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Reducing system of parameters and the Cohen–Macaulay property

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Abstract. Let R be a local ring and let $(x_1,\ldots,x_r)$ be part of a system of parameters of a finitely generated R-module M, where $r < \dim_R M$. We will show that if $(y_1,\ldots,y_r)$ is part of a reducing system of parameters of M with $(y_1,\ldots,y_r)M = (x_1,\ldots,x_r)M$ then $(x_1,\ldots,x_r)$ is already reducing. Moreover, there is such a part of a reducing system of parameters of M iff for all primes $P \in \operatorname{Supp} M \cap V_R(x_1,\ldots,x_r)$ with $\dim_R R/P = \dim_R M - r$ the localization M_P of M at P is an r-dimensional Cohen–Macaulay module over R_P.

Furthermore, we will show that M is a Cohen–Macaulay module iff y_d is a non zero divisor on $M/(y_1,\ldots,y_{d-1})M$, where $(y_1,\ldots,y_d)$ is a reducing system of parameters of M ($d := \dim_R M$).

Keywords. Systems of parameters; Cohen–Macaulay modules.

1. Preliminaries

In what follows, let R be a local ring with maximal ideal $\mathfrak{m}$ and let M be a non zero finitely generated R-module of dimension d. Instead of $\dim_R$, depth_R, Ass_R, Supp_R, … we will write $\dim, \operatorname{depth}, \operatorname{Ass}, \operatorname{Supp}, \ldots$ for short.

We note that $\operatorname{Supp} M/XM = \operatorname{Supp} M \cap V(X)$, where X is a subset of R and that for a prime ideal P of R we have $P \in \operatorname{Ass} M$ iff $PR_P \in \operatorname{Ass} M_P$. Moreover we define $\operatorname{Assh} M := \{P \in \operatorname{Ass} M \mid \dim R/P = d\}$.

For undefined terminology we refer to the standard literature (e.g. [E]).

DEFINITION 1.

A system of parameters $(x_1,\ldots,x_d)$ of M is called *reducing*, if for all $i = 1,\ldots,d-1$ we have

$$x_i \notin P \text{ for all } P \in \operatorname{Ass} M/(x_1,\ldots,x_{i-1})M \text{ with } \dim R/P = d-i.$$

Remark 2. Auslander and Buchsbaum defined in [AB] a system of parameters $(x_1,\ldots,x_d)$ of R to be a reducing system of parameters of M if

$$e_M(x_1,\ldots,x_d) = \operatorname{length}(M/(x_1,\ldots,x_d)M) - \operatorname{length}((x_1,\ldots,x_{d-1})M : x_d/(x_1,\ldots,x_{d-1})M).$$

This definition is equivalent to the definition given above if we pass from R to $\bar{R} := R/\operatorname{Ann}_R M$ and consider M as a $\bar{R}$-module in Definition 1 and use Corollary 4.8 in [AB]. Therefore it is clear that all definitions and results on reducing systems of parameters remain true in this more general context.

Remark 3. For every system of parameters $(x_1,\dots,x_d)$ of M there is a reducing system of parameters $(y_1,\dots,y_d)$ of M such that $(y_1,\dots,y_d)R = (x_1,\dots,x_d)R$, in particular, $(y_1,\dots,y_d)M = (x_1,\dots,x_d)M$ (see Proposition 4.9 in [AB]).

DEFINITION 4.

A sequence $x_1,\dots,x_r$ of elements of $\mathfrak{m}$ is called *part of a* (*reducing*) *system of parameters* of M, if there are elements $x_{r+1},\dots,x_d \in \mathfrak{m}$ such that $(x_1,\dots,x_r,x_{r+1},\dots,x_d)$ is a (reducing) system of parameters of M.

Remark 5.

(1) A sequence $(x_1,\dots,x_r)$ of elements of $\mathfrak{m}$ with $r < d$ is part of a system of parameters of M iff $\dim M/(x_1,\dots,x_r)M = d - r$.
(2) A sequence $(x_1,\dots,x_r)$ of elements of $\mathfrak{m}$ with $r < d$ is part of a reducing system of parameters of M iff for all $i = 1,\dots,r$ we have $x_i \notin P$ for all $P \in \operatorname{Ass} M/(x_1,\dots,x_{i-1})M$ with $\dim R/P \geq d - i$.
(3) Every regular sequence on M is part of a reducing system of parameters of M.

Remark 6.

(1) We note that the following conditions are equivalent:

(i)M is a Cohen–Macaulay module, i.e. $\operatorname{depth} M = d$.
(ii) Every system of parameters of M is a regular sequence on M.
(iii)There exists a system of parameters of M which is a regular sequence on M.

(2) Assume that M is a Cohen–Macaulay module. If $(x_1,\dots,x_r)$ is part of a system of parameters of M then $M/(x_1,\dots,x_r)M$ is unmixed, more precisely, $\dim R/P = d - r$ for all $P \in \operatorname{Ass} M/(x_1,\dots,x_r)M$. Therefore for a sequence $(x_1,\dots,x_r)$ of elements of $\mathfrak{m}$ the following conditions are equivalent:

(i) $(x_1,\dots,x_r)$ is a regular sequence on M.
(ii) $(x_1,\dots,x_r)$ is part of a reducing system of parameters of M.
(iii) $(x_1,\dots,x_r)$ is part of a system of parameters of M.

Let $x_1,\dots,x_r \in \mathfrak{m}$. If $(x_1,\dots,x_r)$ is a regular sequence on M then $(x_1,\dots,x_r)$ is a regular sequence on M_P as well for all primes $P \in \operatorname{Supp} M \cap V(x_1,\dots,x_r)$.

Lemma 7. *Let* $(x_1,\dots,x_r)$ *be part of a* (*reducing*) *system of parameters of* M. *Then* $(x_1,\dots,x_r)$ *is part of a* (*reducing*) *system of parameters of* M_P *for all primes* $P \in \operatorname{Supp} M \cap V(x_1,\dots,x_r)$ *with* $\dim R/P + \dim M_P = d$.

Proof. Let $P \in \operatorname{Supp} M \cap V(x_1,\dots,x_r)$ with $\dim R/P + \dim M_P = d$. An easy induction argument (induction on r) shows that we can restrict ourselves to the case $r = 1$ (and $\dim M_P \geq 2$).

Let $\mathfrak{q} \in \operatorname{Ass} M_P$ with $\dim R_P/\mathfrak{q} = \dim M_P (\dim R_P/\mathfrak{q} \geq \dim M_P - 1)$. Then $\mathfrak{q} = QR_P$ with $Q \in \operatorname{Ass} M$, $Q \subseteq P$, and we obtain

$$\begin{aligned} \dim R/Q &\geq \dim R/P + \dim(R/Q)_P = \dim R/P + \dim R_P/\mathfrak{q} \\ &= \dim R/P + \dim M_P = d \\ &(\geq \dim R/P + \dim M_P - 1 = d-1). \end{aligned}$$

Therefore $x_1 \notin Q$ by our assumption. But then $x_1 \notin \mathfrak{q}$, i.e. (x_1) is part of a (reducing) system of parameters of M_P. □

Lemma 8. *If $x \in R$ is a zero divisor on M, then $P \in \operatorname{Ass} M/xM$ for all minimal primes $P \in \operatorname{Ass} M \cap \mathrm{V}(x)$.*

Proof. Let $P \in \operatorname{Ass} M \cap V(x)$ be minimal. Since $P \in \operatorname{Ass} M/xM$ iff $PR_P \in \operatorname{Ass} M_P/xM_P$ we may assume by localizing at P that $P = \mathfrak{m}$. Then $x \notin Q$ for all $Q \in \operatorname{Ass} M \backslash \{\mathfrak{m}\}$. Since R is noetherian there is an $i \in \mathbb{N}^+$ such that $0 :_M x^i = 0 :_M x^j$ for all $j \geq i$. Let $U := 0 :_M x^i$. Then $U \neq 0$ (otherwise x would be a non zero divisor on M, contradicting our assumption).

Let $Q \in \operatorname{Supp} M \backslash \{\mathfrak{m}\}$. Since $\operatorname{Ass} M_Q = \{Q'R_Q | Q' \in \operatorname{Ass} M, Q' \subseteq Q\}$, we have $x \notin \mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Ass} M_Q$. Therefore $U_Q = 0 :_{M_Q} x^i = 0$ for all $Q \in \operatorname{Supp} M \backslash \{\mathfrak{m}\}$, i.e. $\operatorname{Supp} U = \{\mathfrak{m}\}$. Moreover,

$$U :_M x = 0 :_M x^{i+1} = 0 :_M x^i = U.$$

Let φ: $U \to M/xM$ be the inclusion $U \subseteq M$ followed by the canonical epimorphism MM/xM. Since

$$\ker \varphi = U \cap xM = x(U :_M x) = xU,$$

φ induces a monomorphism $U/xU \to M/xM$. Now $U/xU \neq 0$ by Nakayama's lemma. Therefore $\emptyset \neq \operatorname{Ass} U/xU \subseteq \operatorname{Ass} U = \{\mathfrak{m}\}$, that means $\operatorname{Ass} U/xU = \{\mathfrak{m}\}$. This gives us the existence of a monomorphism $R/\mathfrak{m} \to U/xU \to M/xM$. Thus $\mathfrak{m} \in \operatorname{Ass} M/xM$. □

Lemma 9. *Let $Q \in \operatorname{Supp} M$ and assume that there is an $x \in \mathfrak{m}$ with $x \notin Q$. Then there is a $P \in \operatorname{Supp} M$ such that $x \in P$, $Q \subset P$ and $\dim R/P = \dim R/Q - 1$.*

Proof. Since (x) is part of a system of parameters of R/Q, there is a $P \in \operatorname{Supp}(R/Q)/x(R/Q) = \operatorname{Supp} R/(Q+xR) = V(Q+xR) \subset V(Q)$ with $\dim R/P = \dim R/(Q+xR) = \dim R/Q - 1$. Since $0 \neq M_Q \cong (M_P)_{QR_P}$ we have $M_P \neq 0$, i.e. $P \in \operatorname{Supp} M$.

□

COROLLARY 10.

Let $Q \in \operatorname{Supp} M$ and let $x_1, \ldots, x_r \in \mathfrak{m}$. Then there is a $P \in \operatorname{Supp} M \cap V(x_1, \ldots, x_r)$ such that $Q \subseteq P$ and $\dim R/P \geq \dim R/Q - r$.

The proof follows immediately from Lemma 9 by induction on r.

2. Main results

Theorem 11. *Let $(y_1, \ldots, y_d)$ be a reducing system of parameters of M. M is a Cohen–Macaulay module iff y_d is a non zero divisor on $M/(y_1, \ldots, y_{d-1})M$.*

Proof. The implication '⇒' is clear, since every system of parameters in a Cohen–Macaulay module is a regular sequence (see Remark 6(1)).

We will prove the opposite implication by induction on d, where the case $d = 1$ is clear. Let $d \geq 2$ and assume that the statement is true for modules with a dimension strictly less than d.

Assume that y_d is a non zero divisor on $M/(y_1,\dots,y_{d-1})M$. By our induction hypothesis, M/y_1M is a Cohen–Macaulay module and therefore it remains to show that y_1 is a non zero divisor on M. Suppose this is not the case. Let P be minimal in $\operatorname{Ass} M \cap V(y_1)$. Since (y_1) is part of a reducing system of parameters of M, we have $\dim R/P \leq d-2$. By Lemma 8, $P \in \operatorname{Ass} M/y_1M$ and therefore $\dim R/P = \dim M/y_1M = d-1$ (see Remark 6(2)), a contradiction. □

Lemma 12. *Let (x) be part of a system of parameters of M. If $d \geq 2$, the following conditions are equivalent*:

(i) *(x) is part of a reducing system of parameters of M.*
(ii) *M_P is a one-dimensional Cohen–Macaulay module over R_P for all $P \in \operatorname{Supp} M \cap \operatorname{V}(x)$ satisfying $\dim R/P = d-1$.*
(iii) *There is a $y \in \mathfrak{m}$ such that (y) is part of a reducing system of parameters of M and $yM = xM$.*
(iv) *There is a $y \in \mathfrak{m}$ such that (y) is part of a reducing system of parameters of M and $\operatorname{Supp} M \cap V(x) \subseteq V(y)$.*

Proof. The implications (i) ⇒ (iii) and (iii) ⇒ (iv) are obvious.

(iv) ⇒ (ii): Let (y) be part of a reducing system of parameters of M with $\operatorname{Supp} M \cap V(x) \subseteq V(y)$ and let $P \in \operatorname{Supp} M \cap V(x)$ with $\dim R/P = d-1$. Then $y \in P$. Now $y \notin Q$ for all $Q \in \operatorname{Ass} M$ with $\dim R/Q \geq d-1$ by our assumption. Thus $P \notin \operatorname{Ass} M$ and therefore $PR_P \notin \operatorname{Ass} M_P \ (\neq \emptyset)$ from which

$$0 < \operatorname{depth} M_P \leq \dim M_P \leq \dim M - \dim R/P = 1,$$

i.e. $\operatorname{depth} M_P = \dim M_P = 1$.

(ii) ⇒ (i): Let $P \in \operatorname{Ass} M$ with $\dim R/P \geq d-1$. If $\dim R/P = d$, then $x \notin P$ since (x) is part of a system of parameters of M. Let $\dim R/P = d-1$. If $x \in P$, then M_P is a Cohen–Macaulay module over R_P with $\dim M_P = 1$. Therefore $PR_P \notin \operatorname{Ass} M_P$ contradicting $P \in \operatorname{Ass} M$.

Thus $x \notin P$ for all $P \in \operatorname{Ass} M$ with $\dim R/P \geq d-1$, i.e. (x) is part of a reducing system of parameters of M by Remark 5(2). □

Remark 13. Let $x_1,\dots,x_r,\ y_1,\dots,y_r$ be elements of $\mathfrak{m}$ with

$$\operatorname{Supp} M \cap V(x_1,\dots,x_r) \subseteq V(y_1,\dots,y_r)$$

(which is equivalent to $\operatorname{Supp} M/(x_1,\dots,x_r)M \subseteq \operatorname{Supp} M/(y_1,\dots,y_r)M$).

(a) If $(y_1,\dots,y_r)$ is part of a system of parameters of M then the same is true for $(x_1,\dots,x_r)$. This follows immediately from Remark 5(1).
(b) If $(y_1,\dots,y_r)$ is a regular sequence on M then the same is true for $(x_1,\dots,x_r)$. This follows from Corollary 2 of [PSS].

The equivalence (i) $\Leftrightarrow$ (iv) of our next theorem shows that a similar statement holds for parts of reducing systems of parameters of M, provided $r < d$. (For $r = d$ this is not true in general, see Remark 3.)

Theorem 14. *Let* $(x_1,\ldots,x_r)$ *be part of a system of parameters of* M, *where* $0 \le r < d$. *Then the following conditions are equivalent*:

(i) $(x_1,\ldots,x_r)$ *is part of a reducing system of parameters of* M.
(ii) M_P *is an* r*-dimensional Cohen–Macaulay module over* R_P *for all* $P \in \operatorname{Supp} M \cap V(x_1,\ldots,x_r)$ *satisfying* $\dim R/P = \dim M - r$.
(iii) *There is a part* $(y_1,\ldots,y_r)$ *of a reducing system of parameters of* M *such that* $(y_1,\ldots,y_r)M = (x_1,\ldots,x_r)M$.
(iv) *There is a part* $(y_1,\ldots,y_r)$ *of a reducing system of parameters of* M *such that* $\operatorname{Supp} M \cap \mathrm{V}(x_1,\ldots,x_r) \subseteq V(y_1,\ldots,y_r)$.

Proof. We use induction on r. For $r = 0$, there is nothing to show and for $r = 1$ the statement follows from Lemma 12. So let $r \ge 2$.

The implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (ii): Let $\overline{M} := M/y_1M$. Take $P \in \operatorname{Supp} M \cap V(x_1,\ldots,x_r) \subseteq \operatorname{Supp} M \cap V(y_1,\ldots,y_r)$ with $\dim R/P = d - r$. Since $\dim \overline{M} = d-1$, $\overline{M}_P \cong M_P/y_1M_P$ is an $(r-1)$-dimensional Cohen–Macaulay module (over R_P) by the induction hypothesis ((i) $\Rightarrow$ (ii)). Therefore it is sufficient to show that y_1 is a non zero divisor on M_P.

Suppose this is not the case. Then by Lemma 8 there is a $\mathfrak{q} \in \operatorname{Ass} M_P$ with $\mathfrak{q} \in \operatorname{Ass} \overline{M}_P$. Therefore $\dim R_P/\mathfrak{q} = r-1$ by Remark 6(2). Now $\mathfrak{q} = QR_P$ with $Q \in \operatorname{Supp} \overline{M} = \operatorname{Supp} M \cap V(y_1)$ and $Q \subseteq P$. Then $Q \in \operatorname{Ass} M$ and we have

$$\dim R/Q \ge \dim R/P + \dim (R/Q)_P = \dim R/P + \dim R_P/\mathfrak{q} = d-1.$$

Therefore $y_1 \notin Q$ (since $(y_1,\ldots,y_r)$ is part of a reducing system of parameters of M), a contradiction.

(ii) $\Rightarrow$ (i): Let $\overline{M} := M/x_1M$ and take $P \in \operatorname{Supp} \overline{M} \cap V(x_2,\ldots,x_r) = \operatorname{Supp} M \cap V(x_1,\ldots,x_r)$ with $\dim R/P = \dim \overline{M} - (r-1) = \dim M - r$. Then M_P is an r-dimensional Cohen–Macaulay module (over R_P) by our assumption and $(x_1,\ldots,x_r)$ is a system of parameters of M_P and hence a regular sequence on M_P by Remark 6(1). But then $\overline{M}_P \cong M_P/x_1M_P$ is an $(r-1)$-dimensional Cohen–Macaulay module (over R_P).

By the induction hypothesis $(x_2,\ldots,x_r)$ is part of a reducing system of parameters of $\overline{M}$ and therefore it remains to show that $x_1 \notin Q$ for all $Q \in \operatorname{Ass} M$ with $\dim R/Q = d-1$.

Suppose this is not the case. Choose $Q \in \operatorname{Ass} M$ with $\dim R/Q = d-1$ and $x_1 \in Q$. By Corollary 10 there is a prime $P \in \operatorname{Supp} M \cap V(x_2,\ldots,x_r)$ such that $Q \subseteq P$ and $\dim R/P \ge d-1-(r-1) = d-r$. But then $P \in \operatorname{Supp} M \cap V(x_1,\ldots,x_r)$ (since $x_1 \in Q \subseteq P$) and therefore $\dim R/P \le d-r$, i.e. $\dim R/P = d-r$. By our assumption, M_P is an r-dimensional Cohen–Macaulay module. Since $QR_P \in \operatorname{Ass} M_P$ we therefore have

$$\begin{aligned} r &= \dim M_P = \dim R_P/QR_P = \dim (R/Q)_P \\ &\le \dim R/Q - \dim R/P = d-1-(d-r) \\ &= r-1 \end{aligned}$$

(see Remark 6(2)), a contradiction. $\square$

COROLLARY 15.

Let $(x_1,\dots,x_r)$ *be part of a reducing system of parameters of* M*. If* $r<d$*, then* $(x_{\pi(1)},\dots,x_{\pi(r)})$ *is part of a reducing system of parameters of* M *for any permutation* π *of* $\{1,\dots,r\}$.

We note that the statement of this corollary is not true in general if $r=d$, see the following Example 16.

Example 16. Let $R:=K[\![X,Y,Z]\!]$, where K is a field and X,Y,Z are indeterminates. For

$$M:=R/(XY,XZ)R \quad \text{and} \quad x_1:=Y,\, x_2:=X+Y+Z,$$

(x_1,x_2) is a system of parameters of M, but not a reducing system of parameters. (x_2,x_1) is a reducing system of parameters of M (not a regular sequence of M).

Finally we define the following.

DEFINITION 17.

$$\mathscr{CM}(M):=\{P\in \operatorname{Supp} M \mid \dim R/P+\dim M_P=d \text{ and } M_P \text{ is}$$
$$\text{a Cohen–Macaulay module over } R_P\}$$

(the *strong Cohen–Macaulay locus of* $\operatorname{Supp} M$) and for $0\le r\le d$

$$\mathscr{CM}_r(M):=\{P\in \mathscr{CM}(M) \mid \dim M_P=r\}.$$

Remark 18.

(1) We have

(i) $\mathscr{CM}_0(M)=\operatorname{Assh} M$ and, if $d\ge 1$,
(ii) $\mathscr{CM}_1(M)=\{P\in \operatorname{Supp} M \mid \dim R/P=d-1\}\setminus \operatorname{Ass} M$,
(iii) $\mathscr{CM}(M)=\bigcup_{r=0}^{d}\mathscr{CM}_r(M)$.

(2) The following conditions are equivalent

(i) M is a Cohen–Macaulay module,
(ii) $\mathscr{CM}(M)=\operatorname{Supp} M$,
(iii) $\mathfrak{m}\in \mathscr{CM}(M)$.

(3) If $\operatorname{Supp} M$ is equidimensional and catenarian then $\mathscr{CM}(M)$ coincides with the ordinary Cohen–Macaulay locus of $\operatorname{Supp} M$. This is the case, for example, when $\dim M\le 1$ or when R is an epimorphic image of a local Cohen–Macaulay ring and M is equidimensional.

PROPOSITION 19.

For $r\in\mathbb{N}$, $r<d$, *we have*

$$\mathscr{CM}_r(M)=\{P \mid P\in \operatorname{Ass} M/(x_1,\dots,x_r)M,\ \dim R/P=d-r,$$
$$(x_1,\dots,x_r) \textit{ part of a reducing system of parameters of } M\}.$$

Proof. By Theorem 14 we have '$\supseteq$' and equality holds (trivially) for $r=0$. Therefore it remains to verify the validity of the inclusion '$\subseteq$' for $r\ge 1$.

Let $P \in \mathscr{C}\mathscr{M}_r(M)$. Since M_P is a Cohen–Macaulay module with $\dim M_P = r \geq 1$, we have $PR_P \notin \operatorname{Ass} M_P$ and hence $P \notin \operatorname{Ass} M$. Moreover, $\dim R/P = d - \dim M_P = d - r$.

Let $Q \in \operatorname{Ass} M$ with $\dim R/Q \geq d-1$. Then $P \not\subseteq Q$ since $P = Q$ is impossible ($P \notin \operatorname{Ass} M$) and $P \subset Q$ would imply $\dim R/P = d$ contradicting again '$P \notin \operatorname{Ass} M$'. Therefore we can find an $x_1 \in P$ with $x_1 \notin Q$ for all $Q \in \operatorname{Ass} M$ with $\dim R/Q \geq d-1$. By construction, (x_1) is part of a reducing system of parameters of M and a regular sequence on M_P by Lemma 7 and Remark 6(2).

If $r > 1$ we continue this procedure by passing to M/x_1M and we can construct elements $x_1, \ldots, x_r \in P$ inductively on r such that $(x_1, \ldots, x_r)$ forms a part of a reducing system of parameters of M and a regular sequence on M_P. Let $\overline{M} := M/(x_1, \ldots, x_r)M$. Since $\dim \overline{M}_P = \dim M_P/(x_1, \ldots, x_r)M_P = \dim M_P - r = 0$, P is minimal in $\operatorname{Supp} \overline{M}$ and therefore $P \in \operatorname{Ass} \overline{M}$. □

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