A REMARK ON FRACTIONAL INTEGRALS ON MODULATION SPACES

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1. Introduction

The fractional integral operator I_α is defined by

$$I_\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad \gamma(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)},$$

where $0 < \alpha < n$. The well known Hardy-Littlewood-Sobolev theorem says that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $1 < p < q < \infty$ and $1/q = 1/p - \alpha/n$ (see [6, Chapter 5, Theorem 1]). We can regard this theorem as information on how the operation of I_α changes the decay property of functions. On the other hand, the operator I_α can be understood as a differential operator of $(-\alpha)$-th order since $\widehat{I_\alpha f} = |\xi|^{-\alpha} \widehat{f}$ ([6, Chapter 5, Lemma 1]), and we can expect an increase in the smoothness by acting it to functions.

The purpose of this paper is to investigate the effect of I_α on both decay and smoothness properties. To study these two properties simultaneously, we consider the operation of I_α on the modulation spaces $M^{p,q}$, which were introduced by Feichtinger [3] (see also Triebel [8]). We say that f belongs to $M^{p,q}$ if its short short-time Fourier transform

$$V_\varphi f(x,\xi) = e^{-ix\cdot\xi}[f * (M_\xi \varphi)](x) = (2\pi)^{-n/2}[\widehat{f} * (M_{-x}\varphi)](\xi)$$

is in L^p (resp. L^q) with respect to x (resp. ξ), where φ is the Gauss function $\varphi(t) = e^{-|t|^2/2}$. Although the exact definition will be given in the next section, we can see here that the decay of $V_\varphi f(x,\xi)$ with respect to x is determined by that of f, and the one with respect to ξ is determined by that of $\widehat{f}$, that is, the smoothness of f. Hence, the first index p of $M^{p,q}$ measures the decay of f, and the second index q of $M^{p,q}$ measures the smoothness of f. To understand it, we remark that $C_1(1+|t|)^a \le f(t) \le C_2(1+|t|)^b$ implies $\widetilde{C_1}(1+|t|)^a \le f * \varphi(t) \le \widetilde{C_2}(1+|t|)^b$, where a, b are arbitrary real numbers, since the Gauss function is rapidly decreasing. These explanations can be found in Gröchenig [5, Chapter 11].

Since the fractional integral operator I_α is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ of convolution type, it is easy to see that I_α is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$ when

$$1/p_2 = 1/p_1 - \alpha/n \quad \text{and} \quad q_1 = q_2 \tag{1.1}$$

([7, Theorem 3.2]). This boundedness says that the smoothness does not change but the decay of $I_\alpha f$ is worse than that of f since $M^{p_1,q_1}(\mathbb{R}^n) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^n)$ in this case (see Section 2 for this embedding). However, as we have discussed in the

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above, we can expect an increase in the smoothness. Furthermore, since I_α is not bounded on $L^2(\mathbb{R}^n)$ and $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, we can easily prove that I_α is not bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$ when $p_1 \geq p_2$ and $q_1 \geq q_2$ by using duality and interpolation (see Remark 4.1). This means that both decay and smoothness do not increase, simultaneously.

On the other hand, Tomita [7] essentially proved that I_α is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$ when

$$1/p_2 < 1/p_1 - \alpha/n \quad \text{and} \quad 1/q_2 < 1/q_1 + \alpha/n. \tag{1.2}$$

This boundedness says that the decay of $I_\alpha f$ is worse than that of f by the order α/n, but the smoothness of $I_\alpha f$ is better than that of f up to the order α/n. This result seems to be reasonable but there still remain the problems whether the order α/n is the best possible one or not and what about the critical cases $1/p_2 = 1/p_1 - \alpha/n$ or $1/q_2 = 1/q_1 + \alpha/n$. The following theorem is the complete answers to these questions:

Theorem 1.1. *Let $0 < \alpha < n$ and $1 < p_1, p_2, q_1, q_2 < \infty$. Then the fractional integral operator I_α is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$ if and only if*

$$1/p_2 \leq 1/p_1 - \alpha/n \quad \textit{and} \quad 1/q_2 < 1/q_1 + \alpha/n.$$

Theorem 1.1 says that the boundedness of I_α holds even if $1/p_2 = 1/p_1 - \alpha/n$, $1/q_2 < 1/q_1 + \alpha/n$ and $q_1 > q_2$. This is a strictly improvement of (1.1) and (1.2). However, the boundedness does not hold if the second index is critical, that is, $1/q_2 = 1/q_1 + \alpha/n$. We remark that [7] did not treat the necessary condition for the boundedness.

In order to consider the detailed behavior of the first and second indices, we introduce the more general operator $I_{\alpha,\beta}$ defined by $I_{\alpha,\beta} = I_\alpha + I_\beta$, that is,

$$I_{\alpha,\beta} f = \mathcal{F}^{-1}\left[\left(|\xi|^{-\alpha} + |\xi|^{-\beta}\right)\widehat{f}\right],$$

where $0 < \beta \leq \alpha < n$. We note that $|\xi|^{-\alpha} + |\xi|^{-\beta} \sim |\xi|^{-\alpha}$ in the case $|\xi| \leq 1$, and $|\xi|^{-\alpha} + |\xi|^{-\beta} \sim |\xi|^{-\beta}$ in the case $|\xi| \geq 1$. Since $I_{\alpha,\alpha} = 2I_\alpha$, we have Theorem 1.1 as a corollary of the following main result in this paper:

Theorem 1.2. *Let $0 < \beta \leq \alpha < n$ and $1 < p_1, p_2, q_1, q_2 < \infty$. Then $I_{\alpha,\beta}$ is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$ if and only if*

$$1/p_2 \leq 1/p_1 - \alpha/n \quad \textit{and} \quad 1/q_2 < 1/q_1 + \beta/n.$$

Finally we mention some related results. Cowling, Meda and Pasquale [2] proved that $I_{\alpha,\beta}$ is bounded from (L^{p_1}, ℓ^{q_1}) to (L^{p_2}, ℓ^{q_2}) when

$$1/p_2 \geq 1/p_1 - \beta/n \quad \text{and} \quad 1/q_2 \leq 1/q_1 - \alpha/n,$$

where (L^{p_i}, ℓ^{q_i}), $i = 1, 2$, are amalgam spaces defined by

$$\|f\|_{(L^p,\ell^q)} = \left(\sum_{k\in\mathbb{Z}^n} \|\varphi(\cdot - k)f\|_{L^p}^q\right)^{1/q}$$

with an appropriate (see (3.1)) cut-off function φ. The result between $I_{\alpha,\beta}$ and amalgam spaces of Lorentz type can be also found in Cordero and Nicola [1]. The definition of amalgam spaces is based on a similar idea to that of modulation spaces since we have the equivalence

$$\|f\|_{M^{p,q}} \sim \left(\sum_{k\in\mathbb{Z}^n} \|\mathcal{F}^{-1}[\varphi(\cdot - k)\widehat{f}]\|_{L^p}^q \right)^{1/q}.$$

Roughly speaking, amalgam spaces are defined by a decomposition of the function f while the modulation spaces by the same decomposition of $\widehat{f}$. Theorem 1.2 also shows a difference between the modulation spaces and amalgam spaces, because the boundedness of $I_{\alpha,\beta}$ on the modulation spaces does not hold if the second index is critical.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi.$$

We introduce the modulation spaces based on Gröchenig [5]. Fix a function $\varphi \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ (called the *window function*). Then the short-time Fourier transform $V_\varphi f$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to φ is defined by

$$V_\varphi f(x,\xi) = (f, M_\xi T_x \varphi) \qquad \text{for } x,\xi \in \mathbb{R}^n,$$

where $M_\xi \varphi(t) = e^{i\xi\cdot t}\varphi(t)$, $T_x\varphi(t) = \varphi(t-x)$ and $(\cdot,\cdot)$ denotes the inner product on $L^2(\mathbb{R}^n)$. We note that, for $f \in \mathcal{S}'(\mathbb{R}^n)$, $V_\varphi f$ is continuous on $\mathbb{R}^{2n}$ and $|V_\varphi f(x,\xi)| \le C(1+|x|+|\xi|)^N$ for some constants $C, N \ge 0$ ([5, Theorem 11.2.3]). Let $1 \le p,q \le \infty$. Then the modulation space $M^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M^{p,q}} = \|V_\varphi f\|_{L^{p,q}} = \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\varphi f(x,\xi)|^p\,dx \right)^{q/p} d\xi \right\}^{1/q} < \infty,$$

with usual modification when $p = \infty$ or $q = \infty$. We note that $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ ([5, Proposition 11.3.1]), $M^{p,q}(\mathbb{R}^n)$ is a Banach space ([5, Proposition 11.3.5]), $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}(\mathbb{R}^n)$ if $1 \le p,q < \infty$ ([5, Proposition 11.3.4]), and $M^{p_1,q_1}(\mathbb{R}^n) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^n)$ if $p_1 \le p_2$ and $q_1 \le q_2$ ([5, Theorem 12.2.2]). The definition of $M^{p,q}(\mathbb{R}^n)$ is independent of the choice of the window function $\varphi \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$, that is, different window functions yield equivalent norms ([5, Proposition 11.3.2]). Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp}\varphi$ is compact and $\left|\sum_{k\in\mathbb{Z}^n}\varphi(\xi-k)\right| \ge C > 0$ for all $\xi \in \mathbb{R}^n$. Then it is well known that

$$\|f\|_{M^{p,q}} \sim \left(\sum_{k\in\mathbb{Z}^n} \|\varphi(D-k)f\|_{L^p}^q \right)^{1/q}, \tag{2.1}$$

where $\varphi(D-k)f = \mathcal{F}^{-1}[\varphi(\cdot-k)\widehat{f}]$ (see, for example, [8]). The following two lemmas will be used in the sequel.

Lemma 2.1 ([9, Proposition, 1.3.2],[10, Lemma 3.1])**.** *Let* $1 \le p \le q \le \infty$ *and* $\Omega \subset \mathbb{R}^n$ *be a compact set with* $\operatorname{diam}\Omega < R$. *Then there exists a constant* $C > 0$

such that $\|f\|_{L^q} \le C\|f\|_{L^p}$ *for all* $f \in \mathcal{S}(\mathbb{R}^n)$ *with* $\operatorname{supp}\widehat{f} \subset \Omega$*, where* C *depends only on* p, q, n *and* R*. In particular,*

$$\|\varphi(D-k)f\|_{L^q} \le C\|\varphi(D-k)f\|_{L^p} \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n) \text{ and } k \in \mathbb{Z}^n,$$

where φ *is the Schwartz function with compact support.*

Lemma 2.2 ([6, Chapter 4, Theorem 3])**.** *Let* $1 < p < \infty$*. If* $m \in C^{[n/2]+1}(\mathbb{R}^n \setminus \{0\})$ *satisfies*

$$|\partial^\gamma m(\xi)| \le C_\gamma |\xi|^{-|\gamma|} \qquad \text{for all } |\gamma| \le [n/2]+1,$$

then there exists a constant $C > 0$ *such that*

$$\|m(D)f\|_{L^p} \le C\|f\|_{L^p} \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n),$$

where C *depends only on* p, n *and* C_γ*,* $|\gamma| \le [n/2]+1$*.*

3. Sufficient condition for the boundedness of fractional integral operators

In this section, we prove the "if" part of Theorem 1.2. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\varphi = 1 \text{ on } [-1/2, 1/2]^n, \quad \operatorname{supp}\varphi \subset [-3/4, 3/4]^n, \quad \left|\sum_{k\in\mathbb{Z}^n} \varphi(\xi - k)\right| \ge C > 0 \tag{3.1}$$

for all $\xi \in \mathbb{R}^n$.

Lemma 3.1. *Let* $1 < p < \infty$*,* $\alpha \in \mathbb{R}$ *and*

$$m_k^\alpha(\xi) = |k|^\alpha |\xi|^{-\alpha} \varphi(\xi - k), \tag{3.2}$$

where $k \in \mathbb{Z}^n \setminus \{0\}$ *and* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *is as in* (3.1)*. Then* $\sup_{k\neq 0} \|m_k^\alpha(D)\|_{\mathcal{L}(L^p)} < \infty$*.*

Proof. Our proof is similar to that of [4, Theorem 20]. Since $\|m_k^\alpha(D)\|_{\mathcal{L}(L^p)} = \|m_k^\alpha(D+k)\|_{\mathcal{L}(L^p)}$, by Lemma 2.2, it is enough to show that there exists a constant $C > 0$ such that

$$\sup_{\xi\neq 0} |\xi|^{|\gamma|} |\partial^\gamma m_k^\alpha(\xi + k)| = \sup_{\xi \neq 0} |\xi|^{|\gamma|} \left|\partial^\gamma \left(|k|^\alpha |\xi + k|^{-\alpha} \varphi(\xi)\right)\right| \le C \tag{3.3}$$

for all $k \neq 0$ and $|\gamma| \le [n/2]+1$. Since $\operatorname{supp}\varphi \subset [-3/4, 3/4]^n$, we see that $|\xi + k| \ge 1/4$ on $\operatorname{supp}\varphi$ for all $k \neq 0$. Hence, $|k| \sim |\xi + k|$ on $\operatorname{supp}\varphi$ for all $k \neq 0$. This gives (3.3). □

We are now ready to prove the "if" part of Theorem 1.2.

Proof of "if" part of Theorem 1.2. Let $0 < \beta \le \alpha < n$, $1 < p_1, p_2, q_1, q_2 < \infty$, $1/p_2 \le 1/p_1 - \alpha/n$ and $1/q_2 < 1/q_1 + \beta/n$. We first consider the case $1/p_2 = 1/p_1 - \alpha/n$ and $q_1 > q_2$. In view of (2.1),

$$\begin{aligned} \|I_{\alpha,\beta} f\|_{M^{p_2,q_2}} &\le C \left(\sum_{k\in\mathbb{Z}^n} \|\varphi(D-k)(I_{\alpha,\beta}f)\|_{L^{p_2}}^{q_2}\right)^{1/q_2} \\ &\le \|\varphi(D)(I_{\alpha,\beta}f)\|_{L^{p_2}} + \left(\sum_{k\neq 0} \|\varphi(D-k)(I_{\alpha,\beta}f)\|_{L^{p_2}}^{q_2}\right)^{1/q_2} \end{aligned} \tag{3.4}$$

where φ is as in (3.1). Since $0 < 1/p_2 + \beta/n \le 1/p_2 + \alpha/n = 1/p_1 < 1$, we can take $1 < \widetilde{p_1} < \infty$ such that $1/p_2 = 1/\widetilde{p_1} - \beta/n$. Note that $p_1 \le \widetilde{p_1}$. By the Hardy-Littlewood-Sobolev theorem and Lemma 2.1, we have

$$
\begin{aligned}
\|\varphi(D)(I_{\alpha,\beta}f)\|_{L^{p_2}} &\le \|\varphi(D)(I_\alpha f)\|_{L^{p_2}} + \|\varphi(D)(I_\beta f)\|_{L^{p_2}} \\
&= \|I_\alpha(\varphi(D)f)\|_{L^{p_2}} + \|I_\beta(\varphi(D)f)\|_{L^{p_2}} \\
&\le C_\alpha \|\varphi(D)f\|_{L^{p_1}} + C_\beta \|\varphi(D)f\|_{L^{\tilde{p_1}}} \le C\|\varphi(D)f\|_{L^{p_1}} \\
&\le C\left(\sum_{k\in\mathbb{Z}^n} \|\varphi(D-k)f\|_{L^{p_1}}^{q_1}\right)^{1/q_1} \le C\|f\|_{M^{p_1,q_1}}
\end{aligned}
\tag{3.5}
$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Assume that $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\psi = 1$ on $\operatorname{supp}\varphi$, $\operatorname{supp}\psi$ is compact and $\left|\sum_{k\in\mathbb{Z}^n} \psi(\xi - k)\right| \ge C > 0$ for all $\xi \in \mathbb{R}^n$. Then,

$$
\begin{aligned}
\varphi(D-k)(I_{\alpha,\beta}f) &= I_{\alpha,\beta}(\varphi(D-k)f) = I_{\alpha,\beta}(\varphi(D-k)\psi(D-k)f) \\
&= [I_\alpha\, \varphi(D-k)](\psi(D-k)f) + [I_\beta\, \varphi(D-k)](\psi(D-k)f) \\
&= |k|^{-\alpha} m_k^\alpha(D)(\psi(D-k)f) + |k|^{-\beta} m_k^\beta(D)(\psi(D-k)f)
\end{aligned}
$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $k \ne 0$, where m_k^α and m_k^β are defined by (3.2). Hence, by Lemmas 2.1 and 3.1, we have

$$
\begin{aligned}
\|\varphi(D-k)(I_{\alpha,\beta}f)\|_{L^{p_2}} &\le C(|k|^{-\alpha} + |k|^{-\beta})\|\psi(D-k)f\|_{L^{p_2}} \\
&\le C|k|^{-\beta}\|\psi(D-k)f\|_{L^{p_2}} \le C|k|^{-\beta}\|\psi(D-k)f\|_{L^{p_1}}
\end{aligned}
\tag{3.6}
$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $k \ne 0$. Set $a(k) = |k|^{-\beta}$ if $k \ne 0$, and $a(0) = 1$. Note that $\{a(k)\} \in \ell^r(\mathbb{Z}^n)$, where $1/r = 1/q_2 - 1/q_1$. Therefore, by (3.6) and Hörder's inequality, we see that

$$
\begin{aligned}
&\left(\sum_{k\ne 0} \|\varphi(D-k)(I_{\alpha,\beta}f)\|_{L^{p_2}}^{q_2}\right)^{1/q_2} \le \left\{\sum_{k\in\mathbb{Z}^n} (a(k)\|\psi(D-k)f\|_{L^{p_1}})^{q_2}\right\}^{1/q_2} \\
&\le \|\{a(k)\}\|_{\ell^r} \left(\sum_{k\in\mathbb{Z}^n} \|\psi(D-k)f\|_{L^{p_1}}^{q_1}\right)^{1/q_1} \le C\|f\|_{M^{p_1,q_1}}
\end{aligned}
\tag{3.7}
$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Combining (3.4), (3.5) and (3.7), we obtain the desired result with $1/p_2 = 1/p_1 - \alpha/n$ and $q_1 > q_2$.

We next consider the case $1/p_2 = 1/p_1 - \alpha/n$ and $q_1 \le q_2$. Since $\beta/n > 0$, we can take $1 < \widetilde{q_2} < \infty$ such that $q_1 > \widetilde{q_2}$ and $1/\widetilde{q_2} < 1/q_1 + \beta/n$. Note that $q_2 > \widetilde{q_2}$. Then, by the preceding case, we see that $I_{\alpha,\beta}$ is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,\widetilde{q_2}}(\mathbb{R}^n)$. This implies that $I_{\alpha,\beta}$ is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$, since $M^{p_2,\widetilde{q_2}}(\mathbb{R}^n) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^n)$.

Finally, we consider the case $1/p_2 < 1/p_1 - \alpha/n$. Since $0 < 1/p_1 - \alpha/n < 1$, we can take $1 < \widetilde{p_2} < \infty$ such that $1/\widetilde{p_2} = 1/p_1 - \alpha/n$. Note that $p_2 > \widetilde{p_2}$. Then, by the preceding cases, we see that $I_{\alpha,\beta}$ is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{\widetilde{p_2},q_2}(\mathbb{R}^n)$. This implies that $I_{\alpha,\beta}$ is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$, since $M^{\widetilde{p_2},q_2}(\mathbb{R}^n) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^n)$. The proof is complete.

4. Necessary condition for the boundedness of fractional integral operators

Before proving the "only if" part of Theorem 1.2, we give the following remark:

Remark 4.1. Let $p_1 \geq p_2$ and $q_1 \geq q_2$. In Introduction, we have stated that I_α is not bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$. In fact, since $M^{p_2,q_2}(\mathbb{R}^n) \hookrightarrow M^{p_1,q_1}(\mathbb{R}^n)$, if I_α is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$ then I_α is bounded on $M^{p_1,q_1}(\mathbb{R}^n)$. Then, by duality, I_α is also bounded on $M^{p_1',q_1'}(\mathbb{R}^n)$. By interpolation, the boundedness on $M^{p_1,q_1}(\mathbb{R}^n)$ and on $M^{p_1',q_1'}(\mathbb{R}^n)$ implies that I_α is bounded on $M^{2,2}(\mathbb{R}^n)$. However, since I_α is not bounded on $L^2(\mathbb{R}^n)$ ([6, p.119]), this is a contradiction. Hence, I_α is not bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$.

In the rest of the paper, we prove the "only if" part of Theorem 1.2.

Lemma 4.2. *Let $0 < \beta \leq \alpha < n$ and $1 < p_1, p_2, q_1, q_2 < \infty$. If $I_{\alpha,\beta}$ is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$, then $1/p_2 \leq 1/p_1 - \alpha/n$.*

Proof. We only consider the case $\alpha > \beta$, since the proof in the case $\alpha = \beta$ is simpler. Let $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ be such that $\operatorname{supp}\psi \subset [-1,1]^n$. Set $\Psi = \mathcal{F}^{-1}\psi$ and $\Psi_\lambda(x) = \Psi(\lambda x)$, where $\lambda > 0$. Then

$$\varphi(D-k)\Psi_\lambda = \begin{cases} \Psi_\lambda & \text{if } k = 0, \\ 0 & \text{if } k \neq 0 \end{cases} \tag{4.1}$$

for all $0 < \lambda < 1/4$, where φ is as in (3.1). Similarly,

$$\varphi(D-k)(I_{\alpha,\beta}\Psi_\lambda) = I_{\alpha,\beta}(\varphi(D-k)\Psi_\lambda) = \begin{cases} I_\alpha\Psi_\lambda + I_\beta\Psi_\lambda & \text{if } k = 0, \\ 0 & \text{if } k \neq 0 \end{cases} \tag{4.2}$$

for all $0 < \lambda < 1/4$. By (2.1) and (4.1), we see that

$$\|\Psi_\lambda\|_{M^{p_1,q_1}} \leq C \left(\sum_{k\in\mathbb{Z}^n} \|\varphi(D-k)\Psi_\lambda\|_{L^{p_1}}^{q_1} \right)^{1/q_1} = C\|\Psi_\lambda\|_{L^{p_1}} = C\lambda^{-n/p_1} \tag{4.3}$$

for all $0 < \lambda < 1/4$. Since $\alpha > \beta$, we can take $0 < \lambda_0 < 1/4$ such that $\|I_\alpha\Psi\|_{L^{p_2}}\lambda_0^{-\alpha} > 2\|I_\beta\Psi\|_{L^{p_2}}\lambda_0^{-\beta}$. Note that $\|I_\alpha\Psi\|_{L^{p_2}}\lambda^{-\alpha} > 2\|I_\beta\Psi\|_{L^{p_2}}\lambda^{-\beta}$ for all $0 < \lambda \leq \lambda_0$. Since $I_\alpha\Psi_\lambda(x) = \lambda^{-\alpha}(I_\alpha\Psi)(\lambda x)$, by (2.1) and (4.2), we see that

$$\begin{aligned} \|I_{\alpha,\beta}\Psi_\lambda\|_{M^{p_2,q_2}} &\geq C \left(\sum_{k\in\mathbb{Z}^n} \|\varphi(D-k)(I_{\alpha,\beta}\Psi_\lambda)\|_{L^{p_2}}^{q_2} \right)^{1/q_2} \\ &= C\|I_\alpha\Psi_\lambda + I_\beta\Psi\|_{L^{p_2}} \geq C\left(\|I_\alpha\Psi_\lambda\|_{L^{p_2}} - \|I_\beta\Psi_\lambda\|_{L^{p_2}}\right) \\ &= C\lambda^{-n/p_2}\left(\lambda^{-\alpha}\|I_\alpha\Psi\|_{L^{p_2}} - \lambda^{-\beta}\|I_\beta\Psi\|_{L^{p_2}}\right) \\ &\geq C\lambda^{-n/p_2}\left(\lambda^{-\alpha}\|I_\alpha\Psi\|_{L^{p_2}}/2\right) = C\lambda^{-n/p_2-\alpha} \end{aligned} \tag{4.4}$$

for all $0 < \lambda < \lambda_0$. Hence, by (4.3) and (4.4), if $I_{\alpha,\beta}$ is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$, then

$$C_1\lambda^{-n/p_2-\alpha} \leq \|I_{\alpha,\beta}\Psi_\lambda\|_{M^{p_2,q_2}} \leq \|I_{\alpha,\beta}\|_{\mathrm{op}}\|\Psi_\lambda\|_{M^{p_1,q_1}} \leq C_2\lambda^{-n/p_1}$$

for all $0 < \lambda < \lambda_0$. This implies $-n/p_2 - \alpha \geq -n/p_1$, that is, $1/p_2 \leq 1/p_1 - \alpha/n$. The proof is complete. □

Remark 4.3. Let $0 < p < \infty$ and N be a sufficiently large number. Then

$$\begin{cases} |x|^{-n/p}(\log|x|)^{-\alpha/p}\chi_{\{|x|>N\}} \in L^p(\mathbb{R}^n), & \text{if } \alpha > 1, \\ |x|^{-n/p}(\log|x|)^{-\alpha/p}\chi_{\{|x|>N\}} \notin L^p(\mathbb{R}^n), & \text{if } \alpha \leq 1. \end{cases}$$

In fact, by a change of variables,

$$\int_{|x|>N} |x|^{-n} (\log|x|)^{-\alpha}\, dx = C_n \int_N^\infty r^{-n} (\log r)^{-\alpha} r^{n-1} dr = C_n \int_{\log N}^\infty t^{-\alpha}\, dt.$$

Lemma 4.4. *Let $0 < \beta \le \alpha < n$ and $1 < p_1, p_2, q_1, q_2 < \infty$. If $1/q_2 = 1/q_1 + \beta/n$, then $I_{\alpha,\beta}$ is not bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$.*

Proof. We only consider the case $\alpha > \beta$, since the proof in the case $\alpha = \beta$ is simpler. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be as in (3.1). Set $C_\alpha = \sup_{k\ne 0} \|m_k^\alpha(D)\|_{\mathcal{L}(L^{p_2})}$ and $C_\beta = \sup_{k\ne 0} \|m_k^{-\beta}(D)\|_{\mathcal{L}(L^{p_2})}$, where m_k^α and $m_k^{-\beta}(D)$ are defined by (3.2) with φ. Since $\alpha > \beta$, we can take a sufficiently large natural number N such that $C_\beta^{-1} N^{-\beta} > 2C_\alpha N^{-\alpha}$. Then

$$C_\beta^{-1}|k|^{-\beta} > 2C_\alpha |k|^{-\alpha} \qquad \text{for all } |k| \ge N. \tag{4.5}$$

Since $1/q_2 > 1/q_1$, we can take $\epsilon > 0$ such that $(1+\epsilon)q_2/q_1 < 1$. For these ϵ and N, set

$$f(x) = \sum_{|\ell|>N} |\ell|^{-n/q_1} (\log|\ell|)^{-(1+\epsilon)/q_1} e^{i\ell\cdot x}\, \Psi(x),$$

where $\psi \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ satisfies $\operatorname{supp}\psi \subset [-1/4, 1/4]^n$ and $\Psi = \mathcal{F}^{-1}\psi$. Since $\varphi = 1$ on $[-1/2, 1/2]^n$ and $\operatorname{supp}\varphi \subset [-3/4, 3/4]^n$,

$$\varphi(D-k)f(x) = \begin{cases} |k|^{-n/q_1} (\log|k|)^{-(1+\epsilon)/q_1} e^{ik\cdot x}\, \Psi(x) & \text{if } |k| > N, \\ 0 & \text{if } |k| \le N. \end{cases} \tag{4.6}$$

Similarly,

$$\varphi(D-k)I_{\alpha,\beta}f(x) = \begin{cases} |k|^{-n/q_1} (\log|k|)^{-(1+\epsilon)/q_1} I_{\alpha,\beta}(M_k\Psi)(x) & \text{if } |k| > N, \\ 0 & \text{if } |k| \le N, \end{cases} \tag{4.7}$$

where $M_k\Psi(x) = e^{ik\cdot x}\Psi(x)$. By (4.6), we have

$$\|\varphi(D-k)f\|_{L^{p_1}} = \begin{cases} \|\Psi\|_{L^{p_1}} |k|^{-n/q_1} (\log|k|)^{-(1+\epsilon)/q_1} & \text{if } |k| > N, \\ 0 & \text{if } |k| \le N. \end{cases}$$

Then, by Remark 4.3, we see that $f \in M^{p_1,q_1}(\mathbb{R}^n)$. On the other hand, since

$$\begin{aligned} I_\alpha(M_k\Psi) &= \mathcal{F}^{-1}\left[|\xi|^{-\alpha}\psi(\xi-k)\right] \\ &= |k|^{-\alpha}\mathcal{F}^{-1}\left[\left(|k|^\alpha|\xi|^{-\alpha}\varphi(\xi-k)\right)\psi(\xi-k)\right] = |k|^{-\alpha} m_k^\alpha(D)(M_k\Psi) \end{aligned}$$

and

$$\begin{aligned} |k|^{-\beta}M_k\Psi &= \mathcal{F}^{-1}\left[|k|^{-\beta}\psi(\xi-k)\right] \\ &= \mathcal{F}^{-1}\left[\left(|k|^{-\beta}|\xi|^{\beta}\varphi(\xi-k)\right)\left(|\xi|^{-\beta}\psi(\xi-k)\right)\right] = m_k^{-\beta}(D) I_\beta(M_k\Psi), \end{aligned}$$

by Lemma 3.1, we have

$$\|I_\alpha(M_k\Psi)\|_{L^{p_2}} \le |k|^{-\alpha}\|m_k^\alpha(D)\|_{\mathcal{L}(L^{p_2})}\|M_k\Psi\|_{L^{p_2}} \le C_\alpha|k|^{-\alpha}\|M_k\Psi\|_{L^{p_2}}$$

and

$$\|M_k\Psi\|_{L^{p_2}} \le |k|^{\beta}\|m_k^{-\beta}(D)\|_{\mathcal{L}(L^{p_2})}\|I_\beta(M_k\Psi)\|_{L^{p_2}} \le C_\beta|k|^{\beta}\|I_\beta(M_k\Psi)\|_{L^{p_2}}$$

for all $|k| > N$. Hence, by (4.5),

$$\begin{aligned} \|I_{\alpha,\beta}(M_k\Psi)\|_{L^{p_2}} &\geq \|I_\beta(M_k\Psi)\|_{L^{p_2}} - \|I_\alpha(M_k\Psi)\|_{L^{p_2}} \\ &\geq \left(C_\beta^{-1}|k|^{-\beta} - C_\alpha|k|^{-\alpha}\right)\|M_k\Psi\|_{L^{p_2}} \geq \left(C_\beta^{-1}|k|^{-\beta}/2\right)\|\Psi\|_{L^{p_2}} = C|k|^{-\beta} \end{aligned} \tag{4.8}$$

for all $|k| > N$. Then, it follows from (4.7) and (4.8) that

$$\begin{aligned} \|\varphi(D-k)(I_{\alpha,\beta}f)\|_{L^{p_2}} &\geq C|k|^{-n/q_1-\beta}\left(\log|k|\right)^{-(1+\epsilon)/q_1} \\ &= C|k|^{-n/q_2}\left(\log|k|\right)^{-\{(1+\epsilon)q_2/q_1\}/q_2} \end{aligned}$$

for all $|k| > N$. Also, $\|\varphi(D-k)(I_{\alpha,\beta}f)\|_{L^{p_2}} = 0$ if $|k| \leq N$. Since $(1+\epsilon)q_2/q_1 < 1$, by Remark 4.3, we have $\{|k|^{-n/q_2}(\log|k|)^{-\{(1+\epsilon)q_2/q_1\}/q_2}\}_{|k|>N} \notin \ell^{q_2}(\mathbb{Z}^n)$. This implies $\left(\sum_{k\in\mathbb{Z}^n}\|\varphi(D-k)(I_{\alpha,\beta}f)\|_{L^{p_2}}^{q_2}\right)^{1/q_2} = \infty$, that is, $I_{\alpha,\beta}f \notin M^{p_2,q_2}(\mathbb{R}^n)$. Therefore, $I_{\alpha,\beta}$ is not bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$. The proof is complete. □

We are now ready to prove the "only if" part of Theorem 1.2.

Proof of "only if" part of Theorem 1.2. Let $0 < \beta \leq \alpha < n$ and $1 < p_1, p_2, q_1, q_2 < \infty$. Assume that $I_{\alpha,\beta}$ is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,q_2}(\mathbb{R}^n)$. Then, by Lemma 4.2, we see that $1/p_2 \leq 1/p_1 - \alpha/n$. On the other hand, if $1/q_2 \geq 1/q_1 + \beta/n$ then $I_{\alpha,\beta}$ is bounded from $M^{p_1,q_1}(\mathbb{R}^n)$ to $M^{p_2,\widetilde{q}_2}(\mathbb{R}^n)$, since $M^{p_2,q_2}(\mathbb{R}^n) \hookrightarrow M^{p_2,\widetilde{q}_2}(\mathbb{R}^n)$, where $1/\widetilde{q}_2 = 1/q_1 + \beta/n$. However, this contradicts Lemma 4.4. Hence, $1/q_2 < 1/q_1 + \beta/n$. The proof is complete.

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