Calculation of local Fourier transforms for formal connections

Jiangxue Fang
Chern Institute of Mathematics, Nankai University, Tianjin 300071, P. R. China
fangjiangx@yahoo.com.cn

Abstract We calculate the local Fourier transforms for formal connections. In particular, we verify an analogous conjecture of Laumon and Malgrange ([6] 2.6.3).

Mathematics Subject Classification (2000): Primary 14F40.

1 Introduction

Let k be an algebraic closed field of characteristic zero and let $k((t))$ be the field of formal Laurent series in the variable t. A formal connection on $k((t))$ is a pair $(M, t\partial_t)$ consisting of a finite dimensional $k((t))$-vector space M and a k-linear map $t\partial_t : M \to M$ satisfying

$$t\partial_t(fm) = t\partial_t(f)m + ft\partial_t(m)$$

for any $f \in k((t))$ and $m \in M$. In [2], S. Bloch and H. Esnault define local Fourier transforms $\mathcal{F}^{(0,\infty)}$, $\mathcal{F}^{(\infty,0)}$, $\mathcal{F}^{(\infty,\infty)}$ for formal connections, by analogy with the ℓ-adic local Fourier transform considered in [6]. In [6], 2.6.3, Laumon and Malgrange give conjectural formulas of local Fourier transforms for a class of $\mathbb{Q}_\ell$-sheaf. This results are proved by Lei Fu ([4]). In this paper, we prove an analogous conjecture of local Fourier transform for formal connections. Actually, we can calculate local Fourier transforms for any formal connections.

A key technical tool for the definitions of local Fourier transforms of formal connections is the notion of good lattices pairs. By definition in [3], Lemma 6.21, *a pair of good lattices* $\mathcal{V}$, $\mathcal{W}$ of M is a pair of lattices in M satisfying the following conditions

(1) $\mathcal{V} \subset \mathcal{W} \subset M$

(2) $t\partial_t(\mathcal{V}) \subset \mathcal{W}$

(3) For any $k \in \mathbb{N}$, the natural inclusion of complexes

$$(\mathcal{V} \xrightarrow{t\partial_t} \mathcal{W}) \to (\frac{1}{t^k}\mathcal{V} \xrightarrow{t\partial_t} \frac{1}{t^k}\mathcal{W})$$

is a quasi-isomorphism.

Good lattices pairs $\mathcal{V}$, $\mathcal{W}$ exist. The number $\dim_k \mathcal{W}/\mathcal{V}$ is independent of the choice of good lattices pairs of M, and is called the irregularity of M.

For any $f \in k((t))$, denote by $[f]$ the formal connection on $k((t))$ consisting of a one dimensional $k((t))$-vector space with a basis e and a k-linear map $t\partial_t : k((t))e \to k((t))e$ satisfying

$$t\partial_t(ge) = (t\partial_t(g) + fg)e$$

for any $g \in k((t))$. Two such objects $[f]$ and $[f']$ are isomorphic if and only if $f - f' \in tk[[t]] + \mathbb{Z}$. Therefore the non-negative integer

$$\max(0, -\mathrm{ord}_t(f))$$

is a well-defined invariant of the isomorphic class of $[f]$, and is called the slope of $[f]$. Let p be the slope of $[f]$. One can verify $k[[t]]e,\ t^{-p}k[[t]]e$ is a good lattices pair of $[f]$. So the irregularity coincides with the slope for any one dimensional formal connection. The definition of slopes for arbitrary formal connections is given in [5], (2.2.5). The irregularity of a formal connection coincide with the sum of its slopes. Any formal connection has a unique slope decomposition. So the slope of an irreducible formal connection is equal to its irregularity divided by its dimension. A formal connection is called regular if the irregularity of this connection is equal to 0.

Throughout this paper, r and s are to be positive integers. Let t' be the Fourier transform coordinate of t. Write $z = \frac{1}{t}$ and $z' = \frac{1}{t'}$. Let

$$[r] : k((t)) \hookrightarrow k((\sqrt[r]{t}))$$

be the natural inclusion of fields. Let $T = \sqrt[r]{t}$ and let α be a formal Laurent series in $k((T))$ of order $-s$ with respect to T. Let R be a regular formal connection on $k((T))$. In this paper, we calculate the local Fourier transform

$$\mathcal{F}^{(0,\infty)}\Big([r]_*\Big([T\partial_T(\alpha)] \otimes_{k((T))} R\Big)\Big).$$

Similarly, let $k((z))$ be the field of formal Laurent series in the variable z. Let

$$[r] : k((z)) \hookrightarrow k((\frac{1}{\sqrt[r]{t}}))$$

be the natural inclusion of fields. Let $Z = \frac{1}{\sqrt[r]{t}}$ and let α be a formal Laurent series in $k((Z))$ of order $-s$ with respect to Z. Let R be a regular formal connection on $k((Z))$. We also calculate the local Fourier transforms

$$\mathcal{F}^{(\infty,0)}\Big([r]_*\Big([Z\partial_Z(\alpha)]\otimes_{k((Z))}R\Big)\Big) \text{ if } r>s;$$
$$\mathcal{F}^{(\infty,\infty)}\Big([r]_*\Big([Z\partial_Z(\alpha)]\otimes_{k((Z))}R\Big)\Big) \text{ if } r<s.$$

We refer the reader to [2] for the definitions and properties of local Fourier transforms. The main results of this paper are the following three theorems.

Theorem 1. *Given a formal Laurent series α in $k((\sqrt[r]{t}))$ of order $-s$ with respect to $\sqrt[r]{t}$, consider the following system of equations*

$$\begin{cases} \partial_t(\alpha(\sqrt[r]{t}))+t'=0, \\ \alpha(\sqrt[r]{t})+tt'=\beta(\frac{1}{\sqrt[r+s]{t'}}). \end{cases} \tag{1.1}$$

Using the first equation, we find an expression of $\sqrt[r]{t}$ in terms of $\frac{1}{\sqrt[r+s]{t'}}$. We then substitute this expression into the second equation to get $\beta(\frac{1}{\sqrt[r+s]{t'}})$, which is a formal Laurent series in $k((\frac{1}{\sqrt[r+s]{t'}}))$ of order $-s$ with respect to $\frac{1}{\sqrt[r+s]{t'}}$. Let $T=\sqrt[r]{t}$ and let $Z'=\frac{1}{\sqrt[r+s]{t'}}$. For any regular formal connection R on $k((T))$, we have

$$\mathcal{F}^{(0,\infty)}\Big([r]_*\Big([T\partial_T(\alpha)]\otimes_{k((T))}R\Big)\Big)=[r+s]_*\Big([Z'\partial_{Z'}(\beta)+\frac{s}{2}]\otimes_{K((Z'))}R\Big),$$

where the right R means the formal connection on $k((Z'))$ after replacing the variable T with Z'.

Theorem 2. *Suppose $r>s$. Given a formal Laurent series α in $k((\frac{1}{\sqrt[r]{t}}))$ of order $-s$ with respect to $\frac{1}{\sqrt[r]{t}}$, consider the following system of equations*

$$\begin{cases} \partial_t(\alpha(\frac{1}{\sqrt[r]{t}}))+t'=0, \\ \alpha(\frac{1}{\sqrt[r]{t}})+tt'=\beta(\sqrt[r-s]{t'}). \end{cases} \tag{1.2}$$

Using the first equation, we find an expression of $\frac{1}{\sqrt[r]{t}}$ in terms of $\sqrt[r-s]{t'}$. We then substitute this expression into the second equation to get $\beta(\sqrt[r-s]{t'})$, which is formal Laurent series in $k((\sqrt[r-s]{t'}))$ of order $-s$ with respect to $\sqrt[r-s]{t'}$. Let $Z=\frac{1}{\sqrt[r]{t}}$ and let $T'=\sqrt[r-s]{t'}$. For any regular formal connection R on $k((Z))$, we have

$$\mathcal{F}^{(\infty,0)}\Big([r]_*\Big([Z\partial_Z(\alpha)]\otimes_{k((Z))}R\Big)\Big)=[r-s]_*\Big([T'\partial_{T'}(\beta)+\frac{s}{2}]\otimes_{k((T'))}R\Big),$$

where the right R means the formal connection on $k((T'))$ after replacing the variable Z with T'.

Theorem 3. *Suppose $r < s$. Given a formal Laurent series α in $k((\frac{1}{\sqrt[r]{t}}))$ of order $-s$ with respect to $\frac{1}{\sqrt[r]{t}}$, consider the following system of equations*

$$\begin{cases} \partial_t(\alpha(\frac{1}{\sqrt[r]{t}})) + t' = 0, \\ \alpha(\frac{1}{\sqrt[r]{t}}) + tt' = \beta(\frac{1}{\sqrt[s-r]{t'}}). \end{cases} \tag{1.3}$$

Using the first equation, we find an expression of $\frac{1}{\sqrt[r]{t}}$ in terms of $\frac{1}{\sqrt[s-r]{t'}}$. We then substitute this expression into the second equation to get $\beta(\frac{1}{\sqrt[s-r]{t'}})$, which is a formal Laurent series in $k((\frac{1}{\sqrt[s-r]{t'}}))$ of order $-s$ with respect to $\frac{1}{\sqrt[s-r]{t'}}$. Let $Z = \frac{1}{\sqrt[r]{t}}$ and let $Z' = \frac{1}{\sqrt[s-r]{t'}}$. For any regular formal connection R on $k((Z))$, we have

$$\mathcal{F}^{(\infty,\infty)}\Big([r]_*\Big([Z\partial_Z(\alpha)] \otimes_{k((Z))} R\Big)\Big) = [s-r]_*\Big([Z'\partial_{Z'}(\beta) + \frac{s}{2}] \otimes_{k((Z'))} R\Big),$$

where the right R means the formal connection on $k((Z'))$ after replacing the variable Z with Z'.

When R is trivial, the above three theorems are conjectured by Laumon and Malgrange ([6] 2.6.3) except the term $\frac{s}{2}$ is missing in the conjecture. Any formal connection on $k((t))$ is a direct sum of indecomposable connections. As in [1], section 5.9, any indecomposable connection $M = N \otimes R$, where R is regular and $N = [d]_* L$ where L is a one dimensional connection on a finite extension $[d] : k((t)) \to k((t^{\frac{1}{d}}))$. So we can calculate local Fourier transform for all formal connections.

Acknowledgements. It is a great pleasure to thank my advisor Lei Fu for his guidance and support during my graduate studies. In [8], Claude Sabbah proves these results of local Fourier transforms for formal connections with a geometric method. Our method is elementary and directly.

2 Proofs of Theorems 1, 2

Given a formal Laurent series α in the variable $\sqrt[r]{t}$ of order $-s$, consider the system of equations (1.1). We express $\sqrt[r]{t}$ as a formal Laurent series in $\frac{1}{\sqrt[r+s]{t'}}$ of order 1 using the first equation and then substitute this expression into the second equation to get $\beta \in k((\frac{1}{\sqrt[r+s]{t'}}))$. We have

$$\begin{aligned} \partial_{t'}(\beta) &= \partial_{t'}\big(\alpha(\sqrt[r]{t}) + tt'\big) = \partial_t\big(\alpha(\sqrt[r]{t})\big)\frac{dt}{dt'} + t'\frac{dt}{dt'} + t \\ &= \Big(\partial_t\big(\alpha(\sqrt[r]{t})\big) + t'\Big)\frac{dt}{dt'} + t = t. \end{aligned} \tag{2.1}$$

It follows that β is a formal Laurent series in $\frac{1}{\sqrt[r+s]{t'}}$ of order $-s$. Let $T = \sqrt[r]{t}$ and $Z' = \frac{1}{\sqrt[r+s]{t'}}$. Set

$$a(T) = -T^s t\partial_t(\alpha) \text{ and } b(Z') = Z'^s t'\partial_{t'}(\beta).$$

Then $a(T)$ is a formal power series in T of order 0 and $b(Z')$ is a formal power series in Z' of order 0. From the system of equations (1.1) and (2.1), we get

$$\begin{cases} a(T) = (\frac{T}{Z'})^{r+s} \\ b(Z') = (\frac{T}{Z'})^{r}. \end{cases} \tag{2.2}$$

To prove Theorem 1, it suffices to prove the following theorem.

Theorem 1′. *Given a formal power series $a(T) = \sum_{i\geq 0} a_i T^i$ with $a_i \in k$ and $a_0 \neq 0$, solve the system of equations (2.2) to get $b(Z') = \sum_{i\geq 0} b_i Z'^i$ for some $b_i \in k$. Then $b_s = \frac{r}{r+s} a_s$ and*

$$\begin{aligned} & \mathcal{F}^{(0,\infty)}\Big([r]_*[-r(a_0T^{-s} + a_1T^{1-s} + \ldots + a_s)]\Big) \\ = \quad & [r+s]_*[-(r+s)(b_0Z'^{-s} + b_1Z'^{1-s} + \ldots + b_s) + \frac{s}{2}]. \end{aligned}$$

In fact, suppose Theorem $1'$ holds. Let c be an element in k. By remark 2.2 we shall prove later, for $a(T) = -T^s t\partial_t(\alpha) - \frac{c}{r}T^s$, we can get a solution $b(Z')$ of the system of equations (2.2) such that

$$b(Z') \equiv Z'^s t'\partial_{t'}(\beta) - \frac{c}{r+s} Z'^s \text{ mod. } Z'^{s+1}.$$

Then

$$\begin{aligned} & \mathcal{F}^{(0,\infty)}\Big([r]_*[T\partial_T(\alpha) + c\Big) \\ = \quad & \mathcal{F}^{(0,\infty)}\Big([r]_*[-rT^{-s}(-T^s t\partial_t(\alpha) - \frac{c}{r}T^s)]\Big) \\ = \quad & [r+s]_*[-(r+s)Z'^{-s}(Z'^s t'\partial_{t'}(\beta) - \frac{c}{r+s}Z'^s)] \\ = \quad & [r+s]_*[Z'\partial_{Z'}(\beta) + c]. \end{aligned}$$

So Theorem 1 holds for $R = [c]$. As in [1], section 5.9, every irreducible regular formal connection N on $k((T))$ is $[d]_*L$, where L is a one dimensional formal connection on a finite extension $[d] : k((T)) \to k((T^{\frac{1}{d}})$. So L is regular, we have $L = [c]$ for some $c \in k$. Then $N = [d]_*[c] = \oplus_{1\leq i\leq d}[c + \frac{i}{d}]$. We have $d = 1$ because N is irreducible. This shows that every irreducible regular formal connection is isomorphic to the one dimensional connection $[c]$ for some $c \in k$. So every regular formal connection is a successive extension of connections of the type $[c]$. Since $\mathcal{F}^{(0,\infty)}$ is functoriel and exact, Theorem 1 holds for any regular formal connection R on $k((T))$.

Remark 2.1. If $a_s = 0$, then there exists $\alpha \in k((\sqrt[r]{t}))$ such that $a(T) = -T^s t\partial_t(\alpha)$. Using the first equation of (2.2), we find an expression of T in terms of Z'. We then substitute this expression into the second equation of (2.2) to get $b(Z')$. This expression also satisfies the first equation of

(1.1). We then substitute this expression into the second equation of (1.1) to get $\beta(Z')$. By (2.1), we have

$$b(Z') = \sum_{i\geq 0} b_i Z'^i = Z'^s t' \partial_{t'}(\beta).$$

This shows $b_s = 0$.

Remark 2.2. Solving the first equation of (2.2), we get $T = \sum_{i\geq 0} \lambda_i Z'^{i+1}$ with $\lambda_0 = \sqrt[r+s]{a_0}$. The solution is not unique and different solutions differ by an $r+s$-th root of unity. As long as λ_0 is chosen to be an $r+s$-th root of a_0, for each i, λ_i depends only on $a_0, \ldots, a_i$. We have $b(Z') = (\sum_{i\geq 0} \lambda_i Z'^i)^r$, and for each i, b_i depends only on $\lambda_0, \ldots, \lambda_i$. Therefore as long as we fix an $r+s$-th root of a_0, for each i, b_i depends only on $a_0, \ldots, a_i$. So to prove Theorem $1'$, we can assume $a(T) = \sum_{0\leq i\leq s} a_i T^i$.

Remark 2.3. Solving the first equation of (2.2), we get $T = \sum_{i\geq 0} \lambda_i Z'^{i+1}$ for some $\lambda_j \in k$. Then λ_0 is an $r+s$-th root of a_0. Then $\sum_{i\geq 0} b_i Z'^i = (\sum_{i\geq 0} \lambda_i Z'^i)^r$. Choose $a'_0, \ldots, a'_s \in k$ such that $a'_i = a_i$ for all $0 \leq i < s$ and $a'_s = 0$. For $a(T_1) = \sum_{0\leq i\leq s} a'_i T_1^i$, consider the system of equations (2.2) if the variable T is changed by T_1. Using the first equation, we can express T_1 as $\sum_{i\geq 0} \lambda'_i Z'^{i+1}$ with $\lambda'_0 = \lambda_0$. Then we have $\sum_{i\geq 0} b'_i Z'^i = (\sum_{i\geq 0} \lambda'_i Z'^i)^r$. Remark 2.1 shows $b'_s = 0$. Since $a_i = a'_i$ for $0 \leq i < s$, we have $\lambda_i = \lambda'_i$ for all $0 \leq i < s$. That is,

$$T \equiv T_1 \text{ mod. } Z'^{s+1} \text{ and } T \equiv T_1 \equiv \lambda_0 Z' \text{ mod. } Z'^2.$$

Comparing coefficients of Z'^s on both sides of

$$\sum_{i\geq 0} a_i T^i = \Big(\sum_{i\geq 0} \lambda_i Z'^i\Big)^{r+s} \text{ and } \sum_{0\leq i\leq s} a'_i T_1^i = \Big(\sum_{i\geq 0} \lambda'_i Z'^i\Big)^{r+s},$$

we have

$$a_s \lambda_0^s = (a_s - a'_s)\lambda_0^s = (r+s)(\lambda_s - \lambda'_s)\lambda_0^{r+s-1}.$$

Comparing coefficients of Z'^s on both sides of

$$\sum_{i\geq 0} b_i Z'^i = \Big(\sum_{i\geq 0} \lambda_i Z'^i\Big)^r \text{ and } \sum_{i\geq 0} b'_i Z'^i = \Big(\sum_{i\geq 0} \lambda'_i Z'^i\Big)^r,$$

we have

$$b_s = b_s - b'_s = r(\lambda_s - \lambda'_s)\lambda_0^{r-1}.$$

This proves $b_s = \frac{r}{r+s} a_s$.

Remark 2.4. Set $f = a_0T^{-s} + a_1T^{1-s} + \ldots + a_s$. Let

$$H = \{\sigma \in \mathrm{Gal}\big(k((T))/k((t))\big) | \sigma(f) = f\}.$$

We call f is irreducible with respect to the Galois extension $k((T))/k((t))$ if $\#H = 1$. Then f is irreducible if and only if the connection $[r]_*[-rf]$ is irreducible.

Lemma 2.5. *If Theorem* $1'$ *holds for irreducible* f*, then it holds for all* f*.*

Proof. By Remark 2.2, we can assume $a(T) = \sum_{0\leq i\leq s} a_iT^i$. Keep the notation in Remark 2.4. Set $p = \#H$. Then $p|r$. Let η be a primitive r-th root of unity in k. Then $a_i\eta^{\frac{r}{p}(i-s)} = a_i$ for all $0 \leq i \leq s$. So $a_i = 0$ or $p|i-s$. In particular, $p|s$ since $a_0 \neq 0$. Let $\tau = T^p$ and $\tau' = Z'^p$. Then

$$f = a_0\tau^{-\frac{s}{p}} + a_p\tau^{1-\frac{s}{p}} + \ldots + a_s$$

and it is irreducible with respect to the Galois extension $k((\tau))/k((t))$. For $a(\tau) = \sum_{0\leq i\leq \frac{s}{p}} a_{pi}\tau^i$, suppose $b(\tau') = \sum_{i\geq 0} b_{pi}\tau'^i$ is a solution of the following system of equation

$$\begin{cases} a(\tau) = (\frac{\tau}{\tau'})^{\frac{r+s}{p}} \\ b(\tau') = (\frac{\tau}{\tau'})^{\frac{r}{p}}. \end{cases} \tag{2.3}$$

Then $b_s = \frac{r}{r+s}a_s$ and $b(Z') = \sum_{i\geq 0} b_{pi}Z'^{pi}$ is a solution of the system of equations (2.2). For $a(\tau) = \sum_{0\leq i\leq \frac{s}{p}} a_{pi}\tau^i - \frac{j}{r}\tau^{\frac{s}{p}}$ $(1 \leq j \leq p)$, by Remark 2.2 and 2.3, we can find a solution $b(\tau')$ of the system of equations (2.3) such that

$$b(\tau') \equiv \sum_{0\leq i\leq \frac{s}{p}} b_{pi}\tau'^i - \frac{j}{r+s}\tau'^{\frac{s}{p}} \text{ mod. } \tau'^{\frac{s}{p}+1}.$$

Applying Theorem $1'$ to the system of equations (2.3) for $a(\tau) = \sum_{0\leq i\leq \frac{s}{p}} a_{pi}\tau^i - \frac{j}{r}\tau^{\frac{s}{p}}$ $(1 \leq j \leq p)$, we have

$$\begin{aligned}
& \mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big) \\
= \;& \mathcal{F}^{(0,\infty)}\Big([\frac{r}{p}]_*[p]_*[-r(a_0T^{-s} + a_pT^{p-s} + \ldots + a_s)]\Big) \\
= \;& \bigoplus_{1\leq j\leq p} \mathcal{F}^{(0,\infty)}\Big([\frac{r}{p}]_*[-\frac{r}{p}(a_0\tau^{-\frac{s}{p}} + a_p\tau^{\frac{p-s}{p}} + \ldots + a_s) + \frac{j}{p}]\Big) \\
= \;& \bigoplus_{1\leq j\leq p} [\frac{r+s}{p}]_*[-\frac{r+s}{p}(b_0\tau'^{-\frac{s}{p}} + b_p\tau'^{\frac{p-s}{p}} + \ldots + b_s) + \frac{j}{p} + \frac{s}{2p}] \\
= \;& [\frac{r+s}{p}]_*[p]_*[-(r+s)(b_0Z'^{-s} + b_pZ'^{p-s} + \ldots + b_s) + \frac{s}{2}] \\
= \;& [r+s]_*[-(r+s)(b_0Z'^{-s} + b_pZ'^{p-s} + \ldots + b_s) + \frac{s}{2}].
\end{aligned}$$

$\square$

From now on, we assume f is irreducible.

Let's describe the connection $\mathcal{F}^{(0,\infty)}\big([r]_*[-rf]\big)$ on $k((z'))$.

The formal connection $[-rf]$ on $k((T))$ consist of a one dimensional $k((T))$-vector space with a basis e and a k-linear map $T\partial_T : k((T))e \to k((T))e$ satisfying

$$T\partial_T(ge) = (T\partial_T(g) - rfg)e$$

for any $g \in k((T))$. Since the formal connection $[-rf]$ on $k((T))$ has slope s, we get $k[[T]]e,\ T^{-s}k[[T]]e$ is a good lattices pair for it. Identify $[r]_*[-rf]$ with $k((T))e$ as $k((t))$-vector spaces. Then the formal connection $[r]_*[-rf]$ has pure slope $\frac{s}{r}$ and $k[[T]]e,\ T^{-s}k[[T]]e$ is a good lattices pair for this connection. The action of the differential operator $t\partial_t$ on $k((T))e$ is given by

$$t\partial_t(ge) = (t\partial_t(g) - fg)e$$

for any $g \in k((T))$. So we have

$$\begin{aligned}
&(\partial_t \circ t)(T^{-i}e) = \frac{r-i}{r}T^{-i}e - (a_0T^{-(s+i)}e + \ldots + a_sT^{-i}e)\ (1 \le i \le r),\\
&t \cdot T^{-i}e = T^{-(i-r)}e\ (r+1 \le i \le r+s).
\end{aligned}$$

By [2], Proposition 3.7, the map

$$\iota : k((T))e \to \mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)$$

is an isomorphism of k-vector spaces. By [2], Lemma 2.4, $(\iota T^{-1}e, \ldots, \iota T^{-(r+s)}e)$ is a basis of $\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)$ over $k((z'))$. Then by the relation $\iota \circ t = -z'^2\partial_{z'} \circ \iota$ and $\iota \circ \partial_t = -\frac{1}{z'} \circ \iota$ in [2], Proposition 3.7, the matrix of the connection $\mathcal{F}^{(0,\infty)}\big([r]_*[-rf]\big)$ with respect to the differential operator $z'\partial_{z'}$ and the basis $(\iota T^{-1}e, \ldots, \iota T^{-(r+s)}e)$ is

$$-\begin{pmatrix}
\overbrace{\begin{matrix} a_s & & \\ a_{s-1} & \ddots & \\ \vdots & \ddots & a_s \\ a_0 & & a_{s-1} \\ & \ddots & \vdots \\ & & a_0 \end{matrix}}^{r} & \overbrace{\begin{matrix} \frac{1}{z'} & & \\ & \ddots & \\ & & \frac{1}{z'} \\ & & \\ & & \\ & & \end{matrix}}^{s}
\end{pmatrix} + \mathrm{diag}\{\frac{r-1}{r}, \ldots, \frac{1}{r}, 0, \ldots, 0\}.$$

Then the matrix of the connection

$$[r+s]^*\Big(\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)\Big)=k((Z'))\otimes_{k((z'))}\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)$$

with respect to the differential operator $Z'\partial_{Z'}$ and the basis $(Z'\otimes\iota T^{-1}e,\ldots,Z'^{r+s}\otimes\iota T^{-(r+s)}e)$ is

$$-\frac{r+s}{Z'^s}\begin{pmatrix} a_sZ'^s & & & 1 & \\ a_{s-1}Z'^{s-1} & \ddots & & & \ddots & \\ \vdots & \ddots & a_sZ'^s & & & 1\\ a_0 & \ddots & a_{s-1}Z'^{s-1} & & & \\ & \ddots & \vdots & & & \\ & & a_0 & & & \end{pmatrix}$$
$$+(r+s)\mathrm{diag}\{\frac{r-1}{r},\ldots,\frac{1}{r},0,\ldots,0\}+\mathrm{diag}\{1,\ldots,r+s\}.$$

We can write this matrix as $(r+s)B-(r+s)\sum_{0\leq i\leq s}Z'^{i-s}A_i$ for some matrices A_i and B with entries in k, where

$$\begin{aligned} A_0 &= \begin{pmatrix} 0 & I_s\\ a_0I_r & 0\end{pmatrix},\\ B &= \mathrm{diag}\{\frac{r-1}{r},\ldots,\frac{1}{r},0,\ldots,0\}+\frac{1}{r+s}\mathrm{diag}\{1,\ldots,r+s\}.\end{aligned}$$

Let V be the k-vector subspace of $[r+s]^*\Big(\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)\Big)$ generated by $Z'^i\otimes\iota T^{-i}e$ $(1\leq i\leq r+s)$. With respect to this basis, V can be identified with the k-vector space of column vectors in k of length $r+s$. The action of the differential operator $Z'\partial_{Z'}$ on elements of V can be written as

$$Z'\partial_{Z'}(v)=(r+s)B(v)-(r+s)\sum_{0\leq i\leq s}Z'^{i-s}A_i(v).$$

Lemma 2.6. *Suppose f is irreducible in the sense of Remark 2.4. Given $\alpha_0,\ldots,\alpha_s\in k$, the following three conditions are equivalent:*

(1) $\mathcal{F}^{(0,\infty)}([r]_*[-rf])=[r+s]_*[-(r+s)\sum_{0\leq i\leq s}\alpha_iZ'^{i-s}]$.

(2) $[-(r+s)\sum_{0\leq i\leq s}\alpha_iZ'^{i-s}]$ *is a subconnection of* $[r+s]^*\Big(\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)\Big)$.

(3) *There exist an integer N and $v_0,\ldots,v_s\in V$ such that $v_0\neq 0$ and*

$$\begin{cases}\sum_{0\leq i\leq k}(A_i-\alpha_i)v_{k-i}=0\ (0\leq k\leq s-1);\\ \sum_{0\leq i\leq s-1}(A_i-\alpha_i)v_{s-i}+(A_s-B-\alpha_s-\frac{N}{r+s})v_0=0.\end{cases}\tag{2.4}$$

Proof. Since f is irreducible, the connection $[r]_*[-rf]$ on $k((t))$ is irreducible with pure slope $\frac{s}{r}$. By [2], Proposition 3.14, the connection $\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)$ on $k((z'))$ is irreducible with pure slope

$\frac{s}{r+s}$. As in the proof of [2], Lemma 3.3, we have

$$\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big) = [r+s]_*[-(r+s)\sum_{0\leq i\leq s}\varrho_i Z'^{i-s}]$$

for some $\varrho_0, \ldots, \varrho_s \in k$ with $\varrho_0 \neq 0$. Let μ be a primitive $(r+s)$-th root of unity in k. Then

$$\begin{aligned} & [r+s]^*\Big(\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)\Big) \\ = & \bigoplus_{1\leq j\leq r+s}[-(r+s)(\mu^{-js}\varrho_0 Z'^{-s} + \mu^{j(1-s)}\varrho_1 Z'^{1-s} + \ldots + \varrho_s)]. \end{aligned}$$

So there are $r+s$ one dimensional subconnections of $[r+s]^*\Big(\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)\Big)$ which are not isomorphic to each other.

(1) $\Rightarrow$ (2) is trivial. For (2) $\Rightarrow$ (1), assume that $[-(r+s)\sum_{0\leq i\leq s}\alpha_i Z'^{i-s}]$ is a subconnection of $[r+s]^*\Big(\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)\Big)$. Then

$$[-(r+s)\sum_{0\leq i\leq s}\alpha_i Z'^{i-s}] = [-(r+s)\sum_{0\leq i\leq s}\mu^{j(i-s)}\varrho_i Z'^{i-s}]$$

for some $1 \leq j \leq r+s$. Then

$$\begin{aligned} \mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big) &= [r+s]_*[-(r+s)\sum_{0\leq i\leq s}\mu^{j(i-s)}\varrho_i Z'^{i-s}] \\ &= [r+s]_*[-(r+s)\sum_{0\leq i\leq s}\alpha_i Z'^{i-s}]. \end{aligned}$$

For (2) $\Rightarrow$ (3), assume that $[-(r+s)\sum_{0\leq i\leq s}\alpha_i Z'^{i-s}]$ is a subconnection of $[r+s]^*\Big(\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)\Big)$. This means that there is a nonzero map of connections

$$\phi : [-(r+s)\sum_{0\leq i\leq s}\alpha_i Z'^{i-s}] \to [r+s]^*\Big(\mathcal{F}^{(0,\infty)}\Big([r]_*[-rf]\Big)\Big).$$

The connection $[-(r+s)\sum_{0\leq i\leq s}\alpha_i Z'^{i-s}]$ consist of a one dimensional $k((Z'))$-vector space with a basis ε and a k-linear map $Z'\partial_{Z'} : k((Z'))\varepsilon \to k((Z'))\varepsilon$ satisfying

$$Z'\partial_{Z'}(g\varepsilon) = \Big(Z'\partial_{Z'}(g) - (r+s)g\sum_{0\leq i\leq s}\alpha_i Z'^{i-s}\Big)\varepsilon$$

for any $g \in k((Z'))$. Suppose $\phi(\varepsilon) = \sum_{0\leq i} Z'^{i+N} v_i$ for some integer N and some $v_i \in V$ with

$v_0 \neq 0$. Then

$$\begin{aligned} & -(r+s)\sum_{0\le i\le s}\alpha_i Z'^{i-s}\sum_{0\le i} Z'^{i+N}v_i = \phi\big(Z'\partial_{Z'}(\varepsilon)\big) \\ = \; & Z'\partial_{Z'}\big(\phi(\varepsilon)\big) = Z'\partial_{Z'}\big(\sum_{0\le i} Z'^{i+N}v_i\big) \\ = \; & \sum_{0\le i} Z'^{i+N}\big((r+s)B+i+N\big)v_i - (r+s)\sum_{0\le i} Z'^{i+N}\sum_{0\le j\le s} Z'^{j-s}A_j(v_i). \end{aligned}$$

Comparing coefficients of Z'^i, for $N-s\le i\le N$ on each side, we get the system of equations (2.4). This proves (2) $\Rightarrow$ (3). So for $\alpha_0=\mu^{-sj}\varrho_0, \alpha_1=\mu^{(1-s)j}\varrho_1,\ldots,\alpha_s=\varrho_s$, the system of equations (2.4) holds for some $N\in\mathbb{Z}$ and some $v_0,\ldots,v_s\in V$ with $v_0\neq 0$. These $(s+1)$-tuples $(\mu^{-sj}\varrho_0,\mu^{(1-s)j}\varrho_1,\ldots,\varrho_s)$ $(1\le j\le r+s)$ are pairwise distinct, since f is irreducible. Lemma 2.7 shows that there are at most $r+s$ $(s+1)$-tuples $(\alpha_0,\ldots,\alpha_s)$ such that the system of equations (2.4) holds for $N=0$ and some $v_0,\ldots,v_s\in V$ with $v_0\neq 0$. This proves (3) $\Rightarrow$ (2). □

Hensel's lemma. *Let E be a finite dimensional k-vector space. Suppose D is a $k[[t]]$-linear endomorphism of $E\otimes_k k[[t]]$. Write the action of D on elements of E:*

$$D(v)=\sum_{i\ge 0} t^i D_i(v), \text{ for unique elements } D_i\in \operatorname{End}_k(E).$$

Suppose the characteristic polynomial of D_0 has a simple root α_0 in k. Then

(1) *The equation*

$$(D-\alpha)(u)=0$$

has a solution $\alpha\in k[[t]]$ with constant term α_0 and $0\neq u\in E\otimes_k k[[t]]$. In this case, α is uniquely determined by α_0.

(2) *Let k be a positive integer. The following systems of equations*

$$\sum_{0\le i\le j}(D_i-\alpha_i)u_{j-i}=0 \;(0\le j\le k)$$

has a solution $\alpha_1,\ldots,\alpha_k\in k$; $u_0,\ldots,u_k\in E$ with $u_0\neq 0$. In this case, $\alpha_1,\ldots,\alpha_k$ are uniquely determined by α_0.

Proof. The proof is similar to that of [9], Proposition 7, p. 34. □

Lemma 2.7. *Given $\alpha_0,\ldots,\alpha_s\in k$, there exist $v_0,\ldots,v_s\in V$ such that $v_0\neq 0$ and*

$$\begin{cases} \sum_{0\le i\le k}(A_i-\alpha_i)v_{k-i}=0 \;(0\le k\le s-1), \\ \sum_{0\le i\le s-1}(A_i-\alpha_i)v_{s-i}+(A_s-B-\alpha_s)v_0=0 \end{cases} \tag{2.5}$$

if and only if there exist $v'_0,\dots,v'_s\in V$ such that $v'_0\neq 0$ and

$$\left\{\begin{array}{l}\sum_{0\le i\le k}(A_i-\alpha_i)v'_{k-i}=0\ (0\le k\le s-1),\\ \sum_{0\le i\le s-1}(A_i-\alpha_i)v'_{s-i}+(A_s-\frac{2r+s}{2r+2s}-\alpha_s)v'_0=0.\end{array}\right. \tag{2.6}$$

Moreover, there are at most $r+s$ $(s+1)$-tuples $(\alpha_0,\dots,\alpha_s)$ in k such that the system of equations (2.5) (resp. (2.6)) holds for some $v_0,\dots,v_s\in V$ with $v_0\neq 0$ (resp. $v'_0,\dots,v'_s\in V$ with $v'_0\neq 0$).

Proof. Let μ be a primitive $(r+s)$-th root of unity in k. We fix an $(r+s)$-th root $a_0^{\frac{1}{r+s}}$ of a_0. For any $1\le j\le r+s$, set e_j to be the column vector $(\mu^j a_0^{\frac{1}{r+s}},\dots,\mu^{j(r+s-1)}a_0^{\frac{r+s-1}{r+s}},a_0)$ and ε_j the row vector $(\mu^{-j}a_0^{-\frac{1}{r+s}},\dots,\mu^{-j(r+s-1)}a_0^{-\frac{r+s-1}{r+s}},a_0^{-1})$. Then

$$A_0\cdot e_j=\mu^{rj}a_0^{\frac{r}{r+s}}\cdot e_j,\ \varepsilon_j\cdot A_0=\mu^{rj}a_0^{\frac{r}{r+s}}\cdot\varepsilon_j,\ \varepsilon_i\cdot e_j=(r+s)\delta_{ij}.$$

Set $d=(r,s)$. We get $\ker(A_0-\mu^{rj}a_0^{\frac{r}{r+s}})$ is generated by those e_k with $r+s|(k-j)d$, and $\mathrm{im}(A_0-\mu^{rj}a_0^{\frac{r}{r+s}})$ is generated by the other e_k's. Then

$$\mathrm{im}(A_0-\mu^{rj}a_0^{\frac{r}{r+s}})=\{v\in V|\varepsilon_k\cdot v=0 \text{ for all } k \text{ satisfying } r+s|(k-j)d\}.$$

For the only if part, suppose the system of equations (2.5) holds for some $v_0,\dots,v_s\in V$ with $v_0\neq 0$. In particular, $(A_0-\alpha_0)v_0=0$. Then $\alpha_0=\mu^{rj}a_0^{\frac{r}{r+s}}$ for some integer j and then $v_0=\sum_{r+s|(i-j)d}\gamma_i e_i$ for some $\gamma_i\in k$. For any $1\le k,\ l\le r+s$, we have

$$\begin{aligned}&\varepsilon_k\cdot(B-\frac{2r+s}{2r+2s})e_l\\ =&\sum_{1\le i\le r}\frac{r-i}{r}\mu^{i(l-k)}+\sum_{1\le i\le r+s}\frac{i}{r+s}\mu^{i(l-k)}-\frac{2r+s}{2r+2s}\sum_{1\le i\le r+s}\mu^{i(l-k)}.\end{aligned}$$

If $k=l$,

$$\varepsilon_k\cdot(B-\frac{2r+s}{2r+2s})e_l=\sum_{1\le i\le r}\frac{r-i}{r}+\sum_{1\le i\le r+s}\frac{i}{r+s}-\frac{2r+s}{2r+2s}\sum_{1\le i\le r+s}1=0.$$

Suppose $k\neq l$ and $r+s|(l-k)d$. Let $\xi=\mu^{l-k}$. Then $\xi^d=1$ and $\xi\neq 1$. For any $d|n$, we have $\sum_{1\le i\le n}\xi^i=0$ and hence $\sum_{1\le i\le n}i\xi^i=\frac{n}{d}\sum_{1\le i\le d}i\xi^i$. So we have

$$\begin{aligned}\varepsilon_k\cdot(B-\frac{2r+s}{2r+2s})e_l&=-\frac{1}{r}\sum_{1\le i\le r}i\xi^i+\frac{1}{r+s}\sum_{1\le i\le r+s}i\xi^i\\ &=-\frac{1}{r}\frac{r}{d}\sum_{1\le i\le d}i\xi^i+\frac{1}{r+s}\frac{r+s}{d}\sum_{1\le i\le d}i\xi^i=0.\end{aligned}$$

So $\varepsilon_k\cdot(B-\frac{2r+s}{2r+2s})v_0=0$ if $r+s|(k-j)d$. Therefore $(B-\frac{2r+s}{2r+2s})v_0=(A_0-\alpha_0)v$ for some $v\in V$. Then $v'_0=v_0,\dots,v'_{s-1}=v_{s-1},v'_s=v_s-v$ satisfy the system of equations (2.6). Reversing the above argument, we get the if part. So for the last assertion, it suffices to show that the same assertion holds for the following system of equations

$$\sum_{0\le i\le k}(A_i-\alpha_i)v_{k-i}=0 \text{ for any } 0\le k\le s. \tag{2.7}$$

Suppose the system of equations (2.7) holds for some $\alpha_0,\ldots,\alpha_s\in k$ and some $v_0,\ldots,v_s\in V$ with $v_0\ne 0$. There exists an integer $1\le j\le r+s$ such that $\alpha_0=\mu^{rj}a_0^{\frac{r}{r+s}}$ and $v_0=\sum_{r+s|(i-j)d}\gamma_i e_i$ for some $\gamma_i\in k$ with $\gamma_j\ne 0$. The system of equations (2.7) is equivalent to the following equation

$$\Big(\sum_{0\le i\le s}A_iZ'^i-\sum_{0\le i\le s}\alpha_iZ'^i\Big)\Big(\sum_{0\le i\le s}v_iZ'^i\Big)\equiv 0 \text{ mod. } Z'^{s+1}.$$

There exist $\rho_0=\mu^j a_0^{\frac{1}{r+s}},\rho_1,\ldots,\rho_s\in k$ such that

$$\sum_{0\le i\le s}\alpha_iZ'^i\equiv\Big(\sum_{0\le i\le s}\rho_iZ'^i\Big)^r \text{ mod. } Z'^{s+1}.$$

Let

$$\Gamma=\begin{pmatrix} 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & & & \ddots & \\ a_sZ'^s & & & & \ddots \\ \vdots & & & & 1 \\ a_0 & & & & 0\end{pmatrix} \text{ and } \Gamma_0=\begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ a_0 & & & 0\end{pmatrix}.$$

Then $\sum_{0\le i\le s}A_iZ'^i=\Gamma^r$, $A_0=\Gamma_0^r$ and hence

$$\Big(\Gamma-\sum_{0\le i\le s}\rho_iZ'^i\Big)\Big(\sum_{0\le k\le r-1}\Big(\sum_{0\le i\le s}\rho_iZ'^i\Big)^k\Gamma^{r-1-k}\Big)\Big(\sum_{0\le i\le s}v_iZ'^i\Big)\equiv 0 \text{ mod. } Z'^{s+1}.$$

Write

$$\Big(\sum_{0\le k\le r-1}\Big(\sum_{0\le i\le s}\rho_iZ'^i\Big)^k\Gamma^{r-1-k}\Big)\Big(\sum_{0\le i\le s}v_iZ'^i\Big)=\sum_{0\le i}u_iZ'^i$$

for some $u_i\in V$. Then

$$\begin{aligned} u_0 &= \sum_{0\le k\le r-1}\rho_0^k\Gamma_0^{r-1-k}\sum_{r+s|(i-j)d}\gamma_ie_i \\ &= \sum_{r+s|(i-j)d}\gamma_i\cdot\sum_{0\le k\le r-1}\mu^{jk}a_0^{\frac{k}{r+s}}\mu^{i(r-1-k)}a_0^{\frac{r-1-k}{r+s}}e_i \\ &= r\mu^{j(r-1)}\gamma_ja_0^{\frac{r-1}{r+s}}e_j\ne 0.\end{aligned}$$

and

$$\Big(\Gamma-\sum_{0\leq i\leq s}\rho_i Z'^i\Big)\Big(\sum_{0\leq i\leq s}u_i Z'^i\Big)\equiv 0 \text{ mod. } Z'^{s+1}. \tag{2.8}$$

Since ρ_0 is a simple root of the characteristic polynomial of Γ_0, by Hensel's lemma, $\rho_1,\ldots,\rho_s$ are uniquely determined by ρ_0. So $\alpha_0,\ldots,\alpha_s$ are uniquely determined by $\rho_0=\mu^j a_0^{\frac{1}{r+s}}$ $(1\leq j\leq r+s)$. This proves the last assertion. □

Now we are ready to prove Theorem 1′. By Remark 2.2, we assume that $a(T)=\sum_{0\leq i\leq s}a_iT^i$. Then the first equation of (2.2) means that $\frac{T}{Z'}$ is a root in $k[[Z']]$ of the polynomial

$$\lambda^{r+s}-\sum_{0\leq i\leq s}a_iZ'^i\lambda^i\in k[[Z']][\lambda].$$

This polynomial is exactly the characteristic polynomial of Γ. The characteristic polynomial of Γ_0 is the polynomial $\lambda^{r+s}-a_0$ which has no multiple roots, then by Hensel's lemma, Γ has an eigenvector $\sum_{i\geq 0}Z'^iv_i$ corresponding this eigenvalue $\frac{T}{Z'}$ with $v_0\neq 0$. Since $\sum_{0\leq i\leq s}Z'^iA_i=\Gamma^r$, we have

$$\Big(\sum_{0\leq i\leq s}Z'^iA_i\Big)\Big(\sum_{0\leq i}Z'^iv_i\Big)=\Big(\frac{T}{Z'}\Big)^r\Big(\sum_{0\leq i}Z'^iv_i\Big)=\Big(\sum_{0\leq i}b_iZ'^i\Big)\Big(\sum_{0\leq i}Z'^iv_i\Big).$$

So

$$\sum_{0\leq i\leq k}(A_i-b_i)v_{k-i}=0 \text{ for any } 0\leq k\leq s.$$

Recall that $\sum_{0\leq i\leq s}a_iT^{i-s}$ is assumed to be irreducible. Then by Lemma 2.6 and 2.7, we have

$$\begin{aligned}&\mathcal{F}^{(0,\infty)}\Big([r]_*[-r(a_0T^{-s}+a_1T^{1-s}+\ldots+a_s)]\Big)\\ =\;&[r+s]_*[-(r+s)(b_0Z'^{-s}+b_1Z'^{1-s}+\ldots+b_s-\frac{2r+s}{2r+2s})]\\ =\;&[r+s]_*[-(r+s)(b_0Z'^{-s}+b_1Z'^{1-s}+\ldots+b_s)+\frac{s}{2}].\end{aligned}$$

Suppose $r>s$. Given a formal Laurent series α in the variable $\frac{1}{\sqrt[r]{t}}$ of order $-s$, consider the system of equations (1.2). We express $\frac{1}{\sqrt[r]{t}}$ as a formal power series in $\sqrt[r-s]{t'}$ of order 1 using the first equation, and then substitute this expression into the second equation to get $\beta\in k((\sqrt[r-s]{t'}))$. Similar to equation (2.1), we have $\partial_{t'}(\beta)=t$. It follows that β is a formal Laurent series in $\sqrt[r-s]{t'}$ of order $-s$. Let $Z=\frac{1}{\sqrt[r]{t}}$ and let $T'=\sqrt[r-s]{t'}$. Set

$$a(Z)=Z^st\partial_t(\alpha) \text{ and } b(T')=-T'^st'\partial_{t'}\beta.$$

Then $a(Z)$ is a formal power series in Z of order 0 and $b(T')$ is a formal power series in T' of order 0. From the system of equations (1.2), we get

$$\begin{cases} a(Z) = -(\frac{T'}{Z})^{r-s} \\ b(T') = -(\frac{T'}{Z})^{r}. \end{cases} \tag{2.9}$$

Similar to Theorem 1 and 1′, to prove Theorem 2, it suffices to show the following theorem.

Theorem 2′. Suppose $r > s$. Given a formal power series $a(Z) = \sum_{i\geq 0} a_i Z^i$ with $a_i \in k$ and $a_0 \neq 0$, suppose $b(T') = \sum_{i\geq 0} b_i T'^i$ with $b_i \in k$ is a solution of the system of equations (2.9). We have $b_s = \frac{r}{r-s} a_s$ and

$$\begin{aligned} & \mathcal{F}^{(\infty,0)}\Big([r]_*[-r(a_0 Z^{-s} + a_1 Z^{1-s} + \ldots + a_s)]\Big) \\ = \quad & [r-s]_*[-(r-s)(b_0 T'^{-s} + b_1 T'^{1-s} + \ldots + b_s) + \frac{s}{2}]. \end{aligned}$$

Proof. The proof of $b_s = \frac{r}{r-s} a_s$ is similar to that of Theorem 1′. Using the first equation of (2.9), we can express Z as a formal power series in the variable T' of order 1. We then substitute this expression into the second equation to get $b(T')$ is a formal power series in T' with nonzero constant term. That is, $b_0 \neq 0$. Let ζ be an r-th root of -1 in k and let $Z = \zeta \cdot Z_1$. Let $[-] : k((z)) \to k((z))$ be the automorphism of k-algebra defined by $z \mapsto -z$. From the system of equations (2.9), we get

$$\begin{cases} \sum_{i\geq 0} b_i T'^i = (\frac{T'}{Z_1})^{r} \\ \sum_{i\geq 0} \zeta^{i-s} a_i Z_1^i = (\frac{T'}{Z_1})^{r-s}. \end{cases}$$

Since $b_0 \neq 0$, by Theorem 1′, we have

$$\begin{aligned} & \mathcal{F}^{(0,\infty)}\Big([r-s]_*[-(r-s)(b_0 T'^{-s} + b_1 T'^{1-s} + \ldots + b_s) + \frac{s}{2}]\Big) \\ = \quad & [r]_*[-r(\zeta^{-s} a_0 Z^{-s} + \zeta^{1-s} a_1 Z^{1-s} + \ldots + a_s) + \frac{s}{2} + \frac{s}{2}] \\ = \quad & [-]^*[r]_*[-r(a_0 Z^{-s} + a_1 Z^{1-s} + \ldots + a_s)] \\ = \quad & \mathcal{F}^{(0,\infty)}\Big(\mathcal{F}^{(\infty,0)}\Big([r]_*[-r(a_0 Z^{-s} + a_1 Z^{1-s} + \ldots + a_s)]\Big)\Big). \end{aligned}$$

The theorem holds by [2], Proposition 3.10. □

3 Proof of Theorem 3

Suppose $r < s$. Given a formal Laurent series α in the variable $\frac{1}{\sqrt[r]{t}}$ of order $-s$, consider the system of equations (1.3). We express $\frac{1}{\sqrt[r]{t}}$ as a formal Laurent series in $\frac{1}{\sqrt[s-r]{t'}}$ of order 1 using the first equation and then substitute this expression into the second equation to get $\beta \in k((\frac{1}{\sqrt[s-r]{t'}}))$. Similar to equation (2.1), we have $\partial_{t'}(\beta) = t$. It follows that β is a formal Laurent series in $\frac{1}{\sqrt[s-r]{t'}}$ of order $-s$. Let $Z = \frac{1}{\sqrt[r]{t}}$ and $Z' = \frac{1}{\sqrt[s-r]{t'}}$. Set

$$a(Z) = Z^s t\partial_t(\alpha) \text{ and } b(Z') = Z'^s t'\partial_{t'}(\beta).$$

Then $a(Z)$ is a formal power series in Z of order 0 and $b(Z')$ is a formal power series in Z' of order 0. From the system of equations (1.3), we get

$$\begin{cases} a(Z) = -(\frac{Z}{Z'})^{s-r} \\ b(Z') = (\frac{Z'}{Z})^r. \end{cases} \tag{3.1}$$

Similar to Theorem 1 and 1′, to prove Theorem 3, it suffices to show the following theorem.

Theorem 3′. Suppose $s > r$. Given a formal power series $a(Z) = \sum_{i\geq 0} a_i Z^i$ with $a_i \in k$ and $a_0 \neq 0$, solve the system of equations (3.1) to get $b(Z') = \sum_{i\geq 0} b_i Z'^i$ for some $b_i \in k$. Then $b_s = \frac{r}{s-r} a_s$ and

$$\begin{aligned} &\mathcal{F}^{(\infty,\infty)}\Big([r]_*[-r(a_0Z^{-s} + a_1Z^{1-s} + \ldots + a_s)]\Big) \\ = \quad & [s-r]_*[-(s-r)(b_0Z'^{-s} + b_1Z'^{1-s} + \ldots + b_s) + \frac{s}{2}]. \end{aligned}$$

Lemma 3.1. *Set $h = a_0Z^{-s} + a_1Z^{1-s} + \ldots + a_s$. We can reduce Theorem 3′ to the case where $s \geq 2r$ and where h is irreducible with respect to the Galois extension $k((Z))/k((z))$.*

Proof. The proof of $b_s = \frac{r}{s-r} a_s$ is similar to that of Theorem 1′ and the proof of the last assertion is similar to that of Lemma 2.5. If $s < 2r$, then $s > 2(s-r)$. Let ζ be an r-th root of -1 in k and let $Z = \zeta \cdot Z_1$. From the system of equations (3.1), we get

$$\begin{cases} \sum_{i\geq 0} b_i Z'^i = -(\frac{Z'}{Z_1})^r \\ \sum_{i\geq 0} \zeta^{i-s} a_i Z_1^i = (\frac{Z_1}{Z'})^{s-r}. \end{cases}$$

We prove $b_0 \neq 0$ similarly as in Theorem 2′. Applying this theorem to $[s-r]_*[-(s-r)(b_0Z'^{-s} + \ldots + b_s) + \frac{s}{2}]$, we have

$$\begin{aligned} &\mathcal{F}^{(\infty,\infty)}\Big([s-r]_*[-(s-r)(b_0Z'^{-s} + b_1Z'^{1-s} + \ldots + b_s) + \frac{s}{2}]\Big) \\ = \quad & [r]_*[-r(\zeta^{-s}a_0Z^{-s} + \zeta^{1-s}a_1Z^{1-s} + \ldots + a_s) + \frac{s}{2} + \frac{s}{2}] \\ = \quad & [-]^*[r]_*[-r(a_0Z^{-s} + a_1Z^{1-s} + \ldots + a_s)] \\ = \quad & \mathcal{F}^{(\infty,\infty)}\Big(\mathcal{F}^{(\infty,\infty)}\big([r]_*[-rh]\big)\Big). \end{aligned}$$

The lemma holds by [2], Proposition 3.12 (iv). □

From now on, we assume h is irreducible.

Let's describe the formal connection $\mathcal{F}^{(\infty,\infty)}\big([r]_*[-rh]\big)$ on $k((z'))$.

The formal connection $[-rh]$ on $k((Z))$ consist of a one dimensional $k((Z))$-vector space with a basis e' and a k-linear map $Z\partial_Z : k((Z))e' \to k((Z))e'$ satisfying

$$Z\partial_Z(ge') = (Z\partial_Z(g) - rhg)e'$$

for any $g \in k((Z))$. Since the formal connection $[-rh]$ on $k((Z))$ has slope s, we get $k[[Z]]e',\ Z^{-s}k[[Z]]e'$ is a good lattices pair for it. Identify $[r]_*[-rh]$ with $k((Z))e'$ as $k((z))$-vector spaces. So the connection $[r]_*[-rh]$ on $k((z))$ has pure slope $\frac{s}{r}$ and $k[[Z]]e',\ Z^{-s}k[[Z]]e'$ is a good lattices pair for this connection. The action of the differential operator $z\partial_z$ on $k((Z))e'$ is given by

$$z\partial_z(ge') = (z\partial_z(g) - hg)e'$$

for any $g \in k((Z))$. Then for any $i \in \mathbb{Z}$, we have

$$z^2\partial_z(Z^{-(r+i)}e') = -\frac{r+i}{r}Z^{-i}e' - (a_0Z^{-(i+s)}e' + \ldots + a_sZ^{-i}e').$$

By [2], Proposition 3.12 (ii), the map

$$\iota : k((Z))e' \to \mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big)$$

is an isomorphism of k-vector spaces. As in [2], Proposition 3.14, $(\iota Z^{-1}e', \ldots, \iota Z^{-(s-r)}e')$ is a basis of $\mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big)$ over $k((Z'))$. By the relation $\iota \circ z^2\partial_z = \frac{1}{z'} \circ \iota$ and $-\iota \circ \frac{1}{z} = z'^2\partial_{z'} \circ \iota$ in [2], Proposition 3.12 (iii), we have

$$\begin{aligned} z'^2\partial_{z'}(\iota Z^{-(i+s-r)}e') &= -\iota Z^{-(i+s)}e' \\ &= \frac{a_s}{a_0}\iota Z^{-i}e' + \ldots + \frac{a_1}{a_0}\iota Z^{-(i+s-1)}e' + \frac{1}{a_0z'}\iota Z^{-(r+i)}e' + \frac{r+i}{ra_0}\iota Z^{-i}e'. \end{aligned}$$

Let

$$A = \begin{pmatrix} 0 & & & & & & -\frac{a_s}{a_0} \\ 1 & & & & & & \vdots \\ & \ddots & & & & & -\frac{a_{s-r+1}}{a_0} \\ & & \ddots & & & & -\frac{a_{s-r}}{a_0} - \frac{1}{a_0z'} \\ & & & \ddots & & & -\frac{a_{s-r-1}}{a_0} \\ & & & & \ddots & & \vdots \\ & & & & & 1 & -\frac{a_1}{a_0} \end{pmatrix}.$$

For any $i\in\mathbb{Z}$, let B_i be the $s\times s$-matrix whose entries are all zero except the $(1,s)$-th entry which is valued by $-\frac{r+i}{ra_0}$. We have

$$(\iota Z^{-(i+1)}e',\ldots,\iota Z^{-(i+s)}e')=(\iota Z^{-i}e',\ldots,\iota Z^{-(i+s-1)}e')(A+B_i).$$

So

$$\begin{aligned} z'^2\partial_{z'}(\iota Z^{-1}e',\ldots,\iota Z^{-s}e') &= -(\iota Z^{-(r+1)}e',\ldots,\iota Z^{-(r+s)}e')\\ &= -(\iota Z^{-1}e',\ldots,\iota Z^{-s}e')\prod_{1\leq i\leq r}(A+B_i). \end{aligned}$$

Consider the connection

$$[s-r]^*\Big(\mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big)\Big)=k((Z'))\otimes_{k((z'))}\mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big).$$

Set $\wedge=\mathrm{diag}\{Z',\ldots,Z'^s\}$ and $\varepsilon'=(Z'\otimes\iota Z^{-1}e',\ldots,Z'^s\otimes\iota Z^{-s}e')$. We have

$$\begin{aligned} Z'\partial_{Z'}(\varepsilon') &= \varepsilon'\cdot\Big(\mathrm{diag}\{1,\ldots,s\}-\frac{s-r}{z'}\wedge^{-1}\Big(\prod_{1\leq i\leq r}(A+B_i)\Big)\wedge\Big)\\ &= \varepsilon'\cdot\Big(\mathrm{diag}\{1,\ldots,s\}-\frac{s-r}{Z'^s}\prod_{1\leq i\leq r}\Big(Z'\wedge^{-1}(A+B_i)\wedge\Big)\Big). \end{aligned}$$

We have

$$Z'\wedge^{-1}A\wedge=\begin{pmatrix} 0 & & & & & -\frac{a_s}{a_0}Z'^s\\ 1 & & & & & \vdots\\ & \ddots & & & & -\frac{a_{s-r+1}}{a_0}Z'^{s-r+1}\\ & & \ddots & & & -\frac{a_{s-r}}{a_0}Z'^{s-r}-\frac{1}{a_0}\\ & & & \ddots & & -\frac{a_{s-r-1}}{a_0}Z'^{s-r-1}\\ & & & & \ddots & \vdots\\ & & & & 1 & -\frac{a_1}{a_0}Z' \end{pmatrix}$$

and $Z'\wedge^{-1}B_i\wedge$ is the $s\times s$-matrix whose entries are all zero except the $(1,s)$-th entry which is valued by $-\frac{r+i}{ra_0}Z'^s$. So we can write

$$\mathrm{diag}\{1,2,\ldots,s\}-\frac{s-r}{Z'^s}\prod_{1\leq i\leq r}\Big(Z'\wedge^{-1}(A+B_i)\wedge\Big)=-(s-r)\sum_{i\geq 0}Z'^{i-s}C_i$$

and

$$\big(Z'\wedge^{-1}A\wedge\big)^r=\sum_{i\geq 0}Z'^iC'_i$$

for some matrices C_i and C'_i with entries in k. Then $C_i=C'_i$ for all $0\leq i\leq s-1$ and

$$C'_s-C_s=\mathrm{diag}\{\frac{1}{s-r},\ldots,\frac{s}{s-r}\}-P$$

where P is the $s \times s$-matrix whose entries are all zero except the $(i, i+s-r)$-th entry which is valued by $-\frac{r+i}{ra_0}$ $(1 \leq i \leq r)$. Let W be the k-vector space of column vectors in k of length s. We have

Lemma 3.2. *Suppose $s \geq 2r$ and h is irreducible with respect to the Galois extension $k((Z))/k((z))$. Given $\alpha_0, \ldots, \alpha_s \in k$ with $\alpha_0 \neq 0$, the following three conditions are equivalent:*

(1) $\mathcal{F}^{(\infty,\infty)}\big([r]_*[-rh]\big) = [s-r]_*[-(s-r)\sum_{0\leq i\leq s} \alpha_i Z'^{i-s}]$.

(2) $[-(s-r)\sum_{0\leq i\leq s} \alpha_i Z'^{i-s}]$ *is a subconnection of* $[s-r]^*\mathcal{F}^{(\infty,\infty)}\big([r]_*[-rh]\big)$.

(3) *There exist $N \in \mathbb{Z}$ and $w_0, \ldots, w_s \in W$ such that $w_0 \neq 0$ and*

$$\begin{cases} \sum_{0\leq i\leq k}(C_i - \alpha_i)w_{k-i} = 0 \ (0 \leq k \leq s-1), \\ \sum_{0\leq i\leq s-1}(C_i - \alpha_i)w_{s-i} + (C_s - \alpha_s - \frac{N}{s-r})w_0 = 0. \end{cases} \tag{3.2}$$

Proof. Set $U = W \otimes_k k((Z'))$ and $\mathcal{W} = W \otimes_k k[[Z']]$. Let $u = (u_1, \ldots, u_s)$ be the canonical basis of W. There exists a unique connection $(U, Z'\partial_{Z'})$ such that the action of $Z'\partial_{Z'}$ on elements of W can be written as

$$Z'\partial_{Z'}(w) = -(s-r)\sum_{i\geq 0} Z'^{i-s} C_i(w).$$

The map of $k((Z'))$-vector spaces

$$U \to [s-r]^*\Big(\mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big)\Big)$$

which maps each u_i to $Z'^i \otimes \iota Z^{-i} e'$ is a surjective morphism of connections. We have $Z'^{s+1}\partial_{Z'}(\mathcal{W}) \subset \mathcal{W}$. Let $\psi : \mathcal{W} \to \mathcal{W}/Z'\mathcal{W} \cong W$ be the canonical map. The k-linear action on $W \cong \mathcal{W}/Z'\mathcal{W}$ induced by $Z'^{s+1}\partial_{Z'}$ is $-(s-r)C_0$. Write

$$Z' \wedge^{-1} A \wedge = \sum_{i\geq 0} Z'^i D_i$$

for some matrices D_i with entries in k. The characteristic polynomial of D_0 is $\lambda^s + \frac{1}{a_0}\lambda^r$. So W is the direct sum of two subspaces W_0 and W_1, invariant under D_0, and such that $D_0|_{W_0}$ is nilpotent, $D_0|_{W_1}$ is invertible. Then $\dim W_0 = r$ and $\dim W_1 = s - r$. Since $C_0 = D_0^r$, we have W_0 and W_1 are C_0-invariant, and then $C_0|_{W_0} = 0$, $C_0|_{W_1}$ is invertible. By the splitting lemma in [7], 2, $\mathcal{W}$ is the direct sum of two free submodules $\mathcal{W}_0$ and $\mathcal{W}_1$, invariant under $Z'^{s+1}\partial_{Z'}$, and such that $W_0 = \psi(\mathcal{W}_0)$, $W_1 = \psi(\mathcal{W}_1)$. Let U_0, U_1 be the subconnections of U generated by $\mathcal{W}_0$, $\mathcal{W}_1$, respectively. Then $U = U_0 \oplus U_1$. The induced action of $Z'^{s+1}\partial_{Z'}$ on W_0 is 0, so the slopes of the connection U_0 are all $< s$. But $[s-r]^*\Big(\mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big)\Big)$ is an $s-r$ dimensional connection on $k((Z'))$ with pure slope s, we have

$$\mathrm{Hom}_{\mathrm{conn.}}\Big(U_0, [s-r]^*\Big(\mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big)\Big)\Big) = (0)$$

and then

$$U_1 \cong [s-r]^*\Big(\mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big)\Big).$$

For any one dimensional formal connection L on $k((Z'))$ with slope s, we have

$$\mathrm{Hom}_{\mathrm{conn.}}(L, U_0) = (0)$$

and then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{conn.}}(L, U) &= \mathrm{Hom}_{\mathrm{conn.}}(L, U_1) \bigoplus \mathrm{Hom}_{\mathrm{conn.}}(L, U_0) \\ &= \mathrm{Hom}_{\mathrm{conn.}}\Big(L, [s-r]^*\Big(\mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big)\Big)\Big).\end{aligned}$$

So to find a one dimensional subconnection in $[s-r]^*\Big(\mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big)\Big)$ is equivalent to finding a one dimensional subconnection in U of slope s. By Lemma 3.3, the remainder proof is similar to that of Lemma 2.6. ☐

Lemma 3.3. *Suppose $s \geq 2r$. Given $\alpha_0, \ldots, \alpha_s \in k$ with $\alpha_0 \neq 0$, there exist $w_0, \ldots, w_s \in W$ such that $w_0 \neq 0$ and*

$$\sum_{0 \leq i \leq k} (C_i - \alpha_i) w_{k-i} = 0 \ (0 \leq k \leq s) \tag{3.3}$$

if and only if there exist $w'_0, \ldots, w'_s \in W$ such that $w'_0 \neq 0$ and

$$\begin{cases} \sum_{0 \leq i \leq k} (C'_i - \alpha_i) w'_{k-i} = 0 \ (0 \leq k \leq s-1), \\ \sum_{0 \leq i \leq s-1} (C'_i - \alpha_i) w'_{s-i} + (C'_s - \alpha_s - \frac{s-2r}{2s-2r}) w'_0 = 0. \end{cases} \tag{3.4}$$

Moreover, there are at most $s-r$ $(s+1)$-tuples $(\alpha_0, \ldots, \alpha_s)$ in k such that $\alpha_0 \neq 0$ and the system of equations (3.3) (resp. (3.4)) holds for some $w_0, \ldots, w_s \in W$ with $w_0 \neq 0$ (resp. $w'_0, \ldots, w'_s \in W$ with $w'_0 \neq 0$).

Proof. Let η be a primitive $(s-r)$-th root of unity in k. We fix an $(s-r)$-th root $(-a_0)^{\frac{1}{s-r}}$ of $-a_0$. For any $1 \leq j \leq s-r$, set e'_j to be the column vector $(0, \ldots, 0, \eta^{(r+1)j}(-a_0)^{\frac{r+1}{s-r}}, \ldots, \eta^{sj}(-a_0)^{\frac{s}{s-r}})$ and ε'_j the row vector $(\eta^{-j}(-a_0)^{-\frac{1}{s-r}}, \ldots, \eta^{-sj}(-a_0)^{-\frac{s}{s-r}})$. We have

$$C_0 \cdot e'_j = \eta^{-rj}(-a_0)^{-\frac{r}{s-r}} \cdot e'_j, \ \varepsilon'_j \cdot C_0 = \eta^{-rj}(-a_0)^{-\frac{r}{s-r}} \cdot \varepsilon'_j, \ \varepsilon'_i \cdot e'_j = (s-r)\delta_{ij}.$$

Set $d=(r,s)$. Let W_0 be as in Lemma 3.2. We have $C_0|_{W_0}=0$. Then $\ker(C_0-\eta^{-rj}(-a_0)^{-\frac{r}{s-r}})$ is generated by those e'_k with $s-r|(k-j)d$, and $\mathrm{im}(C_0-\eta^{-rj}(-a_0)^{-\frac{r}{s-r}})$ is generated by W_0 and the other e'_k's. So

$$\mathrm{im}(C_0-\eta^{-rj}(-a_0)^{-\frac{r}{s-r}})=\{w\in W|\varepsilon'_k\cdot w=0 \text{ for all } k \text{ satisfying } s-r|(k-j)d\}.$$

For the only if part, suppose the system of equations (3.3) holds for some $w_0,\ldots,w_s\in W$ with $w_0\neq 0$. So $\alpha_0=\eta^{-rj}(-a_0)^{-\frac{r}{s-r}}$ for some integer j and then $w_0=\sum_{s-r|(i-j)d}\sigma_i e'_i$ for some $\sigma_i\in k$. Since $s\geq 2r$, for any $1\leq k,\ l\leq s-r$, we have

$$\begin{aligned}&\varepsilon'_k\cdot(\mathrm{diag}\{\frac{1}{s-r},\ldots,\frac{s}{s-r}\}-P-\frac{s-2r}{2s-2r})e'_l\\ =\ &\sum_{r+1\leq i\leq s}\frac{i}{s-r}\eta^{(l-k)i}-\sum_{1\leq i\leq r}\frac{r+i}{r}\eta^{(l-k)i}-\frac{s-2r}{2}\delta_{kl}.\end{aligned}$$

If $k=l$, then

$$\begin{aligned}&\varepsilon'_k\cdot(\mathrm{diag}\{\frac{1}{s-r},\ldots,\frac{s}{s-r}\}-P-\frac{s-2r}{2s-2r})e'_l\\ =\ &\sum_{r+1\leq i\leq s}\frac{i}{s-r}-\sum_{1\leq i\leq r}\frac{r+i}{r}-\frac{s-2r}{2}=0.\end{aligned}$$

If $k\neq l$ and $s-r|(l-k)d$, then $(\eta^{l-k})^d=1$ and $\eta^{l-k}\neq 1$. We have

$$\begin{aligned}&\varepsilon'_k\cdot(\mathrm{diag}\{\frac{1}{s-r},\ldots,\frac{s}{s-r}\}-P-\frac{s-2r}{2s-2r})e'_l\\ =\ &\frac{s-r}{d}\sum_{1\leq i\leq d}\frac{i}{s-r}\eta^{(l-k)i}-\frac{r}{d}\sum_{1\leq i\leq d}\frac{i}{r}\eta^{(l-k)i}=0.\end{aligned}$$

So $\varepsilon'_k\cdot(\mathrm{diag}\{\frac{1}{s-r},\ldots,\frac{s}{s-r}\}-P-\frac{s-2r}{2s-2r})w_0=0$ if $s-r|(k-j)d$. Therefore

$$(\mathrm{diag}\{\frac{1}{s-r},\ldots,\frac{s}{s-r}\}-P-\frac{s-2r}{2s-2r})w_0=(C_0-\alpha_0)w$$

for some $w\in W$. Then $w'_0=w_0,\ldots,w'_{s-1}=w_{s-1},w'_s=w_s-w$ satisfy the system of equations (3.4). Reversing the above argument, we get the if part. The characteristic polynomial of D_0 is $\lambda^s+\frac{1}{a_0}\lambda^r$. Each nonzero root of this polynomial is simple. Since $\sum_{i\geq 0}Z'^iC'_i=(Z'\wedge^{-1}A\wedge)^r$ and $C_0=D_0^r$, the proof of the last assertion is similar to that of Lemma 2.7. □

Now we are ready to prove Theorem 3′. Similar to Remark 2.2, we assume $a(Z)=\sum_{0\leq i\leq s}a_iZ^i$. Then the first equation of (3.1) means that $\frac{Z'}{Z}$ is a root in $k[[Z']]$ with nonzero constant term of the polynomial

$$\lambda^s+\frac{a_1}{a_0}Z'\lambda^{s-1}+\ldots+\frac{a_s}{a_0}Z'^s+\frac{1}{a_0}\lambda^r\in k[[Z']][\lambda].$$

This polynomial is exactly the characteristic polynomial of $Z' \wedge^{-1} A \wedge$. The characteristic polynomial of D_0 is $\lambda^s + \frac{1}{a_0}\lambda^r$ which has no nonzero multiple roots, by Hensel's lemma, $Z' \wedge^{-1} A\wedge$ has an eigenvector $\sum_{i\geq 0} Z'^i w_i$ corresponding this eigenvalue $\frac{Z'}{Z}$ with $w_0 \neq 0$. Since $\sum_{i\geq 0} Z'^i C'_i = (Z' \wedge^{-1} A\wedge)^r$, we have

$$\Big(\sum_{i\geq 0} Z'^i C'_i\Big)\Big(\sum_{i\geq 0} Z'^i w_i\Big) = \Big(\frac{Z'}{Z}\Big)^r \Big(\sum_{i\geq 0} Z'^i w_i\Big) = \Big(\sum_{i\geq 0} b_i Z'^i\Big)\Big(\sum_{i\geq 0} Z'^i w_i\Big).$$

That is,

$$\sum_{0\leq i\leq k} (C'_i - b_i) w_{k-i} = 0 \text{ for any } k \geq 0.$$

Recall that $s \geq 2r$ and $\sum_{0\leq i\leq s} a_i Z^{i-s}$ is assumed to be irreducible. By Lemma 3.2 and 3.3, we have

$$\begin{aligned} & \mathcal{F}^{(\infty,\infty)}\Big([r]_*[-rh]\Big) \\ = \quad & [s-r]_*[-(s-r)(b_0 Z'^{-s} + b_1 Z'^{1-s} + \ldots + b_s - \frac{s-2r}{2s-2r})] \\ = \quad & [s-r]_*[-(s-r)(b_0 Z'^{-s} + b_1 Z'^{1-s} + \ldots + b_s) + \frac{s}{2}]. \end{aligned}$$

References

[1] Beilinson, A.; Bloch, S.; Esnault, H.: ϵ-factors for Gauß-Manin determinants, Moscow Mathematical Journal, vol. 2, **3** (2002), 477-532.

[2] Bloch, S. and Esnault, H.: Local Fourier Transforms and rigidity for $\mathcal{D}$-Modules, Asian J. Math. 8 (2004), no. 4, 587–605.

[3] Deligne, P.: Équations Différentielles à Points Singuliers Réguliers, Lectures Notes **163**, Springer Verlag.

[4] Fu, Lei.: Calculation of ℓ-adic local Fourier transformations, arXiv:math/0702436.

[5] Katz, N.: On the calculation of some differential Galois groups, Inv. Math. **87** (1986), 13-61.

[6] Laumon, G.: Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil, Publ. Math. IHES **65** (1987), 131-210.

[7] Levelt, A.H.M.: Jordan decomposition for a class of singualr differential operators, Ark. Math. 13 (1975), 1-27.

[8] Sabbah, Claude.: An explicit stationary phase formula for the local formal Fourier-Laplace transform, arXiv:0706.3570.

[9] Serre, J-P.: Local Fields, Graduate Texts in Mathematics, Springer Verlag.