PROBABILITY MEASURES AND EFFECTIVE RANDOMNESS

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ABSTRACT. We study the question, "For which reals x does there exist a measure μ such that x is random relative to μ?" We show that for every nonrecursive x, there is a measure which makes x random without concentrating on x. We give several conditions on x equivalent to there being continuous measure which makes x random. We show that for all but countably many reals x these conditions apply, so there is a continuous measure which makes x random. There is a meta-mathematical aspect of this investigation. As one requires higher arithmetic levels in the degree of randomness, one must make use of more iterates of the power set of the continuum to show that for all but countably many x's there is a continuous μ which makes x random to that degree.

1. Introduction

Most studies on algorithmic randomness focus on reals random with respect to the uniform distribution, i.e. the $(1/2, 1/2)$-Bernoulli measure, which is measure theoretically isomorphic to Lebesgue measure on the unit interval. The theory of uniform randomness, with all its ramifications (e.g. computable or Schnorr randomness) has been well studied over the past decades and has led to an impressive theory.

Recently, a lot of attention focused on the interaction of algorithmic randomness with recursion theory: What are the computational properties of random reals? In other words, which computational properties hold effectively for almost every real? This has led to a number of interesting results, many of which will be covered in a forthcoming book by Downey and Hirschfeldt [4].

While the understanding of "holds effectively" varied in these results (depending on the underlying notion of randomness, such as computable, Schnorr, or weak randomness, or various arithmetic levels of Martin-Löf randomness, to name only a few), the meaning of "for almost every" was usually understood with respect to Lebesgue measure. One reason for this can surely be seen in the fundamental relation between uniform Martin-Löf tests and descriptive complexity in terms of (prefix-free) Kolmogorov complexity: A real is not covered by any Martin-Löf test (with respect to the uniform distribution) if and only if all of its initial segments are incompressible (up to a constant additive factor).

However, one may ask what happens if one changes the underlying measure. This question is virtually as old as the theory of randomness. Martin-Löf [15] defined randomness not only for Lebesgue measure but also for arbitrary Bernoulli distributions. Levin's contributions in the 1970's [24, 11, 12, 13] extended this to arbitrary probability measures. He obtained a number of remarkable results and principles such as the existence of uniform tests, conservation of randomness, and the existence of neutral measures.

In this paper we will survey a recent line of research by the authors which dealt with the question *for which reals x does there exist a probability measure which makes x random without concentrating on x*. We consider two kinds measures – arbitrary probability measures, which may have atoms (reals other than x on which the measure concentrates), and *continuous measures*, i.e. non-atomic measures. The investigations exhibit an interesting, and quite unexpected, connection between the randomness properties of a real and its logical complexity, in the sense of recursion or set theoretic hierarchies. In the following we will try to describe this connection in some detail. We will sketch proofs to provide some intuition, but for a full account we have to refer the reader to the forthcoming research papers [19, 20].

2. Measures and Randomness

In this section we introduce the basic notions of measure on the Cantor space 2^ω and define randomness for arbitrary probability measures.

The *Cantor space* 2^ω is the set of all infinite binary sequences, also called *reals*. The topology generated by the *cylinder sets*

$$N_\sigma = \{x : x\lceil n = \sigma\},$$

where σ is a finite binary sequence, turns 2^ω into a compact Polish space. We will occasionally use the notation $N(\sigma)$ in place of N_σ to avoid multiple subscripts. $2^{<\omega}$ denotes the set of all finite binary sequences. If $\sigma, \tau \in 2^{<\omega}$, we use $\subseteq$ to denote the usual prefix partial ordering. This extends in a natural way to $2^{<\omega} \cup 2^\omega$. Thus, $x \in N_\sigma$ if and only if $\sigma \subset x$. Finally, given $U \subseteq 2^{<\omega}$, we write N_U to denote the open set induced by U, i.e. $N_U = \bigcup_{\sigma \in U} N_\sigma$.

2.1. Probability measures.

A *probability measure* on 2^ω is a *countably additive, monotone function* $\mu : \mathcal{F} \to [0,1]$, where $\mathcal{F} \subseteq \mathcal{P}(2^\omega)$ is σ-algebra and $\mu(2^\omega) = 1$. μ is called a *Borel probability measure* if $\mathcal{F}$ is the Borel σ-algebra of 2^ω. It is a basic result of measure theory that a probability measure is uniquely determined by the values it takes on an algebra $\mathcal{A} \subseteq \mathcal{F}$ that generates $\mathcal{F}$. It is not hard to see that the Borel sets are generated by the algebra of *clopen sets*, i.e. finite unions of basic open cylinders. Normalized, monotone, countably additive set functions on the algebra of clopen sets are induced by any function $\rho : 2^{<\omega} \to [0,1]$ satisfying

$$\rho(\epsilon) = 1 \qquad \text{and} \qquad \rho(\sigma) = \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1) \tag{2.1}$$

for all finite sequences σ. Then $\mu(N_\sigma) = \rho(\sigma)$ yields an monotone, additive function on the clopen sets, which in turn uniquely extends to a Borel probability measure on 2^ω. In the following, we will deal exclusively with Borel probability measures, and hence we will identify such measures with the underlying function on cylinders satisfying (2.1), and write, in slight abuse of notation, $\mu(\sigma)$ instead of $\mu(N_\sigma)$. Besides, we will mostly speak of *measures*, understanding Borel probability measures.

The *Lebesgue measure* $\mathcal{L}$ on 2^ω is obtained by distributing a unit mass uniformly along the paths of 2^ω, i.e. by setting $\mathcal{L}(\sigma) = 2^{-|\sigma|}$. A *Dirac measure*, on the other hand, is defined by putting a unit mass on a single real, i.e. for $x \in 2^\omega$, let

$$\delta_x(\sigma) = \begin{cases} 1 & \text{if } \sigma \subset x, \\ 0 & \text{otherwise.} \end{cases}$$

If, for a measure μ and $x \in 2^\omega$, $\mu(\{x\}) > 0$, then x is called an *atom* of μ. Obviously, x is an atom of δ_x. A measure that does not have any atoms is called *continuous*.

3. Martin-Löf Randomness

It was Martin-Löf's fundamental idea to define randomness by choosing a *countable family* of nullsets. For any non-trivial measure, the complement of the union of these sets will have positive measure, and any point in this set will be considered *random*. There are of course many possible ways to pick a countable family of nullsets. In this regard, it is very benefiting to use the framework of recursion theory and effective descriptive set theory.

3.1. **Nullsets.**

Before we go on to define Martin-Löf randomness formally, we note that every nullset is contained in a G_δ-nullset.

Proposition 3.1. *Suppose μ is a measure. Then a set $A \subseteq 2^\omega$ is μ-null if and only if there exists a set $U \subseteq \mathbb{N} \times 2^{<\omega}$ such that for all n,*

$$A \subseteq N(U_n) \quad \text{and} \quad \sum_{\sigma \in U_n} \mu(N_\sigma) \leq 2^{-n}, \tag{3.1}$$

where $U_n = \{\sigma : (n, \sigma) \in U\}$.

Of course, the G_δ-cover of A is given by $\bigcap_n U_n$.

3.2. **Martin-Löf tests and randomness.**

Essentially, a Martin-Löf test is an effectively presented G_δ nullset (relative to some parameter z).

Definition 3.2. Suppose $z \in 2^\omega$ is a real. A *test relative to z*, or simply a *z-test*, is a set $W \subseteq \mathbb{N} \times 2^{<\omega}$ which is recursively enumerable in z. Given a natural number $n \geq 1$, an *n-test* is a test which r.e. in $\emptyset^{(n-1)}$, the $(n-1)$st Turing jump of the empty set. A real x *passes* a test W if $x \notin \bigcap_n N(W_n)$.

Passing a test W means not being contained in the G_δ set given by W. The condition '*r.e. in z*' implies that the open sets given by the sets W_n form a uniform sequence of $\Sigma^0_1(z)$ sets, and the set $\bigcap_n N(W_n)$ is a $\Pi^0_2(z)$ subset of 2^ω.

To test for randomness with respect to a measure, we have to ensure two things: First that a test W actually describes a nullset. Second, that the information present in a measure is available to the test. Te first criterion we call *correctness*.

Definition 3.3. Suppose μ is a measure on 2^ω. A test W is *correct for μ* if

$$\sum_{\sigma \in W_n} \mu(N_\sigma) \leq 2^{-n}. \tag{3.2}$$

To incorporate measures into an effective test for randomness we have to represent it in a form that makes it accessible for recursion theoretic methods. Essentially, this means to code a measure via an infinite binary sequence or a function $f : \mathbb{N} \to \mathbb{N}$. Unfortunately, there are many possible such representations. Hence, strictly speaking, we will deal with *randomness with respect to a representation* of a measure, not the measure itself. However, we will see that for one of our main topics, randomness for continuous measures, representational issues can be resolved quite elegantly.

The most straightforward representation of a measure is the following.

Definition 3.4. Given a measure μ, define its *rational representation* r_ρ by letting, for all $\sigma \in 2^{<\omega}$, $q_1, q_2 \in \mathbb{Q}$,

$$\langle \sigma, q_1, q_2 \rangle \in r_\rho \ \Leftrightarrow \ q_1 < \rho(\sigma) < q_2. \tag{3.3}$$

The rational representation does not reflect the topological properties of the space of probability measures on 2^ω. The space of probability measures $\mathcal{P}$ on 2^ω is a compact polish space (see Parthasarathy [17]). The topology is the *weak topology*, which can be metrized by the *Prokhorov metric*, for instance. There is an *effective dense subset*, given as follows: Let Q be the set of all reals of the form $\sigma^\frown 0^\omega$. Given $\bar{q} = (q_1, \dots, q_n) \in Q^{<\omega}$ and non-negative rational numbers $\alpha_1, \dots, \alpha_n$ such that $\sum \alpha_i = 1$, let

$$\delta_{\bar{q}} = \sum_{k=1}^{n} \alpha_k \delta_{q_k},$$

where δ_x denotes the *Dirac point measure* for x. Then the set of measures of the form $\delta_{\bar{q}}$ is dense in $\mathcal{P}$.

The recursive dense subset $\{\delta_{\bar{q}}\}$ and the effectiveness of the metric d between measures of the form $\delta_{\bar{q}}$ suggests that the representation reflects the topology effectively, i.e. the set of representations should be Π^0_1. However, this is not true for the set of rational representations of probability measures. Instead, we have to resort to other representations in metric spaces, such as Cauchy sequences. Using the framework of *effective descriptive set theory*, as for example presented in Moschovakis [16], one can obtain the following.

Theorem 3.5. *There is a recursive surjection*

$$\pi : 2^\omega \to \mathcal{P}$$

and a Π^0_1 subset P of 2^ω such that $\pi \lceil P$ is one-one and $\pi(P) = \mathcal{P}$.

In the following sections, we will always assume that a measure is either represented by its rational representation or via the the set $P \subseteq 2^\omega$ of the previous theorem. The definition of randomness, however, works for any representation.

Definition 3.6. Suppose μ is a probability measure on 2^ω, $\rho_\mu \in 2^\omega$ is a representation of μ, and $z \in 2^\omega$ is a real. A real is *Martin-Löf n-random for μ relative to ρ_μ and z*, or simply (n, z)*-random for ρ_μ* if it passes all $(\rho_\mu \oplus z)^{(n-1)}$-tests which are correct for μ.

If the representation is clear from the context, we speak of (n, z)-randomness for μ. If μ is Lebesgue measure $\mathcal{L}$, we drop reference to the measure and simply say "x is (n, z)-random". We also drop the index 1 in case of $(1, z)$-randomness and simply speak of "randomness relative to z" or z-randomness.

Since there are only countably many Martin-Löf n-tests, it follows from countable additivity that the set of Martin-Löf n-random reals for μ has μ-measure 1. Hence there always exist (n, z)-random reals for any measure μ.

3.3. **Image measures and conservation of randomness.** One can obtain new measures from given measures by transforming them with respect to a sufficiently regular function. Let $f : 2^\omega \to 2^\omega$ be a Borel (measurable) function, i.e. for every Borel set A, $f^{-1}(A)$ is Borel, too. If μ is a measure on 2^ω and f is Borel, then the *image measure* μ_f is defined by

$$\mu_f(A) = \mu(f^{-1}(A)).$$

It can be shown that every probability measure can be obtained from Lebesgue measure $\mathcal{L}$ by means of a measurable transformation.

Theorem 3.7 (folklore, see e.g. Billingsley [1])**.** *If μ is a Borel probability measure on 2^ω, then there exists a measurable $f : 2^\omega \to 2^\omega$ such that $\mu = \mathcal{L}_f$.*

If the transformation of $\mathcal{L}$ is effective, then f maps an $\mathcal{L}$-random real to a $\mathcal{L}_f$-random real. This principle is called *conservation of randomness*, first introduced by Levin. We can use it to construct measures for which a given real is random, as we will see in the next sections.

4. Randomness of Non-Recursive Reals

If x is an atom of some probability measure μ, it is trivially μ-random. Interestingly, the recursive reals are exactly those for which this is the only way to become random.

Theorem 4.1 (Reimann and Slaman [20])**.** *For any real x, the following are equivalent.*

(i) There exists a (representation of a) probability measure μ such that $\mu(\{x\}) = 0$ and x is μ-random.
(ii) x is not recursive.

Proof sketch. If x is recursive and μ is a measure with $\mu(\{x\}) = 0$, then we can obviously construct a μ-test that covers x, by computing (recursively in μ) the measure of initial segments of x, which tends to 0.

Now assume x is not recursive. A fundamental result by Kučera [10] ensures that every Turing degree above $\emptyset'$ contains a $\mathcal{L}$-random real. This result relativizes. Hence one can combine it with the *Posner-Robinson Theorem* [18], which says that for every non-recursive real x there exists a z such that $x \oplus z =_{\mathrm{T}} z'$. This way we obtain a real R which is

(1) $\mathcal{L}$-random relative to some $z \in 2^\omega$, and
(2) $\mathrm{T}(z)$-equivalent to x.

There are Turing functionals Φ and Ψ recursive in z such that

$$\Phi(R) = x \quad \text{and} \quad \Psi(x) = R.$$

We can use the functionals to define a class of measures that are possible candidates to render x random. Given $\sigma \in 2^\omega$, define the set $\mathrm{Pre}(\sigma)$ to be the set of minimal elements of

$$\{\tau \in 2^{<\omega} : \ \Phi(\tau) \supseteq \sigma \ \text{ and } \ \Psi(\sigma) \subseteq \tau\}.$$

We define a set of measures M by requiring that $\mu \in M$ if and only if

$$\forall \sigma [\mathcal{L}(\mathrm{Pre}(\sigma)) \leq \mu(\sigma) \leq \mathcal{L}(\Psi(\sigma))]. \tag{4.1}$$

The first inequality ensures that μ *dominates* an image measure induced by Φ. This will ensure that any Martin-Löf random real is mapped by Φ to a μ-random real. The second inequality guarantees that μ is non-atomic on the domain of Ψ.

One can show the topological representations of the measures in M (Theorem 3.5) form a non-empty Π^0_1 class M in 2^ω relative to z.

In order to apply conservation of randomness, we have to know that one of the measures in M, when given as an additional information to a $\mathcal{L}$-test, will not destroy the randomness of R. This is ensured by the following basis result for Π^0_1 sets regarding relative randomness (essentially a consequence of *compactness*). □

Theorem 4.2 (Reimann and Slaman [20], Downey, Hirschfeldt, Miller, and Nies [3])**.** *Let S be $\Pi^0_1(z)$. If R is $\mathcal{L}$-random relative to z, then there exists $y \in S$ such that R is $\mathcal{L}$-random relative to $y \oplus z$.*

5. Randomness for continuous measures

A natural question arising in the context Theorem 4.1 is whether the measure making a real random can be ensured to have certain regularity properties; in particular, can it be chosen *continuous*?

Reimann and Slaman [20] gave an explicit construction of a non-recursive real not random with respect to any continuous measure. Call such reals 1-*ncr*. In general, let NCR_n be the set of reals which are not n-random with respect to any continuous measure.

Kjos-Hanssen and Montalban [8] observed that any member of a countable Π^0_1 class is an element of NCR_1.

Proposition 5.1. *If $A \subseteq 2^\omega$ is Π^0_1 and countable, then no member of A can be in NCR_1.*

Proof idea. If μ is a continuous measure, then obviously $\mu(A) = 0$. One can use a recursive tree T such that $[T] = A$ to obtain a μ-test for A. □

It follows from results of Cenzer, Clote, Smith, Soare, and Wainer [2] that members of NCR_1 can be found throughout the hyperarithmetical hierarchy of Δ^1_1, whereas Kreisel [9] had shown earlier that each member of a countable Π^0_1 class is in fact hyperarithmetical.

Quite surprisingly, Δ^1_1 turned out to be the precise upper bound for NCR_1. An analysis of the proof of Theorem 4.1 shows that if x is *truth-table* equivalent to a $\mathcal{L}$-random real, then the "pull-back" procedure used to devise a measure for x yields a continuous measure. More generally, we have the following.

Theorem 5.2 (Reimann and Slaman [20])**.** *Let x be a real. For any $z \in 2^\omega$ and any $n \geq 1$, the following are equivalent.*

(i) x is (n, z)-random for a continuous measure μ recursive in z.
(ii) x is (n, z)-random for a continuous dyadic measure ν recursive in z.
(iii) There exists a functional Φ recursive in z which is an order-preserving homeomorphism of 2^ω such that $\Phi(x)$ is (n, z)-random.
(iv) x is truth-table equivalent relative to z to a (n, z)-random real.

Here *dyadic* measure means that the values of μ on the open cylinders are of the form $\mu(\sigma) = m/2^n$ with $m, n \in \mathbb{N}$. The theorem can be seen as an effective version of the *classical isomorphism theorem* for continuous probability measures (see for instance Kechris [7]).[1]

Woodin [23], using a variation on Prikry forcing, was able to prove that if $x \in 2^\omega$ is not hyperarithmetic, then there is a $z \in 2^\omega$ such that $x \oplus z \equiv_{\mathrm{tt}(z)} z'$, i.e. outside Δ^1_1 the Posner-Robinson theorem holds with truth-table equivalence. Hence we can infer the following result.

Theorem 5.3 (Reimann and Slaman [20])**.** *If a real x is not Δ^1_1, then there exists a continuous measure μ such that x is μ-random.*

[1] The theorem suggests that for continuous randomness representational issues do not really arise, since there is always a measure with a computationally minimal representation.

It is on the other hand an open problem whether every real in NCR_1 is a member of a countable Π^0_1 class.

One may ask how the complexity and size of NCR_n grows with n. It turned out all levels of NCR are countable.

Theorem 5.4 (Reimann and Slaman [19]). *For all n, NCR_n is countable.*

Proof idea. The first step is to use *Borel determinacy* to show that the complement of NCR_n contains an upper Turing cone. This follows from the fact that the complement of NCR_n contains a Turing invariant and cofinal (in the Turing degrees) Borel set, which can be seen as follows.

If for two reals x, y, $x \equiv_{\mathrm{T}(z)} y$, then $x \equiv_{\mathrm{tt}(z')} y$. Suppose $x \equiv_{\mathrm{T}(z)} R$ where R is $(n+1)$-random relative to z. Then, since R is n-random relative to z', it follows from Theorem 5.2 that x is random with respect to some continuous measure.

So if we let $B \subseteq 2^\omega$ be the set

$$\{x \in 2^\omega : \exists z\, \exists R\, (x \equiv_{\mathrm{T}} z \oplus R \ \ \& \ \ R \text{ is } (n+1)\text{-random relative to } z)\},$$

B is a Turing invariant Borel set cofinal in the Turing degrees. It follows from Borel Determinacy [14] that B contains an upper cone in the Turing degrees.

The next step is to show that the elements of NCR_n show up at a *countable level of the constructible universe* L. It holds that $\mathrm{NCR}_n \subseteq L_{\beta_n}$, where β_n is the least ordinal such that

$$L_{\beta_n} \models \mathsf{ZFC}^-_n,$$

where ZFC^-_n is ZFC with the power set axiom replaced by the existence of n iterates of the power set of ω. Note that L_{β_n} is the level of constructibility capturing Martin's construction of a winning strategy in a Σ^0_n-game.

Given $x \notin L_{\beta_n}$, construct a set G such that $L_{\beta_n}[G]$ is a model of ZFC^-_n, and for all $y \in L_{\beta_n}[G] \cap 2^\omega$, $y \leq_{\mathrm{T}} x \oplus G$. G is constructed by *Kumabe-Slaman forcing* (see [22]). This notion of forcing provides a method to extend the Posner-Robinson Theorem to higher levels of the jump and beyond. The existence of G allows to conclude: If x is not in L_{β_n}, it will belong to every cone with base in $L_{\beta_n}[G]$. In particular, it will belong to the cone given by the Borel Turing determinacy argument (relativized to G, here one has to use absoluteness), i.e. the cone avoiding NCR_n. Hence x is random relative to G for some continuous μ, an thus in particular μ-random. □

The proof of the countability of NCR_n makes essential use of Borel determinacy. It is known from a result by Friedman [5] that the use of ω_1-many iterates of the power set of ω are necessary to prove Borel determinacy. In the simplest case, Friedman showed that ZFC^- does not prove the statement "All Σ^0_5-games on countable trees are determined." The proof works by showing that there is a model of ZFC^- for which Σ^0_5-determinacy does not hold. This model is just L_{β_0}. The analysis extends to higher levels of the Borel hierarchy, applying to more and more iterates of the power set.

The question suggests itself whether the proof of the countability of NCR_n requires a similar set theoretic complexity.

Theorem 5.5 (Reimann and Slaman [19]). *For every k, the statement*

For every n, NCR_n is countable.

cannot be proven in ZFC^-_k.

Proof sketch. We show that for every fixed k, some NCR_n is cofinal in the Turing degrees of L_{β_k}. In fact, Jensen's *master codes* [6] for L, the universe of constructible sets, are the cofinal set.

L is generated by transfinite recursion in which the recursion steps are closing under first order definability and forming unions. The master codes represent the initial segments of L and are generated by iterating the Turing jump and taking L-least representations of direct limits. In short, a master code is either definable relative to an earlier master code or is the code for the well-founded limit of structures each of which is coded by an earlier master code. The number of iterates of the power set present in the initial segment of L which is being coded is linked to the complexity of describing the direct limit used to form its master code. For α less than β_k, there is a fixed bound on this complexity. We let M_α denote the master code for L_β.

Neither of these cases is consistent with randomness, as indicated by the following lemmas.

Lemma 5.6. *Suppose that $n \geq 2$, $y \in 2^\omega$, and x is n-random for μ. If $i < n$, y is recursive in $(x \oplus \mu)$ and recursive in $\mu^{(i)}$, then y is recursive in μ.*

Lemma 5.7. *Suppose that x is $(n+5)$-random for μ, $\prec$ is a linear ordering that is Δ^0_{n+1} relative to μ, and I is the largest initial segment of $\prec$ which is well-founded. If $i < n$ and I is Σ^0_i in $(x \oplus \mu)$, then I is recursive in μ.*

Suppose α less than β_k and a continuous measure μ are given so that M_α is random relative to μ. Heuristically, we argue as follows. We proceed by induction on $\beta \leq \alpha$ to prove that M_β is recursive in μ. If β is a successor, then M_β is arithmetic in some earlier master code, with a uniform upper bound on the complexity of the definition depending on k. Then, M_β is uniformly arithmetic in μ and recursive in M_α, Lemma 5.6 applies. Otherwise, M_β is the well-founded direct limit of structures recursive in μ and recursive in in M_α, so Lemma 5.7 applies. In either case, M_β is recursive in μ. By induction, M_α is itself recursive in μ and not μ-random, a contradiction. ☐

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