# New Generalizations of Poisson Algebras

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#### Abstract

We introduce many new generalizations of Poisson algebras which can be constructed inside the associative algebra of linear transformations over a vector space.

## 1 Introduction

Poisson algebras play a fundamental role in symplectic geometry. Recently, different generalizations of Poisson algebras have been introduced by several people ([1], [2], [3] and [5]). A Poisson algebra P is a vector space equipped with a square bracket [, ] and a dot product  $\cdot$  such that the following three conditions hold:

**Condition 1** *P* is a Lie algebra with respect to the square bracket [, ];

Condition 2 P is a commutative associative algebra with respect to the dot product  $\cdot$ ;

**Condition 3** The square bracket [, ] and the dot product  $\cdot$  satisfy the Leibniz rule:  $[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z]$  for x, y and  $z \in P$ .

In this paper, we introduce many new generalizations of Poisson algebras. All of the generalizations of Poisson algebras can be divided into two groups. One group consists of the tailless Poisson algebras which do not use a derivative to generalize the Leibniz rule. The other group consists of the tailed Poisson algebras which use a derivative to generalize the Leibniz rule. There are a few kinds of tailless Poisson algebras, but there are many kinds of tailed Poisson algebras. The details of constructing these generalizations of Poisson algebras will be given in [4]. The important facts about these generalizations are that each of these generalizations of Poisson algebras can be constructed inside the associative algebra of linear transformations over a vector space, and some of these generalizations of Poisson algebras can be constructed inside the associative algebra of linear transformations over a vector space by using many different ways.

## 2 Tailless Poisson Algebras

After dropping off the commutative property in the Condition 2, we get the following

**Definition 2.1** A square-circle Poisson algebra P is a vector space equipped with a square bracket [, ] and a circle product  $\circ$  which have three properties:

- (i) P is a Lie algebra with respect to the square bracket [, ];
- (ii) P is an associative algebra with respect to the circle product  $\circ$ ;
- (iii) The square bracket [,] and the circle product  $\circ$  satisfy the Leibniz rule:

$$[x, y \circ z] = [x, y] \circ z + y \circ [x, z] \qquad \text{for } x, y, z \in P.$$

$$\tag{1}$$

A square-circle Poisson algebra is called a non-commutative Poisson algebra in [3]. Recall from [5] that a **right Leibniz algebra** L is a vector space equipped with a **angle bracket**  $\langle, \rangle$  satisfying the **right Leibniz identity**; that is,

$$\langle x, \langle y, z \rangle \rangle = \langle \langle x, y \rangle, z \rangle - \langle \langle x, z \rangle, y \rangle \quad \text{for } x, y, z \in L.$$
(2)

**Definition 2.2** Let P be a vector space equipped with a **angle bracket**  $\langle , \rangle$  and a **circle product**  $\circ$  such that P is a right Leibniz algebra with respect to the angle bracket  $\langle , \rangle$  and P is an associative algebra with respect to the circle product  $\circ$ .

 (i) P is called a left angle-circle Poisson algebra if the angle bracket ⟨, ⟩ and the circle product ∘ satisfy the left Leibniz rule:

$$\langle x, y \circ z \rangle = \langle x, y \rangle \circ z + y \circ \langle x, z \rangle \qquad \text{for } x, y, z \in P.$$
(3)

(ii) P is called a right angle-circle Poisson algebra if the angle bracket (, ) and the circle product 
 o satisfy the right Leibniz rule:

$$\langle x \circ y, z \rangle = x \circ \langle y, z \rangle + \langle x, z \rangle \circ y \qquad for \ x, y, z \in P.$$
(4)

A right angle-circle Poisson algebra is called a non-commutative Leibniz-Poisson algebra in [2].

The next proposition gives a summary of our ways of constructing tailless Poisson algebras.

**Proposition 2.1** Let End(V) be the associative algebra of linear transformations over a vector space V. If V has a proper nonzero subspace, then there exists a subspace  $\mathcal{M}$  of End(V) such that the following hold:

(i) There are 46 ways of making  $\mathcal{M}$  into a square-circle Poisson algebra;

- (ii) There are 6 ways of making  $\mathcal{M}$  into a left angle-circle Poisson algebra;
- (iii) There are 51 ways of making  $\mathcal{M}$  into a right angle-circle Poisson algebra.

A remark about left angle-circle Poisson algebras is that each of the 6 ways of making  $\mathcal{M}$  into a left angle-circle Poisson algebra in Proposition 2.1 simultaneously makes  $\mathcal{M}$  into a right angle-circle Poisson algebra.

#### 3 Tailed Square-Circle Poisson Algebras

We begin this section with the following

**Definition 3.1** Let A be an associative algebra with respect to an associative product  $\circ$ , and let x, y and z are elements of A.

(i) If A is a Lie algebra with respect to a square bracket [, ], then

$$J[x, y, \circ, z] := [x, y \circ z] - [x, y] \circ z - y \circ [x, z]$$
(5)

is called the square-circle Jacobian.

(ii) If A is a right Leibniz algebra with respect to a angle bracket  $\langle , \rangle$ , then

$$J_{\ell}\langle x, y, \circ, z \rangle := \langle x, y \circ z \rangle - \langle x, y \rangle \circ z - y \circ \langle x, z \rangle$$
(6)

is called the left angle-circle Jacobian, and

$$J_r\langle x, \circ, y, z \rangle := \langle x \circ y, z \rangle - x \circ \langle y, z \rangle - \langle x, z \rangle \circ y \tag{7}$$

is called the right angle-circle Jacobian.

We now introduce 8 different tailed square-circle Poisson algebras in the following

**Definition 3.2** Let P be a vector space equipped with a square bracket [, ]and a circle product  $\circ$  such that P is a Lie algebra with respect to the square bracket [, ] and P is an associative algebra with respect to the circle product  $\circ$ . Let D be a derivation with respect to both the square bracket [, ] and the circle product  $\circ$ . Let x, y and z be arbitrary three elements of P.

(i) P is called a tailed 1-st square-circle Poisson algebra if

$$J[x, y, \circ, z] = x \circ y \circ D(z) - y \circ x \circ D(z).$$
(8)

(ii) P is called a tailed 2-nd square-circle Poisson algebra if

$$J[x, y, \circ, z] = y \circ D(x) \circ z.$$
(9)

(iii) P is called a tailed  $^{3-rd}$  square-circle Poisson algebra if there exists  $\alpha \in \mathbf{k}$  such that

$$J[x, y, \circ, z] = D(y) \circ z \circ x - D(y) \circ x \circ z + y \circ (\alpha D)(x) \circ z.$$
(10)

(iv) P is called a tailed 4-th square-circle Poisson algebra if

$$J[x, y, \circ, z] = D(y) \circ z \circ x - D(y) \circ x \circ z \tag{11}$$

and

$$x \circ y \circ D(z) = x \circ D(y) \circ z = 0.$$
<sup>(12)</sup>

(v) P is called a tailed <sup>5-th</sup> square-circle Poisson algebra if there exists a nonzero scalar  $\alpha \in \mathbf{k}$  such that

$$J[x, y, \circ, z] = y \circ (\alpha D)(x) \circ z + D(y) \circ z \circ x - D(y) \circ x \circ z.$$
(13)

(vi) P is called a tailed  $^{6-th}$  square-circle Poisson algebra if there exists a scalar  $\alpha \in \mathbf{k}$  such that

$$J[x, y, \circ, z] = (\alpha D)(y) \circ z \circ x - (\alpha D)(y) \circ x \circ z + + x \circ y \circ D(z) - y \circ x \circ D(z).$$
(14)

(vii) P is called a tailed <sup>7-th</sup> square-circle Poisson algebra if there exists a nonzero scalar  $\alpha \in \mathbf{k}$  such that

$$J[x, y, \circ, z] = y \circ (\alpha D)(x) \circ z + x \circ y \circ D(z) - y \circ x \circ D(z).$$
(15)

(viii) P is called a tailed <sup>8-th</sup> square-circle Poisson algebra if

$$J[x, y, \circ, z] = x \circ y \circ D(z) - y \circ x \circ D(z)$$
<sup>(16)</sup>

and

$$D(x) \circ y \circ z = x \circ D(y) \circ z = 0.$$
<sup>(17)</sup>

Note that if the circle product  $\circ$  is commutative, then the tail  $y \circ D(x) \circ z$  of a tailed<sup>2-nd</sup> square-circle Poisson algebra becomes the tail  $D(x) \circ y \circ z$  of a generalized Poisson superalgebra introduced in [1].

The next proposition gives a summary of our ways of constructing tailed square-circle Poisson algebras.

**Proposition 3.1** Let End(V) be the associative algebra of linear transformations over a vector space V. If V has a proper nonzero subspace, then there exists a subspace  $\mathcal{M}$  of End(V) such that the following hold:

- (i) There are 2 ways of making M into a tailed<sup>1-st</sup> square-circle Poisson algebra;
- (ii) There are 4 ways of making M into a tailed<sup>2-nd</sup> square-circle Poisson algebra;
- (iii) There are 2 ways of making M into a tailed<sup>3-rd</sup> square-circle Poisson algebra;
- (iv) There are 7 ways of making M into a tailed<sup>4-th</sup> square-circle Poisson algebra;
- (v) There are 1 way of making M into a tailed<sup>i-th</sup> square-circle Poisson algebra for i = 5, 6 and 7;
- (vi) There are 6 ways of making  $\mathcal{M}$  into a tailed<sup>8-th</sup> square-circle Poisson algebra.

## 4 Tailed Left Angle-Circle Poisson Algebras

In this section, we introduce 6 different tailed left angle-circle Poisson algebras. Their definitions are as follows.

**Definition 4.1** Let P be a vector space equipped with a **angle bracket**  $\langle , \rangle$  and a **circle product**  $\circ$  such that P is a right Leibniz algebra with respect to the angle bracket  $\langle , \rangle$  and P is an associative algebra with respect to the circle product  $\circ$ . Let D be a derivation with respect to both the angle bracket  $\langle , \rangle$  and the circle product  $\circ$ . Let x, y and z be arbitrary three elements of P.

#### (i) P is called a tailed 1-st left angle-circle Poisson algebra if

$$J_{\ell}\langle x, y, \circ, z \rangle = x \circ y \circ D(z) - y \circ x \circ D(z).$$
(18)

(ii) P is called a tailed 2-nd left angle-circle Poisson algebra if

$$J_{\ell}\langle x, y, \circ, z \rangle = x \circ y \circ D(z) - y \circ x \circ D(z)$$
<sup>(19)</sup>

and

$$x \circ D(y) \circ z = D(x) \circ y \circ z = 0.$$
<sup>(20)</sup>

(iii) P is called a tailed 3-rd left angle-circle Poisson algebra if

$$J_{\ell}\langle x, y, \circ, z \rangle = D(y) \circ x \circ z - D(y) \circ z \circ x.$$
(21)

(iv) P is called a tailed 4-th left angle-circle Poisson algebra if

$$J_{\ell}\langle x, y, \circ, z \rangle = D(y) \circ x \circ z - D(y) \circ z \circ x \tag{22}$$

and

$$x \circ y \circ D(z) = x \circ D(y) \circ z.$$
<sup>(23)</sup>

(v) P is called a tailed 5-th left angle-circle Poisson algebra if

$$J_{\ell}\langle x, y, \circ, z \rangle = x \circ y \circ D(z) - y \circ x \circ D(z) + -D(y) \circ x \circ z + D(y) \circ z \circ x$$
(24)

(vi) P is called a tailed  $^{6-th}$  left angle-circle Poisson algebra if

$$J_{\ell}\langle x, y, \circ, z \rangle = x \circ y \circ D(z) - y \circ x \circ D(z)$$
(25)

and

$$x \circ D(y) \circ z = D(x) \circ y \circ z. \tag{26}$$

The next proposition gives a summary of our ways of constructing tailed left angle-circle Poisson algebras.

**Proposition 4.1** Let End(V) be the associative algebra of linear transformations over a vector space V. If V has a proper nonzero subspace, then there exists a subspace  $\mathcal{M}$  of End(V) such that the following hold:

- (i) There are 3 ways of making M into a tailed<sup>1-st</sup> left angle-circle Poisson algebra;
- (ii) There are 5 ways of making M into a tailed<sup>2-nd</sup> left angle-circle Poisson algebra;
- (iii) There are 2 ways of making M into a tailed<sup>3-rd</sup> left angle-circle Poisson algebra;
- (iv) There are 6 ways of making M into a tailed<sup>4-th</sup> left angle-circle Poisson algebra;
- (v) There are 1 way of making  $\mathcal{M}$  into a tailed<sup>*i*-th</sup> left angle-circle Poisson algebra for *i* = 5 and 6.

## 5 Tailed Right Angle-Circle Poisson Algebras

We now introduce 4 different tailed right angle-circle Poisson algebras.

**Definition 5.1** Let P be a vector space equipped with a **angle bracket**  $\langle , \rangle$ and a **circle product**  $\circ$  such that P is a right Leibniz algebra with respect to the angle bracket  $\langle , \rangle$  and P is an associative algebra with respect to the circle product  $\circ$ . Let D be a derivation with respect to both the angle bracket  $\langle , \rangle$  and the circle product  $\circ$ . Let x, y and z be arbitrary three elements of P.

(i) P is called a tailed 1-st right angle-circle Poisson algebra if

$$J_r \langle x, \circ, y, z \rangle = x \circ z \circ D(y) - z \circ x \circ D(y).$$
<sup>(27)</sup>

(ii) P is called a tailed 2-nd right angle-circle Poisson algebra if

$$J_r\langle x, \circ, y, z \rangle = x \circ z \circ D(y) - z \circ x \circ D(y)$$
(28)

and

$$x \circ D(y) \circ z = D(x) \circ y \circ z. \tag{29}$$

(iii) P is called a tailed 3-rd right angle-circle Poisson algebra if

$$J_r \langle x, \circ, y, z \rangle = D(x) \circ y \circ z - D(x) \circ z \circ y.$$
(30)

(iv) P is called a tailed 4-th right angle-circle Poisson algebra if

$$J_r \langle x, \circ, y, z \rangle = D(x) \circ y \circ z - D(x) \circ z \circ y \tag{31}$$

and

$$x \circ y \circ D(z) = x \circ D(y) \circ z = 0.$$
(32)

The next proposition gives a summary of our ways of constructing tailed right angle-circle Poisson algebras.

**Proposition 5.1** Let End(V) be the associative algebra of linear transformations over a vector space V. If V has a proper nonzero subspace, then there exists a subspace  $\mathcal{M}$  of End(V) such that the following hold:

- (i) There are 2 ways of making M into a tailed<sup>1-st</sup> right angle-circle Poisson algebra;
- (ii) There are 3 ways of making M into a tailed<sup>2-nd</sup> right angle-circle Poisson algebra;
- (iii) There are 2 ways of making M into a tailed<sup>3-rd</sup> right angle-circle Poisson algebra;
- (iv) There are 3 ways of making M into a tailed<sup>4-th</sup> right angle-circle Poisson algebra.

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