lambda^c}(x,y) := J(y-x), \text{ for } x\in\Lambda \text{ and } y \in \Lambda^c. \tag{22}$$

Proof. A simple computation shows that for H^Λ defined by (2),

$$H^\Lambda(\eta) = \frac{1}{2}(\eta_\Lambda' A^\Lambda \eta_\Lambda - 2\eta_\Lambda' B^{\Lambda,\Lambda^c}\eta_{\Lambda^c} + \Psi(\eta_{\Lambda^c})). \tag{23}$$

where the function $\Psi(\eta_{\Lambda^c}) = \sum_{x\in\Lambda}\sum_{y\in\Lambda^c} J(y-x))\eta(y)^2$ does not depend on η_Λ. If A^Λ is positive definite, this shows the proposition because, using (23),

$$\begin{aligned} H^\Lambda(\eta_\Lambda\gamma_{\Lambda^c}) &= \frac{1}{2}\left(\eta_\Lambda' A^\Lambda \eta_\lambda - 2\eta_\Lambda' A^\Lambda((A^\Lambda)^{-1}B^{\Lambda,\Lambda^c}\gamma_{\Lambda^c}) + \Psi(\gamma_{\Lambda^c})\right) \\ &= \frac{1}{2}(\eta_\Lambda - \mathbf{m}_\Lambda^\gamma)' A^\Lambda(\eta_\Lambda - \mathbf{m}_\Lambda^\gamma) + R(\gamma), \end{aligned}$$

where $\mathbf{m}_\Lambda^\gamma = (A^\Lambda)^{-1} B^{\Lambda,\Lambda^c} \gamma_{\Lambda^c}$ and $R(\gamma)$ does not depend on η_Λ.

If J satisfies the conditions of Theorem 1, then A^Λ is positive definite. Indeed, A^Λ can be decomposed as the sum of a positive semidefinite matrix and a linear combination of positive matrices, as follows

$$A^\Lambda = \sum_{x\in\Lambda} \left(\sum_{y\in\Lambda\setminus\{x\}} J(y-x) \right) E_\Lambda^{xx} + \left(\sum_{z\in\mathbb{Z}_+^d\setminus\Delta_+} 2J(z) \right) I_\Lambda + \sum_{z\in\Delta_+} J(z) T_z^\Lambda, \tag{24}$$

where I_Λ is the identity matrix and $T_z^\Lambda : z \in \{y - x : (x,y) \in \Lambda \times \Lambda\} \setminus \{0\}$ is given by

$$T_z^\Lambda(x,y) = 2\,\mathbf{1}\{x = y\} - \mathbf{1}\{y - x \in \{-z, z\}\}, \tag{25}$$

$E_\Lambda^{xx}(z,w) = \mathbf{1}\{(z,w) = (x,x)\}$ and $(\mathbb{Z}_+^d, \mathbb{Z}_-^d)$ is a partition of $\mathbb{Z}^d \setminus \{0\}$ such that $x \in \mathbb{Z}_+^d \Leftrightarrow -x \in \mathbb{Z}_-^d$ and $\Delta_+ = \Delta \cap \mathbb{Z}_+^d$.

Finally, let's prove that for each $z \in \Delta_+$, the matrix T_z^Λ given by (25) is positive definite. We say that sites $x, y \in \Lambda$ are *z-connected* if there exists $x = x_0, x_1, \ldots, x_n = y$ in Λ such that $x_m - x_{m-1} \in \{-z, z\}$ for all $m = 1, \ldots, n$. Since z-connected is an equivalence relation, Λ is decomposed in the equivalent classes $\Lambda_1, \ldots, \Lambda_n$ given by $\Lambda_\ell = \{x_\ell + mz : m = 0, \ldots, m_\ell\}$ for some m_ℓ non negative integer.

Take a non-null vector $\eta \in \mathbb{R}^\Lambda$ and use the previous notation to get

$$\begin{aligned} \eta' T_z^\Lambda \eta \;&=\; \sum_{\ell:\, m_\ell \geq 1} \sum_{m=0}^{m_\ell - 1} (\eta(x_\ell + mz) - \eta(x_\ell + (m+1)z))^2 \\ &\quad + 2 \sum_{\ell:\, m_\ell = 0} \eta(x_\ell)^2 + \sum_{\ell: m_\ell \geq 1} (\eta(x_\ell)^2 + \eta(x_\ell + m_\ell z)^2) \;>\; 0\,. \end{aligned} \tag{26}$$

This proves that T_z^Λ is positive definite and the lemma. □

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