INVARIANT WEIGHTED ALGEBRAS $\mathcal{L}_{p}^{w}(G)$

YU. N. KUZNETSOVA

ABSTRACT. We deal with weighted spaces $\mathcal{L}_p^w(G)$ on a locally compact group G. If w is a positive measurable function on G then we define the space $\mathcal{L}_p^w(G)$, $p \ge 1$, by equality $\mathcal{L}_p^w(G) = \{f : fw \in \mathcal{L}_p(G)\}$. We consider weights w such that these weighted spaces are algebras with respect to usual convolution. We show that for p > 1 such weights exists on any sigma-compact group. We prove also under minimal requirements a criterion known earlier in special cases: $\mathcal{L}_1^w(G)$ is an algebra if and only if w is submultiplicative.

Throughout the paper G is a locally compact group, all integrals are taken with respect to a left Haar measure μ , $p \ge 1$, 1/p + 1/q = 1 (if p = 1 then $q = \infty$). We call any positive measurable function a weight. Weighted space $\mathcal{L}_p^w(G)$ with the weight w is defined as $\{f : fw \in \mathcal{L}_p(G)\}$, norm of a function f being $||f||_{p,w} = (\int |fw|^p)^{1/p}$. Indices p, w are sometimes omitted.

Sufficient conditions on a weight function to define an algebra $\mathcal{L}_p^w(G)$ with respect to usual convolution, $f * g(s) = \int f(t)g(t^{-1}s)dt$, are well-known. For p = 1 it is submultiplicativity:

$$w(st) \leqslant w(s)w(t),\tag{1}$$

and for p > 1 the following inequality (pointwise almost everywhere):

$$w^{-q} * w^{-q} \leqslant w^{-q}. \tag{2}$$

Note that if (1) or (2) holds with a constant C (after \leq sign) then for the weight $w_1 = Cw$ the same inequality holds without any constant. Multiplication of a weight by a number changes by the same number the norm of $\mathcal{L}_p^w(G)$, preserving all the properties of the space. Thus we introduce the notion of equivalent weights: w_1 and w_2 are equivalent if with some C_1 , C_2 locally almost everywhere

$$C_1 \leqslant \frac{w_1}{w_2} \leqslant C_2. \tag{3}$$

For p > 1 it is convenient to introduce a dual function $u = w^{-q}$, then the inequality (2) takes the following form, independent on p and q:

$$u * u \leqslant u. \tag{4}$$

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It is easy to notice that any function u satisfying (4) defines a family of weighted algebras $\mathcal{L}_p^{w_p}(G)$ for all $p \in (1, +\infty)$: $w_p = u^{-1/q}$.

1. CRITERION FOR THE GROUP

For p = 1 weighted algebras exist, of course, on any locally compact group (at least with a unit weight). For p > 1 we cannot take an arbitrary group, and more precisely, the following theorem holds:

Theorem 1.1. For a locally compact group G the following conditions are equivalent:

- (i) G is σ -compact;
- (ii) for some p > 1 there exist a weight w satisfying (2) (the space $\mathcal{L}_p^w(G)$ is then a convolution algebra);
- (iii) for any p > 1 there exist a weight w satisfying (2).

For an abelian G these conditions are also equivalent to the following:

- (iv) for some p > 1 there exist a weight w such that $\mathcal{L}_p^w(G)$ is a convolution algebra;
- (v) for any p > 1 there exist a weight w such that $\mathcal{L}_p^w(G)$ is a convolution algebra.

Proof. Implications (iii) \Rightarrow (ii) \Rightarrow (iv) and (iii) \Rightarrow (v) \Rightarrow (iv) are obvious and do not depend on commutativity of G. We prove that (ii) \Rightarrow (i), (i) \Rightarrow (iii) and for an abelian group (iv) \Rightarrow (i).

(ii) \Rightarrow (i). If (2) holds then for some x the integral $(w^{-q} * w^{-q})(x) = \int w^{-q}(y)w^{-q}(y^{-1}x)dy$ of a strictly positive function is finite. This implies that G is σ -compact.

(iv) \Rightarrow (i). By [8, theorem 3] there exists an algebra $\mathcal{L}_p^w(G)$ where $w^{-q} \in \mathcal{L}_1(G)$. Since w^{-q} is positive, G must be σ -compact.

(i) \Rightarrow (iii). We construct a function on G satisfying (4). Pick a positive function $u_1 \in \mathcal{L}_1(G)$ (it exists because G is σ -compact). We may assume that $||u_1||_1 = 1$. Define inductively functions $u_n, n \in \mathbb{N}$:

$$u_{n+1} = u_1 * u_n$$

Clearly $||u_n||_1 \leq 1$ for all n. We put now $u = \sum n^{-2}u_n$ and prove that (4) holds. Note the following elementary fact:

$$\sum_{n=1}^{m-1} \frac{1}{n^2 (m-n)^2} \leqslant 8\zeta(2) \frac{1}{m^2}.$$

Estimate now the convolution u * u:

$$u * u = \sum_{n,k=1}^{\infty} \frac{u_n * u_k}{n^2 k^2} = \sum_{n,k=1}^{\infty} \frac{u_{n+k}}{n^2 k^2} =$$
$$= \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \frac{u_m}{n^2 (m-n)^2} \leqslant 8\zeta(2) \sum_{m=1}^{\infty} \frac{u_m}{m^2} \equiv Cu.$$

Changing u to u/C, we get (4).

2. Technical Lemmas

In general any positive measurable function can be taken as a weight. Some authors assume also that w^p is locally summable, but this is redundant if $\mathcal{L}_p^w(G)$ is an algebra:

Lemma 2.1. If the space $\mathcal{L}_p^w(G)$, $p \ge 1$, is a convolution algebra then w^p is locally summable.

Proof. Take a compact set $A \subset G$ of positive measure. Here and further I_A denotes the characteristic function of a set A. Consider the functions $\varphi = I_A / \max\{1, w\}$ and $\psi = I_{A^{-1} \cdot A} / \max\{1, w\}$. As $\varphi, \psi \in \mathcal{L}_p(w)$, then also $\tau = \varphi * \psi \in \mathcal{L}_p(w)$. At the same time $\varphi \in \mathcal{L}_1$ and $\psi \in \mathcal{L}_\infty$, so that τ is continuous. Since also $\tau|_A > 0$ we have $\min_A \tau = \tau_0 > 0$. Thus, $I_A \leq \tau / \tau_0$, what implies $I_A \in \mathcal{L}_p(w)$ or, equivalently, $w \in \mathcal{L}_p(A)$.

Lemma 2.2. The set $B_0(G)$ of all bounded compactly supported functions is a dense subspace of every algebra $\mathcal{L}_p^w(G)$.

Proof. Inclusion $B_0(G) \subset \mathcal{L}_p^w(G)$ follows from the previous lemma. It is known that $B_0(G)$ is dense in $\mathcal{L}_p(G)$, therefore $w^{-1}B_0(G)$ is dense in $\mathcal{L}_p^w(G) = w^{-1}\mathcal{L}_p(G)$. Let now $f \in w^{-1}B_0(G)$ be compactly supported, but not necessarily bounded. Changing it on a set of arbitrary small measure ε , we can make f bounded; ε is to be chosen according to continuity of the integral $||f||_{p,w} = ||fw||_p$ as a set function. \Box

The property of submultiplicativity (1) is essential in the weighted algebras theory because exactly this property posess the weights of $L_1^w(G)$ algebras (see theorem 3.3 below). We need the following lemma on submultiplicative functions (it is in fact proposition 1.16 of [2]):

Lemma 2.3. Let a measurable function $L : G \to \mathbb{R}$ be submultiplicative, i.e. satisfy (1), and positive. Then L is bounded and bounded away from zero on any compact set.

The following condition studied first by R. Edwards [2] is also important for the weighted spaces theory. A weight w is of moderate growth if for all $s \in G$

$$L_s = \operatorname{ess\,sup}_t \frac{w(st)}{w(t)} < \infty. \tag{5}$$

This condition is equivalent to the space $L_p^w(G)$ (for any $p \ge 1$) being translation-invariant [2, 1.13]. In the non-commutative case, (5) corresponds to left translations; taking w(ts) indtead of w(st), we get a condition for right translations, in general not equivalent to the former. Immediate calculations show that

$$\sup_{f \neq 0} \frac{\|^{s}f\|_{p,w}}{\|f\|_{p,w}} = L_{s^{-1}}.$$
(6)

The condition (5) implies that $L_s > 0$, $L_{st} \leq L_s L_t$, and

$$\operatorname{ess\,inf}_{t} \frac{w(st)}{w(t)} = 1/L_{s^{-1}} > 0.$$

Lemma 2.4. If (5) holds for locally almost all $s \in G$ then it holds for all $s \in G$.

Proof. Let $S \subset G$ be the set of s for which the inequality (5) holds. By assumption S and hence S^{-1} is locally of full measure. Pick a set $T \subset S \cap S^{-1}$ of positive finite measure. Then $T \cdot T^{-1}$ contains a neighborhood of identity U. As L (finite or infinite) is submultiplicative, S is closed under multiplication and therefore $U \subset T \cdot T^{-1} \subset S \cdot S \subset S$. By the same reason $SU \subset S$, and since S (being locally of full measure) is everywhere dense, then S = G.

Lemma 2.5. [3, th. 2.7] If a weight w satisfies (5) and is locally summable then it is equivalent to a continuous function.

Corollary 2.1. Let $L_p^w(G)$ be an algebra with a weight w satisfying (5). Then w is equivalent to a continuous function.

Proof. By lemma 2.1 for any compact set F we have $w \in \mathcal{L}_p(F) \subset \mathcal{L}_1(F)$, therefore we can apply lemma 2.5.

On a compact group any continuous function is equivalent to a constant function, thus on a compact group all translation-invariant weighted algebras are isomorphic to the usual algebra $\mathcal{L}_p(G)$. Converse of the corollary does not hold:

Example 2.1. There exist an algebra $L_2^w(\mathbb{R})$ such that w is continuous but does not satisfy (5). Let $0 < \alpha_n < 1$, $A_n = [n + \alpha_n, n + 1]$. We put $w|_{A_n} = 1 + n^2$, $w(n + \alpha_n/2) = 1 + |n|$ and extend w piecewise linearly.

For $\alpha_n = n^{-2}$ the condition (2) is satisfied but (5) does not hold in any neighborhood of zero.

3. CRITERION FOR THE ALGEBRA $\mathcal{L}_1^w(G)$

In the case when p = 1 the class of weights defining convolution algebras $\mathcal{L}_p^w(G)$ admits a complete description, and it turns out that every weight is equivalent to a continuous function. The following theorem was proved by Grabiner [5] in the case of the real half-line (without statement of continuity which is false on the half-line). Edwards [2] proved equivalence of (i) and (ii) on a locally compact group under assumption of upper-semicontinuity of w, and later Feichtinger [3] for translation-invariant algebras $\mathcal{L}_1^w(G)$. Our theorem generalizes these results.

Theorem 3.1. For a weight w the following conditions are equivalent:

- (i) w is equivalent (in the sense of (3)) to a continuous submultiplicative function;
- (ii) $\mathcal{L}_1^w(G)$ is a convolution algebra;
- (iii) for some $p, 1 \leq p < \infty$, the inclusion $\mathcal{L}_1^w(G) * \mathcal{L}_p^w(G) \subset \mathcal{L}_p^w(G)$ holds.

Proof. Implications (i) \Rightarrow (ii) and (i) \Rightarrow (iii) are immediate whereas (ii) \Rightarrow (i) is a special case of (iii) \Rightarrow (i) with p = 1. We prove therefore only (iii) \Rightarrow (i).

Inclusion (iii) implies (cf. [6, 38.27]) that with some constant C

 $||f * g||_{p,w} \leq C ||f||_{1,w} ||g||_{p,w}.$

Repeating argument of lemma 2.1 we conclude that w^p together with w are locally summable. Thus the spaces $\mathcal{L}_1^w(G)$, $\mathcal{L}_p^w(G)$ contain characteristic functions of all sets of finite measure. For such sets A, B and arbitrary s, t we have pointwise

$$\mu(A)I_{stB} \leqslant I_{sA} * I_{A^{-1}tB},\tag{7}$$

whence

$$\mu(A) \| I_{stB} \|_{p,w} \leqslant C \| I_{sA} \|_{1,w} \| I_{A^{-1}tB} \|_{p,w}.$$
(8)

We need here a generalization of the Lebesgue differentiation theorem. On a locally compact group one may state the theorem as follows (see a general statement in the review [1] and specifications for the group case in [7]): there exists a family \mathcal{V} of sets of positive measure directed by downward inclusion such that for any locally summable function f

$$\lim_{V \in \mathcal{V}} \frac{1}{\mu(V)} \int_{xV} f(t) dt = f(x)$$
(9)

for locally almost all $x \in G$. At that every $V \in \mathcal{V}$ contains the identity, and every neighborhood of identity contains eventually all $V \in \mathcal{V}$ [7, VIII, 1-2].

So, for locally almost all $s \in G$ (9) holds with f = w, x = s. For each such s (9) holds both with $f = w^p$ and $f = {}^{s}w^p$ for locally almost all x = t. For such s, t and any $\varepsilon > 0$ for sufficiently small $V \in \mathcal{V}$

$$\|I_{stV}\|_{p,w}^{p} = \int_{tV} ({}^{s}w(r))^{p} dr > w^{p}(st)\mu(V)/(1+\varepsilon),$$
$$\|I_{tV}\|_{p,w}^{p} = \int_{tV} w^{p}(r) dr < w^{p}(t)\mu(V)(1+\varepsilon).$$

Fix V such that these inequalities hold. Since the integral of ${}^{s}w^{p}$ are continuous as functions of a set, there is a compact set $B \subset V$ such that

$$\|I_{stV}\|_{p,w} < (1+\varepsilon) \|I_{stB}\|_{p,w}, \qquad \mu(V) < (1+\varepsilon)\mu(B).$$

Moreover, there exists a neighborhood of identity (with compact closure) V_0 such that

$$\|I_{V_0^{-1}tB}\|_{p,w} < (1+\varepsilon)\|I_{tB}\|_{p,w}.$$

And, finally, for sufficiently small $A \in \mathcal{V}, A \subset V_0$ holds

$$||I_{sA}||_{1,w} < (1+\varepsilon)\mu(A)w(s)$$

Obviously, $||I_{A^{-1}tB}||_{p,w} \leq ||I_{V_0^{-1}tB}||_{p,w}$ and $||I_{tB}||_{p,w} \leq ||I_{tV}||_{p,w}$. Uniting all these inequalities with (8), we get:

$$\mu(A)\mu(B)^{1/p}w(st) < C(1+\varepsilon)^{3/p+3}\mu(A)\mu(B)^{1/p}w(s)w(t).$$

and in the limit as $\varepsilon \to 0$

$$w(st) \leqslant Cw(s)w(t). \tag{10}$$

This inequality is obtained for locally almost all t with fixed s for locally almost all s. But by lemma 2.4 for w the condition (5) holds, and by lemma 2.5 w is equivalent to a continuous function w_1 . For w_1 (10) (with another constant) holds for all t and s. Finally, multiplying w_1 by this constant, we get a continuous submultiplicative weight. \Box

As a corollary we obtain a description of multipliers of the algebra $\mathcal{L}_1^w(G)$. Gaudry [4] proved under assumption of upper-semicontinuity that multipliers of $\mathcal{L}_1^w(G)$ may be identified with the weighted space $\mathcal{M}^w(G)$ of regular Borel measures such that $\int wd|\mu| < \infty$. As the weight can be always chosen continuous, statement of the theorem is simplified:

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Theorem 3.2. A bounded operator T on an algebra $\mathcal{L}_1^w(G)$ commutes with right translations if and only if it is a convolution with a measure $\mu \in \mathcal{M}^w(G)$: $Tf = \mu * f$ for all $f \in \mathcal{L}_1^w(G)$.

If p > 1, the weight of an algebra cannot in general be chosen continuous:

Example 3.1. $G = \mathbb{T}$, p = 2. Parametrize the circle by $t \in [-1, 1]$ and take $w(t) = |t|^{-1/4}$ (another example: $w(t) = |t|^{1/4}$). $\mathcal{L}_2^w(\mathbb{T})$ is an algebra but the weight is not equivalent to a continuous function. At that $L_2^w(\mathbb{T})$ is not invariant under translations and $L_1^w(\mathbb{T})$ with the same weight is not an algebra.

The fact that $L_1^w(G)$ is an algebra implies the weight is submultiplicative and all the spaces $L_p^w(G)$, $p \ge 1$, are translation-invariant (i.e. (5) holds). Converse is true for abelian groups:

Theorem 3.3. Let G be an abelian locally compact group, $\mathcal{L}_p^w(G)$ an algebra with a weight w satisfying (5). Then w is equivalent to a submultiplicative function.

Proof. By lemma 2.5 we may assume w is continuous, and by lemma 2.3 the function L is bounded on any compact set. Pick a compact set $D = D^{-1}$ of positive measure, and let $L(r) \leq N$ for $r \in D$. Then for $s \in G, r \in D$ we have

$$\frac{w(s)}{N} \leqslant w(sr) \leqslant Nw(s). \tag{11}$$

Take now arbitrary $s, t \in G$ and use inequality (7) with V = U = D. We get then

$$\mu(D) \|I_{stD}\| \leq \|I_{sD}\| \cdot \|I_{tD^2}\|,$$

whence by (11)

$$\mu(D)^{1+1/p}w(st)/N \leqslant N^2 \mu(D)^{1/p} \mu(D^2)^{1/p}w(s)w(t),$$

i.e. $w(st) \leq C_1 w(s) w(t)$, what completes the proof.

Note that on a discrete group weight of any algebra for all $p \ge 1$ is submultiplicative. Inequality (1) obtains when passing from $I_{st} = I_s * I_t$ to the norms in $\mathcal{L}_p^w(G)$.

4. Approximate units

Algebras $\mathcal{L}_p^w(G)$, as well as those without weight, have a unit iff G is discrete. In the classical case (of a compact group) $\mathcal{L}_p(G)$ always have approximate units. In the weighted case moderate growth of the weight (see th. 4.1) is sufficient for an algebra to have an a.u. If

the weight is not moderate, this theorem may not hold, see example 4.1. Invariant algebras $\mathcal{L}_p^w(G)$ do not have bounded approximate units (theorem 4.2). It follows these algebras are not amenable [9]. In this section, theorems are proved for left a.u., but the same is true for right a.u. with interchange of s and t in the condition (5).

Lemma 4.1. Let $\mathcal{L}_p^w(G)$, $p \ge 1$ be an invariant algebra. Then for any $f \in \mathcal{L}_p^w(G)$ and $\varepsilon > 0$ there exists a neighborhood of identity U such that for all $t \in U ||f - {}^tf||_{p,w} < \varepsilon$.

Proof. Suppose first that $f \in \mathcal{L}_p^w(G)$ is compactly supported, $F = \sup f$. Pick a relatively compact neighborhood of identity U, then $\sup t^t f \subset UF$ when $t \in U$. By lemma 2.5 we may assume w is continuous, thus bounded on every compact set, so that $C = \sup_{UF} w < \infty$. Now

$$||f - {}^{t}f||_{p,w} = \left(\int_{UF} |f(x) - f(tx)|^{p} w^{p}(x) dx\right)^{1/p} \leq C ||f - {}^{t}f||_{p},$$

where the latter norm is less than ε for t in a sufficiently small neighborhood of identity $V \subset U$.

Let now $f \in \mathcal{L}_p^w(G)$ be arbitrary. For any $\varepsilon > 0$ it may be approximated by a compactly supported function $\varphi \in \mathcal{L}_p^w(G)$, $||f - \varphi||_{p,w} < \varepsilon$ (lemma 2.2). Pick again a symmetric relatively compact neighborhood of identity U. By lemma 2.3 the function L defined in the formula (5) is bounded on every compact set, so that $D = \sup_U L < \infty$. Now by equality (6) for $t \in U$

$$\|{}^{t}f - {}^{t}\varphi\|_{p,w} = \|{}^{t}(f - \varphi)\|_{p,w} \leq L_{t^{-1}}\|f - \varphi\|_{p,w} \leq D\|f - \varphi\|_{p,w},$$

and

$$\|f - {}^t f\|_{p,w} \leq \|f - \varphi\|_{p,w} + \|\varphi - {}^t \varphi\|_{p,w} + \|{}^t f - {}^t \varphi\|_{p,w} < \varepsilon + D\varepsilon + C\|\varphi - {}^t \varphi\|_{p,w}$$

what is less than $\varepsilon(1+C+D)$ for t in a sufficiently small neighborhood of identity $V \subset U$.

Theorem 4.1. Let $\mathcal{L}_p^w(G)$, $p \ge 1$ be an invariant algebra. The net $\xi_{\nu} = I_{\nu}/\mu(\nu)$, where ν runs over the net of all relatively compact neighborhoods of identity, is a left approximate unit in $\mathcal{L}_p^w(G)$.

Proof. Note first that all ξ_{ν} belong to $\mathcal{L}_{p}^{w}(G)$ (prop. 2.2). Convergence $\xi_{\nu} * f \to f$ for every $f \in \mathcal{L}_{p}^{w}(G)$ is proved in a standard way. By lemma 4.1 exists a neighborhoods of identity U such that $||f - t^{-1}f||_{p,w} < \varepsilon$ for $t \in U$. We can estimate the norm $||\xi_{\nu} * f - f||_{p,w}, \nu \subset U$ using

functionals of the conjugate space: for every $\varphi \in \mathcal{L}_q^{w^{-1}}(G)$

$$\begin{aligned} |\langle \xi_{\nu} * f - f, \varphi \rangle| &= \Big| \int_{G} \left[(\xi_{\nu} * f)(x) - f(x) \right] \varphi(x) dx \Big| \leq \\ &\leq \int_{G} \int_{\nu} \frac{|f(t^{-1}x)dt - f(x)|}{\mu(\nu)} dt \, |\varphi(x)| dx \leq \frac{1}{\mu(\nu)} \int_{\nu} \langle |t^{-1}f - f|, |\varphi| \rangle dt \leq \\ &\leq \sup_{t \in U} ||t^{-1}f - f||_{p,w} ||\varphi||_{q,w^{-1}} < \varepsilon ||\varphi||_{q,w^{-1}}, \end{aligned}$$

i.e. $\|t^{-1}f - f\|_{p,w} < \varepsilon$, what proves the theorem.

Example 4.1. If the weight fails to satisfy (5), the statement of theorem 4.1 may not be true. The following algebra is a counterexample. Take $G = \mathbb{R}$, p = 2. We denote $\bar{n} = \max(|n|, 1)$ and define the weight in the following way:

$$w(t) = \begin{cases} \bar{n}, & t \in [n, n+1/\bar{n}^2), \\ \bar{n}^2, & t \in [n+1/\bar{n}^2, n+1). \end{cases}$$

We show first that (2) holds for w after a multiplication by some constant, i.e. $\mathcal{L}_2^w(\mathbb{R})$ is an algebra. Denote

$$I_n = I_{[n,n+1/\bar{n}^2)}, \quad I'_n = I_{[n+1/\bar{n}^2,n+1)}$$

In these notations $w = \sum (\bar{n}I_n + \bar{n}^2 I'_n)$. Using a trivial estimate $I_A * I_B \leq \min\{\mu(A), \mu(B)\}I_{A+B}$ and inequality

$$\sum_{n=-\infty}^{\infty} \frac{1}{\bar{n}^{\alpha} (\overline{m-n})^{\alpha}} \leqslant \frac{2^{\alpha+1}}{\bar{m}^{\alpha}} (2\sum_{n=1}^{\infty} \frac{1}{\bar{n}^{\alpha}} + 1) = \frac{C_{\alpha}}{\bar{m}^{\alpha}}$$

for $\alpha = 2, 4$ and integer *m*, we can estimate convolution in (2):

$$w^{-2} * w^{-2} = \sum_{n,m} \left(\frac{1}{\bar{n}^2 \bar{m}^2} I_n * I_m + 2 \frac{1}{\bar{n}^2 \bar{m}^4} I_n * I'_m + \frac{1}{\bar{n}^4 \bar{m}^4} I'_n * I'_m \right) \leqslant$$

$$\leqslant \sum_{n,m} \left(\frac{1}{\bar{n}^2 \bar{m}^2 \max(\bar{n}^2, \bar{m}^2)} I_{n+m+[0,2)} + 2 \frac{1}{\bar{n}^2 \bar{m}^4 \bar{n}^2} I_{n+m+[0,2)} + \frac{1}{\bar{n}^4 \bar{m}^4} I_{n+m+[0,2)} \right) \leqslant$$

$$\leqslant \sum_k I_{k+[0,2)} \left(\sum_n \frac{3}{\bar{n}^4 (\bar{k} - n)^4} + \frac{4}{\bar{k}^2} \sum_n \frac{1}{\bar{n}^2 (\bar{k} - n)^2} \right) \leqslant$$

$$\leqslant \sum_k I_{k+[0,2)} \left(\frac{3C_4}{\bar{k}^4} + \frac{4}{\bar{k}^2} \cdot \frac{C_2}{\bar{k}^2} \right) \leqslant C \sum \frac{1}{\bar{k}^4} I_{[k,k+1)} \leqslant C w^{-2}.$$

Obviously $\mathcal{L}_2^w(\mathbb{R})$ is not translation invariant.

Now we prove that this algebra has no a.u. consisting of nonnegative functions. Suppose the opposite, i.e. that $e_{\alpha} \ge 0$ are an a.u.

First we show that $i_{\alpha} = \int_{-1/4}^{1/4} e_{\alpha} \not\to 0$. Let $I_{\delta} = I_{[0,\delta]}$ be the indicator function of $[0, \delta], 0 < \delta < 1/4$. As

$$(I_{\delta} * e_{\alpha})(t) = \int_{t-\delta}^{t} e_{\alpha},$$

then $I_{\delta} * e_{\alpha} \leq i_{\alpha}$ if $t \in [0, \delta]$. But if $i_{\alpha} \to 0$, then

$$\|I_{\delta} - I_{\delta} * e_{\alpha}\|_{2,w}^{2} \ge \int_{0}^{\delta} (I_{\delta} - I_{\delta} * e_{\alpha})^{2} \ge \delta \cdot (1 - i_{\alpha})^{2} \to \delta \neq 0.$$

Thus, $i_{\alpha} \not\rightarrow 0$, what means that integral of e_{α} either over [-1/4, 0] or over [0, 1/4] does not tend to zero. Suppose the latter (otherwise we should define \tilde{I}_n below as left shifts of I_n instead of right ones).

Introduce functions

$$f = \sum_{n=1}^{\infty} \alpha_n I_{2^n}, \quad g = \sum_{n=1}^{\infty} \gamma_n \tilde{I}_{2^n},$$

where $\tilde{I}_n = I_{n+[1/\bar{n}^2, 2/\bar{n}^2)}$. We will choose α_n , γ_n so that $f \in \mathcal{L}_2^w(\mathbb{R})$, $g \in \mathcal{L}_2^{w^{-1}}(\mathbb{R})$. According to the definition of weight

$$\|f\|_{2,w}^{2} = \sum_{n=1}^{\infty} \alpha_{n}^{2} 2^{2n} \frac{1}{2^{2n}} = \sum \alpha_{n}^{2},$$
$$\|g\|_{2,w^{-1}}^{2} = \sum_{n=1}^{\infty} \gamma_{n}^{2} 2^{-4n} \frac{1}{2^{2n}} = \sum \gamma_{n}^{2} 2^{-6n}.$$

Thus we can put $\gamma_n = 2^{3n}\beta_n$ and take any sequences $\alpha, \beta \in \ell_2$.

Now we show that $f * e_{\alpha} \not\to f$. It is sufficient to show that $\langle f * e_{\alpha}, g \rangle \not\to \langle f, g \rangle$. Since $\langle f, g \rangle = 0$, we should show that

$$\langle f \ast e_{\alpha}, g \rangle = \langle e_{\alpha}, f^{\nabla} \ast g \rangle \not\to 0$$

(here $f^{\nabla}(t) = f(-t)$). Let us calculate the convolution

$$f^{\nabla} * g = \sum \alpha_n \gamma_k I_{2^n}^{\nabla} * \tilde{I}_{2^k}.$$

For fixed n, k

$$\operatorname{supp} I_{2^n}^{\nabla} * \tilde{I}_{2^k} = \operatorname{supp} I_{2^n}^{\nabla} + \operatorname{supp} \tilde{I}_{2^k} = 2^k - 2^n + [-2^{-2n}, 2^{-2k}].$$

We will be interested below in the segment [0, 1] only, i.e. convolutions with n = k. These we calculate explicitly:

$$I_{2^n}^{\nabla} * \tilde{I}_{2^n} = 2^{-2n} J_n,$$

where

$$J_n = \min(1 + 2^{2n}t, 2) - \max(2^{2n}t, 1) = \begin{cases} 2^{2n}t, & t \in [0, 2^{-2n}], \\ 2 - 2^{2n}t, & t \in [2^{-2n}, 2^{1-2n}]. \end{cases}$$

This function is piecewise linear, and $J_n(0) = J_n(2^{1-2n}) = 0$, $J_n(2^{-2n}) = 1$. Thus,

$$(f^{\triangledown} * g)|_{[0,1]} = \sum \alpha_n \gamma_n 2^{-2n} J_n = \sum \alpha_n \beta_n 2^n J_n.$$

Put now $\alpha_n = \beta_n = 2^{-n/2}$. As required, $\alpha, \beta \in \ell_2$, and

$$(f^{\nabla} * g)|_{[0,1]} = \sum J_n = J_n$$

Next we estimate J:

$$J(2^{-2n}) = \sum_{k=1}^{n} 2^{2k} \cdot 2^{-2n} = \frac{2^{2n} - 1}{3} 2^{2-2n} = \frac{4}{3} (1 - 2^{-2n}),$$
$$J(2^{1-2n}) = \sum_{k=1}^{n-1} 2^{2k} \cdot 2^{1-2n} = \frac{2^{2n-2} - 1}{3} 2^{3-2n} = \frac{2}{3} (1 - 2^{1-2n})$$

(in the latter case n > 1). Remember that all summands are piecewise linear, therefore $1/3 \leq J \leq 4/3$ on (0, 1/4]. It follows that

$$\frac{1}{3}\int_0^{1/4} e_\alpha \leqslant \int e_\alpha J \leqslant \frac{4}{3}\int_0^{1/4} e_\alpha$$

i.e. $\int e_{\alpha}J \not\to 0$. Since $\langle e_{\alpha}, f^{\nabla} * g \rangle \ge \int e_{\alpha}J$, we proved that e_{α} is not an approximate unit.

Theorem 4.2. Let G be a non-discrete group, p > 1, and let $\mathcal{L}_p^w(G)$ be an invariant algebra. Then this algebra has no bounded approximate unit.

Proof. Suppose that $\mathcal{L}_p^w(G)$ has a left b.a.u. Then [6, th. 32.22] $\mathcal{L}_p^w(G) \circ X$ is a closed linear subspace in X for any left module X over $\mathcal{L}_p^w(G)$ with a multiplication \circ . Take $X = \mathcal{L}_q^{w^{-1}}(G)$. This is a left module over $\mathcal{L}_p^w(G)$ with multiplication $f \circ g = g * f^{\triangledown}, f^{\triangledown}(x) = f(x^{-1})$.

Show that $Y = \mathcal{L}_p^w(G) \circ X$ is dense in X. For this purpose, it is sufficient to show that closure of Y contains indicator functions of all compact sets. Let $A \subset G$ be a compact set. Take a relatively compact neighborhood of identity $V = V^{-1}$, then $I_V^{\nabla} = I_{V^{-1}} \in \mathcal{L}_p^w(G)$, $I_{AV} \in X$ because both V and AV are relatively compact. Put $f_V =$ $I_{AV} * I_V/\mu(V) = I_V^{\nabla} \circ I_{AV}/\mu(V)$, $f_V \in Y$. Easy to check that

$$I_A \leqslant f_V \leqslant I_{AV^2},$$

whence $||I_A - f_V|| \leq ||I_A - I_{AV^2}||$. At the same time

$$||I_A - I_{AV^2}||_{p,w}^p = \int_{AV^2 \setminus A} w^p(t) dt,$$

what tends to zero as $\mu(V) \to 0$ due to the fact that w^p is locally summable. This means that I_A belongs to the closure of Y, and it follows that Y = X.

Thus, for every $\varphi \in Y$ there are $f \in \mathcal{L}_q^{w^{-1}}(G)$, $g \in \mathcal{L}_p^w(G)$ such that $\varphi = g \circ f = f * g^{\nabla}$. But now

$$|\varphi(x)| = |(f * g^{\nabla})(x)| = \left| \int f(t)g(x^{-1}t)dt \right| \leq \\ \leq ||f||_{q,w^{-1}} ||^{x^{-1}}g||_{p,w} \leq L_x ||f||_{q,w^{-1}} ||g||_{p,w}.$$

This means that $\varphi/L \in \mathcal{L}_{\infty}(G)$. As $Y = \mathcal{L}_q^{w^{-1}}(G) = w\mathcal{L}_q(G)$, we get for all $\psi \in \mathcal{L}_q(G)$ that $\psi w/L \in \mathcal{L}_{\infty}$. Since on every compact set F the function w is bounded away from zero and L is bounded, we get that $\mathcal{L}_q(F) \subset \mathcal{L}_{\infty}(F)$, what is possible for finite F only. This contradicts assumption that G is not discrete. \Box

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VINITI, MATHEMATICS DEPARTMENT, USIEVICHA STR. 20, MOSCOW, 125190 E-mail address: jkuzn@mccme.ru