

Gromov–Witten invariants and quantum cohomology

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Dedicated to Professor K B Sinha on the occasion of his 60th birthday

Abstract. This article is an elaboration of a talk given at an international conference on Operator Theory, Quantum Probability, and Noncommutative Geometry held during December 20–23, 2004, at the Indian Statistical Institute, Kolkata. The lecture was meant for a general audience, and also prospective research students, the idea of the quantum cohomology based on the Gromov–Witten invariants. Of course there are many important aspects that are not discussed here.

Keywords. J -holomorphic curves; moduli spaces; Gromov–Witten classes; quantum cohomology.

1. Introduction

Quantum cohomology is a new mathematical discipline influenced by the string theory as a joint venture of physicists and mathematicians. The notion was first proposed by Vafa [V], and developed by Witten [W] and others [B]. The theory consists of some new approaches to the problem of constructing invariants of compact symplectic manifolds and algebraic varieties. The approaches are related to the ideas of a $(1 + 1)$ -dimensional topological quantum field theory, which indicate that the general principle of constructing invariants should be as follows: The invariants of a manifold M should be obtained by integrating cohomology classes over certain moduli space \mathcal{M} associated to M . In our case the manifold is a symplectic manifold (M, ω) , and the moduli space \mathcal{M} is the space of certain J -holomorphic spheres $\sigma: \mathbb{C}P^1 \rightarrow M$ in a given homology class $A \in H_2(M; \mathbb{Z})$. The relevant cohomology classes on \mathcal{M} are the pullbacks e^*a of the cohomology classes $a \in H^*(M, \mathbb{Z})$ under the evaluation maps $e: \mathcal{M} \rightarrow M$ given by $e(\sigma) = \sigma(z)$ for fixed $z \in \mathbb{C}P^1$. Then the integration of a top dimensional product of such classes (or equivalently, the evaluation of the top dimensional form on the fundamental class $[\mathcal{M}]$) gives rise to the Gromov–Witten invariant

$$\int_{\mathcal{M}} e^*a_1 \wedge \cdots \wedge e^*a_p = \langle e^*a_1 \wedge \cdots \wedge e^*a_p, [\mathcal{M}] \rangle.$$

These invariants are independent of the choices of z_1, \dots, z_p in $\mathbb{C}P^1$ used in their definitions, and can be interpreted as homomorphisms

$$\Phi_A: H_*(M, \mathbb{Z}) \otimes \cdots \otimes H_*(M, \mathbb{Z}) \rightarrow \mathbb{C}$$

given by

$$\Phi_A(\alpha_1, \dots, \alpha_p) = \int_{\mathcal{M}} e^*a_1 \wedge \cdots \wedge e^*a_p,$$

where a_j is the Poincaré dual of α_j .

The invariant Φ_A counts the number of intersection points (with signs of their orientations) of the image of the p -fold evaluation map $\sigma \mapsto (\sigma(z_1), \dots, \sigma(z_p)) \in M^p$ with the cycles representing the α_j , where the dimension of the homology class

$$\alpha_1 \times \cdots \times \alpha_p$$

is chosen so that if all the intersections were transversal, there would be only a finite number of such points. This is simply the number of J -holomorphic spheres in the given homology class A which meets the cycles representing $\alpha_1, \dots, \alpha_p$.

The importance of the J -holomorphic spheres and the Gromov–Witten invariants is that they may be used to define a quantum deformation of the cup product in the cohomology ring $H^*(M)$ of a compact symplectic manifold M making it a quantum cohomology ring $QH^*(M)$.

The description of the Gromov–Witten invariants can be given in terms of a general Riemann surface Σ (see [RT]). However, we have made this expository introduction somewhat simpler by taking $\Sigma = S^2$. The results that guided our approach are to be seen in the work of MacDuff and Salamon [MS2].

2. J -Holomorphic curves

A symplectic manifold (M, ω) is a smooth manifold M of dimension $2n$ with a symplectic structure ω on it which is a closed differential 2-form ω such that the volume form ω^n is nowhere vanishing on M . The basic example is the Euclidean space \mathbb{R}^{2n} with the constant symplectic form

$$\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n,$$

where $(x_1, \dots, x_n, y_1, \dots, y_n)$ are coordinates in \mathbb{R}^{2n} . The next basic example is provided by the phase space of a Hamiltonian system, that is, by the cotangent bundle T^*N of any n -manifold N with a symplectic structure which is locally the pullback of the structure ω_0 on \mathbb{R}^{2n} . A symplectic manifold cannot be odd dimensional.

A symplectic diffeomorphism $\phi: (M_1, \omega_1) \rightarrow (M_2, \omega_2)$ between two symplectic manifolds is a diffeomorphism $\phi: M_1 \rightarrow M_2$ such that $\phi^*\omega_2 = \omega_1$. Symplectic geometry is quite different from Riemannian geometry, and also from Kählerian geometry. The Darboux theorem says that locally any two symplectic manifolds of the same dimension are diffeomorphic. Therefore locally all symplectic manifolds are the same, and there is no local invariant in symplectic geometry, like, for example, the curvature in Riemannian geometry. The only possible invariants have to be global.

The Darboux theorem makes it difficult to study the global structure on a symplectic manifold. Although variational techniques may be employed to tackle some global problems, it is the theory of J -holomorphic curves of Gromov that applies to many problems of symplectic manifolds. We have no other theory to investigate these global questions.

An almost complex structure on a manifold M is a complex structure on its tangent bundle TM , that is, an endomorphism $J: TM \rightarrow TM$ such that $J^2 = -\text{Id}$. Then, J makes TM a complex vector bundle, where the complex vector space structure on each fibre $T_x M$ is given by $(a + \sqrt{-1}b) \cdot v = av + bJv$. If M is already a complex manifold, which is a manifold with holomorphic changes of coordinates, then the tangent bundle TM is a

complex vector bundle, and its almost complex structure J is just multiplication by $\sqrt{-1}$. The standard almost complex structure j on \mathbb{R}^{2n} is given by

$$j(\partial/\partial x_k) = \partial/\partial y_k, \quad j(\partial/\partial y_k) = -\partial/\partial x_k,$$

where $z_k = x_k + \sqrt{-1}y_k$ are the coordinates in \mathbb{C}^n .

An almost complex structure J on a symplectic manifold (M, ω) is called ω -tamed if $\omega(v, Jv) > 0$ for every nonzero vector $v \in T_x M$. This means that the restriction of ω to the complex line in $T_x M$ spanned by v and Jv verifies the same condition, and so ω restricts to a non-degenerate form on each such line. An almost complex structure J is called ω -compatible if it is ω -tamed, and

$$\omega(Jv, Jw) = \omega(v, w) \quad \text{for all } v, w \in TM.$$

The space $\mathcal{J}(M, \omega)$ of all ω -compatible almost complex structures on (M, ω) is non-empty and contractible, because associated with the tangent bundle TM we have a bundle $\mathcal{J}(M, \omega) \rightarrow M$ with contractible fibre $\text{Sp}(2n, \mathbb{R})/U(n)$. Since $\mathcal{J}(M, \omega)$ is pathwise connected, the complex bundles (TM, J) are isomorphic for different choices of $J \in \mathcal{J}(M, \omega)$. Therefore the Chern classes $c_i(M)$ of these bundles do not depend on J . The assertions also apply to ω -tamed almost complex structures (in this case the associated bundle has fibre $GL(2n, \mathbb{R})/U(n)$).

A smooth map $\phi: (M, J) \rightarrow (M', J')$ between almost complex manifolds is called (J, J') -holomorphic if $d\phi_x: T_x M \rightarrow T_{\phi(x)} M'$ is complex linear, that is, $d\phi_x \circ J_x = J'_{\phi(x)} \circ d\phi_x$ for all $x \in M$. These conditions are exactly the Cauchy–Riemann equations in the case when (M, J) and (M', J') are subsets of \mathbb{C}^n . An almost complex structure J on M is called integrable if it arises from a complex structure on M ; in other words, if M admits an atlas whose coordinate charts are (J, j) -holomorphic maps, where j denotes the standard complex structure on \mathbb{C}^n . If $\dim M = 2$, a fundamental theorem says that any almost complex structure J on M is integrable. However, the theorem is not true in higher dimensions. The non-integrability of J is measured by the Nijenhuis tensor N_J (see [MS1]).

A J -holomorphic curve in (M, J) is a (J_0, J) -holomorphic map $\sigma: \Sigma \rightarrow M$, where (Σ, J_0) is a Riemann surface (complex manifold of dimension 1) with complex structure J_0 . Very often we take (Σ, J_0) as the Riemann sphere S^2 , and in this case a J -holomorphic curve is referred to as a J -holomorphic sphere.

If σ is an embedding and C is the image of σ , then σ is called a J -holomorphic parametrization of C . In this case C is 2-submanifold of M with J -invariant tangent bundle TC so that each tangent space is a complex line in TM . Conversely, any 2-submanifold C of M with a J -invariant tangent bundle has a J -holomorphic parametrization σ , because the restriction of J to C is integrable.

For an ω -tamed almost complex structure J on a symplectic manifold (M, ω) , the image of J -holomorphic parametrization is a symplectic 2-submanifold of M with J -invariant tangent spaces. Conversely, given an oriented 2-submanifold C of M , one can construct an ω -tamed J such that TC is J -invariant (first define J on TC and then extend it to TM). One may contrast this situation with that in complex geometry where one often defines a curve as the set of common zeros of a number of holomorphic polynomials. Such an approach makes no sense in the case when the almost complex structure is non-integrable, since there may not exist holomorphic functions $(M, J) \rightarrow \mathbb{C}$ when J is non-integrable.

3. Moduli spaces

Let (M, J) be an almost complex manifold without boundary, and Σ be a Riemann surface of genus g with complex structure J_0 . Then a moduli space $\mathcal{M}(A, J)$ is the space of all simple J -holomorphic curves $\sigma: (\Sigma, J_0) \rightarrow (M, J)$ which represent a given homology class $A \in H_2(M; \mathbb{Z})$ (i.e. $\sigma_*[\Sigma] = A$), with the C^r -topology, $r \geq 0$. Our first problem is to provide a finite dimensional smooth structure on this space.

Note that a curve σ is simple, if it is not multiply-covered, that is, it is not a composition of a holomorphic branched covering $(\Sigma, J_0) \rightarrow (\Sigma', J'_0)$ of degree > 1 and a J -holomorphic map $\Sigma' \rightarrow M$. We avoid multiply-covered curves because they may be singular points in the moduli space $\mathcal{M}(A, J)$. Every simple curve σ has an injective point $z \in \Sigma$, which is a regular point of σ (i.e. $d\sigma_z \neq 0$) such that $\sigma^{-1}\sigma(z) = \{z\}$. Moreover, the set of injective points is open and dense in Σ [MS2].

The space $\mathcal{S} = C^\infty(\Sigma, M, A)$ of smooth maps $\sigma: \Sigma \rightarrow M$ that are somewhere injective, and represent $A \in H_2(M; \mathbb{Z})$ may be looked upon as an infinite dimensional manifold whose tangent space at $\sigma \in \mathcal{S}$ is given by

$$T_\sigma \mathcal{S} = C^\infty(\sigma^* TM),$$

which is the vector space of all smooth vector fields of M along σ .

We can view $\sigma^* TM$ as a complex vector bundle. Therefore we have a splitting of the space of 1-forms

$$\Omega^1(\sigma^* TM) = \Omega^{1,0}(\sigma^* TM) \oplus \Omega^{0,1}(\sigma^* TM),$$

where $\Omega^{1,0}$ and $\Omega^{0,1}$ are respectively vector spaces of J -linear and J -anti-linear 1-forms with values in $\sigma^* TM$. Since $d\sigma \in \Omega^1(\sigma^* TM)$, we can decompose $d\sigma = \partial_J(\sigma) + \bar{\partial}_J(\sigma)$, where

$$\begin{aligned} \partial_J(\sigma) &= \frac{1}{2}(d\sigma - J \circ d\sigma \circ J_0), \\ \bar{\partial}_J(\sigma) &= \frac{1}{2}(d\sigma + J \circ d\sigma \circ J_0) \end{aligned}$$

are respectively J -linear and J -anti-linear parts of $d\sigma$.

If $\mathcal{E} \rightarrow \mathcal{S}$ is the infinite dimensional vector bundle whose fibre \mathcal{E}_σ over $\sigma \in \mathcal{S}$ is the space $\Omega^{0,1}(\sigma^* TM)$, then $\bar{\partial}_J$ is a section of the bundle $\mathcal{E} \rightarrow \mathcal{S}$. Moreover, the J -holomorphic curves are the zeros of the section $\bar{\partial}_J$, that is, if Z denotes the zero section of the bundle, then

$$\mathcal{M}(A, J) = (\bar{\partial}_J)^{-1}(Z).$$

This will be a manifold if $\bar{\partial}_J: \mathcal{S} \rightarrow \mathcal{E}$ is transversal to Z , that is, the image of

$$d\bar{\partial}_J(\sigma): T_\sigma \mathcal{S} \rightarrow T_{(\sigma, 0)} \mathcal{E}$$

is complementary to the tangent space of the zero-section Z for every $\sigma \in \mathcal{M}(A, J)$; in other words, the linear operator $D_\sigma = \pi_\sigma \circ d\bar{\partial}_J(\sigma)$, where

$$\pi_\sigma: T_{(\sigma, 0)} \mathcal{E} = T_\sigma \mathcal{S} \oplus \mathcal{E}_\sigma \rightarrow \mathcal{E}_\sigma$$

is the projection, is surjective for every $\sigma \in \mathcal{M}(A, J)$.

Explicit expression of the operator

$$D_\sigma: C^\infty(\sigma^*TM) \longrightarrow \Omega^{0,1}(\sigma^*TM),$$

can be obtained by differentiating the local expressions of $\bar{\partial}_J(\sigma)$ in the direction of a vector field along σ . These expressions show that the first order terms make up the usual Cauchy–Riemann operator for maps $\mathbb{C} \longrightarrow \mathbb{C}^n = \mathbb{R}^{2n}$. Therefore D_σ is a first-order elliptic differential operator, and hence it is Fredholm. Recall that a bounded operator $F: X \longrightarrow Y$ between Banach spaces X and Y is a Fredholm operator if F has finite dimensional kernel and cokernel, and $F(X)$ is closed. The index of F is defined by

$$\text{index } F = \dim \ker F - \dim \text{coker } F.$$

These operators form an open subset $\mathcal{F}(X, Y)$ of the space of bounded operators $\mathcal{B}(X, Y)$ with the norm topology. The Fredholm index is constant on each connected component of $\mathcal{F}(X, Y)$, and therefore $\text{index } F$ is not altered if F varies continuously.

Although the domain and range of the Fredholm operator D_σ are complex vector spaces, D_σ is not complex linear, because J is not integrable. It will appear from the computations for D_σ mentioned above that the complex anti-linear part of D_σ has order 0. Then, by multiplying the anti-linear part by a constant which tends to 0, we can find a homotopy of D_σ through Fredholm operators. The final Fredholm operator of the homotopy commutes with J , and is a Cauchy–Riemannian operator. It determines a holomorphic structure on the complex vector bundle σ^*TM . Therefore we have by the Riemann–Roch theorem ([GH], p. 243)

$$\text{index } D_\sigma = n(2 - 2g) + 2c_1(\sigma^*TM)[\Sigma] = n(2 - 2g) + 2c_1(A),$$

where c_1 is the first Chern class of the complex bundle (TM, J) , and $c_1(\sigma^*TM)[\Sigma] = (\sigma^*c_1)[\Sigma] = c_1(\sigma_*[\Sigma]) = c_1(A)$.

If the operator D_σ is surjective for every $\sigma \in \mathcal{M}(A, J)$, then it follows from the infinite dimensional implicit function theorem that $\mathcal{M}(A, J)$ is a finite dimensional manifold whose tangent space at σ is $\ker D_\sigma$.

We suppose that the space of ω -compatible almost complex structures $\mathcal{J} = \mathcal{J}(M, \omega)$ has been endowed with the C^∞ -topology. Let \mathcal{J}_r be the subspace of \mathcal{J} consisting of those structures J for which D_σ is surjective for all $\sigma \in \mathcal{M}(A, J)$.

Theorem 3.1.

(a) *If $J \in \mathcal{J}_r$, then $\mathcal{M}(A, J)$ is a smooth manifold with a natural orientation such that*

$$\dim \mathcal{M}(A, J) = n(2 - 2g) + 2c_1(A).$$

(b) *The subset \mathcal{J}_r is residual in \mathcal{J} .*

Recall that a subset of a topological space X is residual if it is the intersection of a countable family of open dense subsets of X . A point of X is called generic if it belongs to some residual subset of X .

Proof. Part (a) follows from the above discussion, except for the orientation. The orientation follows from the fact that a Fredholm operator D between complex Banach spaces induces a canonical orientation on its determinant line

$$\det D = \Omega^p(\ker D) \otimes \Omega^q(\ker D^*),$$

where $p = \dim \ker D$ and $q = \dim \operatorname{coker} D$, provided D is complex linear. As described above, we may suppose by using a homotopy of order 0 that our Fredholm operator D_σ is complex linear with $\operatorname{coker} D_\sigma = 0$. Therefore its determinant line, and hence $\ker D_\sigma = T_\sigma \mathcal{M}(A, J)$, has a canonical orientation. These arguments are due to Ruan [R], also note that earlier Donaldson [D] used similar arguments for the orientation of Yang–Mills moduli spaces.

Part (b) uses an infinite dimensional version of Sard–Smale theorem which is due to Smale [S]. A non-linear smooth map $f: X \rightarrow Y$ between Banach spaces is a Fredholm map of index k , if the derivative $df_x: X \rightarrow Y$ is a linear Fredholm operator of index k for each $x \in X$. A point $y \in Y$ is a regular value of f if df_x is surjective for each $x \in f^{-1}(y)$, otherwise y is called a critical value of f . Then the Sard–Smale theorem says that if $f: X \rightarrow Y$ is a C^k Fredholm map between separable Banach spaces and $k > \max(0, \text{index } f)$, then the set of regular values of f is residual in Y . The theorem remains true if X and Y are Banach manifolds, instead of Banach spaces. It follows from the implicit function theorem for Banach spaces that if $y \in Y$ is a regular value then $f^{-1}(y)$ is a smooth submanifold of X . Moreover, if $f^{-1}(y)$ is finite dimensional, then its dimension is equal to the Fredholm index of f .

For the completion of the proof of the theorem, we need to refine the space \mathcal{S} using the Sobolev $W^{k,p}$ -norm which is given by the sum of the L^p -norms of all derivatives of $\sigma \in \mathcal{S}$ up to order k ,

$$\|\sigma\|_{k,p} = \sum_{|r| \leq k} L^p(\partial^r \sigma),$$

where r is a multi-index and $|r|$ is its order. It can be shown that the Sobolev space $W^{k,p}(\Sigma, M)$, which is the space consisting of all maps $\Sigma \rightarrow M$ whose k -th order derivatives are of class L^p (and which represent the class $A \in H_2(M; \mathbb{Z})$), is the completion of the space \mathcal{S} with respect to the Sobolev $W^{k,p}$ -norm (see Appendix B in [MS2]). It appears that we must assume the condition $kp > 2$ in order for the space $W^{k,p}(\Sigma, M)$ to be well-defined. Under this condition, the Sobolev embedding theorem says that there is a continuous embedding of $W^{k,p}(\Sigma, M)$ into the space of continuous maps $C^0(\Sigma, M)$, and the multiplication theorem says that the product of two maps of class $W^{k,p}$ is again a map of the same class.

At the same time we restrict the space of almost complex structures $\mathcal{J}(M, \omega)$, introduced earlier. Let \mathcal{J}^ℓ , $\ell \geq 1$, be the space of all almost complex structures of class C^ℓ which are compatible with ω , with the C^ℓ topology. We shall choose ℓ later according to our requirement.

Then \mathcal{J}^ℓ is a smooth separable Banach manifold. Let $\operatorname{End}(TM, J, \omega) \rightarrow M$ be the bundle whose fibre over $p \in M$ is the space of linear endomorphisms $X: T_p M \rightarrow T_p M$ such that

$$XJ + JX = 0, \quad \omega(Xv, w) + \omega(v, Xw) = 0, \quad \text{for } v, w \in T_p M.$$

Then the tangent space $T_J \mathcal{J}^\ell$ at J is the space of sections of this bundle.

It can be proved by elliptic bootstrapping methods (see [MS2], Appendix B for details) that if $J \in \mathcal{J}^\ell$ with $\ell \geq 1$, then a J -holomorphic curve $\sigma: \Sigma \rightarrow M$ of class $W^{\ell,p}$ with $p > 2$ is also of class $W^{\ell+1,p}$. In particular, if J is smooth and σ is of class C^ℓ , then σ is also smooth. Thus if $k \leq \ell + 1$ and $J \in \mathcal{J}^\ell$, then the moduli space of J -holomorphic curves of class $W^{k,p}$ does not depend on k .

In the context of the Sobolev space of $W^{k,p}$ -maps $\sigma: \Sigma \rightarrow M$ for some fixed $p > 2$, we have the Banach space bundle $\mathcal{E}^p \rightarrow W^{k,p}(\Sigma, M)$ whose fibre over $\sigma \in W^{k,p}(\Sigma, M)$ is the space

$$\mathcal{E}_\sigma^p = L^p(\Lambda^{0,1} T^* \Sigma \otimes_J \sigma^* TM)$$

of complex anti-linear 1-forms on Σ of class L^p taking values in $\sigma^* TM$. The non-linear Cauchy–Riemann equations determine a section $\bar{\partial}_J$ of this bundle, and the derivative of $\bar{\partial}_J$ at σ gives rise to the operator

$$D_\sigma: W^{k,p}(\sigma^* TM) \rightarrow W^{k-1,p}(\Lambda^{0,1} T^* \Sigma \otimes_J \sigma^* TM).$$

The explicit formula for D_σ is given by

$$D_\sigma \xi = \frac{1}{2}(\nabla \xi + J(\sigma) \nabla \xi \circ J_0) + \frac{1}{8} N_J(\partial_J(\sigma), \xi),$$

where ∇ is the Hermitian connection on M , and N_J is the Nijenhuis tensor (see [M1]). The first part has order 1 and commutes with J , while the second has order 0 and anti-commutes with J .

The ellipticity of D_σ can be established from the estimate

$$\|\xi\|_{W^{1,p}} \leq c_0(\|D_\sigma \xi\|_{L^p} + \|\xi\|_{L^p}),$$

which follows from the L^p -estimate for Laplace operator (the Calderon–Zygmund inequality) (see Appendix B in [MS2]). Therefore D_σ is a Fredholm operator of positive index, by a previous argument in a similar situation.

The following space is also a smooth Banach manifold

$$\mathcal{M}^\ell(A, \mathcal{J}^\ell) = \{(\sigma, J) \in W^{k,p}(\Sigma, M) \times \mathcal{J}^\ell \mid \bar{\partial}_J(\sigma) = 0\}.$$

The tangent space $T_{(\sigma, J)} \mathcal{M}^\ell(A, \mathcal{J}^\ell)$ is the space of all pairs (X, Y) such that

$$D_\sigma X + \frac{1}{2} Y(\sigma) \circ d\sigma \circ J_0 = 0.$$

Let $\pi: \mathcal{M}^\ell(A, \mathcal{J}^\ell) \rightarrow \mathcal{J}^\ell$ be the projection. Then $\pi^{-1}(J) = \mathcal{M}^\ell(A, J)$, and the derivative of π at (σ, J) ,

$$d\pi(\sigma, J): T_{(\sigma, J)} \mathcal{M}^\ell(A, \mathcal{J}^\ell) \rightarrow T_J \mathcal{J}^\ell$$

is just the projection $(X, Y) \mapsto Y$. It follows that $d\pi(\sigma, J)$ is a Fredholm operator having the same index as D_σ . Moreover, a regular value J of π is an almost complex structure such that D_σ is surjective for all J -holomorphic spheres $\sigma \in \pi^{-1}(J)$.

We denote the set of regular values of π by \mathcal{J}_r^ℓ . By the Sard–Smale theorem (stated earlier), the set \mathcal{J}_r^ℓ is residual in \mathcal{J}^ℓ with respect to the C^ℓ topology whenever $\ell - 2 \geq \text{index } D_\sigma = \text{index } \pi$, because π is of class $C^{\ell-1}$.

Let λ be a positive number, and $\mathcal{J}_{r,\lambda}^\ell$ be the set of almost complex structures $J \in \mathcal{J} = \mathcal{J}(M, \omega)$ such that D_σ is surjective for every J -holomorphic sphere σ with $\|\sigma\|_{L^\infty} \leq \lambda$. Clearly, the intersection of the sets $\mathcal{J}_{r,\lambda}^\ell$ over all $\lambda > 0$ is the set \mathcal{J}_r of part (b) of the theorem.

The set \mathcal{J}_r is residual, because each $\mathcal{J}_{r,\lambda}^\ell$ is open and dense in \mathcal{J} with respect to the C^∞ topology. We omit the details which may be found in §3.4 of [MS2]. \square

The above theorem can be extended further in order to understand how the manifolds $\mathcal{M}(A, J)$ depend on $J \in \mathcal{J}_r$. Two almost complex structures J_0 and J_1 in \mathcal{J}_r are called smoothly homotopic if there is a smooth path $[0, 1] \rightarrow \mathcal{J}, t \mapsto J_t$, from J_0 to J_1 .

Theorem 3.2. *Let \mathcal{J} be path-connected, and $J_0, J_1 \in \mathcal{J}_r$. Let $\mathcal{J}(J_0, J_1)$ be the space of all smooth homotopies from J_0 to J_1 . Then there is a dense set*

$$\mathcal{J}_r(J_0, J_1) \subset \mathcal{J}(J_0, J_1)$$

such that for every $\{J_t\} \in \mathcal{J}_r(J_0, J_1)$, the space

$$\mathcal{M}(A, \{J_t\}_{t \in [0,1]}) = \{(t, \sigma) \mid \sigma \in \mathcal{M}(A, J_t)\}$$

is a smooth manifold of dimension $n(2 - 2g) + 2c_1(A) + 1$ with a natural orientation and with a smooth boundary which is given by

$$\partial \mathcal{M}(A, \{J_t\}_{t \in [0,1]}) = \mathcal{M}(A, J_1) - \mathcal{M}(A, J_0),$$

where the negative sign indicates the reversed orientation.

Thus moduli spaces $\mathcal{M}(A, J_0)$ and $\mathcal{M}(A, J_1)$ are oriented cobordant. We may call the elements of the set $\mathcal{J}_r(J_0, J_1)$ regular homotopies.

4. Compactness

The manifold $\mathcal{M}(A, J)$ will not serve any purpose unless some kind of compactness is established for it.

For simplification we suppose that $\mathcal{M}(A, J)$ is the moduli space of J -holomorphic spheres. The Rellich's theorem says that the inclusion map

$$W^{k+1,p}(S^2, M) \longrightarrow W^{k,p}(S^2, M)$$

is compact for all k and p (this means that a sequence $\{\sigma_n\}$ which is bounded in the domain $W^{k+1,p}(S^2, M)$ possesses a subsequence which is convergent in the range $W^{k,p}(S^2, M)$). Moreover, if $k - 2/p > m + \alpha$ where $0 < \alpha < 1$, then $W^{k,p}(S^2, M)$ embeds compactly into the Hölder space $C^{m+\alpha}(S^2, M)$. Using this one gets the main elliptic regularity theorem, which contains a result of compactness.

Theorem 4.1. *If $k \geq 1$, $p > 2$, and $\sigma \in W^{k,p}(S^2, M)$ with $\bar{\partial}_J \sigma = 0$, then $\sigma \in C^\infty(S^2, M)$. Moreover, for every integer $m > 0$, every subset of $\bar{\partial}_J^{-1}(0)$ which is bounded in $W^{k,p}(S^2, M)$ has compact closure in $C^m(S^2, M)$.*

The details are in [M2].

An ω -tamed almost complex structure J determines a Riemannian metric on M ,

$$\langle v, w \rangle_J = \frac{1}{2} [\omega(v, Jw) + \omega(w, Jv)].$$

The energy of a J -holomorphic sphere $\sigma: S^2 \rightarrow M$ with respect to this metric is

$$E(\sigma) = \int_{S^2} |\mathrm{d}\sigma|_J^2.$$

The group $G = PSL(2, \mathbb{C})$ acts on $\mathbb{C} \cup \{\infty\} = \mathbb{C}_\infty$ by Möbius transformations $\phi_L: \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$,

$$\phi_L(z) = \frac{az+b}{cz+d}, \quad L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

We may identify S^2 with $\mathbb{C}P^1 \simeq \mathbb{C}_\infty$ by a stereographic projection π , and different choices of π correspond to the action of $SO(3) \simeq SU(2)/\{\pm Id\} \subset PSL(2, \mathbb{C}) = G$ on $\mathbb{C}P^1 = \mathbb{C}_\infty$. Then a J -holomorphic sphere $S^2 \rightarrow M$ gets identified with a smooth J -holomorphic curve $\sigma: \mathbb{C} \rightarrow M$ such that the map $\mathbb{C} - \{0\} \rightarrow M$ given by $z \mapsto \sigma(1/z)$ extends to a smooth map $\mathbb{C} \rightarrow M$. The space of such maps remain invariant under composition with Möbius transformations $\phi_L: \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$. We say that a sequence of such J -holomorphic curves $\sigma_n: \mathbb{C} \rightarrow M$ converges on \mathbb{C}_∞ if both the sequences $\{\sigma_n(z)\}$ and $\{\sigma_n(1/z)\}$ converge uniformly with all derivatives on compact subsets of \mathbb{C} .

It can be shown that

$$E(\sigma) = \int_{\mathbb{C}} \sigma^* \omega = \omega(A)$$

for all J -holomorphic curves $\sigma: \mathbb{C} \rightarrow M$ (J is ω -tamed), where ω is considered as an integral valued form. Thus the L^2 -norm of the derivative of σ satisfies a uniform bound which depends only on the homology class A represented by σ . This does not imply compactness of the moduli space by the Sobolev estimate, because here $p = 2$ (a uniform bound on the L^p -norms of $d\sigma$ with $p > 2$ would guarantee the compactness).

It may be noted that the space $\mathcal{M}(A, J)$ can never be both compact and non-empty, unless $A = 0$ in which case all σ are constant maps. Because, the group $G = PSL(2, \mathbb{C})$ of holomorphic maps $S^2 \rightarrow S^2$ is non-compact and it acts on S^2 by reparametrization $\sigma \mapsto \sigma \circ \phi$, $\phi \in G$, and so any $\sigma \in \mathcal{M}(A, J)$ has a non-compact orbit. However, it is possible to compactify the quotient $\mathcal{M}(A, J)/G$ sometimes, if A satisfies a certain condition.

One can show that if σ_n is a sequence in $\mathcal{M}(A, J)$ without any limit point in $\mathcal{M}(A, J)$, then there is a point $z \in S^2$ such that the derivatives $d\sigma_n(z)$ are unbounded. This implies after passing to a subsequence that there is a decreasing sequence of neighbourhoods U_n of z in S^2 such that the images $\sigma_n(U_n)$ converge to a J -holomorphic sphere. If B is the homology class of this sphere, then either $\omega(B) = \omega(A)$, or else $0 < \omega(B) < \omega(A)$. In the first case, the maps σ_n can be reparametrized so that they converge in $\mathcal{M}(A, J)$. The second case is referred to as the phenomenon of ‘bubbling off’. Here one must proceed with more care. The phenomenon was discovered by Sacks and Uhlenbeck [SU] in the context of minimal surfaces.

The following theorem gives a criterion for the moduli space $\mathcal{M}(A, J)/G$ to be compact. This is the simplest version of Gromov’s compactness theorem.

A homology class $B \in H_2(M, \mathbb{Z})$ is called spherical if it lies in the image of the Hurewicz homomorphism $\pi_2(M) \rightarrow H_2(M, \mathbb{Z})$. It is customary to write $B \in \pi_2(M)$ if B is a spherical homology class.

Theorem 4.2. *If there is no spherical homology class $B \in H_2(M; \mathbb{Z})$ such that $0 < \omega(B) < \omega(A)$, then the moduli space $\mathcal{M}(A, J)/G$ is compact.*

The proof consists of showing that if $\sigma_n: \mathbb{C} \cup \{\infty\} \rightarrow M$ is a sequence of J -holomorphic A -spheres, then there is a sequence of matrices $L_n \in SL(2, \mathbb{C})$ such that the sequence $\sigma_n \circ \phi_{L_n}$ has a convergent subsequence. Therefore if $\omega(A)$ is already the smallest positive value taken by ω , then the moduli space is compact.

If the criterion of the theorem is not satisfied, it is still possible sometimes to compactify $\mathcal{M}(A, J)/G$ by adding suitable pieces. This we shall explain in the next section in a more general context.

5. Evaluation maps

The Gromov–Witten invariants are constructed from the evaluation map

$$\mathcal{M}(A, J) \times S^2 \longrightarrow M$$

given by $(\sigma, z) \mapsto \sigma(z)$. The group $G = PSL(2, \mathbb{C})$ acts on the space $\mathcal{M}(A, J) \times S^2$ by $\phi \cdot (\sigma, z) = (\sigma \circ \phi^{-1}, \phi(z))$. Therefore we get a map by passing to the quotient

$$e = e_J: \mathcal{W}(A, J) = \mathcal{M}(A, J) \times_G S^2 \longrightarrow M.$$

For example, suppose that $M = \mathbb{C}P^1 \times V$ with a product symplectic form, and $A = [\mathbb{C}P^1 \times \{\text{point}\}]$. If $\pi_2(V) = 0$, then A generates a spherical 2-class in M , and so $\omega(A)$ is necessarily the smallest value assumed by ω on the spherical classes. Therefore by Theorems 3.1 and 4.2, the space $\mathcal{W}(A, J)$ is a compact manifold for generic J . Since $c_1(A) = 2$, $\dim \mathcal{W}(A, J) = 2n$ which is the dimension of M . It can be shown that different choices of J give rise to cobordant maps e_J . Since the cobordant maps have the same degree, $\deg e_J$ is independent of all choices. In the case when $J = J_0 \times J'$ is a product, where J_0 is the standard complex structure on $\mathbb{C}P^1$, it can be seen that the elements of $\mathcal{M}(A, J)$ have the form $\sigma(z) = (\phi(z), v_0)$, where $v_0 \in V$ and $\phi \in G$. It follows that the map e_J has degree 1 for this choice of J and hence for every J .

In general, we have a p -fold evaluation map

$$e_p: \mathcal{W}(A, J, p) = \mathcal{M}(A, J) \times_G (\mathbb{C}P^1)^p \longrightarrow M^p$$

defined by

$$e_p(\sigma, z_1, \dots, z_p) = (\sigma(z_1), \dots, \sigma(z_p)).$$

Here, for a space X , X^p denotes the p -fold product $X \times \dots \times X$.

For a generic almost complex structure J , the space $\mathcal{W}(A, J, p)$ is a manifold with

$$\dim \mathcal{W}(A, J, p) = 2n + 2c_1(A) + 2p - 6.$$

This manifold is not compact in general. However, in many cases the image

$$\mathcal{X}(A, J, p) = e_p(\mathcal{W}(A, J, p)) \subset M^p$$

can be compactified by adding suitable pieces of dimensions at most equal to $\dim \mathcal{W}(A, J, p) - 2$. These pieces are called cusp-curves (the terminology is due to Gromov [G]), and they are connected unions of certain J -holomorphic spheres. By the Gromov compactness theorem (which is a convergence theorem leading to compactness, see [MS2]), the closure of $\mathcal{X}(A, J, p)$ contains points that lie on some cusp-curves representing the class A in a sense that we shall describe in a moment little later. Therefore in order to compactify $\mathcal{X}(A, J, p)$ we must add all simple cusp-curves in the class A to the moduli space $\mathcal{M}(A, J)$. The compactification is important because we want $\mathcal{X}(A, J, p)$ to carry a fundamental homology class. We describe below some features of a cusp-curve.

A cusp-curve σ in (M, ω) , which represents the homology class A , is a collection $\sigma = (\sigma_1, \dots, \sigma_N)$ of J -holomorphic spheres $\sigma_i: \mathbb{C}P^1 \rightarrow M$ such that $C_1 \cup \dots \cup C_N$ is a connected set, where $C_i = \sigma_i(\mathbb{C}P^1)$ and $A = A_1 + \dots + A_N$, A_i being the homology class represented by σ_i . The σ_i are called the components of σ . A cusp-curve σ is called simple if its components σ_i are simple J -holomorphic spheres such that $\sigma_i \neq \sigma_j \circ \phi$ for $i \neq j$ and any $\phi \in G$. Any cusp-curve can be simplified to a simple cusp-curve by replacing each multiply covered component by its underlying simple curve. Of course this operation will change the homology class A , but not the set of points that lie on the curve. Also one can order the components of σ so that $C_1 \cup \dots \cup C_k$ is connected for all $k \leq N$. This means that there exist integers j_2, \dots, j_N with $1 \leq j_i < i$ such that each C_i must intersect some C_{j_i} , that is, there exist $w_i, z_i \in \mathbb{C}P^1$ such that $\sigma_{j_i}(w_i) = \sigma_i(z_i)$.

A framing or intersection pattern D of an ordered simple cusp-curve

$$\sigma = (\sigma_1, \dots, \sigma_N)$$

is a collection

$$D = \{A_1, \dots, A_N, j_2, \dots, j_N\},$$

where $A_i = [C_i] \in H_i(M, \mathbb{Z})$ and j_i are integers with $1 \leq j_i \leq i-1$ chosen so that C_i intersects C_{j_i} (i.e. $C_i \cap C_{j_i} \neq \emptyset$). Then $\omega(A_i) \leq \omega(A)$, and so there are only a finite number framings D associated to σ .

For a fixed framing $D = \{A_1, \dots, A_N, j_2, \dots, j_N\}$, and a $J \in \mathcal{J}(M, \omega)$, let

$$\mathcal{M}(A_1, \dots, A_N, J) = \mathcal{M}(A_1, J) \times \dots \times \mathcal{M}(A_N, J).$$

Let $\mathcal{M}(D, J)$ be the moduli space

$$\mathcal{M}(D, J) \subset \mathcal{M}(A_1, \dots, A_N, J) \times (\mathbb{C}P^1)^{2N-2}$$

consisting of all (σ, w, z) where $\sigma = (\sigma_1, \dots, \sigma_N)$, $\sigma_i \in \mathcal{M}(A_i, J)$,

$$w = (w_2, \dots, w_N) \in (\mathbb{C}P^1)^{N-1} \quad \text{and} \quad z = (z_2, \dots, z_N) \in (\mathbb{C}P^1)^{N-1},$$

such that σ is a simple cusp-curve with $\sigma_{j_i}(w_i) = \sigma_i(z_i)$ for $i = 2, \dots, N$.

For a generic J , $\mathcal{M}(D, J)$ will be an oriented manifold of dimension

$$2 \sum_{j=1}^N c_1(A_j) + 2n + 4(N-1).$$

The proof uses the extended evaluation map

$$e_D: \mathcal{M}(A_1, \dots, A_N, J) \times (\mathbb{C}P^1)^{2N-2} \rightarrow M^{2N-2}$$

given by

$$e_D(\sigma, w, z) = (\sigma_{j_2}(w_2), \sigma_2(z_2), \dots, \sigma_{j_N}(w_N), \sigma_N(z_N)).$$

The map e_D is transversal to the multi-diagonal set

$$\Delta_N = \{(x_2, y_2, \dots, x_N, y_N) \in M^{2N-2} | x_j = y_j\},$$

and therefore the inverse image $e_D^{-1}(\Delta_N) = \mathcal{M}(D, J)$ is a manifold of the above dimension.

The group $G^N = G \times \cdots \times G$ acts freely on $\mathcal{M}(D, J)$ by

$$\phi \cdot (\sigma_j, w_i, z_i) = (\sigma_j \circ \phi_j^{-1}, \phi_{j_i}(w_i), \phi_i(z_i)), \phi = (\phi_1, \dots, \phi_N) \in G^N.$$

The quotient space $\mathcal{M}(D, J)/G^N$ for a generic J is a manifold of dimension

$$2c_1(A_1 + \cdots + A_N) + 2n - 2N - 4.$$

This is precisely our previous moduli space $\mathcal{M}(A, J)/G$ when $N = 1$ and $A_1 = A$.

Let T denote a function $\{1, \dots, p\} \mapsto \{1, \dots, N\}$. This function will indicate which of the N components of $C = C_1 \cup \cdots \cup C_N$ will be evaluated to get a point of M^p . Define

$$\mathcal{W}(D, T, J, p) = \mathcal{M}(D, J) \times_{G^N} (\mathbb{C}P^1)^p,$$

where the j th component of $\phi = (\phi_1, \dots, \phi_N) \in G^N$ acts on $\mathcal{M}(D, J)$ as above, and it acts on the i th factor of $(\mathbb{C}P^1)^p$ if and only if $T(i) = j$. Then $\mathcal{W}(D, T, J, p)$ will be a manifold of dimension

$$2 \sum_{j=1}^N c_1(A_j) + 2n + 2p - 2N - 4.$$

We have an evaluation map $e_{D,T}: \mathcal{W}(D, T, J, p) \rightarrow M^p$ defined by

$$e_{D,T}(\sigma, w, z, \xi) = (\sigma_{T(1)}(\xi_1), \dots, \sigma_{T(p)}(\xi_p)),$$

where $(\sigma, w, z) \in \mathcal{M}(D, J)$ and $\xi = (\xi_1, \dots, \xi_p) \in (\mathbb{C}P^1)^p$.

We shall now choose J suitably so that $\mathcal{X}(A, J, p)$ has a fundamental homology class.

A manifold (M, ω) is weakly monotone if every spherical homology class $B \in H_2(M, \mathbb{Z})$ with $\omega(B) > 0$ and $c_1(B) < 0$ must satisfy the condition $c_1(B) \leq 2 - n$. Here c_1 is the first Chern class of the complex bundle (TM, J) . This means that there are no J -holomorphic spheres in homology classes with negative first Chern number. The manifold (M, ω) is monotone if there is a $\lambda > 0$ such that $\omega(B) = \lambda c_1(B)$ for every spherical $B \in H_2(M, \mathbb{Z})$. It can be shown that a monotone manifold is weakly monotone, and conversely.

Let R be a positive number. Then an ω -compatible almost complex structure J is called R -semi-positive if for every J -holomorphic sphere $\sigma: \mathbb{C}P^1 \rightarrow M$ with energy $E(\sigma) \leq R$ has Chern number $\int_{\mathbb{C}P^1} \sigma^* c_1 \geq 0$. Let $\mathcal{J}_+(M, \omega, R)$ be the set of all ω -compatible R -semi-positive J . This set may be empty. However, if (M, ω) is a weakly monotone compact symplectic manifold, then $\mathcal{J}_+(M, \omega, R)$ is a path connected open dense set for every R .

Theorem 5.1. *Let (M, ω) be a weakly monotone compact symplectic manifold, and $A \in H_2(M, \mathbb{Z})$.*

(a) *For every $J \in \mathcal{J}(M, \omega)$ there is a finite number of evaluation maps*

$$e_{D,T}: \mathcal{W}(D, T, J, p) \rightarrow M^p$$

such that

$$\bigcap e_p(\overline{\mathscr{W}(A, J, p) - K}) \subset \bigcup_{D, T} e_{D, T}(\mathscr{W}(D, T, J, p)),$$

where the intersection is over all compact subsets K in $\mathscr{W}(A, J, p)$, and the union is over all effective framings D and all functions $T: \{1, \dots, p\} \rightarrow \{1, \dots, N\}$.

- (b) There is a residual subset \mathscr{J}_r in $\mathscr{J}(M, \omega)$ such that for every $J \in \mathscr{J}_r$, the spaces $\mathscr{W}(D, T, J, p)$ are smooth oriented σ -compact manifolds with

$$\dim \mathscr{W}(D, T, J, p) = 2n + 2c_1(D) + 2p - 2N - 4.$$

- (c) Suppose that A is not a multiple class λB where $\lambda > 1$ and $c_1(B) = 0$. If $J \in \mathscr{J}_+(M, \omega, R) \cap \mathscr{J}_r$, then

$$\dim \mathscr{W}(D, T, J, p) \leq \dim \mathscr{W}(A, J, p) - 2.$$

Recall that a manifold is σ -compact if it is the union of a countable family of compact sets.

The proof may be found in [MS2]. To understand the essence of the theorem, we need to look at some facts about pseudo-cycles.

If a subset X of a manifold M is within the image of a smooth map $g: V \rightarrow M$ defined on a smooth manifold V of dimension k , then X is said to be of dimension at most k . The boundary of the set $g(V)$, denoted by $g(V^\infty)$, is defined to be the set

$$g(V^\infty) = \bigcap \overline{g(V - K)},$$

where the intersection is over all compact subsets K of V . This is the set of all limit points of sequences $\{g(x_n)\}$ where $\{x_n\}$ has no convergent subsequence in V .

A k -pseudo-cycle in M is a smooth map $f: W \rightarrow M$ defined on a smooth manifold W of dimension k such that $\dim f(W^\infty) \leq k - 1$. Two k -pseudo-cycles $f_0: W_0 \rightarrow M$ and $f_1: W_1 \rightarrow M$ are bordant if there is a $(k + 1)$ -pseudo-cycle $F: W \rightarrow M$ with $\partial W = W_1 - W_0$ such that

$$F|_{W_0} = f_0, \quad F|_{W_1} = f_1, \quad \text{and} \quad \dim F(W^\infty) \leq k - 1.$$

Every singular homology class $\alpha \in H_k(M)$ can be represented by a k -pseudo-cycle $f: W \rightarrow M$. This can be seen in the following way. First represent α by a map $f: X \rightarrow M$ defined on a finite oriented k -simplicial complex X without boundary so that $\alpha = f_*[X]$, where $[X]$ is the fundamental class. Then approximate f by a map which is smooth on each simplex of X . Finally, consider the union of the k - and $(k - 1)$ -faces of X as a smooth manifold W of dimension k and approximate f by a map that is smooth across the $(k - 1)$ -simplexes.

It also follows that bordant k -pseudo-cycles are in the same homology class. However, two pseudo-cycles representing the same homology class may not be bordant.

Theorem 5.1 says that the evaluation map $e_p: \mathscr{W}(A, J, p) \rightarrow M^p$ is a pseudo-cycle, and that the boundary of its image can be covered by the sets

$$e_{D, T}(\mathscr{W}(D, T, J, p)).$$

Therefore the image of e_p carries a fundamental homology class. It can be shown that this class is independent of the choice of the point $z \in (\mathbb{C}P^1)^p$ and the almost complex structure J .

6. Gromov–Witten invariants and quantum cohomology

For the definition of the quantum cohomology, the symplectic manifold (M, ω) is required to satisfy the following (mutually exclusive) conditions:

- (a) M is monotone, that is, $\langle \omega, A \rangle = \lambda \langle c_1, A \rangle$ for every $A \in \pi_2(M)$, where $\lambda > 0$ and $c_1 = c_1(TM, J)$.
- (b) $\langle c_1, A \rangle = 0$ for every $A \in \pi_2(M)$, or $\langle \omega, A \rangle = 0$ for every $A \in \pi_2(M)$.
- (c) The minimal Chern number N , defined by $\langle c_1, \pi_2(M) \rangle = N\mathbb{Z}$ where $N \geq 0$, is greater than or equal to $n - 2$.

It can be shown that a manifold (M, ω) is weakly monotone if and only if one of the above conditions is satisfied.

Let (M, ω) be a weakly monotone compact symplectic manifold with a fixed $A \in H_2(M, \mathbb{Z})$. Then the p -fold evaluation map

$$e_p: \mathcal{W}(A, J, p) \longrightarrow M^p$$

represents a well-defined homology class in M^p , which is independent of J .

If A is a spherical homology class, and $p \geq 1$, then define a homomorphism

$$\Phi_{A,p}: H_d(M^p, \mathbb{Z}) \longrightarrow \mathbb{Z},$$

where $d = 2np - \dim \mathcal{W}(A, J, p)$, in the following way. Let $\alpha \in H_d(M^p, \mathbb{Z})$ so that $\alpha = \alpha_1 \times \cdots \times \alpha_p$, where $\alpha_j \in H_{d_j}(M, \mathbb{Z})$ with $d_1 + \cdots + d_p = d$. One can find a cycle representing the homology class α , which is denoted by the same notation α , such that it intersects the image $\mathcal{X}(A, J, p)$ of the map e_p transversely in a finite number of points. Then the Gromov–Witten invariant $\Phi_A(\alpha_1, \dots, \alpha_p)$ is the intersection number $e_p \cdot \alpha$, which is the number of intersection points counted with signs according to their orientations. This is the number of J -holomorphic spheres σ in the homology class A which intersect each of the cycles $\alpha_1, \dots, \alpha_p$. If the dimension condition for d is not satisfied, then one sets $\Phi_A(\alpha_1, \dots, \alpha_p) = 0$.

The quantum cohomology is obtained by defining a quantum deformation of the cup product on the cohomology of a symplectic manifold (M, ω) . Before going into this, let us review the ordinary cup product in singular cohomology.

We denote by $H^*(M)$ the free part of $H^*(M, \mathbb{Z})$. We may consider $H^*(M)$ as de Rham cohomology consisting of classes which take integral values on all cycles:

$$H^*(M) = H_{\text{DR}}^*(M, \mathbb{Z}).$$

Next, we let $H_*(M)$ denote $H_*(M, \mathbb{Z})/\text{Torsion}$. Then we can identify $H^k(M)$ with $\text{Hom}(H_k(M), \mathbb{Z})$ by the pairing of $a \in H^k(M)$ and $\beta \in H_k(M)$ given by

$$a(\beta) = \int_{\beta} a.$$

In the same way, the intersection pairing $\alpha \cdot \beta$ of $\alpha \in H_{2n-k}(M)$ and $\beta \in H_k(M)$ gives rise to the homomorphism

$$\text{PD}: H_{2n-k}(M) \longrightarrow H^k(M),$$

where $\text{PD}(\alpha) = a$ if

$$a(\beta) = \int_{\beta} a = \alpha \cdot \beta \quad \text{for } \beta \in H_k(M).$$

The Poincaré duality theorem says that PD is an isomorphism. Then the cup product $a \cup b \in H^{k+\ell}(M)$ of $a \in H^k(M)$ and $b \in H^\ell(M)$ is defined by the triple intersection

$$\int_{\gamma} a \cup b = \alpha \cdot \beta \cdot \gamma, \quad \text{for } \gamma \in H_{k+\ell}(M),$$

where $\alpha = \text{PD}^{-1}(a) \in H_{2n-k}(M)$ and $\beta = \text{PD}^{-1}(b) \in H_{2n-\ell}(M)$. This is well-defined, because if the cycles representing α and β are in general position, then they intersect a pseudo-cycle of codimension $k + \ell$.

Next note that by our assumption (M, ω) is monotone with minimal Chern number $N \geq 2$.

The quantum multiplication $a * b$ of classes $a \in H^k(M)$ and $b \in H^\ell(M)$ is defined as follows. Let $\alpha = \text{PD}(a)$ and $\beta = \text{PD}(b)$ denote the Poincaré duals of a and b so that $\deg(\alpha) = 2n - k$ and $\deg(\beta) = 2n - \ell$. Then $a * b$ is the formal sum

$$a * b = \sum_A (a * b)_A \cdot q^{c_1(A)/N},$$

where q is an auxiliary variable supposed to be of degree $2N$, and the cohomology class $(a * b)_A \in H^{k+\ell-2c_1(A)}(M)$ is defined in terms of the Gromov–Witten invariant Φ_A by

$$\int_{\gamma} (a * b)_A = \Phi_A(\alpha, \beta, \gamma),$$

for $\gamma \in H_{k+\ell-2c_1(A)}(M)$. Here α, β, γ satisfy the following dimension condition required for the definition of the invariant Φ_A ,

$$2c_1(A) + \deg(\alpha) + \deg(\beta) + \deg(\gamma) = 4n.$$

The condition shows that $0 \leq c_1(A) \leq 2n$, and therefore only finitely many powers of q occur in the above sum defining $a * b$. Since M is monotone, the classes A which contribute to the coefficient of q^d satisfy $\omega(A) = c_1(A)/N = d$, and therefore only finitely many can be represented by J -holomorphic spheres. Therefore the sum is finite. Since only nonnegative powers of q occur in the sum, it follows that $a * b$ is an element of the group

$$\widetilde{QH}^*(M) = H^*(M) \otimes \mathbb{Z}[q],$$

where $\mathbb{Z}[q]$ is the polynomial ring in the variable q of degree $2N$. Then we get a multiplication by linear extension

$$\widetilde{QH}^*(M) \otimes \widetilde{QH}^*(M) \longrightarrow \widetilde{QH}^*(M).$$

The quantum cup product is skew-commutative in the sense that

$$a * b = (-1)^{\deg a \cdot \deg b} b * a$$

for $a, b \in QH^*(M)$. Moreover, the product is distributive over the sum, and associative. The skew-symmetry and the distributive properties follow easily. But the associative property is a bit complicated and depends on a certain gluing argument for J -holomorphic spheres.

The quantum cohomology $\widetilde{QH}^k(M)$ vanish for $k \leq 0$, and are periodic with period $2N$ for $k \geq 2n$. To get the full periodicity, we consider the group

$$QH^*(M) = H^*(M) \otimes \mathbb{Z}[q, q^{-1}],$$

where $\mathbb{Z}[q, q^{-1}]$ is the ring of Laurent polynomials, which consists of polynomials in the variables q, q^{-1} with the obvious relation $q \cdot q^{-1} = 1$. With this definition $QH^k(M)$ is non-zero for positive and negative values of k , and there is a natural isomorphism

$$QH^k(M) \longrightarrow QH^{k+2N}(M)$$

given by multiplication with q , for every $k \in \mathbb{Z}$.

If $A = 0$, then all J -holomorphic spheres in the class A are constant. It follows then that $\Phi_A(\alpha, \beta, \gamma)$ is just the usual triple intersection $\alpha \cdot \beta \cdot \gamma$. Since $\omega(A) > 0$ for all other A which have the J -holomorphic representatives, the constant term in $a * b$ is just the usual cup product.

The product in $QH^*(M)$ is also distributive over the sum, and skew-commutative. It commutes with the action of $\mathbb{Z}[q, q^{-1}]$. If $a \in H^0(M)$ or $H^1(M)$, then $a * b = a \cup b$ for all $b \in H^*(M)$. Also the canonical generator $\mathbf{1} \in H^0(M)$ is the unit in quantum cohomology.

As an example, let M be the complex projective n -space $\mathbb{C}P^n$ with the standard Kähler form. Let L be the standard generator of $H_2(\mathbb{C}P^1)$ represented by the line $\mathbb{C}P^1$. Then the first Chern class of $\mathbb{C}P^n$ is given by $c_1(L) = n + 1$. Therefore, by the dimension condition, the invariant $\Phi_{mL}(\alpha, \beta, \gamma)$ is non-zero only when $m = 0$ and 1 . Clearly, the case $m = 0$ corresponds to constant curves, and gives the ordinary cup product. Since the minimal Chern number is $N = n + 1$, the quantum cohomology groups are given by $QH^k(M) \simeq \mathbb{Z}$ when k is even, and $QH^k(M) = 0$ when k is odd.

Next, let $a \in H^\ell(M)$ and $b \in H^k(M)$. If $\ell + k \leq 2n$, then the quantum cup product is the same as the ordinary cup product $a * b = a \cup b$. So consider the case when a is the standard generator p of $H^2(M)$ defined by $p(L) = 1$, and $b = p^n \in H^{2n}(M)$. Then the quantum cup product $p * p^n$ is the generator q of $QH^{2n+2}(M)$, because

$$\int_{pt} (p * p^n)_L = \Phi_L([\mathbb{C}P^{n-1}], pt, pt) = 1,$$

where $[\mathbb{C}P^{n-1}] = PD(p)$ and $pt = PD(p^n)$, and all other classes $(p * p^n)_A$ vanish. Therefore the quantum cohomology of $\mathbb{C}P^n$ is given by

$$\widetilde{QH}^*(\mathbb{C}P^n) = \frac{\mathbb{Z}[p, q]}{(p^{n+1} = q)}.$$

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References

- [B] Behrend K *et al*, Quantum cohomology, Springer Lecture Notes in Math. No. 1776 (2002)
- [D] Donaldson S, The orientation of Yang–Mills moduli spaces and 4-manifold topology, *J. Diff. Geom.* **26** (1987) 397–428
- [GH] Griffiths P and Harris J, Introduction of algebraic geometry (Wiley) (1978)
- [G] Gromov M, Pseudo holomorphic curves in symplectic manifolds, *Invent. Math.* **82** (1985) 307–347
- [M1] McDuff D, Examples of symplectic structures, *Invent. Math.* **89** (1987) 13–36
- [M2] McDuff D, Elliptic methods in symplectic geometry, *Bull. Am. Math. Soc.* **23** (1990) 311–358
- [MS1] McDuff D and Salamon D, Introduction to symplectic topology (Oxford: Clarendon Press) (1995)
- [MS2] McDuff D and Salamon D, *J*-holomorphic curves and quantum cohomology, University Lecture Series, vol. 6, *Am. Math. Soc.* (1994)
- [RT] Ruan Y and Tian G, A mathematical theory of quantum cohomology, *Math. Research Lett.* **1** (1994) 269–278
- [R] Ruan Y, Topological sigma model and Donaldson-type invariants in Gromov theory, *Duke Math. J.* **83** (1996) 461–500
- [SU] Sacks J and Uhlenbeck K, The existence of minimal immersions of 2-spheres, *Ann. Math.* **113** (1981) 1–24
- [S] Smale S, An infinite dimensional version of Sard’s theorem, *Am. J. Math.* **87** (1965) 861–866
- [V] Vafa C, Topology mirrors and quantum rings, in: Essays on mirror manifolds (ed.) S-T Yau (Hong Kong: International Press) (1992)
- [W] Witten E, Two dimensional gravity and intersection theory on moduli space, *Surveys in Diff. Geom.* **1** (1991) 243–310