The Norm Index Theorem (An Analytic Proof)

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Introduction

A key result of class field theory for abelian field extensions L/K, concerning the idele norm map $N : \mathbb{I}_L \to \mathbb{I}_K$, is the norm index theorem

$$[\mathbb{I}_K : K^* N(\mathbb{I}_L)] = [L : K] \ .$$

The inequality $[\mathbb{I}_K : K^* N(\mathbb{I}_L)] \leq [L : K]$ is known to hold in general. A rather easy and well known analytic proof ([H1]) employs analytic properties of L-series near the border $s = 1$ of convergency, obtained by expressing Hecke L-series in the form of strongly converging integrals using the Poisson formula (see [H2], [T1]).

The second inequality $[\mathbb{I}_K : K^* N(\mathbb{I}_L)] \geq [L : K]$ holds for abelian extensions L/K only. It can be easily reduced to the case of cyclic extensions, for which it is usually proved by nonanalytic methods (see [T2]).

This note provides a purely analytic proof of the second inequality via the trace formula for the compact multiplicative group $L^* Z_L \backslash \mathbb{I}_L$. The Poisson formula, used for the first inequality, is the trace formula for the additive compact group $\mathbb{A}_K/K$ enhanced by the information, that the Pontryagin dual $(\mathbb{A}_K/K)^D$ has dimension 1 as a K-vector space. Hence the spectral theory of the multiplicative and the additive theory combined prove the norm index theorem by analytic methods.

Review of the trace formula

For the number field L let $\mathbb{I}_L$ be the group of ideles. Let denote $Z_L \cong \mathbb{R}^*_{>0}$ the image of $\mathbb{R}^*_{>0} \subseteq \mathbb{I}_{\mathbb{Q}}$ in $\mathbb{I}_L$. The quotient $X_L = L^* Z_L \backslash \mathbb{I}_L$ is a compact group. For Haar measures dg_L on $\mathbb{I}_L$ and dz_L on Z_L the measure $\frac{dg_L}{dz_L}$ induces a measure dx_L on X_L. Assume $\int_{X_L} dx_L = 1$. The corresponding Hilbert space $L^2(X_L)$ is spanned by the characters $\eta \in (X_L)^D$ of X_L. Let L/K be cyclic with Galois group $\langle \sigma \rangle$.

For functions $\prod_v f_v(x_v)$ in $C_c^\infty(\mathbb{I}_L)$ define f by integration over Z_K. For $\varphi \in L^2(X_L)$ define the convolution $R\varphi$ in $L^2(X_L)$ by $\int_{Z_L \backslash \mathbb{I}_L} f(h^{-1}\theta(g))\varphi(g) \frac{dg_L}{dz_L}$ as a function of h. Here $\theta(g) = \kappa \cdot \sigma(g)$ for a fixed $\kappa \in \mathbb{I}_L$ with idele norm $N(\kappa) \in K^*$. Obviously $\theta = \theta_\kappa$ defines an automorphism of X_L of order $[L : K]$.

The operator R has the kernel $K(y, x) = \sum_{\delta \in L^*} f(y^{-1}\delta\theta(x))$. Hence its trace is $\int_{X_L} K(x, x) dx_L$. Using $K^* \backslash L^* \cong (\sigma - 1)L^*$ and $Z_L = Z_K$ the integral defining the trace, for $c = \frac{dz_K}{dz_L}$, therefore becomes

$$\sum_{\delta \in L/(\sigma-1)L^*} \int_{y \in \mathbb{I}_K \backslash \mathbb{I}_L} \Big(c \cdot \int_{x \in Z_K K^* \backslash \mathbb{I}_K} dx_K \Big) f\Big(y^{-1} \delta \theta(y) \Big) \ \frac{dg_L}{dg_K} \ .$$

Abbreviate $O^L_\delta(f_w) = \int_{K_w^*\backslash L_v^*} \left(\prod_{v|w} f_v(g_v^{-1}\delta\theta(g_v)dg_v\right) / dg_w$, so this simplifies to

$$\sum_{\delta\in L^*/(\sigma-1)L^*} c\cdot\prod_v O^L_\delta(f_v)\ .$$

For characters η of X_L put $\eta^\theta(x) = \eta(\sigma^{-1}(x))$ and $\eta(f) = \int_{Z_L\backslash\mathbb{I}_L} f\big(\theta(g)\big)\eta(g)\frac{dg_L}{dz_L}$. Hence up to these constants $\eta(f)$

$$R\eta(h)\ =\ \eta(f)\cdot\eta^\theta(h)\ .$$

In other words, the trace of R becomes the spectral sum $\sum_{\eta=\eta^\theta}\eta(f)$. Comparing with the previous formula we obtain the usual trace formula (as in [KS])

$$\sum_{\eta=\eta^\theta}\eta(f)\ =\ \sum_{\delta\in L^*/(\sigma-1)L^*} c\cdot\prod_w O^L_\delta(f_w)\ .$$

Matching functions

$\mathbb{I}_K = \prod_w K_w^*$ and $\mathbb{I}_L = \prod_w L_w^* = \prod_w\prod_{v|w} L_v^*$. Functions $\prod_w f_w(x)$ in $C_c^\infty(\mathbb{I}_L)$, s.t. $f_w = \prod_{v|w} f_v$, and functions $\prod_w h_w(x)$ in $C_c^\infty(\mathbb{I}_K)$ are said to be matching functions, if $h_w(\gamma_w) = O^L_{\delta_w}(f_w)$ holds for $\gamma_w = N(\delta_w), \delta_w\in L_w^*$ and $h_w(\gamma_w)$ is zero for $\gamma_w\notin N(L_w^*)$. Notice, that h_w is uniquely determined by f_w. Existence is obvious, since the characteristic functions $\prod_{v|w} 1_{\mathfrak{o}_v}$ and $1_{\mathfrak{o}_w}$ of integral elements do match at all unramified nonarchimedean places by the elementary property $N(\mathfrak{o}_v^*) = \mathfrak{o}_w^*$, which is valid for all unramified places w (the fundamental lemma).

Twisted case revisited

Characters $\eta\in(X_L)^D$ on the spectral side of the trace formula are characters $\eta=\eta^\theta$ of X_L trivial on $(\sigma-1)X_L$, hence of the form $\eta=\chi^\sharp\circ N$ for characters $\chi^\sharp$ of $Y^\sharp = N(Z_L\backslash\mathbb{I}_L)/N(L^*) = X_L/(\sigma-1)X_L$ (Hilbert theorem 90). For $\kappa=1$ the trace summands $\eta(f)$ therefore can be written in the form

$$\eta(f) = \int_{Z_L Kern(N)\backslash\mathbb{I}_L}\eta(g)\left(\int_{Kern(N)} f(gn)dn\right)\frac{dg_L}{dndz_L}$$

for $Kern(N) = (\sigma-1)\mathbb{I}_L$ and $dn = \frac{dg_L}{dg_K}(h)$ and $n=\sigma(h)h^{-1}$. By the matching condition and $\eta(g) = \chi^\sharp\big(N(g)\big)$ the last expression giving $\eta(f)$ becomes

$$\tilde c\cdot\chi^\sharp(h) = \int_{Z_K\backslash N(\mathbb{I}_L)}\tilde c\cdot\chi^\sharp\big(N(g)\big)\ h\big(N(g)\big)\ \frac{dg_K}{dz_K}\quad\text{where}\quad\tilde c\cdot\frac{dg_K}{dz_K} = \frac{dg_L}{dndz_L}\ .$$

On the right side of the trace formula we can also apply the matching condition. Hilbert 90 implies $L^*/(\sigma-1)L^* \cong N(L^*)$, hence the still preliminary formula $(\sharp)$

$$\sum_{\chi^\sharp\in(Y^\sharp)^D} \tilde{c}\cdot\chi^\sharp(h) \;=\; \sum_{\gamma\in N(L^*)} c\cdot h(\gamma)\ .$$

The degenerate case $K = L$

For a pair of matching functions we compare the last formula $(\sharp)$ with the trace formula for h in the case $L = K$ and $\kappa = 1$, which simply reduces to

$$\sum_{\chi\in(X_K)^D} \chi(h) = \sum_{\gamma\in K^*} h(\gamma)\ ,$$

since $\prod_w O_\gamma^K(h_w) = h(\gamma)$. By the matching condition the support of h is contained in $N(Z_L\backslash\mathbb{I}_L) \subseteq Z_K\backslash\mathbb{I}_L$. Therefore on the left we may restrict characters χ from X_K to the image $Y^\flat = N(Z_L\backslash\mathbb{I}_L)\big/(K^*\cap N(\mathbb{I}_L))$ of $N(Z_L\backslash\mathbb{I}_L)$ in X_K, which is a subgroup of index $[\mathbb{I}_K : K^*N(\mathbb{I}_L)]$. For the restrictions $\chi^\flat$ of the characters χ, now with $\chi^\flat$ running over the character group $(Y^\flat)^D$ of $Y^\flat$, we thus may restate the trace formula for $L = K$ as the following formula $(\flat)$

$$\#\Big(\frac{\mathbb{I}_K}{K^*N(\mathbb{I}_L)}\Big)\cdot \sum_{\chi^\flat\in(Y^\flat)^D} \chi^\flat(h) \;=\; \sum_{\gamma\in K^*\cap N(\mathbb{I}_L)} h(\gamma)\ .$$

In particular $[\mathbb{I}_K : K^*N(\mathbb{I}_L)] < \infty$.

Comparing the trace formulas

Recall $Y^\sharp = X_L/(\sigma-1)X_L = N(Z_L\backslash\mathbb{I}_L)\Big/N(L^*)$, which gives the exact sequence

$$0 \to \frac{K^*\cap N(\mathbb{I}_L)}{N(L^*)} \to Y^\sharp \to Y^\flat \to 0\ .$$

A system of representatives κ_i for all possible κ modulo L^* is in 1-1 correspondence with the elements of $\frac{K^*\cap N(\mathbb{I}_L)}{N(L^*)}$. Summing up the κ_i-twisted trace formulas for f – these are nothing but the revisited forms of the trace formulas $(\sharp)$ for the translates $f(\kappa_i x)$ of $f(x)$ – we obtain from $(\sharp)$ therefore the final identity

$$\#\Big(\frac{K^*\cap N(\mathbb{I}_L)}{N(L^*)}\Big)\cdot \sum_{\chi^\flat\in(Y^\flat)^D} \tilde{c}\cdot\chi^\flat(h) \;=\; \sum_{\gamma\in K^*\cap N(\mathbb{I}_L)} c\cdot h(\gamma)\ .$$

In particular $[K^* \cap N(\mathbb{I}_L) : N(L^*)] < \infty$. If we compare this last formula with the trace formula $(\flat)$ for $L = K$, we get the crucial formula

$$\frac{[\mathbb{I}_K : K^* N(\mathbb{I}_L)]}{[K^* \cap N(\mathbb{I}_L) : N(L^*)]} = \frac{\tilde{c}}{c} \,.$$

The quotient $\tilde{c}/c$. The constants were defined by $\tilde{c} \cdot \frac{dg_K}{dz_K} = \frac{dg_L}{dndz_L}$ and $c = \frac{dz_K}{dz_L}$, and $dn = \frac{dg_L}{dg_K}$ by abuse of notation. Indeed, the ratio $\tilde{c}/c$ is independent from the particular choice of Haar measures dg_L, dg_K and dz_K, dz_K. Therefore we may choose them freely. Normalizing constants for dg_K, dg_L cancel. Unraveling the definitions in terms of the maps i and N we are thus reduced to consider invariant K-rational differential forms for the K-tori $\mathbb{G}_m$ and $T = Res_{L/K}(\mathbb{G}_m)$ and the exact sequence

$$0 \to \mathbb{G}_m \overset{i}{\to} T \overset{1-\sigma}{\to} T \overset{N}{\to} \mathbb{G}_m \to 0 \,.$$

An easy calculation on the tangent spaces, for $A = i^*(e_1^*)$ and a form B on the tangent space of $V = (1-\sigma)T$, using the formula $e_1^* \wedge (1-\sigma)^*(B) = e_1^* \wedge [(e_2^* - e_1^*) \wedge \cdots (e_n^* - e_{n-1}^*)] = e_1^* \wedge \cdots \wedge e_n^* = [e_1^* \wedge \cdots e_{n-1}^*] \wedge (e_1^* + \cdots + e_n^*) = B \wedge N^*(A)$ proves, that the quotient $\tilde{c}/c$ entirely comes from the measure comparison between dz_L and dz_K similarly arising from the exact sequence

$$0 \to Z_K \overset{i}{\to} Z_L \overset{1-\sigma}{\to} Z_L \overset{N}{\to} Z_K \to 0 \,.$$

$(1-\sigma)$ is the zero map on Z_L. Hence this comparison is trivial. The factor $\tilde{c}/c$ immediately turns out to be $\tilde{c}/c = [L : K]$. This completes the proof. Of course, combined with the first inequality, this a posteriori implies the Hasse norm theorem $K^* \cap N(\mathbb{I}_L) = N(L^*)$, hence the stability of our particular trace formula as in [KS] 6.4 and 7.4.

Bibliography

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