

## Representations of homogeneous quantum Lévy fields

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*Dedicated to K B Sinha on the event of his 60th birthday*

**Abstract.** We study homogeneous quantum Lévy processes and fields with independent additive increments over a noncommutative  $*$ -monoid. These are described by infinitely divisible generating state functionals, invariant with respect to an endomorphic injective action of a symmetry semigroup. A strongly covariant GNS representation for the conditionally positive logarithmic functionals of these states is constructed in the complex Minkowski space in terms of canonical quadruples and isometric representations on the underlying pre-Hilbert field space. This is of much use in constructing quantum stochastic representations of homogeneous quantum Lévy fields on Itô monoids, which is a natural algebraic way of defining dimension free, covariant quantum stochastic integration over a space-time indexing set.

**Keywords.** Representations; quantum Lévy process; independent increments; symmetry; infinite divisibility; monoid; generating functional; cocycle; conditionally positive definite.

### 1. Introduction

The reconstruction theorem due to Kolmogorov is a celebrated result in the theory of stochastic processes. It allows one to build versions of a stochastic process  $Z_t: \Omega \mapsto \mathbb{R}^d$  in the narrow sense from a consistent family of joint probabilities, called a stochastic process in the wide sense, parametrised by  $t$ . There is a quantum multiparameter generalisation of this theorem [1] allowing the construction of quantum stochastic processes and fields given a projective homogeneous system of multikernel maps describing the process in the wide sense. The reconstruction of stationary quantum weak Markov processes [4] and quantum Lévy fields [2] parametrised by  $x \in \mathbb{R}_+ \times \mathbb{R}^d$  instead of  $t \in \mathbb{R}_+$  are particular interesting cases of this construction.

Our concern in this paper will be the quantum stochastic processes and fields having independent increments in a weak differential sense, called wide quantum Lévy fields. No assumption of stationary increments will be made, instead we will consider homogeneous processes with respect to a given action of a symmetry semigroup in the field parameter space  $X$ . Classical nonstationary Lévy processes are characterised by infinitely divisible probability distributions, giving convolution hemigroups [7] of probability measures parametrised by the intervals  $\Delta = [t_0, t_1)$  of  $\mathbb{R}_+$ . In the quantum setting, a commutative convolution hemigroup of states can be defined by restricting the construction of [2] to one dimensional fields (that is, two parameter processes indexed by intervals  $\Delta = [t_0, t_1)$ ). There the coalgebra structure determining the convolution of states was implicitly defined

as the commutative pointwise multiplication of generating state functionals on the non-commutative monoid of a unital  $\star$ -semigroup  $\mathfrak{b}$ , which will also be used here. However, by enriching the enveloping semigroup algebra  $\mathbb{C}\mathfrak{b}$  of the  $\star$ -monoid  $\mathfrak{b}$  with a noncommutative coproduct one could use our results for general Lévy processes and homogeneous fields as in the stationary, one-parameter case [8].

One can encode infinite divisibility without defining convolution explicitly by using the necessary and sufficient exponential form of the characteristic functional, called the generating state functional in the quantum case. This generating state functional is determined by its exponent, called the cumulant generating state functional. This can be any Hermitian, absolutely continuous, conditionally positive state functional on  $\mathfrak{b}$  which has unit  $u \in \mathfrak{b}$  in its kernel. The problem of reconstructing a quantum Lévy process in the narrow sense is then reduced to finding a representation of the cumulant generating state and exponentiating it in some sense. This exponentiation is defined, similarly to the noncommutative convolution, by a variety of independences (Boolean, tensor, monotone and so on) which all coincide with the usual exponential in the weak sense for the commutative coalgebra structure implicit here. For non-commutative convolutions, this will result in the noncommutative exponentiation depending on the choice of independence, as in [6]. Regardless, in all cases the representation of the exponent is the same.

Here we present the first step by constructing ‘differential’ type representations associated with conditionally positive functionals  $\mathfrak{b} \rightarrow \mathbb{C}$  depending covariantly on the field parameter  $x \in X$  with respect to a semigroup of symmetries acting both on  $X$  and  $\mathfrak{b}$ . Later work will consider the exponentiation mentioned above, giving stochastic integral representations of covariant infinitely divisible positive definite functionals. Covariant quantum dynamical semigroups are an example of the noncommutative extension of this, and their representations have been studied in [5]. Our main interest is when  $\mathfrak{b}$  is obtained by a unitization of a noncommutative Itô algebra  $\mathfrak{a}$  as a parameterising algebra for the quantum stochastic differentials of a quantum Lévy process as operator-valued processes with independent increments in a quite general noncommutative sense.

## 2. Representations of homogeneous conditionally positive functionals on $\star$ -semigroups

Let  $(X, \mathfrak{F}, \mu)$  be a measurable space  $X$  with a  $\sigma$ -algebra  $\mathfrak{F}$  and a positive  $\sigma$ -finite atomless measure  $\mu: \mathfrak{F} \ni \Delta \mapsto \mu_\Delta, \mu_{dx} \equiv dx := d\mu(x)$ , and let  $\mathfrak{b}$  be a semigroup with involution

$$b \mapsto b^*, \quad (a \cdot c)^* = c^* \cdot a^*,$$

and neutral element (unit)  $u = u^*$ ,  $u \cdot b = b = b \cdot u$  for any  $b \in \mathfrak{b}$ . Typically  $\mathfrak{b}$  will be a unitization of a noncommutative Itô  $\star$ -algebra  $\mathfrak{a}$ , in which case

$$a \cdot c = a + c + ac.$$

if  $u$  is identified with zero, or simply write  $a \cdot c = ac$  if  $\mathfrak{a}$  is realised as a  $\star$ -subalgebra of a unital algebra by taking  $u = 1$ . However in what follows one can take any group with  $u = 1$  and  $b^* = b^{-1}$  or any  $\star$ -submonoid of an operator algebra  $\mathcal{B}$ , a unit ball of a unital  $C^*$ -algebra say, or even a filter (i.e. a submonoid) of an idempotent, Boolean say, algebra  $\mathfrak{B}$  with trivial involution  $b^* = b$ .

Denote by  $\mathfrak{m}$  the monoid of integrable step-maps  $g: X \rightarrow \mathfrak{b}$ , that is,  $\mathfrak{b}$ -valued functions  $x \mapsto g(x)$  having countable images  $g(X) = \{g(x): x \in X\} \subseteq \mathfrak{b}$  and integrable preimage  $\Delta(b) = \{x \in X: g(x) = b\} \in \mathfrak{F}$  in the sense  $\mu_{\Delta(b)} < \infty$  for all  $b \in \mathfrak{b}$  except  $b = u$ . We define on  $\mathfrak{m}$  an inductive structure of a  $\star$ -monoid with pointwise defined operations  $g^\star(x) = g(x)^\star$ ,  $(f \cdot h)(x) = f(x) \cdot h(x)$  and unit  $e(x) = u$  for all  $x \in X$ , considering  $\mathfrak{m}$  as the union  $\cup \mathfrak{m}_\Delta$  of subsemigroups  $\mathfrak{m}_\Delta$  of functions  $g \in \mathfrak{m}$  having integrable supports

$$\text{supp } g = \{x \in X: g(x) \neq u\}$$

in a  $\Delta \in \mathfrak{F}$  with  $\mu_\Delta < \infty$ .

It is convenient to describe the  $\star$ -monoid  $\mathfrak{b}$  by means of a single Hermitian operation  $a \star c = a \cdot c^\star$ , satisfying the relations

$$b \star u = b, \quad u \star (u \star b) = b \quad \forall b \in \mathfrak{b}$$

defining  $u = u^\star$  as right unit for the composition  $\star$ ,  $b^\star$  as  $u \star b$ , and

$$u \star ((c \star b) \star a) = a \star ((u \star b) \star c)$$

corresponding to  $(a \cdot c^\star)^\star = (a \star c)^\star = c \star a = c \cdot a^\star$  and associativity of the semigroup operation  $a \cdot c$ . This allows us to define both the product and involution in a  $\star$ -monoid  $\mathfrak{m}$  by a single Hermitian binary operation  $f \star h = g$ ,  $g(x) = f(x) \star h(x)$  with left unit  $e \in \mathfrak{m}$  which recovers the involution by  $g^\star(x) = e \star g$  and the associative product by  $f \cdot h = f \star (e \star h)$  for all  $f, h \in \mathfrak{m}$ .

We introduce a semigroup  $S$ , called the *symmetry semigroup*, which has a measurable action on  $X$  given by injections  $x \mapsto sx$ , and denote for each  $x \in sX$  its preimage  $s^{-1}x$  which is the unique element  $x^s \in X$  such that  $sx^s = x$ . The measure  $\mu$  is assumed invariant under this action in the sense that  $\mu_{\Delta^s} = \mu_\Delta$  for each  $s \in S$  and any measurable  $\Delta \subseteq sX$  with  $\Delta^s = s^{-1}\Delta$ . We admit also an action  $b \mapsto b_s$  of  $S$  on  $\mathfrak{b}$  determined by a representation of  $S$  in the semigroup of unital injective  $\star$ -endomorphisms  $\theta_s: \mathfrak{b} \rightarrow \mathfrak{b}$ , so that  $\theta_s(u) = u$ ,  $\theta_s(a \star b) = \theta_s(a) \star \theta_s(b)$ , and therefore

$$(a \cdot c)^s = a^s \cdot c^s, \quad b^{s^\star} = b^{s^\star}, \quad u^s = u \quad \forall a, b, c \in \theta_s(\mathfrak{b})$$

with respect to the inverse action  $b \mapsto \theta_s^{-1}(b) \equiv b^s$  of each  $\theta_s$  on the  $\star$ -submonoid  $\mathfrak{b}_s = \theta_s(\mathfrak{b})$ . These actions induce a representation of  $S$  on  $\mathfrak{m}$  by the injective  $\star$ -endomorphisms  $g \mapsto g_s$ ,

$$g_s(x) = \begin{cases} g(x^s)_s, & x \in sX \\ u, & x \notin sX \end{cases},$$

obviously having the property that for any  $s \in S$  and  $f, g \in \mathfrak{m}$  there exist unique  $f_s, g_s \in \mathfrak{m}_s$  such that

$$(f_s \star g_s)^s(x) := (f_s(sx) \star g_s(sx))^s = (f \star g)(x)$$

for all  $x \in X$ . Here we denoted by  $\mathfrak{m}_s$  the  $\star$ -submonoid of  $\mathfrak{m}$  consisting of functions  $g$  such that  $g(x) \in \mathfrak{b}_s$  if  $x \in sX$  and  $g(x) = u$  if  $x \notin sX$ , of which  $g_s$  is a member.

We say that a complex functional  $\varphi$  on  $\mathfrak{m}$  is a *generating state functional* over the monoid  $\mathfrak{m}$  (briefly, a *state* over  $\mathfrak{m}$ ), if the mapping  $\varphi: \mathfrak{m} \rightarrow \mathbb{C}$  satisfies the normalisation condition  $\varphi(e) = 1$  and positive definiteness

$$\sum_{f,h \in \mathfrak{m}} \kappa_f \varphi(f \star h) \kappa_h^* \geq 0, \quad \forall \kappa_g \in \mathbb{C}: |\text{supp } \kappa| < \infty, \tag{2.1}$$

where  $|\cdot|$  denotes the cardinality of the set  $\text{supp } \kappa = \{g \in \mathfrak{m}: \kappa_g \neq 0\}$ . Every such function is lifted to a positive normalised linear functional on the semigroup enveloping algebra  $\mathfrak{B} = \mathbb{C}\mathfrak{b}$ . The state  $\varphi$  is called *S-homogeneous* if  $\varphi(g_s) = \varphi(g)$  for all  $s \in S$  and  $g \in \mathfrak{m}$ .

Following [2] we introduce on  $\mathfrak{m}$  a commutative and associative partial operation  $f \sqcup h := f \cdot h$  for any functions  $f, h \in \mathfrak{m}$  with disjoint supports  $\text{supp } f \cap \text{supp } h = \emptyset$ . Thus the defined map  $\mathfrak{m}_\Delta \times \mathfrak{m}_{\Delta'} \rightarrow \mathfrak{m}_\Delta \sqcup \mathfrak{m}_{\Delta'}$  for any measurable disjoint  $\Delta, \Delta' \in \mathfrak{F}$  is obviously lifted to the tensor product  $\mathbb{C}\mathfrak{m}_\Delta \otimes \mathbb{C}\mathfrak{m}_{\Delta'}$  of the enveloping semigroup algebras of the  $\star$ -monoids  $\mathfrak{m}_\Delta$  and  $\mathfrak{m}_{\Delta'}$ . The operation  $\sqcup$  is well defined even for an infinite countable family  $\{g_n\}, g_n \in \mathfrak{m}$  with mutually disjoint supports  $\Delta_n = \text{supp } g_n$ , by  $\sqcup g_n(x) = g_m(x)$  for all  $x \in \text{supp } g_m$  and any  $m$ , otherwise  $\sqcup g_n(x) = u$  if  $x \notin \sum \Delta_n$ . Taking any  $g \in \mathfrak{m}$  and the partition  $\text{supp } g = \sum \Delta_n$  into the co-images  $\Delta_n = \Delta(b_n)$  where  $b_n = g(x)$  for any  $x \in \Delta_n$ , we see that any function  $g \in \mathfrak{m}$  can be written as  $\sqcup g_n$ , where  $g_n = (b_n)_{\Delta_n}$ , the  $b_n$ -valued indicator on  $\Delta_n$ . The  $b$ -valued indicator of the subset  $\Delta \subseteq X$  is defined in the usual way:  $b_\Delta(x) = b$  for all  $x \in \Delta$  and  $b_\Delta(x) = u$  for  $x \notin \Delta$ .

We call a state  $\varphi$  over  $\mathfrak{m}$  *chaotic* if it satisfies the  $\sigma$ -multiplicativity condition

$$\varphi \left( \bigsqcup_{n=1}^{\infty} g_n \right) = \prod_{n=1}^{\infty} \varphi(g_n),$$

where  $\prod_{n=1}^{\infty} \varphi(g_n) = \lim_{N \rightarrow \infty} \prod_{n=1}^N \varphi(g_n)$  for any functions  $g_n \in \mathfrak{m}$  with pairwise disjoint supports:  $\text{supp } g_n \cap \text{supp } g_m = \emptyset$  for all  $n \neq m$ . This condition is obviously fulfilled for  $\varphi$  of the exponential form  $\varphi(g) = e^{\lambda(g)}$  with

$$\lambda(g) = \int l(x, g) dx, \quad l(x, g) = l_x(g(x)), \tag{2.2}$$

which corresponds to absolute continuity (for all  $\Delta \in \mathfrak{F}$  we have  $\mu_\Delta = 0 \Rightarrow \lambda_\Delta(b) = 0$ ) of the  $\sigma$ -additive measure  $\lambda_\Delta(b) := \lambda(b_\Delta)$  for each  $b \in \mathfrak{b}$ .

The family  $\varphi_\Delta: \mathfrak{b} \mapsto \mathbb{C}$  defined by any chaotic state  $\varphi$  as  $\varphi_\Delta(b) = \varphi(b_\Delta)$  is called *infinitely divisible* in the sense of the equality  $\varphi_\Delta(b) = \prod \varphi_{\Delta_i}(b)$  which also holds in the limit of any integral sum sequence given by the decomposition  $\Delta = \sum \Delta_i, \mu_{\Delta_i} \searrow 0$  since  $\varphi_{\Delta_i}(b) \rightarrow 1$  for any  $b \in \mathfrak{b}$ . The function  $\varphi_\Delta: \mathfrak{b} \rightarrow \mathbb{C}$  given by

$$\varphi_\Delta(b) = \exp \left\{ \int_{\Delta} l_x(b) dx \right\}, \tag{2.3}$$

is clearly infinitely divisible in this sense.

Note that if the Radon–Nikodym derivative  $l_x(b) = d\lambda(b)/dx$  of the absolutely continuous measure  $d\lambda(b) = \lambda_{dx}(b)$  does not depend on  $x$ , the states  $\varphi_\Delta(b) = e^{l(b)\mu_\Delta} \equiv \varphi^{\mu_\Delta}(b)$  form a continuous Abelian semigroup

$$\{\varphi^t: t \in \mathbb{R}^+\}, \quad \varphi^0(b) = 1, \quad [\varphi^r \cdot \varphi^s](b) = \varphi^{r+s}(b)$$

with respect to the pointwise multiplication of  $\varphi^t$ .

The  $\lambda_\Delta$  introduced above is called the *cumulant generating state functional* and in general these are defined as any function  $b \mapsto \lambda_\Delta(b)$  which is conditionally positive definite

$$\sum_{a,c \in \mathfrak{b}} \kappa_a \lambda_\Delta(a \star c) \kappa_c^* \geq 0, \forall \kappa: |\text{supp } \kappa| < \infty, \quad \sum_{b \in \mathfrak{b}} \kappa_b = 0, \quad (2.4)$$

such that  $\lambda_\Delta(u) = 0$ ,  $\lambda_\Delta(b^*) = \lambda_\Delta(b)^*$  for any  $b \in \mathfrak{b}$ . They are called *S-homogeneous* if  $\lambda(g_s) = \lambda(g)$  for all  $g \in \mathfrak{m}$ , which implies the S-homogeneity of the generating state functional  $\varphi$ .

The following theorem shows that, along with some differentiability conditions, the properties of a cumulant generating state functional are necessary and sufficient conditions for  $e^{\lambda_\Delta}$  to be an S-homogeneous infinitely divisible state. They are also necessary and sufficient to allow the construction of a covariant Minkowski space dilation. We assume that  $X$  admits a net of decompositions of the Vitali system in which  $\mu_\Delta \searrow 0$ ,  $x \in \Delta$ , as  $\Delta \searrow \{x\}$ .

**Theorem 1.** *Consider an arbitrary functional  $\varphi_\Delta: \mathfrak{b} \mapsto \mathbb{C}$ , defined for any set  $\Delta \in \mathfrak{F}$  of finite measure  $\mu_\Delta < \infty$  as  $\varphi(b_\Delta)$  by an S-homogeneous functional  $\varphi: \mathfrak{m} \rightarrow \mathbb{C}$ . The following are equivalent:*

- (i)  $\varphi_\Delta$  is an infinitely divisible state over  $\mathfrak{b}$ , and is an absolutely continuous multiplicative measure in the sense that  $\mu_\Delta = 0 \Rightarrow \varphi_\Delta(b) = 1$  for all measurable  $\Delta \in \mathfrak{F}$ ,  $b \in \mathfrak{b}$ , and the limit

$$l_x(b) = \lim_{\Delta \downarrow \{x\}} \frac{1}{\mu_\Delta} (\varphi_\Delta(b) - 1) \quad (2.5)$$

exists in the Lebesgue–Vitali sense.

- (ii) The functional  $\lambda(g) = \ln \varphi(g)$  is defined, is absolutely continuous in the sense that  $\mu_\Delta = 0 \Rightarrow \lambda_\Delta(b) = 0$  for all measurable  $\Delta \in \mathfrak{F}$ ,  $b \in \mathfrak{b}$ , and is an S-homogeneous cumulant generating functional with S-covariant Radon–Nikodym derivative

$$l_x(b) = \lim_{\Delta \downarrow \{x\}} \frac{\lambda_\Delta(b)}{\mu_\Delta} = l_{sx}(b_s)$$

for each  $b \in \mathfrak{b}$ .

- (iii) There exists:

(1) an S-homogeneous integral  $\star$ -functional  $\lambda(g) = \int l(x, g) dx = \ln \varphi(g)$  which has complex S-covariant density  $l(x, g) = l_x(g(x)) = l(sx, g_s)$  such that  $l(x, g)^* = l(x, g^*)$  and whose values  $l(x, g) = 0$  for all  $g(x) = u$  and  $l(x, b_\Delta) = l_x(b)$  with  $x \in \Delta$  are independent of  $\Delta$ ;

(2) a vector map  $k: g \mapsto \int^\oplus k(x, g) dx$  into a pre-Hilbert subspace  $K \subseteq \int^\oplus K_x dx$  of square integrable functions  $x \mapsto k(x, g) = k_x(g(x)) \in K_x$  which are S-covariant

$$k(sx, g_s) = k_{sx}(g(x)_s) = V_s k(x, g)$$

in terms of an isometric representation  $s \mapsto V_s$  of  $S$  with respect to the scalar products  $\langle k | k \rangle \equiv \int k_x^* k_x dx = \|V_s k\|^2$ . It has values  $k(x, b_\Delta) = k_x(b) \in K_x$  independent of  $\Delta \ni x$  and  $k(x, b_\Delta) = 0$  if  $x \notin \Delta$  such that  $k(x, g) = 0$  if  $g(x) = u$ . The map  $k$ ,

together with the adjoint functions  $k^*(x, g) = k(x, g^*)^*$  as the linear functionals  $k^*(g) = \int^\oplus k^*(x, g) dx \in K^*$ , satisfies the condition

$$k^*(f)k(h) = \lambda(f \cdot h) - \lambda(f) - \lambda(h), \quad \forall f, h \in \mathfrak{m}; \quad (2.6)$$

(3) a unital  $*$ -representation  $j: g \mapsto G := \int^\oplus j(x, g) dx$ ,

$$j(g \cdot h) = j(g)j(h), \quad j(g^*) = j(g)^*, \quad j(e) = I,$$

of the  $\star$ -monoid  $\mathfrak{m}$  in the  $*$ -algebra of decomposable operators  $G: K \ni k \mapsto \int^\oplus j(x, g)k(x) dx$  with  $j(x, b_\Delta) = j_x(b)$  independent of  $\Delta \ni x$  and  $j(x, b_\Delta) = I_x$  if  $x \notin \Delta$ , which are  $S$ -covariant in the sense  $V_s j(g) = j(g_s) V_s$ , where  $j(x, g_s) = j_x(g(x^s)_s)$ ,  $x \in sX$ , otherwise  $j(x, g_s) = I_x$ , satisfy the cocycle property

$$j(g)k(h) = k(g \cdot h) - k(g), \quad k^*(f)j(g) = k^*(f \cdot g) - k^*(g) \quad (2.7)$$

for all  $f, g, h \in \mathfrak{m}$ , and are continuous in  $K$  with respect to the polynorm

$$\|k\|^h = \left( \int \|j(x, h)k(x)\|_x^2 dx \right)^{1/2}, \quad h \in \mathfrak{m}. \quad (2.8)$$

(iv) For each integrable  $\Delta \in \mathfrak{F}$  there exists a unital  $\dagger$ -representation  $\mathbf{j}_\Delta: \mathfrak{b} \rightarrow \mathfrak{b}(K_\Delta)$ ,

$$\mathbf{j}_\Delta(a \cdot b) = \mathbf{j}_\Delta(a)\mathbf{j}_\Delta(b), \quad \mathbf{j}_\Delta(b^*) = \mathbf{j}_\Delta(b)^\dagger, \quad \mathbf{j}_\Delta(u) = \mathbf{I}_\Delta$$

in the algebra  $\mathfrak{b}(K_\Delta)$  of linear triangular operators  $\mathbf{L} = \mathbf{j}_\Delta(b)$ . The integrals

$$\lambda_\Delta(b) = \int_\Delta l_x(b) dx = \ln \varphi_\Delta(b), \quad j_\Delta(b) = \int_\Delta^\oplus j_x(b) dx,$$

which correspond to  $\sigma$ -additivity and absolute continuity with respect to the  $S$ -homogeneous measure  $\mu$ , appear in the explicit form of  $\mathbf{j}_\Delta$  given by

$$\mathbf{j}_\Delta(b) = \begin{bmatrix} 1 & k_\Delta^*(b) & \lambda_\Delta(b) \\ 0 & j_\Delta(b) & k_\Delta(b) \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{j}_\Delta(b)^\dagger = \begin{bmatrix} 1 & k_\Delta(b)^* & \lambda_\Delta(b^*) \\ 0 & j_\Delta(b)^* & k_\Delta(b^*) \\ 0 & 0 & 1 \end{bmatrix}$$

on the pseudo-Hilbert space  $\mathbb{K}_\Delta = \mathbb{C} \oplus K_\Delta \oplus \mathbb{C}$  defined by a pre-Hilbert space  $K_\Delta \subseteq \int_\Delta^\oplus K_x dx$  with respect to the Minkowski scalar product

$$(\mathbf{k}|\mathbf{k})_\Delta := k_-^* k_+ + \int_\Delta \langle k_x | k_x \rangle + k_+^* k_- \equiv \langle \mathbf{k}_\Delta^\dagger, \mathbf{k}_\Delta \rangle. \quad (2.9)$$

$\mathbf{L}^\dagger$  is the pseudo-Hermitian adjoint  $(\mathbf{k}|\mathbf{L}^\dagger \mathbf{k}) = (\mathbf{L}\mathbf{k}|\mathbf{k})$ ,  $\mathbf{k} \in \mathbb{K}_\Delta$  and  $\lambda_\Delta(b) = \langle \mathbf{e}^\dagger, \mathbf{j}_\Delta(b)\mathbf{e} \rangle$  with respect to the row-vector  $\mathbf{e}^\dagger = (1, 0, 0) \in \mathbb{K}_\Delta^\dagger$  of zero pseudo-norm  $\langle \mathbf{e}^\dagger, \mathbf{e} \rangle_\Delta = 0$ . The family of representations  $\{\mathbf{j}_\Delta\}$  is  $S$ -covariant in the sense that there exists a representation  $\mathbf{V}_s$  of  $S$  in the pseudo-isometric operators  $\mathbb{K}_\Delta \rightarrow \mathbb{K}_{s\Delta}$  such that  $\mathbf{V}_s \mathbf{j}_\Delta(b) = \mathbf{j}_{s\Delta}(b_s) \mathbf{V}_s$  for any  $b \in \mathfrak{b}$  and integrable  $\Delta \subseteq X$ . The pseudo-isometry  $\mathbf{V}_s$  is block-diagonal with  $[\mathbf{V}_s]_- = [\mathbf{V}_s]_+^\dagger = 1$ ,  $[\mathbf{V}_s]_0^\circ = V_s$  for some direct integral pseudo-isometry  $V_s: K_\Delta \mapsto K_{s\Delta}$ , with all other components zero.

We will prove that (iv)  $\Rightarrow$  (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i)  $\Rightarrow$  (iv) following [2] (see also [3]), adding homogeneity and covariance.

(iv)  $\Rightarrow$  (iii). First let us define the density  $\mathbf{j}_x(b) = \lim_{\Delta \downarrow \{x\}} \frac{\mathbf{j}_\Delta(b)}{\mu_\Delta}$  with the limit understood in the appropriate sense for each of the blocks. Then  $\mathbf{j}_x(b)$  is a unital  $\dagger$ -representation of  $\mathfrak{b}$  in  $\mathfrak{b}(K_x)$ . Thus we can define  $\mathbf{j}(x, g) = \mathbf{j}_x(g(x))$  which has direct integral  $\mathbf{j}(g) = \int_X^\oplus \mathbf{j}(x, g) dx$ . Furthermore,  $S$ -homogeneity of  $\mu$  gives the covariance

$$\begin{aligned} \mathbf{V}_s \mathbf{j}_x(b) &= \lim_{\Delta \downarrow \{x\}} \frac{\mathbf{V}_s \mathbf{j}_\Delta(b)}{\mu_\Delta} = \lim_{\Delta \downarrow \{x\}} \frac{\mathbf{j}_{s\Delta}(b_s) \mathbf{V}_s}{\mu_\Delta} \\ &= \lim_{\Delta \downarrow \{sx\}} \frac{\mathbf{j}_\Delta(b_s) \mathbf{V}_s}{\mu_\Delta} = \mathbf{j}_{sx}(b_s) \mathbf{V}_s. \end{aligned}$$

Each of the components  $\lambda(g)$ ,  $k(g)$  and  $j(g)$  of  $\mathbf{j}(g)$  is the direct integral of the component densities  $l_x(g(x)) = l(x, g)$ ,  $k_x(g(x)) = k(x, g)$ ,  $j_x(g(x)) = j(x, g)$  existing due to the  $\sigma$ -additivity and absolute continuity assumed in (iv). We can now use the properties of  $\mathbf{j}_x$  to verify those required by (iii).

Starting with  $\lambda(g) = \int_X^\oplus l(x, g) dx$  we have  $l(x, g) = \langle \mathbf{e}^\dagger, \mathbf{j}(x, g) \mathbf{e} \rangle$  so that  $l(x, g^*) = \langle \mathbf{e}^\dagger, \mathbf{j}(x, g^*) \mathbf{e} \rangle = \langle \mathbf{e}^\dagger, \mathbf{j}(x, g)^\dagger \mathbf{e} \rangle = l(x, g)^*$  and therefore  $l(x, e) = \langle \mathbf{e}^\dagger, \mathbf{j}(x, u) \mathbf{e} \rangle = 0$ . Hence  $l(x, g) = l_x(u) = 0$  for any  $x$  in the kernel of  $g$ , and also  $l(x, b_\Delta)$  is independent of  $\Delta \ni x$ . The covariance of  $\mathbf{j}_x$  gives

$$l(x, g) = \langle \mathbf{e}^\dagger, \mathbf{j}_x(g) \mathbf{e} \rangle = \langle \mathbf{e}^\dagger, \mathbf{V}_s \mathbf{j}_{sx}(g(x)_s) \mathbf{V}_s \mathbf{e} \rangle = l(sx, g_s)$$

as required. The  $S$ -homogeneity  $\lambda(g) = \lambda(g_s)$  follows straight away by integrating and noting that  $l(x, g_s) = 0$  for  $x \notin sX$ .

The vector map density  $k(x, g)$  is given by the density  $[\mathbf{j}_x(g(x))]^\circ_+$ . Due to the unitality of  $\mathbf{j}_x$ ,  $k(x, g(x)) = 0$  for  $x$  in the kernel of  $g$ , and clearly  $k(x, b_\Delta)$  is independent of  $\Delta \ni x$ . To prove condition (2.6) consider

$$\lambda(f \cdot g) = \langle \mathbf{e}^\dagger, \mathbf{j}(f \cdot g) \mathbf{e} \rangle = \lambda(g) + k^*(f)k(g) + \lambda(f). \tag{2.10}$$

This also shows that  $\|k\|^2 = k^*(f)k(g) < \infty$  so that  $k(x, g)$  is square integrable as required. Covariance again follows trivially by taking the appropriate component of the covariance for  $\mathbf{j}_x$ :

$$V_s k(x, g) = [\mathbf{V}_s \mathbf{j}(x, g)]^\circ_+ = [\mathbf{j}(sx, g_s) \mathbf{V}_s]^\circ_+ = k(sx, g_s).$$

Extending  $V_s$  to  $K$  by its direct integral  $V_s k(g) = \int_X^\oplus V_s k(x, g) dx$  also gives a linear pseudo-isometry.

That the decomposable linear operator  $j(g) = [\mathbf{j}(g)]^\circ$  on  $K$  is a unital  $*$ -representation of  $\mathfrak{m}$  follows from the upper triangular form of  $\mathbf{j}(g)$  and its  $\dagger$ -multiplicativity and unitality. The cocycle property (2.7) is seen straightforwardly by calculating  $k(g \cdot h) = [\mathbf{j}(g) \mathbf{j}(h)]^\circ_+$ . Completing  $K$  with respect to the seminorms (2.8) obviously makes  $j$  continuous. Covariance of  $j$  again follows simply by taking the appropriate component of the covariance condition for  $\mathbf{j}$ .

(iii)  $\Rightarrow$  (ii). It is immediate that the absolutely continuous measure  $\lambda_\Delta(b)$  satisfies the conditions  $\lambda_\Delta(b^*) = \lambda_\Delta(b)^*$  and  $\lambda_\Delta(u) = 0$ , since the functional  $l_x$  satisfies the respective conditions. Also the condition  $l_x(u) = 0$  along with the integral form of  $\lambda_\Delta$  ensure its

$\sigma$ -additivity on disjoint integrable subsets of  $X$ , and so is the cumulant generating functional of an infinitely divisible state if it is conditionally positive definite. The conditional positivity (2.4) follows from (2.6) and the positive definiteness of the inner product on  $K$ :

$$\begin{aligned} \sum_{f,g \in \mathfrak{m}} \kappa_f \lambda(f \star g) \kappa_g^* &= \sum_{f,g \in \mathfrak{m}} \kappa_f (\lambda(g^*) + \lambda(f) + k^*(f)k(g^*)) \kappa_g^* \\ &= \sum_{f,g \in \mathfrak{m}} \kappa_f k^*(f)k(g^*) \kappa_g^* \geq 0 \end{aligned}$$

for  $\sum_f \kappa_f = 0$ .  $S$ -homogeneity of  $\lambda(g)$  and  $l_x$  are trivial.

(ii)  $\Rightarrow$  (i). If the function  $\lambda_\Delta(b)$  is a (complex) absolutely continuous measure, then  $\varphi_\Delta(b) = \exp\{\lambda_\Delta(b)\}$  has the property  $\varphi_{\sqcup \Delta_i}(b) = \prod \varphi_{\Delta_i}(b)$  of infinite divisibility. Moreover the limit (2.5) exists, and by virtue of  $\varphi_\Delta(b) \rightarrow 1$  as  $\Delta \downarrow \{x\}$  it coincides with the Radon–Nikodým derivative  $l_x(b) = d \ln \varphi(b)/dx$  as the limit of the quotient  $\lambda_\Delta(b)/\mu_\Delta$  over a net of subsets  $\Delta \ni x$  of the system of Vitali decompositions of the measurable space  $X$ .  $S$ -homogeneity of  $\varphi_\Delta(b)$  is trivial. For any integrable  $\Delta$  the function  $b \mapsto \varphi_\Delta(b)$  can be shown to be positive in the sense of (2.1) by considering the conditioned complex function  $b \mapsto \kappa_b^\circ$  defined as  $\kappa_u^\circ = \kappa_u - \sum_{b \in \mathfrak{b}} \kappa_b$ ,  $\kappa_b^\circ = \kappa_b$  for all  $b \neq u \in \mathfrak{b}$  for an arbitrary finitely supported complex function  $b \mapsto \kappa_b$ . Hence  $\sum_{b \in \mathfrak{b}} \kappa_b^\circ = 0$  so by conditional positivity

$$0 \leq \sum_{a,c \in \mathfrak{b}} \kappa_a^\circ \lambda_\Delta(a \star c) \kappa_c^{\circ*} = \sum_{a,c \in \mathfrak{b}} \kappa_a (\lambda_\Delta(a \star c) - \lambda_\Delta(a) - \lambda_\Delta(c^*)) \kappa_c^*,$$

where we have taken into account the fact that  $\lambda_\Delta(u) = 0$ . Since the exponential of any positive-definite kernel is a positive definite kernel, we have for any  $\Delta$ ,

$$\begin{aligned} &\sum_{a,c \in \mathfrak{b}} \kappa_a^* \exp\{\lambda_\Delta(a \star c)\} \kappa_c \\ &= \sum_{a,c \in \mathfrak{b}} \kappa_\Delta^{a*} \exp\{\lambda_\Delta(a \star c) - \lambda_\Delta(a) - \lambda_\Delta(c^*)\} \kappa_\Delta^c \geq 0, \end{aligned}$$

where  $\kappa_\Delta^b = \kappa_b \exp\{\lambda_\Delta(b)\}$  and we have used  $\lambda_\Delta(b^*) = \lambda_\Delta(b)^*$ .

(i)  $\Rightarrow$  (iv). Since  $\varphi_\Delta$  is an infinitely divisible state on  $\mathfrak{b}$  and  $\varphi_\Delta(b) \rightarrow 1$  for all  $b$  as  $\mu_\Delta \rightarrow 0$ , the limit  $l_x(b)$  is defined as the logarithmic derivative  $\mu_{dx}^{-1} \ln \varphi_{dx}(b)$  of the measure  $\lambda_\Delta(b) = \ln \varphi_\Delta(b)$  in the Radon–Nikodym sense. Consequently, the function  $x \mapsto l_x(b)$  is integrable and almost everywhere satisfies the conditions  $l_x(a \star c)^* = l_x(c \star a)$ ,  $l_x(u) = 0$  and

$$\sum_{b \in \mathfrak{b}} \kappa_b = 0 \Rightarrow (\kappa' | \kappa)_x := \sum_{a,c \in \mathfrak{b}} \kappa_a l_x(a \star c) \kappa_c^* \geq 0$$

for all  $\kappa$  such that  $|\text{supp } \kappa| < \infty$ , which can easily be verified directly for the difference derivative  $l_\Delta(b) = (\varphi_\Delta(b) - 1)/\mu_\Delta$  passing to the limit  $\Delta \downarrow \{x\}$ . In addition  $\int_\Delta l_x(b) dx = \ln \varphi_\Delta(b) = \lambda_\Delta(b)$  by absolute continuity, and since  $(b_\Delta)_s(x) = (b_{s\Delta}(x))_s$ ,  $S$ -homogeneity becomes  $\varphi_{s\Delta}(b) = \varphi_\Delta(b^s)$  for  $\Delta \subset sX$ , giving  $l_{sx}(b) = l_x(b^s)$  for  $x \in sX$  in the limit.

We consider the space  $\mathfrak{B}$  of complex functions  $\kappa = (\kappa_b)_{b \in \mathfrak{b}}$  on  $\mathfrak{b}$  with finite supports  $\{b \in \mathfrak{b} : \kappa_b \neq 0\}$  as a unital  $\star$ -algebra with respect to the product  $\kappa' \star \kappa$  defined as  $\kappa' \star \kappa^*$  by the Hermitian convolution



$$(\kappa' \star \kappa)_b = \sum_{a \star c = b} \kappa'_a \kappa_c^*, \quad \delta_u \star \kappa = \kappa^*, \quad \kappa \star \delta_u = \kappa$$

with right identity  $\delta_u$ . Here  $\delta_a = (\delta_{a,b})_{b \in \mathfrak{b}}$  is the Kronecker delta and it defines a  $\star$ -representation  $a \mapsto \delta_a$  of the monoid  $\mathfrak{b}$  in  $\mathfrak{B}$ ,

$$\delta_a \star \delta_c = \delta_{a \star c}, \quad \delta_u \star \delta_b = \delta_b, \quad \delta_b \star \delta_u = \delta_{b^*},$$

with respect to the involution  $\kappa^* = (\kappa_{b^*}^*)_{b \in \mathfrak{b}}$ . The linear subspace  $\mathfrak{A} \subset \mathfrak{B}$  of distributions  $\kappa$  such that the sum  $\kappa_- := \sum_{b \in \mathfrak{b}} \kappa_b$  equals zero, is a  $\star$ -ideal since

$$\sum_{b \in \mathfrak{b}} (\kappa' \star \kappa)_b = \sum_{b \in \mathfrak{b}} \sum_{a \star c = b} \kappa'_a \kappa_c^* = \sum_{a \in \mathfrak{b}} \kappa'_a \sum_{c \in \mathfrak{b}} \kappa_c^* = 0.$$

Let us equip  $\mathfrak{B}$  for every  $x \in X$  with the Hermitian form  $(\kappa' | \kappa)_x$  of the kernel  $l_x(a \star c)$  which is positive on  $\mathfrak{A}$  and can be written in terms of the kernel  $\langle \delta_a, \delta_c^* \rangle_x^\circ = l_x(a \star c) - l_x(a) - l_x(c^*)$  as

$$(\kappa' | \kappa)_x = \kappa'_- \kappa_+^* + \langle \kappa', \kappa^* \rangle_x^\circ + \kappa'_+ \kappa_-^*,$$

where  $\kappa_\pm^\pm := \sum_b \kappa_b l_x(b)$ . We notice that the Hermitian form

$$\langle \kappa' \star \kappa^* \rangle_x^\circ := \sum_{a, c \in \mathfrak{b}} \kappa'_a \langle \delta_a, \delta_c \rangle_x^\circ \kappa_c^* \equiv \langle \kappa', \kappa^* \rangle_x^\circ$$

is non-negative if  $\kappa_- = 0$  or  $\kappa'_- = 0$  as  $\langle \kappa', \kappa^* \rangle_x^\circ = \sum_a \kappa'_a \langle \delta_a, \delta_c^* \rangle_x^\circ \kappa_c^* \geq 0$ , coinciding with  $(\kappa' | \kappa)_x$ . Since  $(\kappa' | \kappa)_x = \sum_b (\kappa' \star \kappa)_b l_x(b)$ , the form  $(\kappa' | \kappa)_x$  has right associativity property

$$(\kappa' \cdot \kappa | \kappa)_x = (\kappa' | \kappa \star \kappa)_x = (\kappa' | \kappa \cdot \kappa^*)_x,$$

for all  $\kappa, \kappa' \in \mathfrak{B}$ , and therefore its kernel  $\mathfrak{R}_x = \{\kappa: (\kappa' | \kappa)_x = 0, \forall \kappa'\}$  is the right ideal

$$\mathfrak{R}_x = \{\kappa' \in \mathfrak{B}: (\kappa' \cdot \kappa | \kappa)_x = 0, \forall \kappa \in \mathfrak{B}\}$$

belonging to  $\mathfrak{A}$ . We factorise  $\mathfrak{B}$  by this right ideal putting  $\kappa \approx 0$  if  $\kappa \in \mathfrak{R}_x^* := \{\kappa^*: \kappa \in \mathfrak{R}_x\}$  and denoting the equivalence classes of the left factor-space  $\mathbb{K}_x = \mathfrak{B}/\mathfrak{R}_x^*$  as the ket-vectors  $|\kappa\rangle = \{\kappa': \kappa' - \kappa^* \in \mathfrak{R}_x^*\}$ . The condition  $\kappa \in \mathfrak{R}_x$  means in particular that  $\kappa_x^- := (\delta_u | \kappa)_x = 0$ , and therefore

$$(\kappa | \kappa)_x = \sum_{a, c \in \mathfrak{b}} \kappa_a \langle \delta_a, \delta_c^* \rangle_x^\circ \kappa_c^* = \langle \kappa^\circ | \kappa^\circ \rangle_x = 0,$$

where  $\kappa^\circ = (\kappa_b^\circ)_{b \in \mathfrak{b}}$  denotes an element of  $\mathfrak{A}$  obtained as  $\kappa_b^\circ = \kappa_b^*$  for all  $b \neq u$  and  $\kappa_u^\circ = \kappa_u^* - \sum_{b \in \mathfrak{b}} \kappa_b^*$  such that  $\langle \kappa^\circ | \kappa^\circ \rangle_x = \langle \kappa, \kappa^* \rangle_x^\circ$ . Therefore it follows also that  $\kappa^+ := \sum_b \kappa_b^*$  is also zero for any  $\kappa \in \mathfrak{R}_x$  since

$$0 = (\kappa' | \kappa)_x = \kappa'_- \kappa_+^* + \langle \kappa', \kappa^* \rangle_x^\circ + \kappa'_+ \kappa_-^* = \kappa_-^* = \kappa^+$$

for any  $\kappa' \in \mathfrak{B}$  with  $\kappa_+^{\prime x} = 1$  by virtue of  $\kappa_+^{\kappa^*} = \kappa^- = 0$  and also due to the Schwartz inequality  $(\kappa' | \kappa)_x = \langle \kappa', \kappa^* \rangle_x^\circ = 0$ . This allows us to represent the left equivalence classes  $|\kappa\rangle_x$  by the columns  $\mathbf{k} = [k^\mu]$  with  $k^\mp = \kappa^\mp$  and  $k^\circ = |\kappa^\circ\rangle_x$  in the Euclidean component  $K_x \subset \mathbb{K}_x$  as the subspace of the left equivalence classes  $|\kappa^\circ\rangle_x = |\kappa_\circ\rangle_x$  of the elements

$\kappa_\circ = (\kappa_b - \delta_{u,b}\kappa_-)_{b \in \mathfrak{b}} \in \mathfrak{A}$  such that  $\kappa_\circ^* = \kappa^\circ$ . These columns are pseudo-adjoint to the rows  $\mathbf{k} = (k_-, k_\circ, k_+)$  as the right equivalence classes  ${}_x(\kappa := |\kappa|_x)^\dagger \in \mathfrak{B}/\mathfrak{A}_x$  with  $k_\pm = \kappa_\pm$  and  $k_\circ = {}_x(\kappa_\circ)$  defining the indefinite product in terms of the canonical pairing

$$k \cdot k = k_- k^- + \langle k_\circ, k^\circ \rangle_x + k_+ k^+ = (\mathbf{k}|\mathbf{k}^\dagger)_x,$$

where  $k^\circ = k_\circ^* \in K_x$ ,  $k^\pm = k_\mp^* \in \mathbb{C}$  with respect to the Euclidean scalar product  $\langle k_\circ, k^\circ \rangle_x = \langle k_\circ^* | k^\circ \rangle_x$  of the Euclidean space  $K_x = \{k^\circ = |\kappa^\circ|_x : \kappa_\circ \in \mathfrak{A}\}$ , and

$$\kappa_+^{x*} = \sum_{b \in \mathfrak{b}} l_x(b^*) \kappa^b = \kappa_x^-, \quad \kappa_-^* = \sum_{b \in \mathfrak{b}} \kappa_b^* = \kappa^+.$$

We notice that the representation  $\delta: \mathfrak{b} \ni b \mapsto \delta_b$  is Hermitian:

$$(\kappa \cdot \delta_b | \kappa)_x = \sum_{b \in \mathfrak{b}} l(b) (\kappa \cdot \delta_b \star \kappa)_b = (\kappa | \kappa \cdot \delta_{b^*})_x,$$

and that it is well defined as right representation on  $\mathfrak{B}/\mathfrak{A}_x$  (or left representation on  $\mathfrak{B}/\mathfrak{A}_x^*$ ) since  $\kappa \cdot \delta_b \in \mathfrak{A}_x$  if  $\kappa \in \mathfrak{A}_x$ :

$$(\kappa | \kappa)_x = 0, \quad \forall \kappa \in \mathfrak{B} \Rightarrow (\kappa \cdot \delta_b | \kappa)_x = (\kappa | \kappa \star \delta_b)_x = 0, \quad \forall \kappa \in \mathfrak{B}.$$

This allows us to define for each  $b \in \mathfrak{b}$  an operator  ${}_x(\kappa \mathbf{j}_x(b)) = {}_x(\kappa \cdot \delta_b)$  such that  $\mathbf{j}_x(b^*) = \mathbf{j}_x(b)^\dagger$  with the componentwise action

$$\begin{aligned} (\kappa \cdot \delta_b)_- &= \kappa_-, & (\kappa \cdot \delta_b)_\circ &= \kappa_- (\delta_b - \delta_u) + \kappa_\circ \cdot \delta_b, \\ (\kappa \cdot \delta_b)_+^x &= \kappa_- l_x(b) + (\kappa_\circ | \delta_{b^*} - \delta_u)_x + \kappa_+^x, \end{aligned}$$

given as the right multiplications  $\mathbf{k} \mapsto \mathbf{kL}$ ,  $\mathbf{k} \mapsto \mathbf{kL}^\dagger$  of the triangular matrices  $\mathbf{L} \equiv \mathbf{j}_x(b)$ ,  $\mathbf{L}^\dagger \equiv \mathbf{j}_x(b^*)$ ,

$$\mathbf{L} = \begin{bmatrix} 1 & j_-^-(x, b) & j_+^-(x, b) \\ 0 & j_\circ^\circ(x, b) & j_+^\circ(x, b) \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{L}^\dagger = \begin{bmatrix} 1 & j_+^\circ(x, b)^* & j_+^-(x, b)^* \\ 0 & j_\circ^\circ(x, b)^* & j_-^-(x, b)^* \\ 0 & 0 & 1 \end{bmatrix}$$

by the rows  $\mathbf{k} = (k_-, k_\circ, k_+) \in \mathbb{K}_x^\dagger$  (or as the left multiplications  $\mathbf{Lk}$ ,  $\mathbf{L}^\dagger \mathbf{k}$  by columns  $\mathbf{k} \in \mathbb{K}_x$ ). Here

$$\begin{aligned} j_+^-(x, b) &= l_x(b), & {}_x(\kappa_\circ j_\circ^\circ(x, b)) &= {}_x(\kappa_\circ \cdot \delta_b) = {}_x(\kappa_\circ j_x(b)), \\ j_+^\circ(x, b^*) &= \delta_b^* \rangle_x = k_x(b^*) = k_x^*(b)^* = {}_x(\delta_b^* |^* = j_-^-(x, b)^*, \end{aligned}$$

where  $\delta_b^* \rangle_x = |\delta_b - \delta_u \rangle_x$  and  $L_{-v}^{*\mu} = L_{-v}^{\mu*}$  is pseudo-Euclidean conjugation of the triangular matrix  $\mathbf{L} = [L_v^\mu]$  corresponding to the map  $\mathbf{k} \mapsto \mathbf{k}^\dagger$  into the adjoint columns  $\mathbf{k} = [k^\mu]$  with the components  $k^\mu = k_{-v}^{*\mu}$  given by the pseudo-metric tensor  $g^{\mu\nu} = \delta_{-v}^\mu = g_{\mu\nu}$ :

$$\begin{aligned} \begin{bmatrix} b_-^- & b_\circ^- & b_+^- \\ 0 & b_\circ^\circ & b_+^\circ \\ 0 & 0 & b_+^+ \end{bmatrix}^* &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_-^- & b_\circ^- & b_+^- \\ 0 & b_\circ^\circ & b_+^\circ \\ 0 & 0 & b_+^+ \end{bmatrix}^* \begin{bmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} b_+^{+*} & b_+^{\circ*} & b_+^{-*} \\ 0 & b_\circ^{\circ*} & b_\circ^{-*} \\ 0 & 0 & b_\circ^{-*} \end{bmatrix}. \end{aligned}$$

Constructing the direct integral space  $\mathbb{K} = \int_X^\oplus \mathbb{K}_x dx$  allows the definition of  $\mathbf{j}_\Delta(b)$  as the block-wise direct integral  $\int_\Delta^\oplus \mathbf{j}_x(b) dx$ . Thus we can write the constructed canonical  $\dagger$ -representations  $\mathbf{j}_\Delta(b) = [j_V^\mu(\Delta, b)]$  of the monoid  $\mathfrak{b}$  in the pseudo-Euclidean space  $\mathbb{K}$  of columns  $\mathbf{k} = [k^\mu]$  in terms of the usual matrix multiplication

$$\begin{aligned} & \begin{bmatrix} 1 & k_\Delta^*(a) & \lambda_\Delta(a) \\ 0 & j_\Delta(a) & k_\Delta(a) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k_\Delta^*(b) & \lambda_\Delta(b) \\ 0 & j_\Delta(b) & k_\Delta(b) \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & k_\Delta^*(b) + k_\Delta^*(a)j_\Delta(b) & \lambda_\Delta(b) + k_\Delta^*(a)k_\Delta(b) + \lambda_\Delta(a) \\ 0 & j_\Delta(a)j_\Delta(b) & j_\Delta(a)k_\Delta(b) + k_\Delta(a) \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Clearly this realises the conditionally positive function  $\lambda_\Delta(b)$  as the value of the vector form

$$\mathbf{e}^\dagger \mathbf{j}_\Delta(b) \mathbf{e} = e_\mu j_V^\mu(\Delta, b) e^\nu = j_+^-(\Delta, b) = \lambda_\Delta(b)$$

with the column  $\mathbf{e} = [\delta_+^\mu] = \mathbf{e}^\dagger$  the adjoint to the row  $\mathbf{e} = (1, 0, 0)$  of zero pseudonorm  $\mathbf{e}^\dagger \mathbf{e} = e_\mu e^\mu = 0$  for each  $x$ . A representation  $V_s$  of the symmetry semigroup  $S$  is defined on  $\mathfrak{B}$  as

$$(V_s \kappa)_b = \sum_{a \in \mathfrak{b}} \kappa_a \delta_{a_s, b} = \begin{cases} \kappa_{b^s}, & \text{if } b \in \mathfrak{b}_s \\ 0, & \text{if } b \notin \mathfrak{b}_s \end{cases}.$$

This map is  $(x, sx)$ -isometric on  $\mathfrak{B}$  in the sense that

$$(V_s \kappa' | V_s \kappa)_{sx} = \sum_{a, b \in \mathfrak{b}} \kappa'_a l_{sx}(a_s \star b_s) \kappa_b^* = \sum_{a, b \in \mathfrak{b}} \kappa'_a l_x(a \star b) \kappa_b^* = (\kappa' | \kappa)_x.$$

The pseudo-adjoint  $V_s^*$  is well defined as a surjection from  $\mathfrak{B}_s = \mathbb{C}\mathfrak{b}_s$  onto  $\mathfrak{B}$  so that every  $\kappa' \in \mathfrak{B}$  can be written as  $V_s^* \kappa''$  for some  $\kappa'' \in \mathfrak{B}_s$ . Hence  $V_s$  maps the ideals  $\mathfrak{R}_x$  and  $\mathfrak{R}_x^*$  to  $\mathfrak{R}_{sx}$  and  $\mathfrak{R}_{sx}^*$  respectively since  $(\kappa' | V_s \kappa)_{sx} = (\kappa'' | \kappa)_x = 0$  for  $\kappa \in \mathfrak{R}_x$  or  $\mathfrak{R}_x^*$ . This gives for each  $x$  a well defined linear isometry  $V_s: \mathfrak{B}/\mathfrak{R}_x^* \rightarrow \mathfrak{B}/\mathfrak{R}_{sx}^*$ , denoted by  $\mathbf{V}_s$ , and acting on columns  $\mathbf{k}$  given by the components

$$\begin{aligned} (\mathbf{V}_s \mathbf{k})_x^- &= (V_s \kappa)_x^- = (\delta_u | V_s \kappa)_{sx} = \sum_b l_{sx}(b_s^*) \kappa_b^* = \kappa_x^-, \\ (\mathbf{V}_s \mathbf{k})^\circ &= (V_s \kappa)^\circ = V_s \kappa^\circ, \\ (\mathbf{V}_s \mathbf{k})^+ &= (V_s \kappa)^+ = \sum_{b \in \mathfrak{b}_s} \kappa_{b^s} = \kappa^+. \end{aligned}$$

Hence we may write the action of the semigroup  $S$  on  $\mathbb{K}_x$  as

$$\mathbf{V}_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & V_s & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and extend it to  $\mathbb{K}_\Delta$  as the direct integral. Clearly it leaves  $\mathbf{e}$  invariant. Finally, we have for any  $\kappa \in \mathfrak{B}$ ,

$$(\delta_{b_s} | V_s \kappa)_{sx} = (\delta_b | \kappa)_x, \quad V_s \delta_{b^*} = \delta_{b_s^*}, \quad V_s (\delta_b \cdot \kappa) = \delta_{b_s} \cdot V_s \kappa$$

where the first equality follows from the covariance of  $l_x$ , the second simply from the definition of  $V_s$ , and the third from the surjectivity of  $b \mapsto b^s$ . This shows that  $\mathbf{V}_{s, \mathbf{j}_x}(b) = \mathbf{j}_{sx}(b_s) \mathbf{V}_s$  and hence the required relation for integrable subsets  $\Delta$ .

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