HOLOMORPHY CONDITIONS OF FUJI-SUZUKI COUPLED PAINLEVÉ VI SYSTEM

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Abstract. In this note, we give some holomorphy conditions of Fuji-Suzuki coupled Painlevé VI system. We also give two translation operators acting on the constant parameter η. We note a confluence process from the Fuji-Suzuki system to the Noumi-Yamada system of type $A_5^{(1)}$.

1. Introduction

In this note, we study Fuji-Suzuki coupled Painlevé VI system (see [1, 2, 3]).

Define birational and symplectic transformations r_i $(i = 0, 1, \ldots, 6)$ as follows:

$$
\begin{aligned}
&r_0 : (x_0, y_0, z_0, w_0) = \left(-((q_1 - q_2)p_1 - \alpha_0)p_1, \frac{1}{p_1}, q_2, p_2 + p_1\right),\\
&r_1 : (x_1, y_1, z_1, w_1) = \left(\frac{1}{q_1}, -(q_1p_1 + \alpha_1)q_1, q_2, p_2\right),\\
&r_2 : (x_2, y_2, z_2, w_2) = \left(-((q_1 - t)p_1 - \alpha_2)p_1, \frac{1}{p_1}, q_2, p_2\right),\\
&r_3 : (x_3, y_3, z_3, w_3) = \left(-(q_1p_1 + q_2p_2 - (\alpha_3 - \eta)))p_1, \frac{1}{p_1}, q_2p_1, \frac{p_2}{p_1}\right),\\
&r_4 : (x_4, y_4, z_4, w_4) = \left(q_1, p_1, -((q_2 - 1)p_2 - \alpha_4)p_2, \frac{1}{p_2}\right),\\
&r_5 : (x_5, y_5, z_5, w_5) = \left(q_1, p_1, \frac{1}{q_2}, -(p_2q_2 + \alpha_5)q_2\right),\\
&r_6 : (x_6, y_6, z_6, w_6) = \left(-(XY + ZW - (\eta - \alpha_1 - \alpha_5))Y, \frac{1}{Y}, ZY, \frac{W}{Y}\right),
\end{aligned}
\tag{1}
$$

where the coordinate system (X, Y, Z, W) is given by

$$r_5 \circ r_1 : (X, Y, Z, W) := \left(\frac{1}{q_1}, -(q_1p_1 + \alpha_1)q_1, \frac{1}{q_2}, -(p_2q_2 + \alpha_5)q_2\right).$$

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We note that it was difficult to find the condition r_6. Because this condition is a patching data on the double boundary of the variables q_1, q_2 in 4-dimensional complex manifold $\mathcal{S}$ given in the paper [31], that is, $r_5 \circ r_1 : (X, Y, Z, W) = \left(\frac{1}{q_1}, -(q_1 p_1 + \alpha_1) q_1, \frac{1}{q_2}, -(p_2 q_2 + \alpha_5) q_2\right)$.

There exist a polynomial H_{FS}, such that the Hamiltonian system

$$\frac{dq_1}{dt} = \frac{\partial H_{FS}}{\partial p_1}, \quad \frac{dp_1}{dt} = -\frac{\partial H_{FS}}{\partial q_1}, \quad \frac{dq_2}{dt} = \frac{\partial H_{FS}}{\partial p_2}, \quad \frac{dp_2}{dt} = -\frac{\partial H_{FS}}{\partial q_2} \tag{2}$$

is transformed into a polynomial Hamiltonian system under the action of each r_i $(i = 0, 1, \ldots, 6)$, where a polynomial Hamiltonian H_{FS} is given by

$$\begin{aligned} t(t-1)H_{FS} = \\ H_{VI}(q_1, p_1; \alpha_2, \alpha_0 + \alpha_4, \alpha_3 + \alpha_5 - \eta, \eta\alpha_1) + H_{VI}(q_2, p_2; \alpha_0 + \alpha_2, \alpha_4, \alpha_3 + \alpha_1 - \eta, \eta\alpha_5) \\ + (q_1 - t)(q_2 - 1)\{(q_1 p_1 + \alpha_1) p_2 + p_1 (q_2 p_2 + \alpha_5)\}, \end{aligned} \tag{3}$$

where q_i, p_i $(i = 1, 2)$ denote unknown complex variables, and α_j, η $(j = 0, 1, \ldots, 5)$ are complex constant parameters satisfying the parameter's relation:

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1.$$

It is known that the system (2),(3) admits affine Weyl group symmetry of type $A_5^{(1)}$ as the group of its Bäcklund transformations (see [1]).

The symbol $H_{VI}(q, p; a, b, c, d)$ denotes

$$H_{VI}(q, p; a, b, c, d) := q(q-1)(q-t)p^2 - \{(a-1)q(q-1) + bq(q-t) + c(q-1)(q-t)\}p + dq.$$

We note that the holomorphy condition r_2 should be read that

$$r_2(H_{FS} - p_1)$$

is a polynomial with respect to x_2, y_2, z_2, w_2.

This system admits several Lax pairs (see [1, 2, 3]).

We note that the Hamiltonian system (2),(3) is invariant under the following diagram automorphisms s_8, s_9, s_{10}. With *the notation* $(*) := (q_1, p_1, q_2, p_2, t; \alpha_0, \alpha_1, \ldots, \alpha_5, \eta)$;

$$\begin{aligned} &s_8 : (*) \to \left(\frac{q_2}{t}, tp_2, \frac{q_1}{t}, tp_1, \frac{1}{t}; \alpha_0, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \eta\right), \\ &s_9 : (*) \to \left(\frac{1}{q_1}, -(p_1 q_1 + p_2 q_2 + \eta) q_1, \frac{q_2}{q_1}, p_2 q_1, \frac{1}{t}; \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0, \alpha_5, \eta\right), \\ &s_{10} : (*) \to \left(\frac{t}{q_2}, -\frac{(p_2 q_2 + p_1 q_1 + \eta) q_2}{t}, \frac{q_1}{q_2}, p_1 q_2, t; \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_0, \alpha_1, \eta\right). \end{aligned} \tag{4}$$

We also remark that these transformations s_8, s_9, s_{10} satisfy the following relations:

$$s_8^2 = s_9^2 = 1, \quad s_{10}^3 = 1, \quad s_{10} = s_8 \circ s_9.$$

We remark that we can consider the transformation s_9 as holomorphy condition (see (37), cf. [14]).

Finally, we will give two translation operators acting on the constant parameter η.

Proposition 1.1. *Let us define the following translation operators*;

$$T_1 := (s_2 s_{10} s_{10} s_1)^4, \quad T_2 := s_1 T_1 s_1, \quad T_3 := s_5 T_1 s_5. \tag{5}$$

These translation operators T_k $(k = 1, 2, 3)$ *act on parameters* α_i, η *as follows*:

$$\begin{aligned} T_1(\alpha_0, \alpha_1, \dots, \alpha_5, \eta) =&(\alpha_0, \alpha_1, \dots, \alpha_5, \eta) + (0, -1, 1, 0, -1, 1, 0), \\ T_2(\alpha_0, \alpha_1, \dots, \alpha_5, \eta) =&(\alpha_0, \alpha_1, \dots, \alpha_5, \eta) + (-1, 1, 0, 0, -1, 1, 1), \\ T_3(\alpha_0, \alpha_1, \dots, \alpha_5, \eta) =&(\alpha_0, \alpha_1, \dots, \alpha_5, \eta) + (1, -1, 1, 0, 0, -1, -1). \end{aligned} \tag{6}$$

Here, (see [1])

$$\begin{aligned} &s_1 : (*) \rightarrow \left(q_1 + \frac{\alpha_1}{p_1}, p_1, q_2, p_2, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5, \eta - \alpha_1\right), \\ &s_2 : (*) \rightarrow \left(q_1, p_1 - \frac{\alpha_2}{q_1 - t}, q_2, p_2, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5, \eta + \alpha_2\right), \\ &s_5 : (*) \rightarrow \left(q_1, p_1, q_2 + \frac{\alpha_5}{p_2}, p_2, t; \alpha_0 + \alpha_5, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_5, -\alpha_5, \eta - \alpha_5\right). \end{aligned}$$

In particular, two transformations T_2, T_3 are translation operators acting on the constant parameter η :

$$T_2(\eta) = \eta + 1, \quad T_3(\eta) = \eta - 1. \tag{7}$$

Next, we review a confluence process from the system (2),(3) to the Noumi-Yamada system of type $A_5^{(1)}$ (cf. [28, 2, 1, 33]).

For the system (2),(3), we make a change of parameters and variables

$$\alpha_0 = A_0, \ \alpha_1 = A_1, \dots, \alpha_5 = A_5, \quad \eta = A_1 + A_5 + \frac{1}{\varepsilon}, \tag{8}$$

$$t = 1 + \varepsilon T, \quad q_1 = 1 + \varepsilon Q_1, \quad q_2 = 1 + \varepsilon Q_2, \quad p_1 = \frac{P_1}{\varepsilon}, \quad p_2 = \frac{P_2}{\varepsilon}, \tag{9}$$

$$H_{FS}(\varepsilon) = \varepsilon \left(H_{FS} - \frac{(\alpha_1 + \alpha_5)\eta}{t(t-1)(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)}\right) \tag{10}$$

from $\alpha_0, \alpha_1, \dots, \alpha_5, \eta, t, q_1, p_1, q_2, p_2$ to $A_0, \dots, A_5, \varepsilon, T, Q_1, P_1, Q_2, P_2$. Then the system (2),(3) can also be written in the new variables T, Q_1, P_1, Q_2, P_2 and parameters $A_0, \dots, A_5$, ε as a Hamiltonian system

$$\frac{dQ_1}{dT} = \frac{\partial H_{FS}(\varepsilon)}{\partial P_1}, \quad \frac{dP_1}{dT} = -\frac{\partial H_{FS}(\varepsilon)}{\partial Q_1}, \quad \frac{dQ_2}{dT} = \frac{\partial H_{FS}(\varepsilon)}{\partial P_2}, \quad \frac{dP_2}{dT} = -\frac{\partial H_{FS}(\varepsilon)}{\partial Q_2}. \tag{11}$$

Here, its holomorphy conditions are given by

(12)

$$
\begin{aligned}
&r_0 : (x_0, y_0, z_0, w_0) = \left(-((Q_1 - Q_2)P_1 - A_0)P_1, \frac{1}{P_1}, Q_2, P_2 + P_1\right),\\
&r_1 : (x_1, y_1, z_1, w_1) = \left(\frac{1}{Q_1}, -(Q_1 P_1 + A_1)Q_1, Q_2, P_2\right),\\
&r_2 : (x_2, y_2, z_2, w_2) = \left(-((Q_1 - T)P_1 - A_2)P_1, \frac{1}{P_1}, Q_2, P_2\right),\\
&\tilde{r}_3 : (\tilde{x}_3, \tilde{y}_3, \tilde{z}_3, \tilde{w}_3) = \left(-\left(\left(Q_1 + \frac{1}{\varepsilon}\right)P_1 + \left(Q_2 + \frac{1}{\varepsilon}\right)P_2 - A_3 + A_1 + A_5 + \frac{1}{\varepsilon}\right)P_1, \frac{1}{P_1}, \left(Q_2 + \frac{1}{\varepsilon}\right)P_1, \frac{P_2}{P_1}\right),\\
&\tilde{r}_4 : (\tilde{x}_4, \tilde{y}_4, \tilde{z}_4, \tilde{w}_4) = \left(Q_1, P_1, -(Q_2 P_2 - A_4)P_2, \frac{1}{P_2}\right),\\
&r_5 : (x_5, y_5, z_5, w_5) = \left(Q_1, P_1, \frac{1}{Q_2}, -(P_2 Q_2 + A_5)Q_2\right),\\
&r_6 : (x_6, y_6, z_6, w_6) = \left(-\left(XY + ZW - \frac{1}{\varepsilon}\right)Y, \frac{1}{Y}, ZY, \frac{W}{Y}\right),
\end{aligned}
$$

where the coordinate system (X, Y, Z, W) is given by

$$r_5 \circ r_1 : (X, Y, Z, W) := \left(\frac{1}{Q_1}, -(Q_1 P_1 + A_1)Q_1, \frac{1}{Q_2}, -(P_2 Q_2 + A_5)Q_2\right).$$

This new system tends to the Noumi-Yamada system of type $A_5^{(1)}$ as $\varepsilon \to 0$, where the Noumi-Yamada system of type $A_5^{(1)}$ is explicitly given as follows:

(13)

$$
\begin{aligned}
&\frac{dq_1}{dt} = \frac{\partial H_{NYA5}}{\partial p_1}, \quad \frac{dp_1}{dt} = -\frac{\partial H_{NYA5}}{\partial q_1}, \quad \frac{dq_2}{dt} = \frac{\partial H_{NYA5}}{\partial p_2}, \quad \frac{dp_2}{dt} = -\frac{\partial H_{NYA5}}{\partial q_2},\\
&tH_{NYA5} = H_V(q_1, p_1; \alpha_0 + \alpha_4, \alpha_3 + \alpha_5, \alpha_1) + H_V(q_2, p_2; \alpha_4, \alpha_3 + \alpha_1, \alpha_5) + 2(q_1 - t)p_1 q_2 p_2,
\end{aligned}
$$

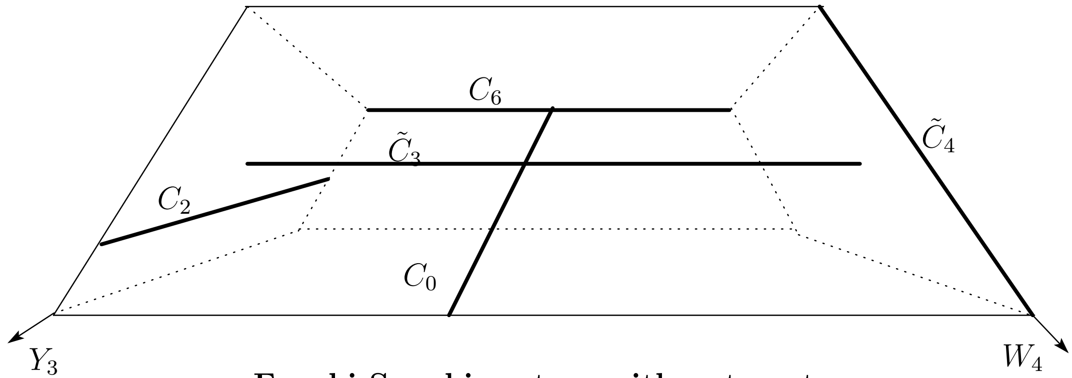
where q_i, p_i $(i = 1, 2)$ denote unknown complex variables, and α_j $(j = 0, 1, \ldots, 5)$ are complex constant parameters satisfying the parameter's relation:

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1.$$

Here, for notational convenience, we have renamed $(Q_1, P_1, Q_2, P_2, T, A_0, \ldots, A_5)$ to $(q_1, p_1, q_2, p_2, t, \alpha_0, \ldots, \alpha_5)$ (which are not the same as the previous $(q_1, p_1, q_2, p_2, t, \alpha_0, \ldots, \alpha_5)$).

The symbol $H_V(q, p; a, b, c)$ denotes

$$H_V(q, p; a, b, c) := q(q - t)p(p + 1) + atp + bqp + cq(p + 1).$$



Fuzuki-Suzuki system with patameter ε

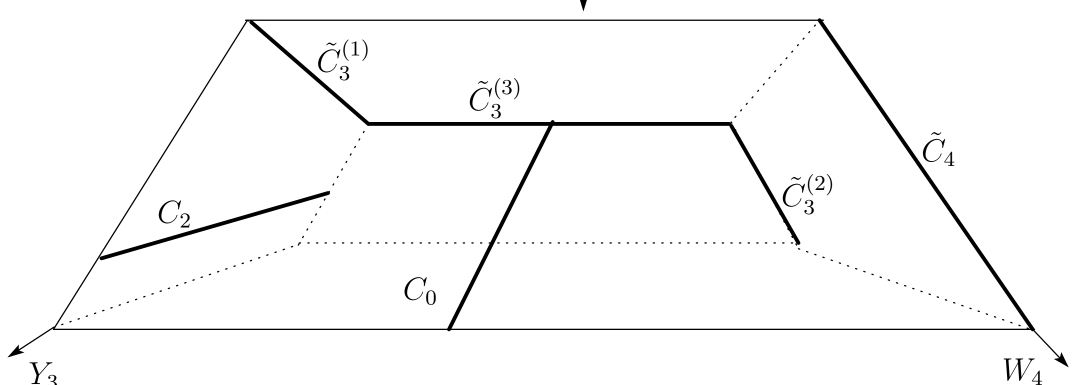
$$\tilde{C}_3 = \{(X_3, Y_3, Z_3, W_3) | X_3 = Z_3 = -\tfrac{1}{\varepsilon}, Y_3 = 0\} \cup \{(X_4, Y_4, Z_4, W_4) | X_4 = Z_4 = -\tfrac{1}{\varepsilon}, W_4 = 0\} \cong \mathbb{P}^1,$$

$$\tilde{C}_4 = \{(X_4, Y_4, Z_4, W_4) | Y_4 = Z_4 = W_4 = 0\} \cup \{(X_7, Y_7, Z_7, W_7) | Y_7 = Z_7 = W_7 = 0\} \cong \mathbb{P}^1.$$

FIGURE 1. This figure denotes the boundary divisor $\mathcal{H}$ of $\mathcal{S}$ (see (23)). The bold lines C_i $i = 0, 2, 6$ (see (22),(23)) and $\tilde{C}_j$ $j = 3, 4$ in $\mathcal{H}$ denote the accessible singular loci of the system (11).

Confluence process

$\varepsilon \to 0$ $\quad\downarrow\quad$ $\tilde{C}_3 \to C_6$



Noumi-Yamada system of type $A_5^{(1)}$

$$\tilde{C}_3^{(1)} = \{(X_6, Y_6, Z_6, W_6) | X_6 = Y_6 = W_6 = 0\} \cup \{(X_8, Y_8, Z_8, W_8) | X_8 = Y_8 = W_8 = 0\} \cong \mathbb{P}^1,$$

$$\tilde{C}_3^{(2)} = \{(X_{11}, Y_{11}, Z_{11}, W_{11}) | Y_{11} = Z_{11} = W_{11} = 0\} \cup \{(X_9, Y_9, Z_9, W_9) | Y_9 = Z_9 = W_9 = 0\} \cong \mathbb{P}^1,$$

$$\tilde{C}_3^{(3)} = \{(X_8, Y_8, Z_8, W_8) | X_8 = Z_8 = W_8 = 0\} \cup \{(X_9, Y_9, Z_9, W_9) | X_9 = Y_9 = Z_9 = 0\} \cong \mathbb{P}^1$$

FIGURE 2. This figure denotes the boundary divisor $\mathcal{H}$ of $\mathcal{S}$ (see (23)). The bold lines C_i $i = 0, 2$ (see (22),(23)) and $\tilde{C}_3^{(1)}, \tilde{C}_3^{(2)}, \tilde{C}_3^{(3)}, \tilde{C}_4$ in $\mathcal{H}$ denote the accessible singular loci of the system (13).

Its holomorphy conditions are given by r_j $(j=0,1,2,5)$, $\tilde{r}_4$ (given in (12)) and $\tilde{r}_3^{(0)}, \tilde{r}_3^{(1)}, \tilde{r}_3^{(2)}, \tilde{r}_3^{(3)}$; (see Figure 2)

(14)

$$r_0 : (x_0,y_0,z_0,w_0) = \left(-((q_1-q_2)p_1-\alpha_0)p_1, \frac{1}{p_1}, q_2, p_2+p_1\right),$$
$$r_1 : (x_1,y_1,z_1,w_1) = \left(\frac{1}{q_1}, -(q_1p_1+\alpha_1)q_1, q_2, p_2\right),$$
$$r_2 : (x_2,y_2,z_2,w_2) = \left(-((q_1-t)p_1-\alpha_2)p_1, \frac{1}{p_1}, q_2, p_2\right),$$
$$\tilde{r}_4 : (\tilde{x}_4,\tilde{y}_4,\tilde{z}_4,\tilde{w}_4) = \left(q_1, p_1, -(q_2p_2-\alpha_4)p_2, \frac{1}{p_2}\right),$$
$$r_5 : (x_5,y_5,z_5,w_5) = \left(q_1, p_1, \frac{1}{q_2}, -(p_2q_2+\alpha_5)q_2\right),$$
$$\tilde{r}_3^{(0)} : (\tilde{x}_3^{(0)},\tilde{y}_3^{(0)},\tilde{z}_3^{(0)},\tilde{w}_3^{(0)}) = \left(\frac{1}{q_1}, -((p_1+p_2+1)q_1+\alpha_3)q_1, q_2-q_1, p_2\right),$$
$$\tilde{r}_3^{(1)} : (\tilde{x}_3^{(1)},\tilde{y}_3^{(1)},\tilde{z}_3^{(1)},\tilde{w}_3^{(1)}) = \left(x_1, y_1+\frac{\alpha_1-\alpha_3}{x_1}-\frac{w_1+1}{x_1^2}, z_1-\frac{1}{x_1}, w_1\right),$$
$$\tilde{r}_3^{(2)} : (\tilde{x}_3^{(2)},\tilde{y}_3^{(2)},\tilde{z}_3^{(2)},\tilde{w}_3^{(2)}) = \left(x_5-\frac{1}{z_5}, y_5, z_5, w_5+\frac{\alpha_5-\alpha_3}{z_5}-\frac{y_5+1}{z_5^2}\right),$$
$$\tilde{r}_3^{(3)} : (\tilde{x}_3^{(3)},\tilde{y}_3^{(3)},\tilde{z}_3^{(3)},\tilde{w}_3^{(3)}) = \left(X, Y-W+\frac{2ZW-\alpha_3+\alpha_1+\alpha_5}{X}-\frac{1}{X^2}, \frac{Z-X}{X^2}, WX^2\right),$$

where the coordinate system (X,Y,Z,W) is given by

$$r_5 \circ r_1 : (X,Y,Z,W) := \left(\frac{1}{q_1}, -(q_1p_1+\alpha_1)q_1, \frac{1}{q_2}, -(p_2q_2+\alpha_5)q_2\right).$$

The Noumi-Yamada system of type $A_5^{(1)}$ can be characterized by four pairs of holomorphy conditions;

$$\{r_0,r_1,r_2,\tilde{r}_4,r_5,\tilde{r}_3^{(0)}\}, \quad \{r_0,r_1,r_2,\tilde{r}_4,r_5,\tilde{r}_3^{(1)}\},$$
$$\{r_0,r_1,r_2,\tilde{r}_4,r_5,\tilde{r}_3^{(2)}\}, \quad \{r_0,r_1,r_2,\tilde{r}_4,r_5,\tilde{r}_3^{(3)}\}. \tag{15}$$

We remark that by making a change of variables (q_i,p_i) and α_j, the following transformation $\tilde{s}_3^{(1)}$ associated with $\tilde{r}_3^{(1)}$ becomes a Bäcklund transformation:

$$\tilde{s}_3^{(1)} : (x_1,y_1,z_1,w_1,t;\alpha_0,\alpha_1,\dots,\alpha_5) \rightarrow$$
$$\left(-x_1, -\left(y_1+\frac{\alpha_1-\alpha_3}{x_1}-\frac{w_1+1}{x_1^2}\right), z_1-\frac{1}{x_1}, w_1, -t; \alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0,\alpha_5\right), \tag{16}$$

and the following transformation $\tilde{s}_3^{(2)}$ associated with $\tilde{r}_3^{(2)}$ becomes a Bäcklund transformation:

$$\begin{aligned}(17)\quad &\tilde{s}_3^{(2)}:(x_5,y_5,z_5,w_5,t;\alpha_0,\alpha_1,\dots,\alpha_5)\to\\ &\left(x_5-\frac{1}{z_5}-t,y_5,-z_5,-\left(w_5+\frac{\alpha_5-\alpha_3}{z_5}-\frac{y_5+1}{z_5^2}\right),-t;\alpha_2,\alpha_1,\alpha_0,\alpha_5,\alpha_4,\alpha_3\right).\end{aligned}$$

Pulling back a diagram automorphism π_1;

(18)
$$\pi_1:(q_1,p_1,q_2,p_2,t;\alpha_0,\alpha_1,\dots,\alpha_5)\to(-q_1,-(p_1+p_2+1),q_2-q_1,p_2,-t;\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0,\alpha_5)$$

by the birational transformation r_1, we can obtain $\tilde{s}_3^{(1)}$, and a diagram automorphism π_2;

(19)
$$\pi_2:(q_1,p_1,q_2,p_2,t;\alpha_0,\alpha_1,\dots,\alpha_5)\to(q_1-q_2-t,p_1,-q_2,-(p_2+p_1+1),-t;\alpha_2,\alpha_1,\alpha_0,\alpha_5,\alpha_4,\alpha_3)$$

by the birational transformation r_5, we can obtain $\tilde{s}_3^{(2)}$.

The system (13) has the following invariant divisors:

parameter's relation	f_i
$\alpha_0=0$	$f_0:=q_1-q_2$
$\alpha_1=0$	$f_1:=p_1$
$\alpha_2=0$	$f_2:=q_1-t$
$\alpha_3=0$	$f_3:=p_1+p_2+1$
$\alpha_4=0$	$f_4:=q_2$
$\alpha_5=0$	$f_5:=p_2$

We note that when $\alpha_1=0$, we see that the system (13) admits a particular solution $p_1=0$, and when $\alpha_3=0$, after we make the birational and symplectic transformation:

$$x_3=q_1,\ y_3=p_1+p_2+1,\ z_3=q_2-q_1,\ w_3=p_2$$

we see that the system (13) admits a particular solution $y_3=0$.

The Bäcklund transformations of the system of type $A_5^{(1)}$ satisfy

$$s_i(g)=g+\frac{\alpha_i}{f_i}\{f_i,g\}+\frac{1}{2!}\left(\frac{\alpha_i}{f_i}\right)^2\{f_i,\{f_i,g\}\}+\cdots\quad(g\in\mathbb{C}(t)[q_1,p_1,q_2,p_2]),$$

where $\{,\}$ is the Poisson bracket such that $\{p_i,q_j\}=\delta_{ij}$ (see [29]).

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

Finally, we list some holomorphy conditions of the system (13).

Hamiltonian $H_1=r_1(H_{NYA5}),\ r_1:x=\frac{1}{q_1},\ y=-(p_1q_1+\alpha_1)q_1,\ z=q_2,\ w=p_2$

$$
\begin{aligned}
&r_0^1: x_0=q_1,\ y_0=p_1-\frac{\alpha_0 q_2}{q_1q_2-1},\ z_0=q_2,\ w_0=p_2-\frac{\alpha_0 q_1}{q_1q_2-1},\\
&r_1^1: x_1=\frac{1}{q_1},\ y_1=-(p_1q_1+\alpha_1)q_1,\ z_1=q_2,\ w_1=p_2,\\
&r_2^1: x_2=-((q_1-1/t)p_1-\alpha_2)p_1,\ y_2=\frac{1}{p_1},\ z_2=q_2\ w_2=p_2,\\
&r_3^1: x_3=q_1,\ y_3=p_1-\frac{\alpha_3-\alpha_1}{q_1}-\frac{p_2+1}{q_1^2},\ z_3=q_2-\frac{1}{q_1},\ w_3=p_2,\\
&r_4^1: x_4=q_1,\ y_4=p_1,\ z_4=-(q_2p_2-\alpha_4)p_2,\ w_4=\frac{1}{p_2},\\
&r_5^1: x_5=q_1,\ y_5=p_1,\ z_5=\frac{1}{q_2},\ w_5=-(p_2q_2+\alpha_5)q_2,
\end{aligned}
$$

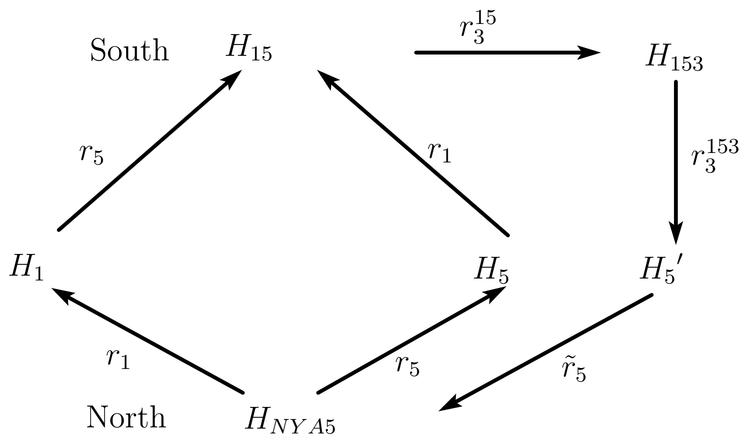
where $r_2^1\left(H_1+\frac{p_1}{t^2}\right)$. Here, for notational convenience, we have renamed (x,y,z,w) to (q_1,p_1,q_2,p_2) (which are not the same as the previous (q_1,p_1,q_2,p_2)).

Hamiltonian $H_5=r_5(H_{NYA5}),\ r_5: x=q_1,\ y=p_1,\ z=\frac{1}{q_2},\ w=-(p_2q_2+\alpha_5)q_2$

$$
\begin{aligned}
&r_0^5: x_0=q_1,\ y_0=p_1-\frac{\alpha_0 q_2}{q_1q_2-1},\ z_0=q_2,\ w_0=p_2-\frac{\alpha_0 q_1}{q_1q_2-1},\\
&r_1^5: x_1=\frac{1}{q_1},\ y_1=-(p_1q_1+\alpha_1)q_1,\ z_1=q_2,\ w_1=p_2,\\
&r_2^5: x_2=-((q_1-t)p_1-\alpha_2)p_1,\ y_2=\frac{1}{p_1},\ z_2=q_2\ w_2=p_2,\\
&r_3^5: x_3=q_1-\frac{1}{q_2},\ y_3=p_1,\ z_3=q_2,\ w_3=p_2-\frac{\alpha_3-\alpha_5}{q_2}-\frac{p_1+1}{q_2^2},\\
&r_4^5: x_4=q_1,\ y_4=p_1,\ z_4=\frac{1}{q_2},\ w_4=-(p_2q_2+\alpha_4+\alpha_5)q_2,\\
&r_5^5: x_5=q_1,\ y_5=p_1,\ z_5=\frac{1}{q_2},\ w_5=-(p_2q_2+\alpha_5)q_2,
\end{aligned}
$$

where $r_2^5\,(H_5-p_1)$.

Hamiltonian $H_{15}=r_{15}(H_{NYA5}),\ r_{15}: x=\frac{1}{q_1},\ y=-(p_1q_1+\alpha_1)q_1,\ z=\frac{1}{q_2},\ w=-(p_2q_2+\alpha_5)q_2$



$\tilde{r}_5 : x = q_1,\ y = p_1 - 1,\ z = \frac{1}{q_2},\ w = -(p_2q_2 + \alpha_5)q_2$

FIGURE 3. Relation between Hamiltonians H_{NYA5} and $H_1, H_5, H_{15}, H_{153}$

$$
\begin{aligned}
&r_0^{15} : x_0 = -((q_1 - q_2)p_1 - \alpha_0)p_1,\ y_0 = \frac{1}{p_1},\ z_0 = q_2,\ w_0 = p_2 + p_1,\\
&r_1^{15} : x_1 = \frac{1}{q_1},\ y_1 = -(p_1q_1 + \alpha_1)q_1,\ z_1 = q_2,\ w_1 = p_2,\\
&r_2^{15} : x_2 = -((q_1 - 1/t)p_1 - \alpha_2)p_1,\ y_2 = \frac{1}{p_1},\ z_2 = q_2\ w_2 = p_2,\\
&r_3^{15} : x_3 = q_1,\ y_3 = p_1 - p_2 + \frac{2q_2p_2 - (\alpha_3 - \alpha_1 - \alpha_5)}{q_1} - \frac{1}{q_1^2},\ z_3 = \frac{q_2 - q_1}{q_1^2},\ w_3 = p_2q_1^2,\\
&r_4^{15} : x_4 = q_1,\ y_4 = p_1,\ z_4 = \frac{1}{q_2},\ w_4 = -(p_2q_2 + \alpha_4 + \alpha_5)q_2,\\
&r_5^{15} : x_5 = q_1,\ y_5 = p_1,\ z_5 = \frac{1}{q_2},\ w_5 = -(p_2q_2 + \alpha_5)q_2,
\end{aligned}
$$

where $r_2^{15}\left(H_{15} + \frac{p_1}{t^2}\right)$.

Hamiltonian $H_{153} = r_3^{15}(H_{15})$

$$
\begin{aligned}
&r_0^{153} : x_0 = q_1,\ y_0 = p_1,\ z_0 = -(q_2p_2 - \alpha_0)p_2,\ w_0 = \frac{1}{p_2},\\
&r_1^{153} : x_1 = q_1,\ y_1 = p_1 - \frac{2q_2p_2 - (\alpha_3 - \alpha_1 - \alpha_5)}{q_1} + \frac{1}{q_1^2},\ z_1 = q_2q_1^2,\ w_1 = \frac{p_2}{q_1^2},\\
&r_2^{153} : x_2 = -((q_1 - 1/t)p_1 - \alpha_2)p_1,\ y_2 = \frac{1}{p_1},\ z_2 = q_2\ w_2 = p_2,\\
&r_3^{153} : x_3 = \frac{1}{q_1},\ y_3 = -\left(\left(p_1 + \frac{p_2}{q_1^2}\right) - 2(q_2q_1 + 1)\frac{p_2}{q_1} + \alpha_3 - \alpha_5\right)q_1,\ z_3 = (q_2q_1 + 1)q_1,\ w_3 = \frac{p_2}{q_1^2},\\
&r_4^{153} : x_4 = q_1,\ y_4 = p_1,\ z_4 = \frac{1}{q_2},\ w_4 = -(p_2q_2 + \alpha_4 + \alpha_5)q_2,\\
&r_5^{153} : x_5 = q_1,\ y_5 = p_1,\ z_5 = \frac{1}{q_2},\ w_5 = -(p_2q_2 + \alpha_5)q_2,
\end{aligned}
$$

where $r_2^{153}\left(H_{153} + \frac{p_1}{t^2}\right)$.

After we review the notion of accessible singularity in next section, we make its holomorphy conditions by resolving the accessible singularities.

2. Accessible singularity and local index

Let us review the notion of *accessible singularity*. Let B be a connected open domain in $\mathbb{C}$ and $\pi : \mathcal{W} \longrightarrow B$ a smooth proper holomorphic map. We assume that $\mathcal{H} \subset \mathcal{W}$ is a normal crossing divisor which is flat over B. Let us consider a rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfying the condition

$$\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$

Fixing $t_0 \in B$ and $P \in \mathcal{W}_{t_0}$, we can take a local coordinate system $(x_1, \dots, x_n)$ of $\mathcal{W}_{t_0}$ centered at P such that $\mathcal{H}_{\mathrm{smooth}}$ can be defined by the local equation $x_1 = 0$. Since $\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H}))$, we can write down the vector field $\tilde{v}$ near $P = (0, \dots, 0, t_0)$ as follows:

$$\tilde{v} = \frac{\partial}{\partial t} + g_1 \frac{\partial}{\partial x_1} + \frac{g_2}{x_1}\frac{\partial}{\partial x_2} + \cdots + \frac{g_n}{x_1}\frac{\partial}{\partial x_n}.$$

This vector field defines the following system of differential equations

$$\frac{dx_1}{dt} = g_1(x_1, \dots, x_n, t),\ \frac{dx_2}{dt} = \frac{g_2(x_1, \dots, x_n, t)}{x_1}, \cdots, \frac{dx_n}{dt} = \frac{g_n(x_1, \dots, x_n, t)}{x_1}. \tag{20}$$

Here $g_i(x_1, \dots, x_n, t),\ i = 1, 2, \dots, n$, are holomorphic functions defined near P.

Definition 2.1. With the above notation, assume that the rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfies the condition

$$(A) \quad \tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$

We say that $\tilde{v}$ has an *accessible singularity* at $P = (0, \dots, 0, t_0)$ if

$$\boxed{x_1 = 0 \text{ and } g_i(0, \dots, 0, t_0) = 0 \text{ for every } i,\ 2 \le i \le n.} \tag{21}$$

If $P \in \mathcal{H}_{\mathrm{smooth}}$ is not an accessible singularity, all solutions of the ordinary differential equation passing through P are vertical solutions, that is, the solutions are contained in the fiber $\mathcal{W}_{t_0}$ over $t = t_0$. If $P \in \mathcal{H}_{\mathrm{smooth}}$ is an accessible singularity, there may be a solution of (20) which passes through P and goes into the interior $\mathcal{W} - \mathcal{H}$ of $\mathcal{W}$.

3. Construction of the holomorphy conditions

In this section, we will give the holomorphy conditions r_i $(i = 0, 1, \dots, 6)$ by resolving some accessible singular loci of the system (2),(3).

In order to consider the singularity analysis for the system (2),(3) as a compactification of $\mathbb{C}^4$ which is the phase space of the system (2),(3), we take 4-dimensional complex manifold $\mathcal{S}$ given in the paper [31]. This manifold can be considered as a generalization of the Hirzebruch surface.

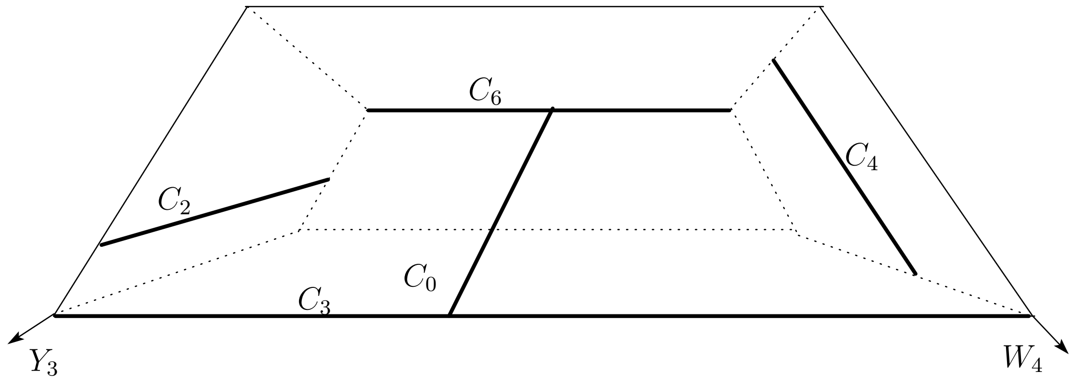


FIGURE 4. This figure denotes the boundary divisor $\mathcal{H}$ of $\mathcal{S}$. The bold lines C_i $i = 0, 2, 3, 4, 6$ in $\mathcal{H}$ denote the accessible singular loci of the system (2),(3).

We easily see that the rational vector field $\tilde{v}$ associated with the system (2),(3) satisfies the condition:

$$\tilde{v} \in H^0(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{H})(\mathcal{H})).$$

LEMMA 3.1. *The rational vector field $\tilde{v}$ associated with the system* (2),(3) *has the following accessible singular loci $C_i \cong \mathbb{P}^1$* $(i = 0, 2, 3, 4, 6)$ (see Figure 4):

$$\begin{cases} C_0 =\{(X_3,Y_3,Z_3,W_3)|X_3 = Z_3, Y_3 = 0, W_3 = -1\} \\ \qquad \cup \{(X_8,Y_8,Z_8,W_8)|X_8 = Z_8, Y_8 = 0, W_8 = -1\} \cong \mathbb{P}^1, \\ C_2 =\{(X_3,Y_3,Z_3,W_3)|X_3 = t, Y_3 = 0, W_3 = 0\}, \\ \qquad \cup \{(X_{10},Y_{10},Z_{10},W_{10})|X_{10} = t, Y_{10} = 0, W_{10} = 0\} \cong \mathbb{P}^1, \\ C_3 =\{(X_3,Y_3,Z_3,W_3)|X_3 = Y_3 = Z_3 = 0\} \\ \qquad \cup \{(X_4,Y_4,Z_4,W_4)|X_4 = Z_4 = W_4 = 0\} \cong \mathbb{P}^1, \\ C_4 =\{(X_4,Y_4,Z_4,W_4)|Y_4 = 0, Z_4 = 1, W_4 = 0\}, \\ \qquad \cup \{(X_7,Y_7,Z_7,W_7)|Y_7 = 0, Z_7 = 1, W_7 = 0\} \cong \mathbb{P}^1, \\ C_6 =\{(X_8,Y_8,Z_8,W_8)|X_8 = Y_8 = Z_8 = 0\} \\ \qquad \cup \{(X_9,Y_9,Z_9,W_9)|X_9 = Z_9 = W_9 = 0\} \cong \mathbb{P}^1. \end{cases} \tag{22}$$

Here, the coordinate systems (X_i, Y_i, Z_i, W_i) $(i = 0, 1, \cdots, 11)$ (see Figure 4, cf. [31]) are explicitly given by

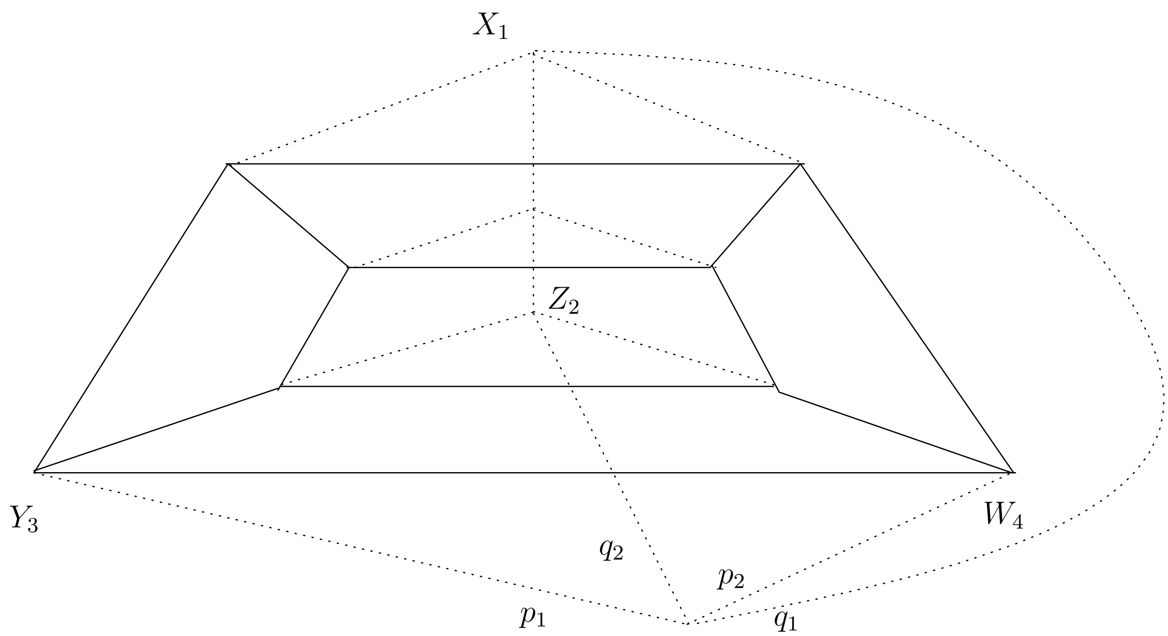


FIGURE 5. This figure denotes 4-dimensional complex manifold $\mathcal{S}$ (see [31]) and its boundary divisor $\mathcal{H}$. $\mathcal{H}$ is drawn by solid line.

$$
\begin{aligned}
&(X_0,Y_0,Z_0,W_0)=(q_1,p_1,q_2,p_2)\,,\\
&(X_1,Y_1,Z_1,W_1)=\left(\frac{1}{q_1},-(p_1q_1+\alpha_1)q_1,q_2,p_2\right),\\
&(X_2,Y_2,Z_2,W_2)=\left(q_1,p_1,\frac{1}{q_2},-(p_2q_2+\alpha_5)q_2\right),\\
&(X_3,Y_3,Z_3,W_3)=\left(q_1,\frac{1}{p_1},q_2,\frac{p_2}{p_1}\right),\\
&(X_4,Y_4,Z_4,W_4)=\left(q_1,\frac{p_1}{p_2},q_2,\frac{1}{p_2}\right),\\
&(X_5,Y_5,Z_5,W_5)=\left(\frac{1}{q_1},-(p_1q_1+\alpha_1)q_1,\frac{1}{q_2},-(p_2q_2+\alpha_5)q_2\right),\\
&(X_6,Y_6,Z_6,W_6)=\left(\frac{1}{q_1},-\frac{1}{(q_1p_1+\alpha_1)q_1},q_2,-\frac{p_2}{(q_1p_1+\alpha_1)q_1}\right),\\
&(X_7,Y_7,Z_7,W_7)=\left(\frac{1}{q_1},-\frac{(p_1q_1+\alpha_1)q_1}{p_2},q_2,\frac{1}{p_2}\right),\\
&(X_8,Y_8,Z_8,W_8)=\left(\frac{1}{q_1},-\frac{1}{(p_1q_1+\alpha_1)q_1},\frac{1}{q_2},\frac{(p_2q_2+\alpha_5)q_2}{(p_1q_1+\alpha_1)q_1}\right),\\
&(X_9,Y_9,Z_9,W_9)=\left(\frac{1}{q_1},\frac{(p_1q_1+\alpha_1)q_1}{(p_2q_2+\alpha_5)q_2},\frac{1}{q_2},-\frac{1}{(p_2q_2+\alpha_5)q_2}\right),\\
&(X_{10},Y_{10},Z_{10},W_{10})=\left(q_1,\frac{1}{p_1},\frac{1}{q_2},-\frac{(p_2q_2+\alpha_5)q_2}{p_1}\right),\\
&(X_{11},Y_{11},Z_{11},W_{11})=\left(q_1,-\frac{p_1}{(p_2q_2+\alpha_5)q_2},\frac{1}{q_2},-\frac{1}{(p_2q_2+\alpha_5)q_2}\right).
\end{aligned}
\tag{23}
$$

Proposition 3.2. *If we resolve the accessible singular loci given in Lemma 3.1 by blowing-ups, then we can obtain the canonical coordinates r_i $(i=0,2,3,4,6)$.*

Proof. By the following steps, we can resolve the accessible singular locus C_3.

Step 0: Around the point $P:=\{(X_3,Y_3,Z_3,W_3)|X_3=Y_3=Z_3=W_3=0\}$, we rewrite the system (2) as follows:

$$\frac{d}{dt}\begin{pmatrix} X_3 \\ Y_3 \\ Z_3 \\ W_3 \end{pmatrix}=\frac{1}{Y_1}\left\{\begin{pmatrix} \frac{2}{t-1} & -\frac{\alpha_3-\eta}{t-1} & 0 & 0 \\ 0 & \frac{1}{t-1} & 0 & 0 \\ \frac{1}{t-1} & 0 & \frac{1}{t-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_3 \\ Y_3 \\ Z_3 \\ W_3 \end{pmatrix}+\cdots\right\}.$$

Step 1: We blow up along the curve C_3.

$$X_3^{(1)}=\frac{X_3}{Y_3},\quad Y_3^{(1)}=Y_3,\quad Z_3^{(1)}=\frac{Z_3}{Y_3},\quad W_3^{(1)}=W_3.$$

Step 2: We blow up along the surface $\{(X_3^{(1)},Y_3^{(1)},Z_3^{(1)},W_3^{(1)})|X_3^{(1)}=-Z_3^{(1)}W_3^{(1)}+(\alpha_3-\eta)\}$

$$X_3^{(2)}=\frac{X_3^{(1)}+Z_3^{(1)}W_3^{(1)}-(\alpha_3-\eta)}{Y_3^{(1)}},\quad Y_3^{(2)}=Y_3^{(1)},\quad Z_3^{(2)}=Z_3^{(1)},\quad W_3^{(2)}=W_3^{(1)}.$$

Now, we have resolved the accessible singularity C_3.

By choosing a new coordinate system as

$$(x_3,y_3,z_3,w_3)=(-X_3^{(2)},Y_3^{(2)},Z_3^{(2)},W_3^{(2)}),$$

we can obtain the coordinate r_3.

Next, by the following steps, we can resolve the accessible singular locus C_6.

Step 0: Around the point $Q:=\{(X_8,Y_8,Z_8,W_8)|X_8=Y_8=Z_8=W_8=0\}$, we rewrite the system (2) as follows:

$$\frac{d}{dt}\begin{pmatrix} X_8 \\ Y_8 \\ Z_8 \\ W_8 \end{pmatrix}=\frac{1}{Y_1}\left\{\begin{pmatrix} \frac{2}{t(t-1)} & -\frac{\eta-(\alpha_1+\alpha_5)}{t(t-1)} & 0 & 0 \\ 0 & \frac{1}{t(t-1)} & 0 & 0 \\ \frac{1}{t(t-1)} & 0 & \frac{1}{t(t-1)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_8 \\ Y_8 \\ Z_8 \\ W_8 \end{pmatrix}+\cdots\right\}.$$

Step 1: We blow up along the curve C_6.

$$X_8^{(1)}=\frac{X_8}{Y_8},\quad Y_8^{(1)}=Y_8,\quad Z_8^{(1)}=\frac{Z_8}{Y_8},\quad W_8^{(1)}=W_8.$$

Step 2: We blow up along the surface $\{(X_8^{(1)},Y_8^{(1)},Z_8^{(1)},W_8^{(1)})|X_8^{(1)}=-Z_8^{(1)}W_8^{(1)}+\eta-(\alpha_1+\alpha_5)\}$

$$X_8^{(2)}=\frac{X_8^{(1)}+Z_8^{(1)}W_8^{(1)}-(\eta-\alpha_1-\alpha_5)}{Y_8^{(1)}},\quad Y_8^{(2)}=Y_8^{(1)},\quad Z_8^{(2)}=Z_8^{(1)},\quad W_8^{(2)}=W_8^{(1)}.$$

Now, we have resolved the accessible singularity C_6.

By choosing a new coordinate system as

$$(x_6, y_6, z_6, w_6) = (-X_8^{(2)}, Y_8^{(2)}, Z_8^{(2)}, W_8^{(2)}),$$

we can obtain the coordinate r_6.

For the remaining accessible singular loci, the proof is similar. Collecting all the cases, we have obtained the canonical coordinates r_i $(i = 0, 2, 3, 4, 6)$, which proves Proposition 3.2. □

Finally, we remark some holomorphy conditions of the system (2),(3).

Hamiltonian $H_{02} = \tilde{r}_{02}(H_{FS} - (p_1+p_2)),\ \tilde{r}_{02} : x = -((q_1-t)(p_1+p_2)-\alpha_2)(p_1+p_2),\ y = \frac{1}{p_1+p_2},\ z = -((q_1-q_2)p_2-\alpha_0)p_2,\ w = \frac{1}{p_2}$

$$\begin{aligned}
&r_0^{02} : x_0 = q_1,\ y_0 = p_1,\ z_0 = -(q_2p_2 - \alpha_0)p_2,\ w_0 = \frac{1}{p_2},\\
&r_1^{02} : x_1 = \frac{1}{q_1},\ y_1 = -((p_1 - p_2)q_1 + \alpha_1)q_1,\ z_1 = q_2 + q_1,\ w_1 = p_2,\\
&r_2^{02} : x_2 = -(q_1p_1 - \alpha_2)p_1,\ y_2 = \frac{1}{p_1},\ z_2 = q_2\ w_2 = p_2,\\
&r_3^{02} : x_3 = q_1 + \frac{q_2p_2 + \alpha_3 - (\alpha_0 + \alpha_2 + \eta)}{p_1} - \frac{t}{p_1^2},\ y_3 = p_1,\ z_3 = q_2p_1,\ w_3 = \frac{p_2}{p_1},\\
&r_4^{02} : x_4 = q_1 - q_2 + \frac{2q_2p_2 + \alpha_4 - (\alpha_0 + \alpha_2)}{p_1} + \frac{1-t}{p_1^2},\ y_4 = p_1,\ z_4 = q_2p_1^2,\ w_4 = \frac{p_2 - p_1}{p_1^2},\\
&r_5^{02} : x_5 = q_1,\ y_5 = p_1,\ z_5 = -(q_2p_2 - \alpha_5 - \alpha_0)p_2,\ w_5 = \frac{1}{p_2},\\
&r_6^{02} : x_6 = -(q_1p_1 + q_2p_2 - (\eta + \alpha_0 + \alpha_2))p_1,\ y_6 = \frac{1}{p_1},\ z_6 = q_2p_1,\ w_6 = \frac{p_2}{p_1},
\end{aligned}$$

where $r_3^{02}\left(H_{02} + \frac{1}{p_1}\right),\ r_4^{02}\left(H_{02} + \frac{1}{p_1}\right)$ (cf. (57)).

Appendix A :Reformulation of Fuji-Suzuki coupled Painlevé VI system

In this Appendix A, we will reformulate the Hamiltonian system (2),(3) by replacing its constant complex parameters α_j $(0, 1, \ldots, 5)$ and η by β_j $(j = 0, 1, \ldots, 6)$.

Define birational and symplectic transformations r_i $(i=0,1,\dots,6)$ as follows:

(24)
$$\begin{aligned}
r_0 &: (x_0,y_0,z_0,w_0)=\left(-((q_1-q_2)p_1-\beta_0)p_1,\frac{1}{p_1},q_2,p_2+p_1\right),\\
r_1 &: (x_1,y_1,z_1,w_1)=\left(\frac{1}{q_1},-(q_1p_1+\beta_1)q_1,q_2,p_2\right),\\
r_2 &: (x_2,y_2,z_2,w_2)=\left(-((q_1-t)p_1-\beta_2)p_1,\frac{1}{p_1},q_2,p_2\right),\\
r_3 &: (x_3,y_3,z_3,w_3)=\left(-(q_1p_1+q_2p_2-\beta_3)p_1,\frac{1}{p_1},q_2p_1,\frac{p_2}{p_1}\right),\\
r_4 &: (x_4,y_4,z_4,w_4)=\left(q_1,p_1,-((q_2-1)p_2-\beta_4)p_2,\frac{1}{p_2}\right),\\
r_5 &: (x_5,y_5,z_5,w_5)=\left(q_1,p_1,\frac{1}{q_2},-(p_2q_2+\beta_5)q_2\right),\\
r_6 &: (x_6,y_6,z_6,w_6)=\left(-(XY+ZW-\beta_6)Y,\frac{1}{Y},ZY,\frac{W}{Y}\right),
\end{aligned}$$

where the coordinate system (X,Y,Z,W) is given by

$$r_5\circ r_1 : (X,Y,Z,W):=\left(\frac{1}{q_1},-(q_1p_1+\alpha_1)q_1,\frac{1}{q_2},-(p_2q_2+\alpha_5)q_2\right).$$

There exist a polynomial $\tilde{H}_1$, such that the Hamiltonian system

(25)
$$\frac{dq_1}{dt}=\frac{\partial \tilde{H}_1}{\partial p_1},\ \frac{dp_1}{dt}=-\frac{\partial \tilde{H}_1}{\partial q_1},\ \frac{dq_2}{dt}=\frac{\partial \tilde{H}_1}{\partial p_2},\ \frac{dp_2}{dt}=-\frac{\partial \tilde{H}_1}{\partial q_2}$$

is transformed into a polynomial Hamiltonian system under the action of each r_i $(i=0,1,\dots,6)$, where a polynomial Hamiltonians $\tilde{H}_1$ is given by

(26)
$$\begin{aligned}
\tilde{H}_1 =&\frac{H_{VI}(q_1,p_1;\beta_2,\beta_0+\beta_4,\beta_3+\beta_5,\beta_1(\beta_1+\beta_5+\beta_6))}{(\beta_0+2\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5+\beta_6)t(t-1)}+\frac{H_{VI}(q_2,p_2;\beta_0+\beta_2,\beta_4,\beta_3+\beta_1,\beta_5(\beta_1+\beta_5+\beta_6))}{(\beta_0+2\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5+\beta_6)t(t-1)}\\
&+\frac{(q_1-t)(q_2-1)\{(q_1p_1+\beta_1)p_2+p_1(q_2p_2+\beta_5)\}}{(\beta_0+2\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5+\beta_6)t(t-1)},
\end{aligned}$$

where q_i,p_i $(i=1,2)$ denote unknown complex variables, and β_j $(j=0,1,\dots,6)$ are complex constant parameters satisfying the parameter's relation:

(27)
$$\beta_0+2\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5+\beta_6=1.$$

The relations between α_i $(i=0,1,\dots,5),\eta$ and β_j $(j=0,1,\dots,6)$ are explicitly given as follows:

(28)
$$\alpha_0=\beta_0,\ \alpha_1=\beta_1,\ \alpha_2=\beta_2,\ \alpha_3=\beta_1+\beta_3+\beta_5+\beta_6,\ \alpha_4=\beta_4,\alpha_5=\beta_5,\ \eta=\beta_1+\beta_5+\beta_6.$$

Of course, α_i and β_j satisfy the relation:

$$\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\beta_0+2\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5+\beta_6=1.$$

We remark that on new constant complex parameters β_j $(j=0,1,\dots,6)$ the Hamiltonian system (25),(26) is invariant under these birational and symplectic transformations $s_0,s_1,\dots,s_{10}$ (cf. Appendix B in [1]), whose generators are defined as follows: with *the notation* $(*):=(q_1,p_1,q_2,p_2,t;\beta_0,\beta_1,\dots,\beta_6)$;

$$
\begin{aligned}
&s_0:(*)\rightarrow\left(q_1,p_1-\frac{\beta_0}{q_1-q_2},q_2,p_2+\frac{\beta_0}{q_1-q_2},t;-\beta_0,\beta_1+\beta_0,\beta_2,\beta_3-\beta_0,\beta_4,\beta_5+\beta_0,\beta_6-\beta_0\right),\\
&s_1:(*)\rightarrow\left(q_1+\frac{\beta_1}{p_1},p_1,q_2,p_2,t;\beta_0+\beta_1,-\beta_1,\beta_2+\beta_1,\beta_3+\beta_1,\beta_4,\beta_5,\beta_6+\beta_1\right),\\
&s_2:(*)\rightarrow\left(q_1,p_1-\frac{\beta_2}{q_1-t},q_2,p_2,t;\beta_0,\beta_1+\beta_2,-\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\right),\\
&s_3:(*)\rightarrow(q_1+\frac{(\beta_1+\beta_3+\beta_5+\beta_6)q_1}{q_1p_1+q_2p_2-\beta_3},p_1-\frac{(\beta_1+\beta_3+\beta_5+\beta_6)p_1}{q_1p_1+q_2p_2+\beta_1+\beta_5+\beta_6},\\
&\qquad q_2+\frac{(\beta_1+\beta_3+\beta_5+\beta_6)q_2}{q_1p_1+q_2p_2-\beta_3},p_2-\frac{(\beta_1+\beta_3+\beta_5+\beta_6)p_2}{q_1p_1+q_2p_2+\beta_1+\beta_5+\beta_6},t;\beta_0,\beta_1,\\
&\qquad \beta_1+\beta_2+\beta_3+\beta_5+\beta_6,-\beta_1-\beta_5-\beta_6,\beta_1+\beta_3+\beta_4+\beta_5+\beta_6,\beta_5,-\beta_1-\beta_3-\beta_5),\\
&s_4:(*)\rightarrow\left(q_1,p_1,q_2,p_2-\frac{\beta_4}{q_2-1},t;\beta_0,\beta_1,\beta_2,\beta_3,-\beta_4,\beta_5+\beta_4,\beta_6\right),\\
&s_5:(*)\rightarrow\left(q_1,p_1,q_2+\frac{\beta_5}{p_2},p_2,t;\beta_0+\beta_5,\beta_1,\beta_2,\beta_3+\beta_5,\beta_4+\beta_5,-\beta_5,\beta_6+\beta_5\right),\\
&s_6:(*)\rightarrow\left(\frac{t}{q_2},-\frac{(p_2q_2+\beta_5)q_2}{t},\frac{t}{q_1},-\frac{(p_1q_1+\beta_1)q_1}{t},t;\beta_0,\beta_5,\beta_4,\beta_6,\beta_2,\beta_1,\beta_3\right),\\
&s_7:(*)\rightarrow\left(\frac{1}{q_1},-(p_1q_1+\beta_1)q_1,\frac{1}{q_2},-(p_2q_2+\beta_5)q_2,\frac{1}{t};\beta_0,\beta_1,\beta_2,\beta_6,\beta_4,\beta_5,\beta_3\right),\\
&s_8:(*)\rightarrow\left(\frac{q_2}{t},tp_2,\frac{q_1}{t},tp_1,\frac{1}{t};\beta_0,\beta_5,\beta_4,\beta_3,\beta_2,\beta_1,\beta_6\right),\\
&s_9:(*)\rightarrow(\frac{1}{q_1},-(p_1q_1+p_2q_2+\beta_1+\beta_5+\beta_6)q_1,\frac{q_2}{q_1},p_2q_1,\frac{1}{t};\beta_4,\beta_1+\beta_3+\beta_5+\beta_6,\beta_2,\\
&\qquad -\beta_5-\beta_6,\beta_0,\beta_5,-\beta_3-\beta_5),\\
&s_{10}:(*)\rightarrow(\frac{t}{q_2},-\frac{(p_2q_2+p_1q_1+\beta_1+\beta_5+\beta_6)q_2}{t},\frac{q_1}{q_2},p_1q_2,t;\beta_2,\beta_1+\beta_3+\beta_5+\beta_6,\beta_4,\\
&\qquad -\beta_1-\beta_6,\beta_0,\beta_1,-\beta_1-\beta_3).
\end{aligned}
\tag{29}
$$

We note that the subgroup $<s_0,s_1,\dots,s_5>$ generated by $s_0,s_1,\dots,s_5$ is isomorphic to the affine Weyl group of type $A_5^{(1)}$ (see Appendix B in [1]), and the transformation s_6 was found by Professor K. Fuji in Kobe university in August 2012.

We also remark that these transformations s_i satisfies the following relations:

$$s_{10}=s_8\circ s_9,\quad s_6=s_8\circ s_7,\quad s_k^2=1\ (k=0,1,\dots,9),\quad s_{10}^3=1.$$

Finally, let us define the following translation operators:

$$T_1:=(s_2s_{10}s_{10}s_1)^4,\quad T_2:=s_1T_1s_1,\quad T_3:=s_5T_1s_5.$$

These translation operators act on parameters β_i as follows:

$$\begin{aligned}
T_1(\beta_0,\beta_1,\dots,\beta_6) =&(\beta_0,\beta_1,\dots,\beta_6)+(0,-1,1,0,-1,1,0),\\
T_2(\beta_0,\beta_1,\dots,\beta_6) =&(\beta_0,\beta_1,\dots,\beta_6)+(-1,1,0,-1,-1,1,-1),\\
T_3(\beta_0,\beta_1,\dots,\beta_6) =&(\beta_0,\beta_1,\dots,\beta_6)+(1,-1,1,1,0,-1,1).
\end{aligned}$$

Appendix B :Searching for the Bäcklund transformation s_3

In this Appendix B, we will make Fuji-Suzuki's Bäcklund transformation s_3 in (29) by our method.

The key property is given as follows (see [34, 35]):

$$\begin{aligned}
&r:\left(-(XY+ZW-\beta),\frac{1}{Y},ZY,\frac{W}{Y}\right) \Longleftrightarrow s:\left(X,Y+\frac{ZW-\beta}{X},\frac{Z}{X},WX\right),\\
&r':\left(\frac{1}{X},-(XY+ZW+\beta)X,\frac{Z}{X},WX\right) \Longleftrightarrow s':\left(X+\frac{ZW+\beta}{Y},Y,ZY,\frac{W}{Y}\right).
\end{aligned} \tag{30}$$

These transformations r, r', s and s' are birational and symplectic, however, these are not auto-Bäcklund transformations. These transformations can be considered as a relation between symmetry and holomorphy conditions appearing in the Garnier system (see [34, 35]).

Equations	Relation between symmetry and holomorphy conditions
Painlevé equations	(34)
Garnier systems	(30)

At first, for the system (25),(26), we will make the above transformation.

Proposition 3.3. *The birational and symplectic transformation*

$$S:(q_1,p_1,q_2,p_2)\rightarrow\left(q_1,p_1+\frac{q_2p_2-\beta_3}{q_1},\frac{q_2}{q_1},p_2q_1\right) \tag{31}$$

takes the system (25),(26) *to a Hamiltonian system*

$$\frac{dq_1}{dt}=\frac{\partial H_2}{\partial p_1},\ \frac{dp_1}{dt}=-\frac{\partial H_2}{\partial q_1},\ \frac{dq_2}{dt}=\frac{\partial H_2}{\partial p_2},\ \frac{dp_2}{dt}=-\frac{\partial H_2}{\partial q_2} \tag{32}$$

with the polynomial Hamiltonian:

(33)

$$
\begin{aligned}
&(\beta_0+2\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5+\beta_6)t(t-1)H_2 = \\
&tp_2p_1-q_1p_1-p_2q_1p_1+q_1^2p_1+tq_1p_1^2-q_1^2p_1^2-tq_1^2p_1^2+q_1^3p_1^2+p_2^2q_2-p_2q_1q_2-tp_2p_1q_2 \\
&+p_2q_1p_1q_2+tp_2q_1p_1q_2-p_2q_1^2p_1q_2-p_2^2q_2^2-tp_2^2q_2^2+p_2q_1q_2^2-tp_2q_1p_1q_2^2+p_2q_1^2p_1q_2^2+tp_2^2q_2^3 \\
&+tq_1p_1\beta_0-q_1^2p_1\beta_0+p_2q_2\beta_0-tp_2q_2\beta_0+p_2q_1q_2\beta_0-p_2q_1q_2^2\beta_0-p_2\beta_1+p_2q_2\beta_1+p_2q_1q_2\beta_1 \\
&-p_2q_1q_2^2\beta_1+q_1\beta_1^2+q_1p_1\beta_2-q_1^2p_1\beta_2+p_2q_1q_2\beta_2-p_2q_1q_2^2\beta_2-\beta_3-p_2\beta_3+q_1\beta_3+tp_1\beta_3 \\
&-q_1p_1\beta_3-tq_1p_1\beta_3+q_1^2p_1\beta_3+p_2q_2\beta_3+tp_2q_2\beta_3-tp_2q_2^2\beta_3+t\beta_0\beta_3-q_1\beta_0\beta_3+\beta_2\beta_3 \\
&-q_1\beta_2\beta_3+tq_1p_1\beta_4-q_1^2p_1\beta_4+p_2q_1q_2\beta_4-p_2q_1q_2^2\beta_4+t\beta_3\beta_4-q_1\beta_3\beta_4+tq_1p_1\beta_5-q_1^2p_1\beta_5 \\
&-tp_2q_2\beta_5+p_2q_1q_2\beta_5-tq_1p_1q_2\beta_5+q_1^2p_1q_2\beta_5+tp_2q_2^2\beta_5-p_2q_1q_2^2\beta_5+q_1\beta_1\beta_5+q_1q_2\beta_1\beta_5 \\
&+t\beta_3\beta_5-q_1\beta_3\beta_5-tq_2\beta_3\beta_5+q_1q_2\beta_3\beta_5+q_1q_2\beta_5^2+q_1\beta_1\beta_6+q_1q_2\beta_5\beta_6.
\end{aligned}
$$

For this system (32),(33), we can find the holomorphy condition:

$$R_3:(X_1,Y_1,Z_1,W_1)=\left(\frac{1}{q_1},-(q_1p_1+\beta_1+\beta_3+\beta_5+\beta_6)q_1,q_2,p_2\right).$$

Next, we will explain how to find the Bäcklund transformation s_3. Here, let us review the relation between symmetry and holomorphy (see [25]):

$$r:\left(\frac{1}{X},-(YX+\beta),Z,W\right)\Longleftrightarrow s:\left(X+\frac{\beta}{Y},Y,Z,W\right). \tag{34}$$

By using this method, we can obtain the following Bäcklund transformation for this system (32),(33).

Proposition 3.4. *The system* (32),(33) *is invariant under the following birational and symplectic transformation* :

$$
\begin{aligned}
S_3:(q_1,p_1,q_2,p_2,t;\beta_0,\beta_1,\dots,\beta_6)\rightarrow&\left(q_1+\frac{\beta_1+\beta_3+\beta_5+\beta_6}{p_1},p_1,q_2,p_2,t;\beta_0,\beta_1,\right. \\
&\beta_1+\beta_2+\beta_3+\beta_5+\beta_6,-\beta_1-\beta_5-\beta_6, \\
&\left.\beta_1+\beta_3+\beta_4+\beta_5+\beta_6,\beta_5,-\beta_1-\beta_3-\beta_5\right).
\end{aligned} \tag{35}
$$

Pulling back the transformation S_3 by the birational transformation (31), we can obtain Fuji-Suzuki's Bäcklund transformation s_3 in (29).

Appendix C :Holomorphy conditions of type III

	Holomorphy cond. of type a	Holomorphy cond. of type b
(37)	r_1,r_3,r_5	r_0,r_2,r_4
(41)	r_0,r_2,r_4	r_1,r_3,r_5
Type of Accessible sing.	$\mathbb{P}^1$	$\mathbb{P}^1$
Type of Local index	$(2,1,0,1)$	$(2,1,1,0)$

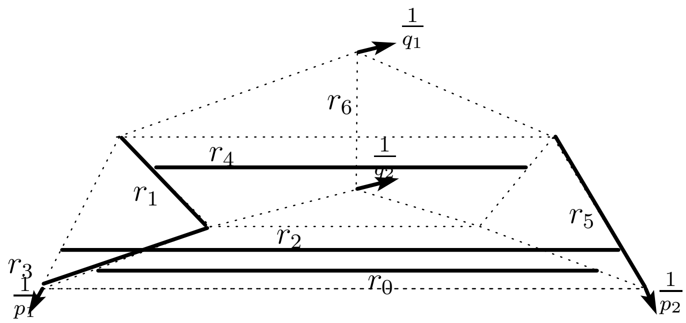
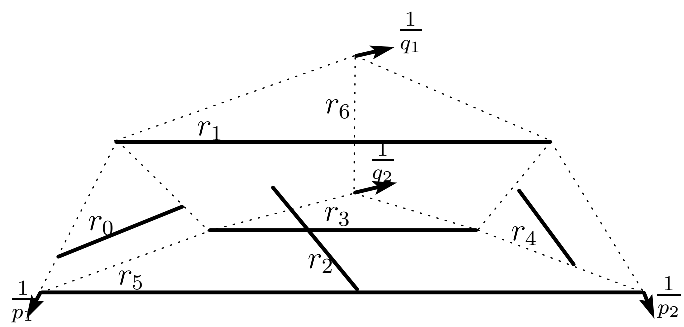
<table>
<tr><th>System</th><th>Accessible singular loci</th><th>Partition of Accessible singular loci</th></tr>
<tr><td>(37)</td><td>
</td><td>$(a, b) = (3, 3)$</td></tr>
<tr><td>(41)</td><td>
</td><td>$(a, b) = (3, 3)$</td></tr>
</table>

FIGURE 6. This figure is the Hirzebruch manifold defined by H. Kimura (see [13]). The bold lines denote some accessible singular loci for each system. Both systems are transformed by the birational transformation (43). We also note that both types a,b are exchanged by the birational transformation (43).

In this appendix, at first we will give some holomorphy conditions for the Hamiltonian system transformed the system (2),(3) by the birational transformation;

$$sr_6 : (Q_1, P_1, Q_2, P_2) = \left(q_1 + \frac{q_2p_2 + \eta}{p_1}, p_1, \frac{p_2}{p_1}, -q_2p_1\right). \tag{36}$$

Define birational and symplectic transformations r_i $(i=0,1,\dots,6)$ as follows:

$$
\begin{aligned}
r_0 &: (x_0,y_0,z_0,w_0)=\left(-(Q_1P_1+(Q_2+1)P_2-\gamma_0)P_1,\frac{1}{P_1},(Q_2+1)P_1,\frac{P_2}{P_1}\right),\\
r_2 &: (x_2,y_2,z_2,w_2)=\left(-((Q_1-t)P_1+Q_2P_2-\gamma_2)P_1,\frac{1}{P_1},Q_2P_1,\frac{P_2}{P_1}\right),\\
r_3 &: (x_3,y_3,z_3,w_3)=\left(-(Q_1P_1-\gamma_3)P_1,\frac{1}{P_1},Q_2,P_2\right),\\
r_5 &: (x_5,y_5,z_5,w_5)=\left(Q_1,P_1,-(Q_2P_2-\gamma_5)P_2,\frac{1}{P_2}\right),\\
r_6 &: (x_6,y_6,z_6,w_6)=\left(\frac{1}{Q_1},-(Q_1P_1+Q_2P_2+\gamma_6)Q_1,\frac{Q_2}{Q_1},P_2Q_1\right),\\
r_1 &: (x_1,y_1,z_1,w_1)=\left(-(x_6y_6-\gamma_1)y_6,\frac{1}{y_6},z_6,w_6\right),\\
r_4 &: (x_4,y_4,z_4,w_4)=\left(-(x_6y_6+(z_6-1)w_6-\gamma_4)y_6,\frac{1}{y_6},(z_6-1)y_6,\frac{w_6}{y_6}\right).
\end{aligned}
\tag{37}
$$

There exist a polynomial $\tilde{H}_1$, such that the Hamiltonian system

$$
\frac{dQ_1}{dt}=\frac{\partial \tilde{H}_1}{\partial P_1},\ \frac{dP_1}{dt}=-\frac{\partial \tilde{H}_1}{\partial Q_1},\ \frac{dQ_2}{dt}=\frac{\partial \tilde{H}_1}{\partial P_2},\ \frac{dP_2}{dt}=-\frac{\partial \tilde{H}_1}{\partial Q_2}
\tag{38}
$$

is transformed into the polynomial Hamiltonian $\tilde{H}_1(Q_1,P_1,Q_2,P_2)=sr_6(H_{FS}(q_1,p_1,q_2,p_2))$.

The relations between α_i $(i=0,1,\dots,5),\eta$ and γ_j $(j=0,1,\dots,6)$ are explicitly given as follows:

$$
\begin{aligned}
&\gamma_0=\alpha_0+\eta, \quad \gamma_1=\alpha_1, \quad \gamma_2=\alpha_2+\eta, \quad \gamma_3=\alpha_3\\
&\gamma_4=\alpha_4+\eta, \quad \gamma_5=\alpha_5, \quad \gamma_6=-\eta.
\end{aligned}
\tag{39}
$$

Next, we will give some holomorphy conditions for the Hamiltonian system transformed the system (2),(3) by the birational transformation;

$$
rr_6 : (\tilde{Q}_1,\tilde{P}_1,\tilde{Q}_2,\tilde{P}_2)=\left(\frac{q_1}{q_2},p_1q_2,\frac{1}{q_2},-(q_2p_2+q_1p_1+\eta)q_2\right).
\tag{40}
$$

Define birational and symplectic transformations r_i $(i = 0, 1, \dots, 6)$ as follows:

$$\begin{aligned}
r_0 &: (x_0, y_0, z_0, w_0) = \left(-((\tilde{Q}_1 - 1)\tilde{P}_1 - (\gamma_0 + \gamma_6))\tilde{P}_1, \frac{1}{\tilde{P}_1}, \tilde{Q}_2, \tilde{P}_2\right), \\
r_2 &: (x_2, y_2, z_2, w_2) = \left(-((\tilde{Q}_1 - t\tilde{Q}_2)\tilde{P}_1 - (\gamma_2 + \gamma_6))\tilde{P}_1, \frac{1}{\tilde{P}_1}, \tilde{Q}_2, \tilde{P}_2 + t\tilde{P}_1\right), \\
r_4 &: (x_4, y_4, z_4, w_4) = \left(\tilde{Q}_1, \tilde{P}_1, -((\tilde{Q}_2 - 1)\tilde{P}_2 - (\gamma_4 + \gamma_6))\tilde{P}_2, \frac{1}{\tilde{P}_2}\right), \\
r_5 &: (x_5, y_5, z_5, w_5) = \left(-(\tilde{Q}_1\tilde{P}_1 + \tilde{Q}_2\tilde{P}_2 - (\gamma_5 + \gamma_6))\tilde{P}_1, \frac{1}{\tilde{P}_1}, \tilde{Q}_2\tilde{P}_1, \frac{\tilde{P}_2}{\tilde{P}_1}\right), \\
r_6 &: (x_6, y_6, z_6, w_6) = \left(\frac{\tilde{Q}_1}{\tilde{Q}_2}, \tilde{P}_1\tilde{Q}_2, \frac{1}{\tilde{Q}_2}, -(\tilde{Q}_2\tilde{P}_2 + \tilde{Q}_1\tilde{P}_1 - \gamma_6)\tilde{Q}_2\right), \\
r_1 &: (x_1, y_1, z_1, w_1) = \left(-(XY + ZW - (\gamma_1 + \gamma_6))Y, \frac{1}{Y}, ZY, \frac{W}{Y}\right), \\
r_3 &: (x_3, y_3, z_3, w_3) = \left(-(x_6y_6 + z_6w_6 - (\gamma_3 + \gamma_6))y_6, \frac{1}{y_6}, z_6y_6, \frac{w_6}{y_6}\right),
\end{aligned} \tag{41}$$

where the coordinate system (X, Y, Z, W) is given by

$$R_6 : (X, Y, Z, W) = \left(\frac{1}{\tilde{Q}_1}, -(\tilde{Q}_1\tilde{P}_1 + \tilde{Q}_2\tilde{P}_2 - \gamma_6)\tilde{Q}_1, \frac{\tilde{Q}_2}{\tilde{Q}_1}, \tilde{P}_2\tilde{Q}_1\right).$$

There exist a polynomial $\tilde{H}_6$, such that the Hamiltonian system

$$\frac{d\tilde{Q}_1}{dt} = \frac{\partial \tilde{H}_6}{\partial \tilde{P}_1}, \ \frac{d\tilde{P}_1}{dt} = -\frac{\partial \tilde{H}_6}{\partial \tilde{Q}_1}, \ \frac{d\tilde{Q}_2}{dt} = \frac{\partial \tilde{H}_6}{\partial \tilde{P}_2}, \ \frac{d\tilde{P}_2}{dt} = -\frac{\partial \tilde{H}_6}{\partial \tilde{Q}_2} \tag{42}$$

is transformed into the polynomial Hamiltonian $\tilde{H}_6(\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) = rr_6(H_{FS}(q_1, p_1, q_2, p_2))$ with parameter relations (39).

We note that the condition r_2 should be read that $r_2(K - \tilde{P}_1\tilde{Q}_2)$ is a polynomial with respect to x_2, y_2, z_2, w_2.

We show that both systems (38),(42) are transformed by the birational transformation;

$$Tr : (\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2) = \left(-\left(Q_2 + \frac{Q_1P_1 + \gamma_6}{P_2}\right), -P_2, -\frac{P_1}{P_2}, Q_1P_2\right). \tag{43}$$

Holomorphy conditions	Parameter of (37)	Parameter of (41)
r_0	γ_0	$\gamma_0 + \gamma_6$
r_1	γ_1	$\gamma_1 + \gamma_6$
r_2	γ_2	$\gamma_2 + \gamma_6$
r_3	γ_3	$\gamma_3 + \gamma_6$
r_4	γ_4	$\gamma_4 + \gamma_6$
r_5	γ_5	$\gamma_5 + \gamma_6$
r_6	γ_6	$-\gamma_6$

4. Appendix D

Holomorphy conditions

Define birational and symplectic transformations r_i $(i=0,1,\dots,6)$ as follows:

$$
\begin{aligned}
&r_0 : (x_0,y_0,z_0,w_0) = \left(-((q_1-q_2)p_1-\beta_0)p_1, \frac{1}{p_1}, q_2, p_2+p_1\right),\\
&r_1 : (x_1,y_1,z_1,w_1) = \left(\frac{1}{q_1}, -(q_1p_1+\beta_1)q_1, q_2, p_2\right),\\
&r_2 : (x_2,y_2,z_2,w_2) = \left(-((q_1-t)p_1-\beta_2)p_1, \frac{1}{p_1}, q_2, p_2\right),\\
&r_3 : (x_3,y_3,z_3,w_3) = \left(-(q_1p_1+q_2p_2-\beta_3)p_1, \frac{1}{p_1}, q_2p_1, \frac{p_2}{p_1}\right),\\
&r_4 : (x_4,y_4,z_4,w_4) = \left(q_1, p_1, -((q_2-s)p_2-\beta_4)p_2, \frac{1}{p_2}\right),\\
&r_5 : (x_5,y_5,z_5,w_5) = \left(q_1, p_1, \frac{1}{q_2}, -(p_2q_2+\beta_5)q_2\right),\\
&r_6 : (x_6,y_6,z_6,w_6) = \left(-((q_1-1)p_1+(q_2-1)p_2-\beta_6)p_1, \frac{1}{p_1}, (q_2-1)p_1, \frac{p_2}{p_1}\right).
\end{aligned}
\tag{44}
$$

There exist two polynomials H_1^{FS} and H_2^{FS}, such that the Hamiltonian system

$$
\begin{cases}
dq_1 = \dfrac{\partial H_1^{FS}}{\partial p_1}dt + \dfrac{\partial H_2^{FS}}{\partial p_1}ds, \quad dp_1 = -\dfrac{\partial H_1^{FS}}{\partial q_1}dt - \dfrac{\partial H_2^{FS}}{\partial q_1}ds,\\
dq_2 = \dfrac{\partial H_1^{FS}}{\partial p_2}dt + \dfrac{\partial H_2^{FS}}{\partial p_2}ds, \quad dp_2 = -\dfrac{\partial H_1^{FS}}{\partial q_2}dt - \dfrac{\partial H_2^{FS}}{\partial q_2}ds
\end{cases}
\tag{45}
$$

is transformed into a polynomial Hamiltonian system under the action of each r_i $(i=0,1,\dots,6)$, where two polynomial Hamiltonians H_1^{FS}, H_2^{FS} are given by (cf. [1, 2, 3])

$$
\begin{aligned}
H_1^{FS} &= H_{VI}(q_1,p_1,t,s;\beta_2,\beta_0+\beta_4,\beta_1,\beta_5+\beta_6,\beta_3+\beta_5)\\
&\quad + H_{VI}(q_2,p_2,t,s;\beta_0+\beta_2,\beta_4,\beta_5,\beta_1+\beta_6,\beta_1+\beta_3)\\
&\quad + \frac{(q_1-t)(q_2-s)\{2(q_1p_1+\beta_1)(q_2p_2+\beta_5)-(q_1p_1+\beta_1)p_2-(q_2p_2+\beta_5)p_1\}}{(\beta_0+2\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5+\beta_6)t(t-1)(t-s)}\\
&\quad - \frac{2\beta_1\beta_5 s}{(\beta_0+2\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5+\beta_6)(t-1)(t-s)},\\
H_2^{FS} &= \pi(H_1^{FS}), \quad \pi = \{q_1 \leftrightarrow q_2,\ p_1 \leftrightarrow p_2,\ t \leftrightarrow s,\ \beta_1 \leftrightarrow \beta_5,\ \beta_2 \leftrightarrow \beta_4\}.
\end{aligned}
\tag{46}
$$

The symbol $H_{VI}(q,p,t,\eta;\gamma_0,\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ denotes (see [31])

(47)
$$\begin{aligned}
&t(t-1)(t-\eta)H_{VI}(q,p,t,\eta;\gamma_0,\gamma_1,\gamma_2,\gamma_3,\gamma_4)\\
&=q(q-1)(q-\eta)(q-t)p^2+\{\gamma_1(t-\eta)q(q-1)+2\gamma_2 q(q-1)(q-\eta)\\
&+\gamma_3(t-1)q(q-\eta)+\gamma_4 t(q-1)(q-\eta)\}p\\
&+\gamma_2\{(\gamma_1+\gamma_2)(t-\eta)+\gamma_2(q-1)+\gamma_3(t-1)+t\gamma_4\}q, \quad (\gamma_0+\gamma_1+2\gamma_2+\gamma_3+\gamma_4=1).
\end{aligned}$$

We note that the holomorphy conditions should be read that in the Hamiltonian H_1^{FS}

$$r_2(H_1^{FS}-p_1)$$

are polynomials with respect to x_2,y_2,z_2,w_2, and in the Hamiltonian H_2^{FS}

$$r_4(H_2^{FS}-p_2)$$

are polynomials with respect to x_4,y_4,z_4,w_4.

We see that the birational and symplectic transformation φ:

$$\left\{\begin{aligned}
Q_1 &=\frac{1-q_1}{q_1}, \quad P_1=-(p_1q_1+\beta_1)q_1, \quad Q_2=\frac{1-q_2}{q_2}, \quad P_2=-(p_2q_2+\beta_5)q_2,\\
T &=\frac{1-t}{t}, \quad S=\frac{1-s}{s}
\end{aligned}\right. \tag{48}$$

takes the system (45) into Fuji-Suzuki system (see [1]) when $S=1$.

We remark that the relations between α_i $(i=0,1,\dots,5),\eta$ (see [1]) and β_j $(j=0,1,\dots,6)$ are explicitly given as follows:

$$\begin{aligned}
&\alpha_3=\beta_1+\beta_3+\beta_5+\beta_6, \quad \eta=\beta_1+\beta_3+\beta_5,\\
&\alpha_0=\beta_0,\ \alpha_1=\beta_1,\ \alpha_2=\beta_2,\ \alpha_4=\beta_4,\alpha_5=\beta_5.
\end{aligned} \tag{49}$$

Of course, α_i and β_j satisfy the relation:

$$\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\beta_0+2\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5+\beta_6=1. \tag{50}$$

Completely integrable

PROPOSITION 4.1. *Setting*

$$K_1:=-H_1+\frac{\beta_3(\beta_1+\beta_5)(\mathrm{Log(s-t)}-\mathrm{Log(s-1)})}{(\beta_0+2\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5+\beta_6)(t-1)^2}, \quad K_2:=-H_2. \tag{51}$$

Two Hamiltonians K_1 and K_2 satisfy

$$\{K_1,K_2\}+\left(\frac{\partial}{\partial s}\right)K_1-\left(\frac{\partial}{\partial t}\right)K_2=0, \tag{52}$$

where $\{,\}$ denotes the Poisson brackets:

$$\{L_1,L_2\}=\frac{\partial L_1}{\partial p_1}\frac{\partial L_2}{\partial q_1}-\frac{\partial L_1}{\partial q_1}\frac{\partial L_2}{\partial p_1}+\frac{\partial L_1}{\partial p_2}\frac{\partial L_2}{\partial q_2}-\frac{\partial L_1}{\partial q_2}\frac{\partial L_2}{\partial p_2}. \tag{53}$$

We remark that on new constant complex parameters β_j $(j=0,1,\dots,6)$ the Hamiltonian system (45) is invariant under these birational and symplectic transformations $s_0,s_1,\dots,s_9$ (cf. Appendix B in [1]), whose generators are defined as follows: with *the notation* $(*):=(q_1,p_1,q_2,p_2,t,s;\beta_0,\beta_1,\dots,\beta_6)$;

(54)

$$
\begin{aligned}
&s_0:(*)\to\left(q_1,p_1-\frac{\beta_0}{q_1-q_2},q_2,p_2+\frac{\beta_0}{q_1-q_2},t,s;-\beta_0,\beta_1+\beta_0,\beta_2,\beta_3-\beta_0,\beta_4,\beta_5+\beta_0,\beta_6-\beta_0\right),\\
&s_1:(*)\to\left(q_1+\frac{\beta_1}{p_1},p_1,q_2,p_2,t,s;\beta_0+\beta_1,-\beta_1,\beta_2+\beta_1,\beta_3+\beta_1,\beta_4,\beta_5,\beta_6+\beta_1\right),\\
&s_2:(*)\to\left(q_1,p_1-\frac{\beta_2}{q_1-t},q_2,p_2,t,s;\beta_0,\beta_1+\beta_2,-\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\right),\\
&s_3:(*)\to(\frac{q_1(-q_1p_1+q_1^2p_1-p_2q_2+p_2q_2^2-\beta_1+q_1\beta_1-\beta_5+q_2\beta_5-\beta_6)}{g_1},\\
&-\frac{g_1(q_1p_1^2-q_1^2p_1^2+p_2p_1q_2-p_2p_1q_2^2+\beta_1^2-p_1\beta_3+q_1p_1\beta_3+\beta_1\beta_3+q_1p_1\beta_5-p_1q_2\beta_5+\beta_1\beta_5+q_1p_1\beta_6+\beta_1\beta_6)}{(-q_1p_1+q_1^2p_1-p_2q_2+p_2q_2^2+q_1\beta_1+\beta_3+q_2\beta_5)(-q_1p_1+q_1^2p_1-p_2q_2+p_2q_2^2-\beta_1+q_1\beta_1-\beta_5+q_2\beta_5-\beta_6)},\\
&\frac{q_2(-q_1p_1+q_1^2p_1-p_2q_2+p_2q_2^2-\beta_1+q_1\beta_1-\beta_5+q_2\beta_5-\beta_6)}{g_2},\\
&-\frac{g_2(p_2q_1p_1-p_2q_1^2p_1+p_2^2q_2-p_2^2q_2^2-p_2q_1\beta_1+p_2q_2\beta_1-p_2\beta_3+p_2q_2\beta_3+\beta_1\beta_5+\beta_3\beta_5+\beta_5^2+p_2q_2\beta_6+\beta_5\beta_6)}{(-q_1p_1+q_1^2p_1-p_2q_2+p_2q_2^2+q_1\beta_1+\beta_3+q_2\beta_5)(-q_1p_1+q_1^2p_1-p_2q_2+p_2q_2^2-\beta_1+q_1\beta_1-\beta_5+q_2\beta_5-\beta_6)},\\
&t,s;\beta_0,\beta_1,\beta_1+\beta_2+\beta_3+\beta_5+\beta_6,-\beta_1-\beta_5-\beta_6,\beta_1+\beta_3+\beta_4+\beta_5+\beta_6,\beta_5,-\beta_1-\beta_3-\beta_5),\\
&s_4:(*)\to\left(q_1,p_1,q_2,p_2-\frac{\beta_4}{q_2-s},t,s;\beta_0,\beta_1,\beta_2,\beta_3,-\beta_4,\beta_5+\beta_4,\beta_6\right),\\
&s_5:(*)\to\left(q_1,p_1,q_2+\frac{\beta_5}{p_2},p_2,t,s;\beta_0+\beta_5,\beta_1,\beta_2,\beta_3+\beta_5,\beta_4+\beta_5,-\beta_5,\beta_6+\beta_5\right),\\
&s_6:(*)\to(1-q_1,-p_1,1-q_2,-p_2,1-t,1-s;\beta_0,\beta_1,\beta_2,\beta_6,\beta_4,\beta_5,\beta_3),\\
&s_7:(*)\to(q_2,p_2,q_1,p_1,s,t;\beta_0,\beta_5,\beta_4,\beta_3,\beta_2,\beta_1,\beta_6),\\
&s_8:(*)\to(1-q_1,\frac{q_1p_1-q_1^2p_1+p_2q_2-p_2q_2^2-\beta_3+q_1\beta_3+q_1\beta_5-q_2\beta_5+q_1\beta_6}{(-1+q_1)q_1},\frac{(-1+s)(-1+q_1)q_2}{sq_1+q_2-sq_2-q_1q_2},\\
&-\frac{(sq_1+q_2-sq_2-q_1q_2)(-sp_2q_1-p_2q_2+sp_2q_2+p_2q_1q_2-\beta_5+s\beta_5+q_1\beta_5)}{(-1+s)s(-1+q_1)q_1},\\
&1-t,1-s;\beta_4,\beta_1+\beta_3+\beta_5+\beta_6,\beta_2,-\beta_5-\beta_6,\beta_0,\beta_5,-\beta_3-\beta_5),\\
&s_9:=s_7\circ s_8\quad((s_9)^6=1),
\end{aligned}
$$

where

$$
\begin{aligned}
g_1&:=-q_1p_1+q_1^2p_1-p_2q_2+p_2q_2^2+\beta_3-q_1\beta_3-q_1\beta_5+q_2\beta_5-q_1\beta_6,\\
g_2&:=-q_1p_1+q_1^2p_1-p_2q_2+p_2q_2^2+q_1\beta_1-q_2\beta_1+\beta_3-q_2\beta_3-q_2\beta_6.
\end{aligned}
$$

We note that the subgroup $<s_0,s_1,\dots,s_5>$ generated by $s_0,s_1,\dots,s_5$ is isomorphic to the affine Weyl group of type $A_5^{(1)}$ (see Appendix B in [1]), and the transformation s_6 was found by Professor K. Fuji in Kobe university in August 2012.

Finally, let us define the following translation operators:

$$T_1:=(s_2s_9s_9s_1)^4,\quad T_2:=s_1T_1s_1,\quad T_3:=s_5T_1s_5. \tag{55}$$

These translation operators act on parameters β_i as follows:

$$\begin{aligned} T_1(\beta_0,\beta_1,\dots,\beta_6) =&(\beta_0,\beta_1,\dots,\beta_6)+(0,-1,1,0,-1,1,0),\\ T_2(\beta_0,\beta_1,\dots,\beta_6) =&(\beta_0,\beta_1,\dots,\beta_6)+(-1,1,0,-1,-1,1,-1),\\ T_3(\beta_0,\beta_1,\dots,\beta_6) =&(\beta_0,\beta_1,\dots,\beta_6)+(1,-1,1,1,0,-1,1). \end{aligned} \tag{56}$$

Finally, we remark some holomorphy conditions of the system (45).

Hamiltonians $H_{04}^{(1)}=r_{04}(H_1^{FS}),\ H_{04}^{(2)}=r_{04}(H_2^{FS}-p_1-p_2),\ r_{04}:x=-((q_1-q_2)p_1-\beta_0)p_1,\ y=\frac{1}{p_1},\ z=-((q_2-s)(p_2+p_1)-\beta_4)(p_2+p_1),\ w=\frac{1}{p_2+p_1}$

$$\begin{aligned}
&r_0^{04}:x_0=-(q_1p_1-\beta_0)p_1,\ y_0=\frac{1}{p_1},\ z_0=q_2,\ w_0=p_2,\\
&r_1^{04}:x_1=-(q_1p_1-\beta_1-\beta_0)p_1,\ y_1=\frac{1}{p_1},\ z_1=q_2,\ w_1=p_2,\\
&r_2^{04}:x_2=q_1-q_2+\frac{2q_2p_2+\beta_2-(\beta_0+\beta_4)}{p_1}+\frac{t-s}{p_1^2},\ y_2=p_1,\ z_2=q_2p_1^2\ w_2=\frac{p_2-p_1}{p_1^2},\\
&r_3^{04}:x_3=q_1p_2,\ y_3=\frac{p_1}{p_2},\ z_3=q_2+\frac{q_1p_1+\beta_3-(\beta_0+\beta_4)}{p_2}-\frac{s}{p_2^2},\ w_3=p_2,\\
&r_4^{04}:x_4=q_1,\ y_4=p_1,\ z_4=-(q_2p_2-\beta_4)p_2,\ w_4=\frac{1}{p_2},\\
&r_5^{04}:x_5=\frac{1}{q_1},\ y_5=-((p_1-p_2)q_1+\beta_5)q_1,\ z_5=q_2+q_1,\ w_5=p_2,\\
&r_6^{04}:x_6=q_1p_2,\ y_6=\frac{p_1}{p_2},\ z_6=q_2+\frac{q_1p_1+\beta_6-(\beta_0+\beta_4)}{p_2}-\frac{s-1}{p_2^2},\ w_6=p_2,
\end{aligned}$$

where $r_2^{04}\left(H_{04}^{(1)}-\frac{1}{p_1}\right),\ r_2^{04}\left(H_{04}^{(2)}+\frac{1}{p_1}\right),\ r_3^{04}\left(H_{04}^{(2)}+\frac{1}{p_2}\right),\ r_6^{04}\left(H_{04}^{(2)}+\frac{1}{p_2}\right)$.

Hamiltonians $H_{045}^{(1)}=r_5^{04}(H_{04}^{(1)}),\ H_{045}^{(2)}=r_5^{04}(H_{04}^{(2)})$

$$
\begin{aligned}
&r_0^{045}: x_0=\frac{1}{q_1},\ y_0=-(p_1q_1+\beta_0+\beta_5)q_1,\ z_0=q_2,\ w_0=p_2,\\
&r_1^{045}: x_1=\frac{1}{q_1},\ y_1=-(p_1q_1+\beta_1+\beta_0+\beta_5)q_1,\ z_1=q_2,\ w_1=p_2,\\
&r_2^{045}: x_2=x_5w_5^2,\ y_2=\frac{y_5}{w_5^2},\ z_2=z_5+\frac{2x_5y_5+\beta_2-(\beta_0+\beta_4)}{w_5}+\frac{t-s}{w_5^2}\ w_2=w_5,\\
&r_3^{045}: x_3=\frac{q_1}{p_2},\ y_3=p_1p_2,\ z_3=q_2-\frac{q_1p_1-\beta_3+\beta_0+\beta_4+\beta_5}{p_2}-\frac{s}{p_2^2},\ w_3=p_2,\\
&r_4^{045}: x_4=q_1,\ y_4=p_1-\frac{\beta_4q_2}{q_1q_2-1},\ z_4=q_2,\ w_4=p_2-\frac{\beta_4q_1}{q_1q_2-1},\\
&r_5^{045}: x_5=\frac{1}{q_1},\ y_5=-(p_1q_1+\beta_5)q_1,\ z_5=q_2,\ w_5=p_2,\\
&r_6^{045}: x_6=\frac{q_1}{p_2},\ y_6=p_1p_2,\ z_6=q_2-\frac{q_1p_1-\beta_6+\beta_0+\beta_4+\beta_5}{p_2}-\frac{s-1}{p_2^2},\ w_6=p_2,
\end{aligned}
$$

where $r_2^{045}\left(r_5^{045}(H_{045}^{(1)})-\frac{1}{w_5}\right),\ r_2^{045}\left(r_5^{045}(H_{045}^{(2)})+\frac{1}{w_5}\right),\ r_3^{045}\left(H_{045}^{(2)}+\frac{1}{p_2}\right),\ r_6^{045}\left(H_{045}^{(2)}+\frac{1}{p_2}\right)$.

Hamiltonians $H_{15}^{(1)}=r_{01}\left(-\frac{1}{T^2}H_1^{FS}\right),\ H_{15}^{(2)}=r_{15}\left(-\frac{1}{S^2}H_2^{FS}\right),\ r_{15}: Q_1=\frac{1-q_1}{q_1},\ P_1=-(p_1q_1+\beta_1)q_1,\ Q_2=\frac{1-q_2}{q_2},\ P_2=-(p_2q_2+\beta_5)q_2,\ T=\frac{1-t}{t},\ S=\frac{1-s}{s}$

$$
\begin{aligned}
&r_0^{15}: x_0=-((q_1-q_2)p_1-\beta_0)p_1,\ y_0=\frac{1}{p_1},\ z_0=q_2,\ w_0=p_2+p_1,\\
&r_1^{15}: x_1=\frac{1}{q_1},\ y_1=-(q_1p_1+\beta_1)q_1,\ z_1=q_2,\ w_1=p_2,\\
&r_2^{15}: x_2=-((q_1-t)p_1-\beta_2)p_1,\ y_2=\frac{1}{p_1},\ z_2=q_2,\ w_2=p_2,\\
&r_3^{15}: x_3=\frac{1}{q_1},\ y_3=-(p_1q_1+p_2q_2+\beta_3+\beta_1+\beta_5)q_1,\ z_3=\frac{q_2}{q_1},\ w_3=p_2q_1,\\
&r_4^{15}: x_4=q_1,\ y_4=p_1,\ z_4=-((q_2-s)p_2-\beta_4)p_2,\ w_4=\frac{1}{p_2},\\
&r_5^{15}: x_5=q_1,\ y_5=p_1,\ z_5=\frac{1}{q_2},\ w_5=-(p_2q_2+\beta_5)q_2,\\
&r_6^{15}: x_6=-(q_1p_1+q_2p_2-\beta_6)p_1,\ y_6=\frac{1}{p_1},\ z_6=q_2p_1,\ w_6=\frac{p_2}{p_1},
\end{aligned}
$$

where $r_2^{15}\left(H_{15}^{(1)}-p_1\right),\ r_4^{15}\left(H_{15}^{(2)}-p_2\right)$.

Hamiltonians $H_{150}^{(1)} = r_0^{15}\left(H_{15}^{(1)}\right),\ H_{150}^{(2)} = r_0^{15}\left(H_{15}^{(2)}\right)$

$$\begin{aligned}
&r_0^{150} : x_0 = -(q_1p_1 - \beta_0)p_1,\ y_0 = \frac{1}{p_1},\ z_0 = q_2,\ w_0 = p_2,\\
&r_1^{150} : x_1 = -(q_1p_1 - \beta_1 - \beta_0)p_1,\ y_1 = \frac{1}{p_1},\ z_1 = q_2,\ w_1 = p_2,\\
&r_2^{150} : x_2 = q_1 + \frac{\beta_2 - \beta_0}{p_1} + \frac{t - q_2}{p_1^2},\ y_2 = p_1,\ z_2 = q_2,\ w_2 = p_2 - \frac{1}{p_1},\\
&r_3^{150} : x_3 = -(q_1p_1 - q_2p_2 - \beta_3 - \beta_0 - \beta_1 - \beta_5)p_1,\ y_3 = \frac{1}{p_1},\ z_3 = \frac{q_2}{p_1},\ w_3 = p_2p_1,\\
&r_4^{150} : x_4 = q_1,\ y_4 = p_1,\ z_4 = -((q_2 - s)p_2 - \beta_4)p_2,\ w_4 = \frac{1}{p_2},\\
&r_5^{150} : x_5 = q_1 + \frac{\beta_5p_2}{p_1p_2 - 1},\ y_5 = p_1,\ z_5 = q_2 + \frac{\beta_5p_1}{p_1p_2 - 1},\ w_5 = p_2,\\
&r_6^{150} : x_6 = \frac{1}{q_1},\ y_6 = -(p_1q_1 - p_2q_2 + \beta_6 - \beta_0)q_1,\ z_6 = q_2q_1,\ w_6 = \frac{p_2}{q_1},
\end{aligned}$$

where $r_2^{150}\left(H_{150}^{(1)} - \frac{1}{p_1}\right),\ r_4^{150}\left(H_{150}^{(2)} - p_2\right)$.

Hamiltonians $H_{1503}^{(1)} = sr_3\left(H_{150}^{(1)}\right),\ H_{1503}^{(2)} = sr_3\left(H_{150}^{(2)}\right),\ sr_3 : Q_1 = q_1,\ P_1 = p_1 - \frac{q_2p_2+\beta_3+\beta_0+\beta_1+\beta_5}{q_1},\ Q_2 = q_2q_1,\ P_2 = \frac{p_2}{q_1}$

$$\begin{aligned}
&r_0^{1503} : x_0 = -(q_1p_1 + q_2p_2 - \delta_0)p_1,\ y_0 = \frac{1}{p_1},\ z_0 = q_2p_1,\ w_0 = \frac{p_2}{p_1},\\
&r_1^{1503} : x_1 = -(q_1p_1 + q_2p_2 - \delta_1 - \delta_0)p_1,\ y_1 = \frac{1}{p_1},\ z_1 = q_2p_1,\ w_1 = \frac{p_2}{p_1},\\
&r_2^{1503} : x_2 = q_1 + \frac{q_2}{t} + \frac{2q_2p_2 + 2\delta_2}{p_1} + \frac{t}{p_1^2},\ y_2 = p_1,\ z_2 = q_2p_1^2,\ w_2 = \frac{p_2 + \frac{p_1}{t}}{p_1^2},\\
&r_3^{1503} : x_3 = \frac{1}{q_1},\ y_3 = -(p_1q_1 + \delta_3)q_1,\ z_3 = q_2,\ w_3 = p_2,\\
&r_4^{1503} : x_4 = -\left(\left(q_1 - \frac{1}{s}q_2\right)p_1 - \delta_4\right)p_1,\ y_4 = \frac{1}{p_1},\ z_4 = q_2,\ w_4 = p_2 + \frac{1}{s}p_1,\\
&r_5^{1503} : x_5 = q_1p_2,\ y_5 = \frac{p_1}{p_2},\ z_5 = q_2 + \frac{q_1p_1 + 2\delta_5}{p_2} - \frac{1}{p_2^2},\ w_5 = p_2,\\
&r_6^{1503} : x_6 = -(q_1p_1 + q_2p_2 - \delta_6 - \delta_1 - \delta_0)p_1,\ y_6 = \frac{1}{p_1},\ z_6 = q_2q_1,\ w_6 = \frac{p_2}{q_1},
\end{aligned} \tag{57}$$

where $r_2^{1503}\left(H_{1503}^{(1)} + \frac{p_1q_2}{t^2} - \frac{1}{p_1}\right),\ r_4^{1503}\left(H_{1503}^{(2)} + \frac{p_1q_2}{s^2}\right)$.

Here, δ_j $(j=0,1,\dots,6)$ are complex constant parameters satisfying the parameter's relation:

$$\begin{aligned}&\beta_0=-\delta_6,\ \beta_1=-\delta_1,\ \beta_2=2\delta_0+2\delta_1+2\delta_2+\delta_6,\\&\beta_3=-2\delta_0-\delta_1-2\delta_5-\delta_6,\ \beta_4=\delta_4,\ \beta_5=\delta_0+\delta_1+2\delta_5+\delta_6,\ \beta_6=\delta_0+\delta_1+\delta_3,\end{aligned} \tag{58}$$

$$3\delta_0+2\delta_1+2\delta_2+\delta_3+\delta_4+2\delta_5+\delta_6=1. \tag{59}$$

We remark that the transformations r_2^{1503},r_5^{1503} are not its auto-Bäcklund transformations. It is still an open question whether the transformations r_2^{1503},r_5^{1503} can be considered as each transformation denoted by the symbol $\odot$ in the Oshima's paper (see [36]).

It is also still an open question whether we can obtain the Hamiltonian system with $H_{1503}^{(1)},H_{1503}^{(2)}$ by solving 3×3 Lax pair (cf. [3, 36]) satisfying the following Riemann scheme:

$$\begin{pmatrix} X=0 & X=1 & X=t & X=\infty\\ 0 & 0 & 0 & \theta_1^{\infty}\\ \theta_1^0 & 0 & 0 & \theta_2^{\infty}\\ \theta_2^0 & \theta^1 & \theta^t & \theta_3^{\infty}\end{pmatrix}$$

Here, we will conjecture the following relations between Riemann data and Holomorphy conditions r_i^{1503} $(i=0,1,\dots,6)$;

$$\begin{pmatrix} X=0\\ 0\\ \delta_3\\ \delta_4\end{pmatrix}\Longleftrightarrow \text{Holomorphy conditions}\begin{pmatrix} r_3^{1503}\\ r_4^{1503}\end{pmatrix},\quad \begin{pmatrix} X=\infty\\ \delta_0\\ \delta_1+\delta_0\\ \delta_6+\delta_1+\delta_0\end{pmatrix}\Longleftrightarrow \text{Holomorphy conditions}\begin{pmatrix} r_0^{1503}\\ r_1^{1503}\\ r_6^{1503}\end{pmatrix}.$$

Appendix E: Holomorphy History

article	Author	Contents
[15, 16]	P. Painlevé	Convergence of meromorphic solution
[17]	K. Okamoto	Patching data of space of initial conditions
[14]	K. Okamoto and H. Kimura	Patching data of Garnier system in n variables
[18, 20]	A. Matumiya and K. Takano	Symplectic structure of space of initial conditions
[24, 32]	H. Kimura and M. Suzuki	Degenerate Garnier System in two variables
[27]	N. Tahara	Augmentation
[25]	Y. Yamada	Relation between symmetry and holomorphy conditions

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