# INVARIANT WEIGHTED ALGEBRAS  ${\mathcal{L}}_p^w(G)$

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ABSTRACT. We deal with weighted spaces  $\mathcal{L}_p^w(G)$  on a locally compact group  $G$ . If  $w$  is a positive measurable function on  $G$  then we define the space  $\mathcal{L}_p^w(G), p \geq 1$ , by equality  $\mathcal{L}_p^w(G) = \{f : fw \in$  $\mathcal{L}_p(G)$ . We consider weights w such that these weighted spaces are algebras with respect to usual convolution. We show that for  $p > 1$  such weights exists on any sigma-compact group. We prove also under minimal requirements a criterion known earlier in special cases:  $\mathcal{L}_{1}^{w}(G)$  is an algebra if and only if w is submultiplicative.

Throughout the paper  $G$  is a locally compact group, all integrals are taken with respect to a left Haar measure  $\mu$ ,  $p \geq 1$ ,  $1/p + 1/q = 1$  (if  $p = 1$  then  $q = \infty$ ). We call any positive measurable function a weight. Weighted space  $\mathcal{L}_p^w(G)$  with the weight w is defined as  $\{f : fw \in$  $\mathcal{L}_p(G)$ , norm of a function f being  $||f||_{p,w} = (\int |fw|^p)^{1/p}$ . Indices p, w are sometimes omitted.

Sufficient conditions on a weight function to define an algebra  $\mathcal{L}_p^w(G)$ with respect to usual convolution,  $f * g(s) = \int f(t)g(t^{-1}s)dt$ , are wellknown. For  $p = 1$  it is submultiplicativity:

<span id="page-0-0"></span>
$$
w(st) \leqslant w(s)w(t),\tag{1}
$$

and for  $p > 1$  the following inequality (pointwise almost everywhere):

<span id="page-0-1"></span>
$$
w^{-q} * w^{-q} \leq w^{-q}.
$$
 (2)

Note that if [\(1\)](#page-0-0) or [\(2\)](#page-0-1) holds with a constant C (after  $\leq$  sign) then for the weight  $w_1 = Cw$  the same inequality holds without any constant. Multiplication of a weight by a number changes by the same number the norm of  $\mathcal{L}_p^w(G)$ , preserving all the properties of the space. Thus we introduce the notion of equivalent weights:  $w_1$  and  $w_2$  are equivalent if with some  $C_1$ ,  $C_2$  locally almost everywhere

<span id="page-0-3"></span>
$$
C_1 \leqslant \frac{w_1}{w_2} \leqslant C_2. \tag{3}
$$

For  $p > 1$  it is convenient to introduce a dual function  $u = w^{-q}$ , then the inequality [\(2\)](#page-0-1) takes the following form, independent on  $p$  and  $q$ :

<span id="page-0-2"></span>
$$
u * u \leqslant u. \tag{4}
$$

Supported by RFBR grant no. 05-01-00982.

It is easy to notice that any function  $u$  satisfying  $(4)$  defines a family of weighted algebras  $\mathcal{L}_p^{w_p}(G)$  for all  $p \in (1, +\infty)$ :  $w_p = u^{-1/q}$ .

## 1. CRITERION FOR THE GROUP

For  $p = 1$  weighted algebras exist, of course, on any locally compact group (at least with a unit weight). For  $p > 1$  we cannot take an arbitrary group, and more precisely, the following theorem holds:

Theorem 1.1. *For a locally compact group* G *the following conditions are equivalent:*

- (i) G *is*  $\sigma$ -compact;
- (ii) *for some*  $p > 1$  *there exist a weight* w *satisfying* [\(2\)](#page-0-1) *(the space*  $\mathcal{L}_{p}^{w}(G)$  *is then a convolution algebra)*;
- (iii) *for any*  $p > 1$  *there exist a weight* w *satisfying* [\(2\)](#page-0-1).

*For an abelian* G *these conditions are also equivalent to the following:*

- (iv) *for some*  $p > 1$  *there exist a weight* w *such that*  $\mathcal{L}_p^w(G)$  *is a convolution algebra;*
- (v) *for any*  $p > 1$  *there exist a weight* w *such that*  $\mathcal{L}_p^w(G)$  *is a convolution algebra.*

*Proof.* Implications (iii)⇒(ii)⇒(iv) and (iii)⇒(v)⇒(iv) are obvious and do not depend on commutativity of G. We prove that  $(ii) \Rightarrow (i)$ ,  $(i) \Rightarrow (iii)$ and for an abelian group (iv) $\Rightarrow$ (i).

(ii)⇒(i). If [\(2\)](#page-0-1) holds then for some x the integral  $(w^{-q} * w^{-q})(x) =$  $\int w^{-q}(y)w^{-q}(y^{-1}x)dy$  of a strictly positive function is finite. This implies that G is  $\sigma$ -compact.

(iv)⇒(i). By [\[8,](#page-11-0) theorem 3] there exists an algebra  $\mathcal{L}_p^w(G)$  where  $w^{-q} \in \mathcal{L}_1(G)$ . Since  $w^{-q}$  is positive, G must be  $\sigma$ -compact.

(i)⇒(iii). We construct a function on G satisfying [\(4\)](#page-0-2). Pick a positive function  $u_1 \in \mathcal{L}_1(G)$  (it exists because G is  $\sigma$ -compact). We may assume that  $||u_1||_1 = 1$ . Define inductively functions  $u_n, n \in \mathbb{N}$ :

$$
u_{n+1} = u_1 * u_n.
$$

Clearly  $||u_n||_1 \leq 1$  for all n. We put now  $u = \sum n^{-2}u_n$  and prove that [\(4\)](#page-0-2) holds. Note the following elementary fact:

$$
\sum_{n=1}^{m-1} \frac{1}{n^2(m-n)^2} \leqslant 8\zeta(2)\frac{1}{m^2}.
$$

Estimate now the convolution  $u * u$ :

$$
u * u = \sum_{n,k=1}^{\infty} \frac{u_n * u_k}{n^2 k^2} = \sum_{n,k=1}^{\infty} \frac{u_{n+k}}{n^2 k^2} =
$$
  
= 
$$
\sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \frac{u_m}{n^2 (m-n)^2} \le 8\zeta(2) \sum_{m=1}^{\infty} \frac{u_m}{m^2} \equiv Cu.
$$

Changing u to  $u/C$ , we get [\(4\)](#page-0-2).

# 2. Technical lemmas

In general any positive measurable function can be taken as a weight. Some authors assume also that  $w^p$  is locally summable, but this is redundant if  $\mathcal{L}_p^w(G)$  is an algebra:

<span id="page-2-0"></span>**Lemma 2.1.** If the space  $\mathcal{L}_p^w(G)$ ,  $p \geq 1$ , is a convolution algebra then w p *is locally summable.*

*Proof.* Take a compact set  $A \subset G$  of positive measure. Here and further  $I_A$  denotes the characteristic function of a set A. Consider the functions  $\varphi = I_A/\max\{1, w\}$  and  $\psi = I_{A^{-1} \cdot A}/\max\{1, w\}$ . As  $\varphi, \psi \in \mathcal{L}_p(w)$ , then also  $\tau = \varphi * \psi \in \mathcal{L}_p(w)$ . At the same time  $\varphi \in \mathcal{L}_1$  and  $\psi \in \mathcal{L}_{\infty}$ , so that  $\tau$  is continuous. Since also  $\tau|_A > 0$  we have  $\min_A \tau = \tau_0 > 0$ . Thus,  $I_A \leq \tau/\tau_0$ , what implies  $I_A \in \mathcal{L}_p(w)$  or, equivalently,  $w \in \mathcal{L}_p(A)$ .  $\Box$ 

<span id="page-2-2"></span>**Lemma 2.2.** *The set*  $B_0(G)$  *of all bounded compactly supported functions is a dense subspace of every algebra*  $\mathcal{L}_p^w(G)$ *.* 

*Proof.* Inclusion  $B_0(G) \subset \mathcal{L}_p^w(G)$  follows from the previous lemma. It is known that  $B_0(G)$  is dense in  $\mathcal{L}_p(G)$ , therefore  $w^{-1}B_0(G)$  is dense in  $\mathcal{L}_p^w(G) = w^{-1} \mathcal{L}_p(G)$ . Let now  $f \in w^{-1}B_0(G)$  be compactly supported, but not necessarily bounded. Changing it on a set of arbitrary small measure  $\varepsilon$ , we can make f bounded;  $\varepsilon$  is to be chosen according to continuity of the integral  $||f||_{p,w} = ||fw||_p$  as a set function.

The property of submultiplicativity [\(1\)](#page-0-0) is essential in the weighted algebras theory because exactly this property posess the weights of  $L_1^w(G)$  algebras (see theorem [3.3](#page-6-0) below). We need the following lemma on submultiplicative functions (it is in fact proposition 1.16 of [\[2\]](#page-11-1)):

<span id="page-2-1"></span>**Lemma 2.3.** Let a measurable function  $L: G \to \mathbb{R}$  be submultiplica*tive, i.e. satisfy* [\(1\)](#page-0-0)*, and positive. Then* L *is bounded and bounded away from zero on any compact set.*

The following condition studied first by R. Edwards [\[2\]](#page-11-1) is also important for the weighted spaces theory. A weight  $w$  is of moderate growth if for all  $s \in G$ 

<span id="page-3-0"></span>
$$
L_s = \operatorname{ess} \sup_t \frac{w(st)}{w(t)} < \infty. \tag{5}
$$

This condition is equivalent to the space  $L_p^w(G)$  (for any  $p \geq 1$ ) being translation-invariant  $[2, 1.13]$ . In the non-commutative case,  $(5)$  corresponds to left translations; taking  $w(ts)$  indtead of  $w(st)$ , we get a condition for right translations, in general not equivalent to the former. Immediate calculations show that

<span id="page-3-3"></span>
$$
\sup_{f \neq 0} \frac{\|sf\|_{p,w}}{\|f\|_{p,w}} = L_{s^{-1}}.\tag{6}
$$

The condition [\(5\)](#page-3-0) implies that  $L_s > 0$ ,  $L_{st} \le L_s L_t$ , and

ess 
$$
\inf_t \frac{w(st)}{w(t)} = 1/L_{s^{-1}} > 0.
$$

<span id="page-3-2"></span>**Lemma 2.4.** *If* [\(5\)](#page-3-0) *holds for locally almost all*  $s \in G$  *then it holds for*  $all \ s \in G.$ 

*Proof.* Let  $S \subset G$  be the set of s for which the inequality [\(5\)](#page-3-0) holds. By assumption S and hence  $S^{-1}$  is locally of full measure. Pick a set  $T \subset$  $S \cap S^{-1}$  of positive finite measure. Then  $T \cdot T^{-1}$  contains a neighborhood of identity  $U$ . As  $L$  (finite or infinite) is submultiplicative,  $S$  is closed under multiplication and therefore  $U \subset T \cdot T^{-1} \subset S \cdot S \subset S$ . By the same reason  $SU \subset S$ , and since S (being locally of full measure) is everywhere dense, then  $S = G$ .

<span id="page-3-1"></span>Lemma 2.5. [\[3,](#page-11-2) th. 2.7] *If a weight* w *satisfies* [\(5\)](#page-3-0) *and is locally summable then it is equivalent to a continuous function.*

**Corollary 2.1.** Let  $L_p^w(G)$  be an algebra with a weight w satisfying [\(5\)](#page-3-0)*. Then* w *is equivalent to a continuous function.*

*Proof.* By lemma [2.1](#page-2-0) for any compact set F we have  $w \in \mathcal{L}_p(F) \subset$  $\mathcal{L}_1(F)$ , therefore we can apply lemma [2.5.](#page-3-1)

On a compact group any continuous function is equivalent to a constant function, thus on a compact group all translation-invariant weighted algebras are isomorphic to the usual algebra  $\mathcal{L}_p(G)$ . Converse of the corollary does not hold:

**Example 2.1.** There exist an algebra  $L_2^w(\mathbb{R})$  such that w is continuous but does not satisfy [\(5\)](#page-3-0). Let  $0 < \alpha_n < 1$ ,  $A_n = [n + \alpha_n, n + 1]$ . We put  $w|_{A_n} = 1 + n^2$ ,  $w(n + \alpha_n/2) = 1 + |n|$  and extend w piecewise linearly.

For  $\alpha_n = n^{-2}$  the condition [\(2\)](#page-0-1) is satisfied but [\(5\)](#page-3-0) does not hold in any neighborhood of zero.

3. CRITERION FOR THE ALGEBRA  $\mathcal{L}_1^w(G)$ 

In the case when  $p = 1$  the class of weights defining convolution algebras  $\mathcal{L}_p^w(G)$  admits a complete description, and it turns out that every weight is equivalent to a continuous function. The following theorem was proved by Grabiner [\[5\]](#page-11-3) in the case of the real half-line (without statement of continuity which is false on the half-line). Edwards [\[2\]](#page-11-1) proved equivalence of (i) and (ii) on a locally compact group under assumption of upper-semicontinuity of  $w$ , and later Feichtinger [\[3\]](#page-11-2) for translation-invariant algebras  $\mathcal{L}_{1}^{w}(G)$ . Our theorem generalizes these results.

Theorem 3.1. *For a weight* w *the following conditions are equivalent:*

- (i) w *is equivalent* (*in the sense of* [\(3\)](#page-0-3)) *to a continuous submultiplicative function;*
- (ii)  $\mathcal{L}_{1}^{w}(G)$  *is a convolution algebra;*
- (iii) *for some*  $p, 1 \leq p < \infty$ *, the inclusion*  $\mathcal{L}_1^w(G) * \mathcal{L}_p^w(G) \subset \mathcal{L}_p^w(G)$ *holds.*

*Proof.* Implications (i)⇒(ii) and (i)⇒(iii) are immediate whereas (ii)⇒(i) is a special case of (iii)⇒(i) with  $p = 1$ . We prove therefore only  $(iii) \Rightarrow (i).$ 

Inclusion (iii) implies (cf.  $[6, 38.27]$ ) that with some constant C

 $||f * g||_{p,w} \leq C ||f||_{1,w} ||g||_{p,w}.$ 

Repeating argument of lemma [2.1](#page-2-0) we conclude that  $w^p$  together with w are locally summable. Thus the spaces  $\mathcal{L}_{1}^{w}(G)$ ,  $\mathcal{L}_{p}^{w}(G)$  contain charasteristic functions of all sets of finite measure. For such sets A, B and arbitrary  $s, t$  we have pointwise

<span id="page-4-2"></span>
$$
\mu(A)I_{stB} \leqslant I_{sA} * I_{A^{-1}tB},\tag{7}
$$

whence

<span id="page-4-1"></span>
$$
\mu(A)\|I_{stB}\|_{p,w} \leqslant C\|I_{sA}\|_{1,w}\|I_{A^{-1}tB}\|_{p,w}.\tag{8}
$$

We need here a generalization of the Lebesgue differentiation theorem. On a locally compact group one may state the theorem as follows (see a general statement in the review [\[1\]](#page-11-5) and specifications for the group case in [\[7\]](#page-11-6)): there exists a family  $\mathcal V$  of sets of positive measure directed by downward inclusion such that for any locally summable function f

<span id="page-4-0"></span>
$$
\lim_{V \in \mathcal{V}} \frac{1}{\mu(V)} \int_{xV} f(t)dt = f(x) \tag{9}
$$

for locally almost all  $x \in G$ . At that every  $V \in V$  contains the identity, and every neighborhood of identity contains eventually all  $V \in \mathcal{V}$  [\[7,](#page-11-6) VIII, 1-2].

So, for locally almost all  $s \in G$  [\(9\)](#page-4-0) holds with  $f = w, x = s$ . For each such s [\(9\)](#page-4-0) holds both with  $f = w^p$  and  $f = w^p$  for locally almost all  $x = t$ . For such s, t and any  $\varepsilon > 0$  for sufficiently small  $V \in \mathcal{V}$ 

$$
||I_{stV}||_{p,w}^p = \int_{tV} ({}^s w(r))^{p} dr > w^p(st)\mu(V)/(1+\varepsilon),
$$
  

$$
||I_{tV}||_{p,w}^p = \int_{tV} w^p(r) dr < w^p(t)\mu(V)(1+\varepsilon).
$$

Fix V such that these inequalities hold. Since the integral of  $w^p$  are continuous as functions of a set, there is a compact set  $B \subset V$  such that

$$
||I_{stV}||_{p,w} < (1+\varepsilon) ||I_{stB}||_{p,w}, \qquad \mu(V) < (1+\varepsilon)\mu(B).
$$

Moreover, there exists a neighborhood of identity (with compact closure)  $V_0$  such that

$$
||I_{V_0^{-1}tB}||_{p,w} < (1+\varepsilon)||I_{tB}||_{p,w}.
$$

And, finally, for sufficiently small  $A \in \mathcal{V}$ ,  $A \subset V_0$  holds

$$
||I_{sA}||_{1,w} < (1+\varepsilon)\mu(A)w(s).
$$

Obviously,  $||I_{A^{-1}tB}||_{p,w} \le ||I_{V_0^{-1}tB}||_{p,w}$  and  $||I_{tB}||_{p,w} \le ||I_{tV}||_{p,w}$ . Uniting all these inequalities with [\(8\)](#page-4-1), we get:

$$
\mu(A)\mu(B)^{1/p}w(st) < C(1+\varepsilon)^{3/p+3}\mu(A)\mu(B)^{1/p}w(s)w(t),
$$

and in the limit as  $\varepsilon \to 0$ 

<span id="page-5-0"></span>
$$
w(st) \leqslant Cw(s)w(t). \tag{10}
$$

This inequality is obtained for locally almost all  $t$  with fixed  $s$  for locally almost all s. But by lemma [2.4](#page-3-2) for  $w$  the condition  $(5)$  holds, and by lemma [2.5](#page-3-1) w is equivalent to a continuous function  $w_1$ . For  $w_1$  [\(10\)](#page-5-0) (with another constant) holds for all t and s. Finally, multiplying  $w_1$ by this constant, we get a continuous submultiplicative weight.  $\Box$ 

As a corollary we obtain a description of multipliers of the algebra  $\mathcal{L}_{1}^{w}(G)$ . Gaudry [\[4\]](#page-11-7) proved under assumption of upper-semicontinuity that multipliers of  $\mathcal{L}_{1}^{w}(G)$  may be identified with the weighted space  $\mathcal{M}^w(G)$  of regular Borel measures such that  $\int w d|\mu| < \infty$ . As the weight can be always chosen continuous, statement of the theorem is simplified:

**Theorem 3.2.** *A bounded operator*  $T$  *on an algebra*  $\mathcal{L}_1^w(G)$  *commutes with right translations if and only if it is a convolution with a measure*  $\mu \in \mathcal{M}^w(G)$ :  $Tf = \mu * f$  for all  $f \in \mathcal{L}_1^w(G)$ .

If  $p > 1$ , the weight of an algebra cannot in general be chosen continuous:

Example 3.1.  $G = \mathbb{T}$ ,  $p = 2$ . Parametrize the circle by  $t \in [-1, 1]$ and take  $w(t) = |t|^{-1/4}$  (another example:  $w(t) = |t|^{1/4}$ ).  $\mathcal{L}_2^w(\mathbb{T})$  is an algebra but the weight is not equivalent to a continuous function. At that  $L_2^w(\mathbb{T})$  is not invariant under translations and  $L_1^w(\mathbb{T})$  with the same weight is not an algebra.

The fact that  $L_1^w(G)$  is an algebra implies the weight is submultiplicative and all the spaces  $L_p^w(G)$ ,  $p \geq 1$ , are translation-invariant (i.e. [\(5\)](#page-3-0) holds). Converse is true for abelian groups:

<span id="page-6-0"></span>**Theorem 3.3.** Let G be an abelian locally compact group,  $\mathcal{L}_p^w(G)$  and *algebra with a weight* w *satisfying* [\(5\)](#page-3-0)*. Then* w *is equivalent to a submultiplicative function.*

*Proof.* By lemma [2.5](#page-3-1) we may assume w is continuous, and by lemma [2.3](#page-2-1) the function  $L$  is bounded on any compact set. Pick a compact set  $D = D^{-1}$  of positive measure, and let  $L(r) \leq N$  for  $r \in D$ . Then for  $s \in G$ ,  $r \in D$  we have

<span id="page-6-1"></span>
$$
\frac{w(s)}{N} \leqslant w(sr) \leqslant Nw(s). \tag{11}
$$

Take now arbitrary  $s, t \in G$  and use inequality [\(7\)](#page-4-2) with  $V = U = D$ . We get then

$$
\mu(D)\|I_{stD}\| \leq \|I_{sD}\| \cdot \|I_{tD^2}\|,
$$

whence by [\(11\)](#page-6-1)

$$
\mu(D)^{1+1/p} w(st)/N \leq N^2 \mu(D)^{1/p} \mu(D^2)^{1/p} w(s) w(t),
$$

i.e.  $w(st) \leq C_1w(s)w(t)$ , what completes the proof.

Note that on a discrete group weight of any algebra for all  $p \geq 1$  is submultiplicative. Inequality [\(1\)](#page-0-0) obtains when passing from  $I_{st} = I_s * I_t$ to the norms in  $\mathcal{L}_p^w(G)$ .

#### 4. Approximate units

Algebras  $\mathcal{L}_p^w(G)$ , as well as those without weight, have a unit iff G is discrete. In the classical case (of a compact group)  $\mathcal{L}_p(G)$  always have approximate units. In the weighted case moderate growth of the weight (see th. [4.1\)](#page-7-0) is sufficient for an algebra to have an a.u. If

the weight is not moderate, this theorem may not hold, see example [4.1.](#page-8-0) Invariant algebras  $\mathcal{L}_p^w(G)$  do not have bounded approximate units (theorem [4.2\)](#page-10-0). It follows these algebras are not amenable [\[9\]](#page-11-8). In this section, theorems are proved for left a.u., but the same is true for right a.u. with interchange of s and t in the condition  $(5)$ .

<span id="page-7-1"></span>**Lemma 4.1.** Let  $\mathcal{L}_p^w(G)$ ,  $p \geq 1$  be an invariant algebra. Then for any  $f \in \mathcal{L}_p^w(G)$  and  $\varepsilon > 0$  there exists a neighborhood of identity U such *that for all*  $t \in U$   $||f - tf||_{p,w} < \varepsilon$ .

*Proof.* Suppose first that  $f \in \mathcal{L}_p^w(G)$  is compactly supported,  $F =$ supp  $f$ . Pick a relatively compact neighborhood of identity  $U$ , then supp  $f \subset UF$  when  $t \in U$ . By lemma [2.5](#page-3-1) we may assume w is continuous, thus bounded on every compact set, so that  $C = \sup w < \infty$ . UF Now

$$
||f - {}^t f||_{p,w} = \left( \int_{UF} |f(x) - f(tx)|^p w^p(x) dx \right)^{1/p} \leq C ||f - {}^t f||_p,
$$

where the latter norm is less than  $\varepsilon$  for t in a sufficiently small neighborhood of identity  $V \subset U$ .

Let now  $f \in \mathcal{L}_p^w(G)$  be arbitrary. For any  $\varepsilon > 0$  it may be approximated by a compactly supported function  $\varphi \in \mathcal{L}_p^w(G)$ ,  $||f - \varphi||_{p,w} < \varepsilon$ (lemma [2.2\)](#page-2-2). Pick again a symmetric relatively compact neighborhood of identity U. By lemma [2.3](#page-2-1) the function  $L$  defined in the formula  $(5)$ is bounded on every compact set, so that  $D = \sup_U L < \infty$ . Now by equality [\(6\)](#page-3-3) for  $t \in U$ 

$$
||^{t}f - {}^{t}\varphi||_{p,w} = ||^{t}(f - \varphi)||_{p,w} \leqslant L_{t^{-1}}||f - \varphi||_{p,w} \leqslant D||f - \varphi||_{p,w},
$$

and

$$
||f - f||_{p,w} \le ||f - \varphi||_{p,w} + ||\varphi - f\varphi||_{p,w} + ||f - f\varphi||_{p,w} < \varepsilon + D\varepsilon + C||\varphi - f\varphi||_p,
$$

what is less than  $\varepsilon(1+C+D)$  for t in a sufficiently small neighborhood of identity  $V \subset U$ .

<span id="page-7-0"></span>**Theorem 4.1.** Let  $\mathcal{L}_p^w(G)$ ,  $p \geq 1$  be an invariant algebra. The net  $\xi_{\nu} = I_{\nu}/\mu(\nu)$ , where  $\nu$  *runs over the net of all relatively compact neigh*borhoods of identity, is a left approximate unit in  $\mathcal{L}_p^w(G)$ .

*Proof.* Note first that all  $\xi_{\nu}$  belong to  $\mathcal{L}_p^w(G)$  (prop. [2.2\)](#page-2-2). Convergence  $\xi_{\nu} * f \to f$  for every  $f \in \mathcal{L}_p^w(G)$  is proved in a standard way. By lemma [4.1](#page-7-1) exists a neighborhoods of identity U such that  $||f - t^{-1}f||_{p,w} < \varepsilon$ for  $t \in U$ . We can estimate the norm  $\|\xi_{\nu} * f - f\|_{p,w}^{\infty}$ ,  $\nu \subset U$  using

functionals of the conjugate space: for every  $\varphi \in \mathcal{L}_q^{w^{-1}}(G)$ 

$$
\left| \langle \xi_{\nu} * f - f, \varphi \rangle \right| = \left| \int_{G} \left[ (\xi_{\nu} * f)(x) - f(x) \right] \varphi(x) dx \right| \le
$$
  

$$
\leq \int_{G} \int_{\nu} \frac{\left| f(t^{-1}x)dt - f(x) \right|}{\mu(\nu)} dt \, |\varphi(x)| dx \leq \frac{1}{\mu(\nu)} \int_{\nu} \langle |t^{-1}f - f|, |\varphi| \rangle dt \leq
$$
  

$$
\leq \sup_{t \in U} \| t^{-1}f - f\|_{p,w} \|\varphi\|_{q,w^{-1}} < \varepsilon \|\varphi\|_{q,w^{-1}},
$$

i.e.  $||^{t^{-1}}f - f||_{p,w} < \varepsilon$ , what proves the theorem.

<span id="page-8-0"></span>**Example 4.1.** If the weight fails to satisfy  $(5)$ , the statement of theorem [4.1](#page-7-0) may not be true. The following algebra is a counterexample. Take  $G = \mathbb{R}$ ,  $p = 2$ . We denote  $\bar{n} = \max(|n|, 1)$  and define the weight in the following way:

$$
w(t) = \begin{cases} \bar{n}, & t \in [n, n + 1/\bar{n}^2), \\ \bar{n}^2, & t \in [n + 1/\bar{n}^2, n + 1). \end{cases}
$$

We show first that  $(2)$  holds for w after a multiplication by some constant, i.e.  $\mathcal{L}_2^w(\mathbb{R})$  is an algebra. Denote

$$
I_n = I_{[n,n+1/\bar{n}^2)}, \quad I'_n = I_{[n+1/\bar{n}^2,n+1)}.
$$

In these notations  $w = \sum_{n=1}^{\infty} (\bar{n} I_n + \bar{n}^2 I'_n)$ . Using a trivial estimate  $I_A * I_B \leq$  $\min\{\mu(A), \mu(B)\}\$   $I_{A+B}$  and inequality

$$
\sum_{n=-\infty}^{\infty} \frac{1}{\bar{n}^{\alpha} (\overline{m-n})^{\alpha}} \leqslant \frac{2^{\alpha+1}}{\bar{m}^{\alpha}} (2 \sum_{n=1}^{\infty} \frac{1}{\bar{n}^{\alpha}} + 1) = \frac{C_{\alpha}}{\bar{m}^{\alpha}}
$$

for  $\alpha = 2, 4$  and integer m, we can estimate convolution in (2):

$$
w^{-2} * w^{-2} = \sum_{n,m} \left( \frac{1}{\bar{n}^2 \bar{m}^2} I_n * I_m + 2 \frac{1}{\bar{n}^2 \bar{m}^4} I_n * I'_m + \frac{1}{\bar{n}^4 \bar{m}^4} I'_n * I'_m \right) \leq
$$
  

$$
\leq \sum_{n,m} \left( \frac{1}{\bar{n}^2 \bar{m}^2 \max(\bar{n}^2, \bar{m}^2)} I_{n+m+[0,2)} + 2 \frac{1}{\bar{n}^2 \bar{m}^4 \bar{n}^2} I_{n+m+[0,2)} + \frac{1}{\bar{n}^4 \bar{m}^4} I_{n+m+[0,2)} \right) \leq
$$
  

$$
\leq \sum_k I_{k+[0,2)} \left( \sum_n \frac{3}{\bar{n}^4 (\overline{k} - \bar{n})^4} + \frac{4}{\bar{k}^2} \sum_n \frac{1}{\bar{n}^2 (\overline{k} - \bar{n})^2} \right) \leq
$$
  

$$
\leq \sum_k I_{k+[0,2)} \left( \frac{3C_4}{\bar{k}^4} + \frac{4}{\bar{k}^2} \cdot \frac{C_2}{\bar{k}^2} \right) \leq C \sum_n \frac{1}{\bar{k}^4} I_{[k,k+1)} \leq C w^{-2}.
$$

Obviously  $\mathcal{L}_2^w(\mathbb{R})$  is not translation invariant.

Now we prove that this algebra has no a.u. consisting of nonnegative functions. Suppose the opposite, i.e. that  $e_{\alpha} \geqslant 0$  are an a.u.

First we show that  $i_{\alpha} = \int_{-1/4}^{1/4} e_{\alpha} \neq 0$ . Let  $I_{\delta} = I_{[0,\delta]}$  be the indicator function of  $[0, \delta]$ ,  $0 < \delta < 1/4$ . As

$$
(I_{\delta} * e_{\alpha})(t) = \int_{t-\delta}^{t} e_{\alpha},
$$

then  $I_{\delta} * e_{\alpha} \leqslant i_{\alpha}$  if  $t \in [0, \delta]$ . But if  $i_{\alpha} \to 0$ , then

$$
||I_{\delta}-I_{\delta}*e_{\alpha}||_{2,w}^2 \geq \int_0^{\delta} (I_{\delta}-I_{\delta}*e_{\alpha})^2 \geq \delta \cdot (1-i_{\alpha})^2 \to \delta \neq 0.
$$

Thus,  $i_{\alpha} \nrightarrow 0$ , what means that integral of  $e_{\alpha}$  either over [-1/4, 0] or over  $[0, 1/4]$  does not tend to zero. Suppose the latter (otherwise we should define  $I_n$  below as left shifts of  $I_n$  instead of right ones).

Introduce functions

$$
f = \sum_{n=1}^{\infty} \alpha_n I_{2^n}, \quad g = \sum_{n=1}^{\infty} \gamma_n \tilde{I}_{2^n},
$$

where  $\tilde{I}_n = I_{n+[1/\bar{n}^2,2/\bar{n}^2)}$ . We will choose  $\alpha_n$ ,  $\gamma_n$  so that  $f \in \mathcal{L}_2^w(\mathbb{R}),$  $g \in \mathcal{L}_2^{w^{-1}}(\mathbb{R})$ . According to the definition of weight

$$
||f||_{2,w}^{2} = \sum_{n=1}^{\infty} \alpha_n^2 2^{2n} \frac{1}{2^{2n}} = \sum \alpha_n^2,
$$
  

$$
||g||_{2,w^{-1}}^2 = \sum_{n=1}^{\infty} \gamma_n^2 2^{-4n} \frac{1}{2^{2n}} = \sum \gamma_n^2 2^{-6n}.
$$

Thus we can put  $\gamma_n = 2^{3n} \beta_n$  and take any sequences  $\alpha, \beta \in \ell_2$ .

Now we show that  $f*e_{\alpha} \nleftrightarrow f$ . It is sufficient to show that  $\langle f*e_{\alpha}, g \rangle \nleftrightarrow$  $\langle f, g \rangle$ . Since  $\langle f, g \rangle = 0$ , we should show that

$$
\langle f * e_{\alpha}, g \rangle = \langle e_{\alpha}, f^{\nabla} * g \rangle \nrightarrow 0
$$

(here  $f^{\triangledown}(t) = f(-t)$ ). Let us calculate the convolution

$$
f^{\nabla} * g = \sum \alpha_n \gamma_k I_{2^n}^{\nabla} * \tilde{I}_{2^k}.
$$

For fixed  $n, k$ 

$$
\operatorname{supp} I_{2^n}^{\nabla} * \tilde{I}_{2^k} = \operatorname{supp} I_{2^n}^{\nabla} + \operatorname{supp} \tilde{I}_{2^k} = 2^k - 2^n + [-2^{-2n}, 2^{-2k}].
$$

We will be interested below in the segment  $[0, 1]$  only, i.e. convolutions with  $n = k$ . These we calculate explicitly:

$$
I_{2^n}^{\nabla} * \tilde{I}_{2^n} = 2^{-2n} J_n,
$$

where

$$
J_n = \min(1 + 2^{2n}t, 2) - \max(2^{2n}t, 1) = \begin{cases} 2^{2n}t, & t \in [0, 2^{-2n}], \\ 2 - 2^{2n}t, & t \in [2^{-2n}, 2^{1-2n}]. \end{cases}
$$

This function is piecewise linear, and  $J_n(0) = J_n(2^{1-2n}) = 0, J_n(2^{-2n}) =$ 1. Thus,

$$
(f^{\nabla} * g)|_{[0,1]} = \sum \alpha_n \gamma_n 2^{-2n} J_n = \sum \alpha_n \beta_n 2^n J_n.
$$

Put now  $\alpha_n = \beta_n = 2^{-n/2}$ . As required,  $\alpha, \beta \in \ell_2$ , and

$$
(f^{\triangledown} * g)|_{[0,1]} = \sum J_n = J.
$$

Next we estimate J:

$$
J(2^{-2n}) = \sum_{k=1}^{n} 2^{2k} \cdot 2^{-2n} = \frac{2^{2n} - 1}{3} 2^{2-2n} = \frac{4}{3} (1 - 2^{-2n}),
$$
  

$$
J(2^{1-2n}) = \sum_{k=1}^{n-1} 2^{2k} \cdot 2^{1-2n} = \frac{2^{2n-2} - 1}{3} 2^{3-2n} = \frac{2}{3} (1 - 2^{1-2n})
$$

(in the latter case  $n > 1$ ). Remember that all summands are piecewise linear, therefore  $1/3 \leqslant J \leqslant 4/3$  on  $(0, 1/4]$ . It follows that

$$
\frac{1}{3} \int_0^{1/4} e_\alpha \leq \int e_\alpha J \leq \frac{4}{3} \int_0^{1/4} e_\alpha,
$$

i.e.  $\int e_{\alpha} J \nightharpoondown 0$ . Since  $\langle e_{\alpha}, f^{\triangledown} * g \rangle \geqslant \int e_{\alpha} J$ , we proved that  $e_{\alpha}$  is not an approximate unit.

<span id="page-10-0"></span>**Theorem 4.2.** Let G be a non-discrete group,  $p > 1$ , and let  $\mathcal{L}_p^w(G)$ *be an invariant algebra. Then this algebra has no bounded approximate unit.*

*Proof.* Suppose that  $\mathcal{L}_p^w(G)$  has a left b.a.u. Then [\[6,](#page-11-4) th. 32.22]  $\mathcal{L}_p^w(G) \circ$ X is a closed linear subspace in X for any left module X over  $\mathcal{L}_p^w(G)$ with a multiplication  $\circ$ . Take  $X = \mathcal{L}_q^{w^{-1}}(G)$ . This is a left module over  $\mathcal{L}_p^w(G)$  with multiplication  $f \circ g = g * f^\nabla$ ,  $f^\nabla(x) = f(x^{-1})$ .

Show that  $Y = \mathcal{L}_p^w(G) \circ X$  is dense in X. For this purpose, it is sufficient to show that closure of  $Y$  contains indicator functions of all compact sets. Let  $A \subset G$  be a compact set. Take a relatively compact neighborhood of identity  $V = V^{-1}$ , then  $I_V^{\triangledown} = I_{V^{-1}} \in \mathcal{L}_p^w(G)$ ,  $I_{AV} \in X$  because both V and AV are relatively compact. Put  $f_V =$  $I_{AV} * I_V/\mu(V) = I_V^{\nabla}$  $V_V^{\triangledown} \circ I_{AV}/\mu(V)$ ,  $f_V \in Y$ . Easy to check that

$$
I_A \leq f_V \leq I_{AV^2},
$$

whence  $||I_A - f_V|| \le ||I_A - I_{AV}||$ . At the same time

$$
||I_A - I_{AV^2}||_{p,w}^p = \int_{AV^2 \backslash A} w^p(t)dt,
$$

what tends to zero as  $\mu(V) \to 0$  due to the fact that  $w^p$  is locally summable. This means that  $I_A$  belongs to the closure of Y, and it follows that  $Y = X$ .

Thus, for every  $\varphi \in Y$  there are  $f \in \mathcal{L}_q^{w^{-1}}(G)$ ,  $g \in \mathcal{L}_p^w(G)$  such that  $\varphi = g \circ f = f * g^{\nabla}$ . But now

$$
|\varphi(x)| = |(f * g^{\nabla})(x)| = |\int f(t)g(x^{-1}t)dt| \leq
$$
  

$$
\leq ||f||_{q,w^{-1}}||^{x^{-1}}g||_{p,w} \leq L_x ||f||_{q,w^{-1}} ||g||_{p,w}.
$$

This means that  $\varphi/L \in \mathcal{L}_{\infty}(G)$ . As  $Y = \mathcal{L}_q^{w^{-1}}(G) = w\mathcal{L}_q(G)$ , we get for all  $\psi \in \mathcal{L}_q(G)$  that  $\psi \psi/L \in \mathcal{L}_{\infty}$ . Since on every compact set F the function  $w$  is bounded away from zero and  $L$  is bounded, we get that  $\mathcal{L}_q(F) \subset \mathcal{L}_{\infty}(F)$ , what is possible for finite F only. This contradicts assumption that G is not discrete.  $\Box$ 

#### **REFERENCES**

- <span id="page-11-5"></span>[1] A. Bruckner, "Differentiation of integrals", Amer. Math. Monthly 78 no. 9 pt. 2 (1971).
- <span id="page-11-1"></span>[2] R. E. Edwards, "The stability of weighted Lebesgue spaces", Trans. Amer. Math. Soc. 93 (1959), 369–394.
- <span id="page-11-2"></span>[3] H. G. Feichtinger, "Gewichtsfunktionen auf lokalkompakten Gruppen", Sitzber. Österr. Akad. Wiss. Abt. II, 188 No. 8–10, 451–471 (1979).
- <span id="page-11-7"></span>[4] G. I. Gaudry, "Multipliers of weighted Lebesgue and measure spaces", Proc. London Math. Soc. (3) 19 (1969), 327–340.
- <span id="page-11-3"></span>[5] S. Grabiner, "Weighted convolution algebras as analogues of Banach algebras of power series", Radical Banach algebras and automatic continuity, Proc. Conf., Long Beach 1981, Lect. Notes Math. 975, 295–300 (1983).
- <span id="page-11-4"></span>[6] E. Hewitt, K. A. Ross, Abstract harmonic analysis I, II. Springer–Verlag, 3rd printing, 1997.
- <span id="page-11-6"></span>[7] Ionescu Tulcea A., Ionescu Tulcea C., "Topics in the theory of lifting", Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 48. Springer—Verlag, 1969.
- <span id="page-11-0"></span>[8] Yu. N. Kuznetsova, "Weighted  $L_p$ -algebras on groups", Funkts. anal. i ego pril., 2006, 40:3, 82-85.
- <span id="page-11-8"></span>[9] Volker Runde, "Lectures on Amenability", Springer—Verlag, 2002.

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