

On the Brauer groups of quasilocal fields and the norm groups of their finite Galois extensions*

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Abstract. This paper shows that divisible abelian torsion groups are realizable as Brauer groups of quasilocal fields. It describes the isomorphism classes of Brauer groups of primarily quasilocal fields and solves the analogous problem concerning the reduced components of the Brauer groups of two basic types of Henselian valued absolutely stable fields. For a quasilocal field E and a finite separable extension R/E , we find two sufficient conditions for validity of the norm group equality $N(R/E) = N(R_0/E)$, where R_0 is the maximal abelian extension of E in R . This is used for deriving information on the arising specific relations between Galois groups and norm groups of finite Galois extensions of E .

Key words: Quasilocal field; Brauer group; Character group; Corestriction; Galois extension; Norm group; Abelian closed class; Transfer; Brauer-Severi variety.

1. Introduction and statements of the main results

This paper is devoted to the study of norm groups and Brauer groups of the fields pointed out in the title, i.e. of fields whose finite extensions are primarily quasilocal (abbr., PQL). Our main result describes, up-to an isomorphism, the abelian groups that can be realized as Brauer groups of several basic types of PQL-fields (see Theorem 1.2, Propositions 2.3 (ii), 3.4 and Section 6). For a quasilocal field E and a finite separable extension R/E , it gives two sufficient conditions that the norm group $N(R/E)$ coincides with $N(R/E)_{\text{Ab}}$, the norm group of the maximal abelian extension of E in R (see Theorem 1.1). When the field E is nonreal, this allows us to clarify essential algebraic and topological aspects of the behaviour of norm groups of finite Galois extensions of E .

The basic notions needed to present this research are the same as those in [7]; the reader is referred to [18; 21; 27; 31 and 15], for any missing definitions concerning simple algebras, Brauer groups, field extensions, Galois cohomology and abelian groups. Simple algebras are supposed to be associative with a unit and finite-dimensional over their centres, and Galois groups are viewed as profinite with respect to the Krull topology. For a central simple algebra A over a field E , we write $[A]$ for the similarity class of A in the Brauer group $\text{Br}(E)$. As usual, E^* denotes the multiplicative group of E , E_{sep} a separable closure of E , $G_E = G(E_{\text{sep}}/E)$ is the absolute Galois group of E , C_E stands for the character group of G_E , and $d(E)$ is the class of central division E -algebras. For an arbitrary field

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extension Λ/E , $\text{Br}(\Lambda/E)$ is its relative Brauer group, $\rho_{E/\Lambda}$ is the scalar extension map, $\text{Im}(E/\Lambda)$ is the image of $\rho_{E/\Lambda}$, and $I(\Lambda/E)$ is the set of intermediate fields of Λ/E . We write $\text{Gal}(E)$ for the set of finite Galois extensions of E in E_{sep} , and put $\Omega(E) = \{M \in \text{Gal}(E): G(M/E) \in \text{Ab}\}$. Throughout, \overline{P} is the set of prime numbers and $\Pi(E)$ consists of those $p \in \overline{P}$, for which G_E is of nonzero cohomological p -dimension $\text{cd}_p(G_E)$. For each $p \in \overline{P}$, $E(p)$ is the maximal p -extension of E in E_{sep} , and $\Omega_p(E) = \{Y \in \Omega(E): Y \subseteq E(p)\}$. When Ψ is a nonempty formation of finite groups in the sense of [34], E_Ψ denotes the compositum of all fields $M \in \text{Gal}(E)$ with $G(M/E) \in \Psi$; in view of Galois theory and the choice of Ψ , E_Ψ is the union of these M . For a finite extension R of E in E_{sep} , we put $N(R/E)_\Psi = N(E_\Psi \cap R/E)$. The formations of abelian, metabelian, nilpotent, solvable, and of all finite groups are denoted by Ab , Met , Nil , Sol and Fin , respectively. A class $\chi \subseteq \text{Fin}$ is called abelian closed, if it is nonempty and closed with respect to taking subgroups, homomorphic images, finite direct products, and group extensions with abelian kernels (a series of typical examples of such classes is given in Remark 6.1). We say that E is formally real, if -1 is not presentable over E as a finite sum of squares; E is called nonreal, otherwise. The field E is said to be PQL, if every cyclic extension F of E embeds as an E -subalgebra in each $D \in d(E)$ of Schur index $\text{ind}(D)$ divisible by the degree $[F:E]$. We say that E is strictly PQL, if it is PQL and the p -component $\text{Br}(E)_p$ of $\text{Br}(E)$ is nontrivial, when p runs through the set $P(E)$ of those elements of \overline{P} , for which $E(p) \neq E$.

Singled out in the process of characterizing basic types of stable fields with Henselian valuations (see [7] and the references there), PQL-fields E deserve interest in their own right because of the arising close relations between the fields in $\Omega(E)$, their norm groups and central simple E -algebras. Firstly, it should be pointed out that strictly PQL-fields admit one-dimensional local class field theory (abbr. LCFT, see (2.1)) and the converse holds in all presently known cases (cf. [8, Theorem 1.1, Remark 4.4 and Sect. 3]). Note also that the field E is strictly quasilocal (SQL), i.e. its finite extensions are strictly PQL, if and only if they admit LCFT [8, Proposition 3.6]. Secondly, this research is motivated by the dependence of some of these relations on the structure of $\text{Br}(E)$ (see [8, Theorems 1.2 and 1.3]), and therefore, by the problem of describing the isomorphism classes of Brauer groups over the main kinds of PQL-fields. It is worth adding that the quasilocal property singles out one of the basic classes of absolutely stable fields (in the sense of Brussel, see [7, I, Proposition 2.3]), and the structure of $\text{Br}(F)$, for an arbitrary absolutely stable field F , is of interest for the theory of central simple algebras in general (see [27, Sects. 14.4 and 19.6]). The choice of our main topic is determined by the fact that the groups $N(M/E): M \in \text{Gal}(E) \setminus \Omega(E)$, reflect more aspects of the specific nature of E than merely the influence of the PQL-property. Our starting point are the following analogues to the norm limitation theorem about local fields (see [17, Ch. 6, Theorem 8]):

(1.1) (i) $N(R/E) = N(R/E)_{\text{Ab}}$, provided that R is a finite separable extension of a field E with LCFT in the sense of Neukirch-Perlis [26], i.e. if the triple $(G_E, \{G(E_{\text{sep}}/F), F \in \text{Fe}(E)\}, E_{\text{sep}}^*)$ is an Artin-Tate class formation (cf. [2, Ch. XIV]), where $\text{Fe}(E)$ is the set of finite extensions of E in E_{sep} ;

(ii) $N(R/E) = N(R/E)_{\text{Ab}}$, if E is PQL and $R \subseteq E_{\text{Nil}}$ [5].

It is known (cf. [26]) that a field E admits LCFT in the sense of Neukirch-Perlis if and

only if it is SQL, $\text{Br}(E)$ embeds in the quotient group \mathbb{Q}/\mathbb{Z} of the additive group of rational numbers by the subgroup of integers, and $\rho_{E/F}: \text{Br}(E) \rightarrow \text{Br}(F)$ is surjective, for every finite extension F of E in E_{sep} . This holds when E has a Henselian discrete valuation with a quasifinite residue field \widehat{E} (see, e.g., [41]). The basis for the present discussion is also formed by the characterization of the PQL-property in the class of algebraic extensions of global fields, which yields the following (see [5, Sects. 1 and 2] and the references there):

(1.2) (i) For each $G \in \text{Fin} \setminus \text{Nil}$, there exist algebraic extensions $E(G)$ and $M(G)$ of the field \mathbb{Q} of rational numbers, such that $E(G)$ is strictly PQL, $M(G) \in \text{Gal}(E(G))$, $G(M(G)/E(G)) \cong G$ and $N(M(G)/E(G)) \neq N(M(G)/E(G))_{\text{Ab}}$;

(ii) If E is an algebraic PQL-extension of a global field E_0 , then $\text{Br}(E)$ embeds in \mathbb{Q}/\mathbb{Z} . Moreover, if R/E is a finite extension, then $N(R/E) = N(\Sigma/E)$, for some $\Sigma \in \Omega(E)$; when E is strictly PQL, Σ is uniquely determined by R/E .

The purpose of this paper is to present two main results which shed an additional light on (1.1) and (1.2), and solve the above-noted problem for Brauer groups of nonreal PQL-fields. The first result is stated as follows:

Theorem 1.1. *Let E be a quasilocal field and R a finite extension of E in E_{sep} . Then $N(R/E) = N(R/E)_{\text{Ab}}$ in the following two cases:*

- (i) *The map $\rho_{E/M}$ is surjective, for some $M \in \text{Gal}(E)$ including R ;*
- (ii) *There exists a field $\Phi(R) \in \Omega(E)$, such that $N(\Phi(R)/E) \subseteq N(R/E)$.*

Theorem 1.1 is deduced in Section 3 from its p -primary analogue stated as Theorem 3.1. This analogue enables us to generalize Theorem 1.1 (i) by proving at the end of Section 3 that $N(R/E) = N(R/E)_{\text{Ab}}$, provided that E is quasilocal and $R \in I(M/E)$, for some $M \in \text{Gal}(E)$ with $\rho_{E/L}$ surjective, where L is the fixed field of the Fitting subgroup of $G(M/E)$. In this setting, it may occur that $\rho_{E/\Phi}$ is not surjective, for any $\Phi \in \text{Gal}(E)$ including R (see the comment preceding Proposition 6.3). Theorem 3.1 has been used in [6] for describing the norm groups of finite separable extensions of SQL-fields with Henselian discrete valuations. Like the description of the norm groups of formally real quasilocal fields, obtained in [9], this yields a generally nonabelian LCFT. Our second main result, combined with [7, I, Theorem 3.1 (ii) and Lemma 3.5], shows that an abelian torsion group T is isomorphic to $\text{Br}(E)$, for some nonreal PQL-field E , if and only if T is divisible (for the formally real case, see Propositions 6.4 (i) and 3.4); this specifies observations made at the end of [37, Sect. 3]. When T is divisible, it states that E can be found among quasilocal fields so as to solve one of the main inverse problems related to (1.1) and (1.2).

Theorem 1.2. *Let E_0 be a field, T a divisible abelian torsion group, T_0 a subgroup of $\text{Br}(E_0)$ embeddable in T , and let χ and χ' be subclasses of Fin , such that $\text{Nil} \subseteq \chi \subseteq \chi'$. Assume also that any class χ, χ' is abelian closed unless it equals Nil . Then there exists a quasilocal and nonreal extension $E = E(T)$ of E_0 with the following properties:*

- (i) *$\text{Br}(E) \cong T$, E_0 is separably closed in E , ρ_{E/E_0} maps T_0 injectively into $\text{Br}(E)$, and each $G \in \text{Fin}$ is realizable as a Galois group over E ;*
- (ii) *For each finite extension R of E in E_χ , $N(R/E) = N(R/E)_{\text{Ab}}$; moreover, if $\chi \neq \text{Nil}$, then $\rho_{E/R}$ is surjective;*
- (iii) *$N(M/E) \neq N(M/E)_{\text{Ab}}$, for every $M \in \text{Gal}(E)$ with $G(M/E) \notin \chi'$;*

(iv) If $G \in \chi' \setminus \chi$, then there are $M_1, M_2 \in \text{Gal}(E)$, such that $G(M_j/E) \cong G$, $j = 1, 2$, $N(M_1/E) = N(M_1/E)_{\text{Ab}}$ and $N(M_2/E) \neq N(M_2/E)_{\text{Ab}}$.

The assertions of Theorem 1.2 (i)-(ii) in the case of $T_0 = \text{Br}(E_0)$, combined with [7, I, Corollary 8.5] and the behaviour of Schur indices under scalar extensions of finite degrees (cf. [27, Sect. 13.4]), imply [37, Theorems 3.7÷3.9]. Since n -dimensional F -algebras embed in the matrix F -algebra $M_n(F)$, for any field F and $n \in \mathbb{N}$, these assertions and well-known properties of tensor products (see [27, Sects. 9.3 and 9.4]) also enable one to deduce [37, Theorem 3.10] from the Skolem-Noether theorem (as in the proof of the double centralizer theorem, for example, in [27, Sect. 12.7]). Thus the noted part of Theorem 1.2 simplifies the proofs of [37, Theorems 2.6 and 2.8]. When $\chi = \text{Fin}$, it admits a Galois cohomological interpretation (see Remark 5.4 and [7, I, Theorem 8.1]) and yields the following result:

(1.3) For each divisible abelian torsion group T , there is a quasilocal field E , such that $\text{Br}(E) \cong T$, all $G \in \text{Fin}$ can be realized as Galois groups over E , and $N(R/E) = N(R/E)_{\text{Ab}}$, for every finite extension R of E in E_{sep} .

When $\chi = \text{Nil}$ or χ is abelian closed with $\text{Nil} \subset \chi \neq \text{Fin}$, the conclusions of Theorem 1.2 in the special cases of $\chi' = \chi$ and $\chi' = \text{Fin}$ amount essentially to the following:

(1.4) For each divisible abelian torsion group T , there exist quasilocal fields E_1 and E_2 with $\text{Br}(E_i) \cong T$, $i = 1, 2$, and such that:

(i) All $G \in \text{Fin}$ are realizable as Galois groups over E_1 and E_2 , and whenever $M_1 \in \text{Gal}(E_1)$, $N(M_1/E_1) = N(M_1/E_1)_{\text{Ab}}$ if and only if $G(M_1/E_1) \in \chi$.

(ii) $N(M_2/E_2) = N(M_2/E_2)_{\text{Ab}}$, provided that $M_2 \in \text{Gal}(E_2)$ and $G(M_2/E_2) \in \chi$. For each $G \in \text{Fin} \setminus \chi$, $\text{Gal}(E_2)$ contains elements $M(G)_1$ and $M(G)_2$ with $G(M(G)_j/E_2) \cong G$, $j = 1, 2$, $N(M(G)_1/E_2) = N(M(G)_1/E_2)_{\text{Ab}}$ and $N(M(G)_2/E_2) \neq N(M(G)_2/E_2)_{\text{Ab}}$.

Theorem 1.1 and statements (1.1) (ii), (1.2) (i)-(ii), (1.3) and (1.4) mark the limit behaviour of norm groups of finite Galois extensions of PQL-fields. By [8, (2.3)], the fields singled out by (1.3) and (1.4) have no Henselian valuations with indivisible value groups. Note also that if (F, v) is a Henselian discrete valued SQL-field, then G_F is prosolvable of special type (see Corollary 6.7 and [8, (2.1) and the comments to (2.4) (ii)]). These facts and the topological interpretation of Theorem 1.2 in Section 6 allow one to appreciate from an algebraic point of view the Neukirch-Perlis generalization of LCFT, and without artificial limitations, to incorporate it in the study of quasilocal fields and other areas.

Here is an overview of the paper: Section 2 includes preliminaries needed in the sequel, such as statements of frequently used projection formulae relating the corestrictions of Brauer and character groups of an arbitrary finite separable extension. The proofs of these formulae (and of Proposition 2.8) given in [10] as well as of Propositions 4.1 and 6.8 show that Theorem 1.2 (i)-(ii), applied $T_0 = \text{Br}(E_0)$ and $\chi = \text{Fin}$, provides useful tools for the study of various aspects of Brauer group theory on a unified basis. Theorems 1.1 and 1.2 are proved in Sections 3 and 5, respectively. The technical preparation for the proof of Theorem 1.2 is made in Section 4 and its results seem to be of independent interest. As an application of Theorem 1.2, we describe in Section 6 the isomorphism classes of Brauer groups of formally real PQL and of strictly PQL-fields, and do the same for the reduced parts of the Brauer groups of two basic types of Henselian valued absolutely stable fields.

2. Preliminaries on norm groups, p -quasilocal fields and corestrictions of Brauer and character groups

(2.1) Let E be a field and $\text{Nr}(E)$ the set of norm groups of its finite extensions in E_{sep} . We say that E admits LCFT, if the mapping $\pi: \Omega(E) \rightarrow \text{Nr}(E)$, by the rule $\pi(F) = N(F/E): F \in \Omega(E)$, is injective, and whenever $M_1, M_2 \in \Omega(E)$, $N(M_1M_2/E) = N(M_1/E) \cap N(M_2/E)$ and $N(M_1 \cap M_2/E) = N(M_1/E)N(M_2/E)$ (as usual, M_1M_2 is the compositum of M_1 and M_2). We call E a field with local p -class field theory (local p -CFT), for some $p \in \overline{P}$, if the restriction of π on the set $\Omega_p(E)$ has the same properties.

The following lemma (proved, e.g., in [5]) implies that a field E admits LCFT if and only if it admits local p -CFT, for every $p \in P(E)$. When E is of this kind, [7, I, Lemma 4.2 (ii)] shows that $\text{Br}(E)_p \neq \{0\}$, $p \in P(E)$.

Lemma 2.1. *Let E, R and M be fields, such that $R \in I(M/E)$, $R \neq E$, $M \in \text{Gal}(E)$ and $G(M/E) \in \text{Nil}$. Let $P(R/E)$ be the set of prime divisors of $[R:E]$, and $R_p = R \cap E(p)$, for each $p \in P(R/E)$. Then R equals the compositum of the fields R_p : $p \in P(R/E)$, $[R:E] = \prod_{p \in P(R/E)} [R_p:E]$, $N(R/E) = \cap_{p \in P(R/E)} N(R_p/E)$ and the quotient group $E^*/N(R/E)$ is isomorphic to the direct group product $\prod_{p \in P(R/E)} E^*/N(R_p/E)$.*

Henceforth, $\text{Syl}_p(M/E)$ denotes the set of Sylow p -subgroups of $G(M/E)$, for any $M \in \text{Gal}(E)$, $p \in \overline{P}$. For the proof of the following lemma, we refer the reader to [7, II].

Lemma 2.2. *Let E and M be fields, $M \in \text{Gal}(E)$ and $P(M/E) = \{p \in \overline{P}: p|[M:E]\}$. Then $N(M/E) \subseteq N(M/F)$, for every $F \in I(M/E)$. Moreover, if E_p is the fixed field of a group $G_p \in \text{Syl}_p(M/E)$, then $N(M/E) = \cap_{p \in P(M/E)} N(M/E_p)$.*

The main results of [7, I] used in the present paper (supplemented by a well-known result on orderings in Pythagorean fields), can be stated as follows:

Proposition 2.3. *Assume that E is a p -quasilocal field, for some $p \in P(E)$, R is a finite extension of E in $E(p)$, and $D \in d(E)$ is an algebra of p -primary index. Then:*

- (i) R is a p -quasilocal field and $\text{ind}(D) = \text{exp}(D)$.
- (ii) $\text{Br}(R)_p$ is a divisible group unless $p = 2$, $R = E$ and E is formally real. In the formally real case, $E(2) = E(\sqrt{-1})$, $\text{Br}(E)_2$ is of order 2 and $\text{Br}(E(\sqrt{-1}))_2 = \{0\}$; this occurs if and only if E is Pythagorean with a unique ordering.
- (iii) $\rho_{E/R}$ maps $\text{Br}(E)_p$ surjectively on $\text{Br}(R)_p$, i.e. $\text{Br}(R)_p \subseteq \text{Im}(E/R)$.
- (iv) R embeds in D as an E -subalgebra if and only if $[R:E]|\text{ind}(D)$.

The following lemma provides an easy method of constructing p -quasilocal fields. Before stating it, recall that a field extension F/F_0 is said to be regular, if F_0 is separably closed in F and $I(F/F_0)$ contains an element F'_0 , such that F'_0/F_0 is rational (i.e. purely transcendental) and F/F'_0 is separable. It is known that the tensor product $\otimes_{F_0} \Lambda_i$, $i \in I$, of such extensions is a domain with a fraction field $F(I)$ regular over F_0 . We call $\Lambda(I)$ a tensor compositum of the fields Λ_i over F_0 , and write $\Lambda(I) = \tilde{\otimes}_{F_0} \Lambda_i$. Recall further that the class $\text{Reg}(F_0)$ of regular extensions of F_0 contains the function fields of the F_0 -varieties (i.e. algebraic varieties defined over F_0 and irreducible over $F_{0,\text{sep}}$) considered in this paper. In what follows, we shall use without an explicit reference the well-known facts

that $F_1 \otimes_{F_0} \Lambda \in \text{Reg}(F_1)$ whenever $\Lambda \in \text{Reg}(F_0)$ and F_1 is a finite extension of F_0 in $F_{0,\text{sep}}$, the image of $\text{Reg}(F_1)$ under the transfer map Tr_{F_1/F_0} (over F_1/F_0) is included in $\text{Reg}(F_0)$, and the compositions $\tilde{\otimes}_{F_0} \circ \text{Tr}_{F_1/F_0}$ and $\text{Tr}_{F_1/F_0} \circ \tilde{\otimes}_{F_1}$ coincide. These are easily obtained from Galois theory and the definition of Tr_{F_1/F_0} (see the beginning of [30, Sect. 3]).

Lemma 2.4. *Let F_0 be a field and $p \in \overline{P}$. Then there exists a field extension F/F_0 , such that F is p -quasilocal, F_0 is algebraically closed in F and $\text{Br}(F/F_0) = \{0\}$.*

Proof. Using [14, Theorem 1], one constructs F as a union $\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} B'_i$ of fields defined inductively as follows:

(2.2) (i) B_0 is a rational function field in one indeterminate over F_0 and $B'_0 = B_0$ (since $p \in P(B_0)$, this implies that $\text{Br}(B'_0)_p \neq \{0\}$).

(ii) For each $i \in \mathbb{N}$, $B_i/B'_{(i-1)}$ is a rational extension with a transcendence basis (abbr, tr-basis) $\{X_{(r_i, c_i)}\}$ indexed by the Cartesian product $R_{p,i} \times C_{p,i}$, where $R_{p,i}$ is a system of representatives of the isomorphism classes of algebras in $d(B'_{(i-1)})$ of index p , and $C_{p,i}$ is the set of extensions of $B'_{(i-1)}$ in $B'_{(i-1)}(p)$ of degree p .

(iii) For any $i \in \mathbb{N}$, let $F(r_i, c_i)$ be the function field of the Brauer-Severi B_i -variety canonically associated with the central simple B_i -algebra $(r_i \otimes_{B'_{(i-1)}} B_i) \otimes ((c_i \otimes_{B'_{(i-1)}} B_i)/B_i, \tilde{c}_i, X_{(r_i, c_i)}^{-1})$, for each $(r_i, c_i) \in R_{p,i} \times C_{p,i}$, \tilde{c}_i being a generator of $G((c_i \otimes_{B'_{(i-1)}} B_i)/B_i)$. Then $B'_i = \tilde{\otimes}_{B_i} F(r_i, c_i)$, $(r_i, c_i) \in R_{p,i} \times C_{p,i}$; in particular, $B'_i \in \text{Reg}(B'_{(i-1)})$.

Throughout this paper, $\text{Cor}_{F/E}$ denotes the corestriction homomorphism of $\text{Br}(F)$ into $\text{Br}(E)$, and $\text{Ker}(F/E)$ stands for the kernel of $\text{Cor}_{F/E}$, for any finite separable extension F/E . The first part of the following statement, complemented by Proposition 3.4, gives evidence of close relations between $\text{Cor}_{F/E}$ and quasilocal nonreal fields:

(2.3) (i) E is p -quasilocal if and only if $\text{Cor}_{R/E}$ maps $\text{Br}(R)_p$ injectively into $\text{Br}(E)_p$ (i.e. $\text{Br}(R)_p \cap \text{Ker}(R/E) = \{0\}$), for each finite extension R of E in $E(p)$ [10, (1.1) (i)];

(ii) If E is p -quasilocal and R is a finite extension of E in $E(p)$, then:

(α) E admits local p -CFT, provided that $\text{Br}(E)_p \neq \{0\}$ [8, Theorem 3.1]; in particular, $N(R/E) \neq E^*$ unless $R = E$;

(β) $N(R/E) = N(R/E)_{\text{Ab}}$ [5, Theorem 3.1];

(γ) $N(R/E) = E^*$ in case $\text{Br}(E)_p = \{0\}$ [7, I, Lemma 4.2].

Statement (2.3) (i), Proposition 3.4 and the noted proofs of (2.3) (ii) help us observe the possibility to apply Theorem 1.2 (i) and the second assertion of Theorem 1.2 (ii) to the study of $\text{Cor}_{F/E}$. This is demonstrated by the alternative proofs in [10] of two known projection formulae (see [33, page 205]). We first state a special case of the first projection formula, which is particularly easy to apply.

Proposition 2.5. *Let E, F and M be fields with $M \cap F = E$, $F \subseteq E_{\text{sep}}$ and M cyclic over E , and let $\tilde{\sigma}$ be a generator of $G(M/E)$. Then MF/F is cyclic, $\tilde{\sigma}$ extends uniquely to an F -automorphism σ of MF , $G(MF/F) = \langle \sigma \rangle$ and $\text{Cor}_{F/E}$ maps the similarity class of the cyclic F -algebra $(MF/F, \sigma, \lambda)$ into $[(M/E, \tilde{\sigma}, N_E^F(\lambda))]$, for each $\lambda \in F^*$.*

Let now E be a field and F a finite extension of E in E_{sep} , $r_{E/F}$ the restriction homomorphism $C_E \rightarrow C_F$, and $\text{cor}_{F/E}$ the corestriction map $C_F \rightarrow C_E$. It is known (cf. [18, Ch.

7, Corollary 5.3]) that C_F is an abelian torsion group and for each $p \in \overline{P}$, its p -component can be identified with the character group $C(F(p)/F)$ of $G(F(p)/F)$. Recall that for each $\chi \in C_F$, the fixed field L_χ of the kernel $\text{Ker}(\chi)$ is cyclic over F ; we denote by σ_χ the generator of $G(L_\chi/F)$ induced by any $\bar{\sigma}_\chi \in G_E$ satisfying the equality $\chi(\bar{\sigma}_\chi) = (1/[L_\chi:F]) + \mathbb{Z}$. Note that $L_{r_{E/F}(\chi)} = L_\chi F$, $\sigma_{r_{E/F}(\chi)}$ is the unique $(L_\chi \cap F)$ -automorphism of $L_\chi F$ extending $\sigma^{d(\chi)}$, and $\rho_{E/(L_\chi \cap F)}$ maps $\text{Br}(L_\chi/E)$ on the set $\{[(L_\chi/(L_\chi \cap F), \sigma^{d(\chi)}, c)]: c \in E^*\}$, where $d(\chi) = [L_\chi \cap F:E]$. These observations enable one to deduce from Proposition 2.5 the first projection formula in general (see the proof of [10, (3.1)]):

$$(2.4) \quad \text{Cor}_{F/E}([(L_\chi F/F, \sigma_{r_{E/F}(\chi)}, \lambda)]) = [(L_\chi/E, \sigma_\chi, N_E^F(\lambda))], \lambda \in F^*.$$

The second projection formula is contained in the following result, which is used for proving Theorem 3.1:

Proposition 2.6. *Let E be a field, F a finite extension of E in E_{sep} , c and χ elements of E^* and C_F , respectively. Then $\text{Cor}_{F/E}([(L_\chi/F, \sigma_\chi, c)]) = [(L_{\tilde{\chi}}/E, \sigma_{\tilde{\chi}}, c)]$, where $\tilde{\chi} = \text{cor}_{F/E}(\chi)$. Also, $L_{\tilde{\chi}} \subseteq M$, provided that $M \in \text{Gal}(E)$ and $L_\chi \subseteq M$.*

The proof of Proposition 2.6 in [10] is based not only on Theorem 1.2 (i) and the second part of Theorem 1.2 (ii), applied to $T_0 = \text{Br}(E_0)$ and $\chi = \text{Fin}$. It also relies on Proposition 2.5, statements (2.3) (ii) (α), (β) and the fact (see [7, II, Lemma 2.3]) that if F is p -quasilocal with a primitive p -th root of unity, for some $p \in P(F)$, then the structure of $C(F(p)/F)$ is determined by the group ${}_p\text{Br}(F) = \{b \in \text{Br}(F): pb = 0\}$ and the group $R_p(F)$ of roots of unity in F of p -primary degrees, as follows:

(2.5) $C(F(p)/F)$ is divisible if and only if $\text{Br}(F)_p = \{0\}$ or $R_p(F)$ is infinite. If $\text{Br}(F)_p \neq \{0\}$, B_p is a basis of ${}_p\text{Br}(F)$ as a vector space over the field \mathbb{F}_p with p elements, and $R_p(F)$ is of finite order p^μ , then the group $p^\mu C(F(p)/F)$ is divisible and $C(F(p)/F)$ is isomorphic to the direct sum $p^\mu C(F(p)/F) \oplus R(F(p)/F)$, where $R(F(p)/F)$ is a subgroup of $C(F(p)/F)$ presentable as a direct sum of cyclic groups of order p^μ , indexed by B_p .

Remark 2.7. Let E be a field and F/E a finite separable extension. Suppose also that E contains a primitive n -th root of unity, for some $n \in \mathbb{N}$, or $\text{char}(E) = p > 0$. Applying Kummer theory and its analogue obtained by Witt (see [21, Ch. VIII, Sect. 8] and [18, Ch. 7, 1.9 and 2.9]), one deduces from Proposition 2.6 the projection formula for symbol F -algebras of indices dividing n , and the one for p^μ -symbol F -algebras, $\mu \in \mathbb{N}$, contained in [35, Theorem 3.2] and [22, Proposition 3 (i)], respectively.

Let us mention that (2.4) and Propositions 2.5 and 2.6 can also be proved by applying group-cohomological technique (see [40, Proposition 4.3.7] and [18, Ch. 7, Corollary 5.3]). Without comparing the approach referred to with the one followed in [10], note that the latter bears an entirely field-theoretic character both technically and conceptually. As shown in [10, Sect. 2] and Section 5, our approach also allows us to prove Theorem 1.2 (i) and the concluding assertion of Theorem 1.2 (ii) together with the following result.

Proposition 2.8. *Let F be a field, M/F a finite separable extension, Λ' a tensor compositum over M of function fields of Brauer-Severi M -varieties, and Λ is the transfer of Λ' over M/F . Then $\text{Br}(\Lambda/F)$ equals the image of $\text{Br}(\Lambda'/M)$ under $\text{Cor}_{M/F}$.*

Proposition 2.8 is a special case of [13, Proposition 2.6] which has been deduced in [13] from [30, Theorem 3.13] and the description of the relative Brauer groups of function fields of generalized Brauer-Severi varieties [3]. In view of the relations between quasilocal nonreal fields and Brauer group corestrictions, and of the preservation of rationality under transfer (see [30, Lemma 3.2 (a)]), one may expect that Theorem 1.2 can be used for simplifying the proofs and the presentations of index reduction formulae, for the function fields of a number of twisted rational varieties like those considered in [30; 24; 23] and [37].

3. p -primary analogue to Theorem 1.1

Let E be a field, R a finite extension of E in E_{sep} , and $H(E)^n = \{h^n: h \in H(E)\}$, for any subgroup $H(E)$ of E^* and each $n \in \mathbb{N}$. For each $p \in \overline{P}$, let $R_{\text{Ab},p} = R \cap E_{\text{Ab}} \cap E(p)$, ρ_p be the greatest divisor of $[R:E]$, and $N_p(R/E) = \{u_p \in E^*: \text{the co-set } u_p N(R/E) \text{ is a } p\text{-element of } E^*/N(R/E)\}$. Clearly, $E^{*\rho_p} \subseteq N_p(R/E)$ and $N(R/E)_{\text{Ab}}^{\rho_p} \subseteq N(R_{\text{Ab},p}/E)$, $p \in \overline{P}$, so Theorem 1.1 can be deduced from the following result (in the case of $\Omega = E_{\text{sep}}$):

Theorem 3.1. *Let E be a field, Ω a Galois extension of E in E_{sep} , and R a finite subextension of E in Ω . Assume that finite extensions of E in Ω are p -quasilocal, for some $p \in \overline{P}$. Then $N(R/E) = N(R_{\text{Ab},p}/E) \cap N_p(R/E)$ in the following cases:*

- (i) *There exists $M \in \text{Gal}(E) \cap I(\Omega/E)$ with $R \in I(M/E)$ and $\text{Br}(M)_p \subseteq \text{Im}(E/M)$;*
- (ii) *$N(R/E)$ includes $N(\Phi(R)/E) \cap N_p(R/E)$, for some $\Phi(R) \in \Omega_p(E) \cap I(\Omega/E)$.*

Proof. The inclusion $N(R/E) \subseteq N(R_{\text{Ab},p}/E) \cap N_p(R/E)$ is obvious. We prove the converse by showing that $p \nmid e_p$, where e_p is the exponent of the $N(R_{\text{Ab},p}/E)/N(R/E)$. Theorem 3.1 (ii) is obtained as a special case of the following lemma.

Lemma 3.2. *Let E , Ω , R and p satisfy the conditions of Theorem 3.1, and let $M \in \text{Gal}(E) \cap I(\Omega/E)$. Then $N_p(M/E)N_p(R/E) = N_p(M \cap R/E)$.*

Proof. It is clearly sufficient to show that p does not divide the exponent e of $L^*/N(M/L)N(R/L)$, where $L = M \cap R$. Hence, by the p -quasilocal property of finite extensions of L , one may assume further that $L = E$. Let \tilde{R} be the normal closure of R in E_{sep} over E , $\tilde{H}_p \in \text{Syl}_p(\tilde{R}/R)$, $\tilde{G}_p \in \text{Syl}_p(\tilde{R}/E)$, $\tilde{H}_p \subseteq \tilde{G}_p$ and $G_p \in \text{Syl}_p(M/E)$. Denote by R_p , Φ_p and E_p the fixed fields of \tilde{H}_p , \tilde{G}_p and G_p , respectively. We prove that $e \mid \varphi_p$, i.e. $E^{*\varphi_p} \subseteq N(M/E)N(R/E)$, where $\varphi_p = [\Phi_p:E][E_p:E]$; this implies the lemma, since $p \nmid \varphi_p$. Let ξ be a primitive element of R/E . By Galois theory and the equality $L = E$, $[MR:R] = [M:E]$, i.e. $[M(\xi):M] = [R:E]$. This means that the minimal polynomial of ξ over E is irreducible over M . Considering E_p , RE_p and M instead of E , R and M , respectively, and using Lemma 2.2, one reduces our proof to the special case where $E_p = E$, i.e. $M \subseteq E(p)$. The choice of R_p guarantees that $p \nmid [R_p:R]$, so it follows from Galois theory, the equality $[M:E] = [MR:R]$ and the inclusion $M \subseteq E(p)$ that $[MR_p:R_p] = [M:E]$ and $R_p \cap MF = F$, for every $F \in I(R_p/E)$. This, applied to the case of $F = \Phi_p$, enables one to deduce from (2.3) (ii) (α) and (γ) that $N(M\Phi_p/\Phi_p)N(R_p/\Phi_p) = \Phi_p^*$. Now the inclusion $E^{*\varphi_p} \subseteq N(M/E)N(R/E)$ becomes obvious, so Lemma 3.2 is proved.

Remark 3.3. Statements (1.2) (i) and (ii) show that the conclusions of Theorem 1.1 (ii) and Lemma 3.2 are not always true, if E is only a PQL-field.

Our next result characterizes the fields whose finite extensions are p -quasilocal, for a given $p \in \overline{P}$. It simplifies the proof of Theorem 3.1 (i) and leads to the idea of constructing quasilocal nonreal fields and formally real PQL-fields by the method followed in this paper.

Proposition 3.4. *Let E be a field and Ω a Galois extension of E in E_{sep} . Then finite extensions of E in Ω are p -quasilocal, for a given $p \in \overline{P}$, if and only if one of the following two conditions is fulfilled:*

(c) $p > 2$ or E is nonreal, and for each pair $(M, M') \in \text{Gal}(E) \times \text{Gal}(E)$ with $M \subseteq \Omega$ and $M' \in I(M(p)/M)$, $\text{Cor}_{M'/M}$ maps $\text{Br}(M')_p$ injectively into $\text{Br}(M)_p$;

(cc) $p = 2$, E is formally real and its formally real finite extensions in Ω are uniquely ordered Pythagorean fields; when $\Omega \neq E$, this holds if and only if $2 \notin P(\Omega)$, $G(\Omega/E(\sqrt{-1}))$ is abelian, $\text{cd}_2(G(\Omega/E(\sqrt{-1}))) = 0$ and $G(\Omega/E)$ is continuously isomorphic to the semidirect product $G(\Omega/E(\sqrt{-1})) \times \langle \sigma \rangle$, where $\sigma^2 = 1$ and $\sigma\tau\sigma = \tau^{-1}$, for all $\tau \in G(\Omega/E(\sqrt{-1}))$.

If (cc) is satisfied and $\Omega = E_{\text{sep}}$, then $N(L/E) = N_2(L/E) \cap N(L/E)_{\text{Ab}}$, $\text{Br}(E)$ is of order 2 and $\text{Im}(E/L) = \text{Br}(L)_2$, for each finite extension L of E in E_{sep} .

Proof. It is clear from (2.3) (i) that if finite extensions of E in Ω are p -quasilocal, then $\text{Cor}_{F'/F}$ maps $\text{Br}(F')_p$ injectively into $\text{Br}(F)_p$ whenever $F \in \text{Gal}(E)$, $F \subseteq \Omega$ and F' is a finite extension of F in $F(p)$. We first prove that the fulfillment of (c) implies that finite extensions of E in Ω are p -quasilocal. To begin with, (2.3) (i) and Proposition 2.3 (ii) guarantee that E is p -quasilocal and $\text{Br}(E)_p$ is divisible. Observe that both properties are preserved by each $M \in \text{Gal}(E)$, $M \subseteq \Omega$. If $p \notin P(M)$, this follows at once from Proposition 2.3 (ii), so we assume further that $p \in P(M)$. Denote by \mathbb{F} the prime subfield of E , and by Γ_p the unique \mathbb{Z}_p -extension of \mathbb{F} in E_{sep} . The divisibility of $\text{Br}(M)_p$ can be deduced from Witt's theorem (see [11, Sect. 15]), if $p = \text{char}(E)$, and from the Merkurjev-Suslin theorem [25, (16.1)] in case $p \neq \text{char}(E)$ and $\Gamma_p \subseteq E$. Assuming that $\Gamma_p \not\subseteq E$, one obtains from Galois theory that $M\Gamma_p/M$ is a \mathbb{Z}_p -extension, and for each $n \in \mathbb{N}$, $\text{Gal}(E)$ contains the field $M_n \in I(M\Gamma_p/M)$ of degree $[M_n:M] = p^n$. Hence, by (c) and the RC-formula, M_n is a splitting field of each $T_n \in d(M_n)$ of exponent dividing p^n . This enables one to deduce the following statements, arguing as in the proof of [7, I, Theorem 3.1]:

(3.1) (i) $\text{ind}(\Delta) = \exp(\Delta)$, for every $\Delta \in d(M)$ of p -primary dimension.

(ii) If $\text{Br}(M)_p \neq \{0\}$, then there exists $\Delta_n \in d(M)$ of index p^n , for each $n \in \mathbb{N}$.

Statement (3.1) the established property of M_n and [27, Sect. 15.1, Corollary b] imply the divisibility of $\text{Br}(M)_p$. Our objective now is to prove that M is p -quasilocal, provided that $p \in P(M)$. Let \widetilde{M} be a finite extension of M in $M(p)$, D a central division M -algebra, such that $\exp(D) = [M':M] = p$. Then it follows from Galois theory that $M(p)$ contains as a subfield the normal closure \widetilde{M}' of \widetilde{M} over E (in E_{sep}). Since $\text{Br}(M)_p$ is divisible, one can find an algebra $\widetilde{D} \in d(M)$ so as to satisfy the equalities $\text{ind}(\widetilde{D}) = p^{\widetilde{m}}$ and $p^{\widetilde{m}-1}[\widetilde{D}] = [D]$, where $p^{\widetilde{m}} = [\widetilde{M}':M]$. Hence, by the RC-formula, $\text{Cor}_{\widetilde{M}'/\widetilde{M}}(\rho_{M/\widetilde{M}'}([\widetilde{D}])) = [D]$, and $\text{Cor}_{\widetilde{M}'/M}([\widetilde{D}]) = 0 = \text{Cor}_{\widetilde{M}/M}(\rho_{M/\widetilde{M}}([D]))$. In view of (c), this means that $\rho_{M/\widetilde{M}'}([\widetilde{D}]) = 0$ and $\rho_{M/\widetilde{M}}([D]) = 0$ (in $\text{Br}(\widetilde{M}')$ and $\text{Br}(\widetilde{M})$, respectively). In other words, \widetilde{D} is split by \widetilde{M}' and D is split by \widetilde{M} , so the p -quasilocal property of M becomes obvious.

Suppose now that R is an arbitrary finite extension of E in Ω and denote by R_1 its normal closure in E_{sep} over E . By definition, R is p -quasilocal, if $p \notin P(R)$ or $\text{Br}(R)_p = \{0\}$, so we assume that $p \in P(R)$ and $\text{Br}(R)_p \neq \{0\}$. Note first that $\text{Br}(R)_p$ is divisible. As in the special case where $R = R_1$, one sees that it is sufficient to prove our assertion under the hypothesis that $\Gamma_p \not\subseteq E$. Applying [27, Sect. 15.1, Corollary b], one concludes that if $\text{Br}(R)_p$ is not divisible, then $\text{Br}(R\Gamma_p/R) \neq \text{Br}(R)_p$ and $\text{Br}(R\Gamma_p)_p$ is infinite and divisible. As $[R_1\Gamma_p: R\Gamma_p] \in \mathbb{N}$, this implies that $\text{Br}(R_1\Gamma_p)_p \neq \{0\}$. On the other hand, R_1 is p -quasilocal and $R_1\Gamma_p/R_1$ is a \mathbb{Z}_p -extension, so it follows from [7, I, Theorem 4.1 (iv)] that $\text{Br}(R_1\Gamma_p)_p = \{0\}$. The obtained contradiction completes the proof of the divisibility of $\text{Br}(R)_p$, so we return to the assumption that $p \in P(R)$ and $\text{Br}(R)_p \neq \{0\}$. Let R' be an extension of R in $R(p)$ of degree p , R'_1 the normal closure of R' in E_{sep} over E , and d_1 an element of $\text{Br}(R'_1)_p$, such that $\text{Cor}_{R'/R}(d_1) = 0$. Then $\text{Br}(R'_1)$ is divisible, so the equation $[R'_1: R']x = d_1$ has a solution $d'_1 \in \text{Br}(R'_1)_p$. Hence, by the RC-formula, $\text{Cor}_{R'_1/R'}(d'_1) = d_1$. In view of the equality $\text{Cor}_{R'_1/R} = \text{Cor}_{R'/R} \circ \text{Cor}_{R'_1/R'}$, this means that $\rho_{R'/R'}(d'_1) \in \text{Ker}(R'_1/R)$. Thus condition (c) yields $\rho_{R/R'}(d_1) = 0$ and $d_1 = 0$. The obtained result indicates that $\text{Cor}_{R'/R}$ maps $\text{Br}(R'_1)_p$ injectively into $\text{Br}(R)_p$, so the assertion that R is p -quasilocal reduces to a consequence of (2.3) (i).

Assume now that E is formally real and $p = 2$. Note first that it suffices for the proof of Proposition 3.4 (cc) to show that if $\Omega \neq E$ and formally real finite extensions of E in Ω are 2-quasilocal, then $2 \notin P(\Omega)$ and $\text{Br}(\Phi)_2 = \{0\}$, for each $\Phi \in \text{Gal}(E) \cap I(\Omega/E)$, $\Phi \neq E$. This follows from Becker's theorem (cf. [4, (3.3)]), [9, Proposition 3.1] and the latter part of Proposition 2.3 (ii). Observe that every admissible Φ is a nonreal field. Indeed, for each primitive element ξ of Φ/E and any $g \in G(\Phi/E)$, the trace $\text{Tr}_E^\Phi(\xi - g(\xi))$ equals zero. Therefore, the hypothesis that Φ is formally real requires that $(\xi - g(\xi)) \notin \Phi^{*2} \cup (-1)\Phi^{*2}$. In view of Proposition 2.3 (ii), when $g \neq 1$, this contradicts the assumption that Φ is 2-quasilocal, so the assertion that Φ is nonreal is proved. Let E_2 be the fixed field of some $G_2 \in \text{Syl}_2(\Phi/E)$. Then $2 \nmid [E_2: E]$, and by the Artin-Schreier theory (cf. [21, Ch. XI, Proposition 2]), E_2 is formally real. Hence, by Proposition 2.3 (ii), $\Phi = E_2(\sqrt{-1})$, $2 \notin P(\Phi)$ and $2 \nmid [\Phi: E(\sqrt{-1})]$, so $\text{Br}(\Phi)_2 = \{0\}$ and $2 \notin P(\Omega)$, which proves (cc).

Henceforth, we assume that $\Omega = E_{\text{sep}}$. This ensures that $P(E) = \{2\}$, $E_{\text{Ab}} = E(\sqrt{-1})$ and $\text{Br}(E)$ is of order 2 (cf. [4, (3.3)] and [9, Lemma 2.4]). Let L be a finite extension of E in E_{sep} . By [9, Proposition 3.1], $L \in \text{Gal}(E)$ if and only if $\sqrt{-1} \in L$. When $L \in \text{Gal}(E)$, the same result shows that $2 \nmid [L: E(\sqrt{-1})]$, which yields $\text{Br}(L)_2 = \{0\}$ and $N_2(L/E(\sqrt{-1})) = E(\sqrt{-1})^*$. Since $E^* = E^{*2} \cup (-1)E^{*2}$ and, by Lemma 2.2, $N_2(L/E) \subseteq N_2(L/E(\sqrt{-1}))$, this means that $N_2(L/E) \cap N(L/E)_{\text{Ab}} = N_2(L/E)^2 = N(L/E)$. Suppose finally that $L \notin \text{Gal}(E)$. Then $2 \nmid [L: E]$ and L is formally real, which implies that $\rho_{E/L}$ is an isomorphism and $N_2(L/E) = N(L/E)$. These results prove Proposition 3.4.

Theorem 3.1 (ii), Propositions 2.3 (ii)-3.4 and our next statement reduce the proof of Theorem 3.1 (i) to the case where $\text{Br}(E)_p$ is divisible and $R = M$.

Corollary 3.5. *Assume that E and M are fields, such that $M \in \text{Gal}(E)$, $\text{Br}(E)_p$ is divisible and $I(M/E)$ consists of p -quasilocal fields, for some $p \in \overline{P}$. Then $\text{Br}(L)_p$ is divisible, for every extension L of E in M , and the following conditions are equivalent:*

- (i) $\rho_{E/M}$ maps $\text{Br}(E)_p$ surjectively on $\text{Br}(M)_p$;
- (ii) $\text{Cor}_{M/E}$ maps $\text{Br}(M)_p$ injectively into $\text{Br}(E)_p$;

(iii) For each $R \in I(M/E)$, $\text{Br}(R)_p \subseteq \text{Im}(E/R)$ and $\text{Br}(R)_p \cap \text{Ker}(R/E) = \{0\}$.

Proof. The conclusion that $\text{Br}(L)_p$ is divisible follows from Proposition 2.3 (i), the divisibility of $\text{Br}(E)_p$ and the p -quasilocal property of E and L . For any pair $(U, V) \in I(M/E) \times I(M/E)$, such that $U \subseteq V$, and put $\text{Im}(U/V)_p = \text{Br}(V)_p \cap \text{Im}(U/V)$ and $\text{Ker}(V/U)_p = \text{Br}(V)_p \cap \text{Ker}(V/U)$. Since $\text{Br}(U)_p$ is divisible, the RC-formula implies $\text{Br}(V)_p = \text{Im}(U/V)_p + \text{Ker}(V/U)_p$. This, applied to (E, M) , proves that (ii) \rightarrow (i).

The rest of the proof relies on the well-known fact (see, e.g. [35]) that for each tower $U_1 \subseteq U_2 \subseteq U_3$ of finite separable extensions, Cor_{U_3/U_1} equals the composition $\text{Cor}_{U_2/U_1} \circ \text{Cor}_{U_3/U_2}$. This implies that if M/E satisfies (ii), then so does M/U . At the same time, it follows from the RC-formula and the divisibility of $\text{Br}(V)_p$ that $\text{Br}(V)_p$ is included in the image of $\text{Br}(V)_p$ under $\text{Cor}_{M/V}$. Considering now the tower $U \subseteq V \subseteq M$, one concludes that if condition (ii) holds, then $\text{Cor}_{V/U}$ maps $\text{Br}(V)_p$ injectively into $\text{Br}(U)_p$.

Let now E_p be the fixed field of some $G_p \in \text{Syl}_p(M/E)$. Then $p \nmid [E_p:E]$, so the RC-formula and the general properties of Schur indices (cf. [27, Sect. 13.4]) imply that the sum $\text{Br}(E_p)_p = \text{Im}(E/E_p)_p + \text{Ker}(E_p/E)_p$ is direct. Since $\rho_{E/M} = \rho_{E_p/M} \circ \rho_{E/E_p}$, one also sees that if condition (i) holds, then $\text{Br}(E_p)_p = \text{Im}(E/E_p)_p + \text{Br}(E_p/M)$. As $\text{Br}(E_p)_p$ is divisible and $\text{Br}(E_p/M)$ is of exponent dividing $[M:E_p]$, these observations yield $\text{Br}(E_p)_p = \text{Im}(E/E_p)_p$ and $\text{Ker}(E_p/E)_p = \{0\}$. It is now easily obtained from (2.3) (i), applied to E_p and M , and the equality $\text{Cor}_{M/E} = \text{Cor}_{E_p/E} \circ \text{Cor}_{M/E_p}$ that (i) \rightarrow (ii). Returning to the beginning of our proof, one also sees that (i) implies $\text{Br}(M)_p = \text{Im}(U/M)_p$ and $\text{Br}(V)_p \subseteq \text{Im}(U/V)_p + \text{Br}(M/V)$. Since $\text{Br}(V)_p$ is divisible and the exponent of $\text{Br}(M/V)$ divides $[M:V]$, this yields $\text{Br}(V)_p = \text{Im}(U/V)_p$, which completes the proof of the implication (i) \rightarrow (iii). As (iii) obviously implies (i), Corollary 3.5 is proved.

Now we prove Theorem 3.1 (i) in the case where $\text{Br}(E)_p = \{0\}$ and $R = M$. Let E_p be the fixed field of a group $G_p \in \text{Syl}_p(M/E)$, and let $[E_p:E] = m_p$. By Corollary 3.5, $\text{Br}(E_p)_p = \{0\}$, so it follows from (2.3) (ii) (γ) that $N(M/E_p) = E_p^*$. Hence, by the norm equality $N_E^M = N_{E_p}^{E_p} \circ N_{E_p}^M$, $N(M/E) = N(E_p/E)$. As $p \nmid m_p$ and $E^{*m_p} \subseteq N(E_p/E)$, this implies that $N_p(M/E) \subseteq N(M/E)$, as claimed. For the proof of Theorem 3.1 (i) in the case of $\text{Br}(E)_p \neq \{0\}$, we need the following lemmas.

Lemma 3.6. *Let E be a field, $M \in \text{Gal}(E)$, $G_p \in \text{Syl}_p(M/E)$, for some $p \in \overline{P}$, E_p the fixed field of G_p , L a cyclic extension of E_p in M , and σ a generator of $G(M/E)$. Assume that p does not divide the index of the commutator subgroup $[G(M/E), G(M/E)]$ in $G(M/E)$. Then $[(L/E_p, \sigma, c)] \in \text{Ker}(E_p/E)$, for every $c \in E^*$.*

Proof. Our assumptions show that $M \cap E(p) = E$. Since $L = L_\chi$, for some $\chi \in C(F(p)/F)$, this reduces our assertion to a consequence of Proposition 2.6.

Lemma 3.7. *With assumptions being as in Theorem 3.1, let $M \cap E(p) = E$. Then $E^* \subseteq N(M/E_p)$ and $N_p(R/E) \subseteq N(R/E)$.*

Proof. Clearly, one may consider only the special case of $R = M \neq E$ and $\text{Br}(E)_p \neq \{0\}$. Take G_p and E_p as in Lemma 3.6 and put $m_p = [E_p:E]$. We show that $E^* \subseteq N(M/E_p)$. As $p \nmid m_p$, ρ_{E/E_p} maps $\text{Br}(E)_p$ injectively into $\text{Br}(E_p)_p$. Therefore, $\text{Br}(E_p)_p \neq \{0\}$ and since E_p is p -quasilocal, (2.3) (ii) (α) and (β) indicate that it is sufficient to prove

the inclusion $E^* \subseteq N(L/E_p)$, for an arbitrary cyclic extension L of E_p in M . By [27, Sect. 15.1, Proposition b], this amounts to showing that $[(L/E_p, \sigma, c)] = 0$ in $\text{Br}(E_p)$, for each $c \in E^*$, where σ is a generator of $G(L/E_p)$. As $\text{g.c.d.}(m_p, p) = 1$ and ρ_{E/E_p} maps $\text{Br}(E)_p$ surjectively on $\text{Br}(E_p)_p$, $\text{Cor}_{E_p/E}$ induces an isomorphism $\text{Br}(E_p)_p \cong \text{Br}(E)_p$ (cf. [35, Theorem 2.5]). This, combined with Lemma 3.6, implies that $[(L/E_p, \sigma, c)] = 0$, $c \in E^*$, as claimed. Hence, $E^* \subseteq N(M/E_p)$ and $E^{*m_p} \subseteq N(M/E)$, which proves Lemma 3.7.

Lemma 3.8. *In the setting of Theorem 3.1 (i), let $M \in \text{Gal}(E)$ and $G(M/E) \in \text{Sol}$. Then $N_p(M/E) \cap N(M_{\text{Ab},p}/E) \subseteq N(M/E)$.*

Proof. It is sufficient to prove the lemma under the hypothesis that $N(M'/E')$ includes $N_p(M'/E') \cap N(M'_{\text{Ab},p}/E')$, whenever E' and p satisfy the conditions of Theorem 3.1, $M' \in \text{Gal}(E')$, $G(M'/E') \in \text{Sol}$ and $[M':E'] < [M:E]$. As in the proof of [5, Theorem 1.1], we first show that one may assume further that $G(M/E)$ is a Miller-Moreno group (i.e. nonabelian whose proper subgroups lie in Ab). Our argument relies on the fact that the class of fields satisfying the conditions of Theorem 3.1 is closed under the formation of finite extensions. Note that if $G(M/E)$ is not Miller-Moreno, then it has a subgroup $H \notin \text{Ab}$ with $[H, H]$ normal in $G(M/E)$. Indeed, one can put $H = [G(M/E), G(M/E)]$ in case $G(M/E) \notin \text{Met}$, and take as H any nonabelian maximal subgroup of $G(M/E)$, otherwise. Let F and L be the fixed fields of H and $[H, H]$, respectively. It follows from Galois theory and the choice of H that $L \in \text{Gal}(E)$, $M \cap E_{\text{Ab}} \subseteq L$ and $E \neq L \neq M$, so our extra assumption and Lemma 2.1 lead to the conclusion that $N_p(L/E) \cap N(M_{\text{Ab},p}/E) \subseteq N(L/E)$ and $N_p(M/F) \cap N(L/F) \subseteq N(M/F)$. Let now μ be an element of $N_p(M/E) \cap N(M_{\text{Ab},p}/E)$, and $\lambda \in L^*$ be of norm $N_E^L(\lambda) = \mu$. Then $N_F^L(\lambda)^k \in N_p(M/F)$, for some $k \in \mathbb{Z}$ such that $p \nmid k$. Therefore, $N_F^L(\lambda)^k \in N(M/F)$ and $\mu^k \in N(M/E)$. As $\mu \in N_p(M/E)$, this implies that $\mu \in N(M/E)$, which gives the desired reduction. In view of (2.3) (ii) (β), Lemma 2.1 and Galois theory, one may assume that $G(M/E)$ is a Miller-Moreno group and $G(M/E) \notin \text{Nil}$. Denote by θ the order of $[G(M/E), G(M/E)]$. The assertion of the lemma is obvious, if $p \nmid \theta$, so we suppose further that $p \mid \theta$. By Miller-Moreno's classification of these groups or by Schmidt's theorem (cf. [29, Theorem 445] and [34, Theorem 26.1]), $G(M/E)$ has the following properties:

(3.2) $[G(M/E), G(M/E)]$ is a minimal normal subgroup of $G(M/E)$, which lies in $\text{Syl}_p(M/E) \cap \text{Ab}$ and has exponent p . Also, $G(M/E)/([G(M/E), G(M/E)]) \cong C_{\pi^n}$, with $C_{\pi^n} \subset G(M/E)$ cyclic of order π^n , for some $\pi \in \overline{\mathbb{P}}$ and $n \in \mathbb{N}$. The centre of $G(M/E)$ equals the subgroup of C_{π^n} of order π^{n-1} , and $\theta = p^k$, where k is the order of p modulo π .

It follows from (3.2) and Galois theory that $[E_p:E] = \pi^n$, where $E_p = M \cap E_{\text{Ab}}$. Hence, by Lemma 3.7, $E^* \subseteq N(M/E_p)$ and $E^{*\pi^n} \subseteq N(M/E)$. Since $N_{E_p}^M(\eta) = \eta^{p^k}$, for every $\eta \in E_p$, we also have $N(M/E)_{\text{Ab}}^{p^k} \subseteq N(M/E)$, so Lemma 3.8 is proved.

It is now easy to complete the proof of Theorem 3.1 (i). Put $M_0 = M \cap E_{\text{Sol}}$, and denote by μ_p and m_p the maximal divisors of $[M_0:E]$ and $[M:E]$, respectively, for which $p \nmid \mu_p m_p$. Using Corollary 3.5 as well as the inclusion $(M \cap E_{\text{Ab}}) \subseteq M_0$, and applying Lemma 3.8 to M_0/E and Lemma 3.7 to M/M_0 , one obtains that $N(M/E)_{\text{Ab}}^{\mu_p} \subseteq N(M_0/E)$ and $M_0^{*\bar{m}_p} \subseteq N(M/M_0)$, where $\bar{m}_p = m_p/\mu_p$. Hence, by the norm identity $N_E^M = N_E^{M_0} \circ N_{M_0}^M$,

we have $N(M/E)_{\text{Ab}}^{m_p} \subseteq N(M_0/E)^{\bar{m}_p} \subseteq N(M/E)$, i.e. m_p is divisible by the exponent of $N(M/E)_{\text{Ab}}/N(M/E)$, so Theorem 3.1 (i) is proved.

Remark 3.9. Theorem 3.1 (i) plays a role in the proof of the first of the following two results (see the references in [6, page 384]):

(i) There exists a nonreal SQL-field E , such that G_E is not pronilpotent but is metabelian and every finite extension R of E in E_{sep} is subject to the alternative $R \subseteq E_{\text{Nil}}$ or $N(R/E) \neq N(\Theta/E)$, $\Theta \in \Omega(E)$.

(ii) Let F be a formally real quasilocal field and Φ a finite extension of F in F_{sep} . We have already proved that then G_F is metabelian, $F_{\text{Nil}} = F(\sqrt{-1})$, $\Phi(\sqrt{-1}) \in \text{Gal}(F)$, and ρ_{F/Φ_0} is surjective, for each formally real field $\Phi_0 \in I(\Phi/F)$. Note also that the following conditions are equivalent: (a) $N(\Phi/F) = N(\Phi/F)_{\text{Ab}}$; (b) $\rho_{F/\Phi(\sqrt{-1})}$ is surjective; (c) $\text{cd}_t(G_F) = 1: t \in \bar{P} \setminus \{2\}$, $t \nmid [\Phi:F]$; when F is not SQL, this holds in infinitely many cases (see [9, Lemma 2.3 and Remark 3.2]). On the contrary, it follows from [9, Theorems 1.1 and 1.2] that if F is SQL, then $N(\Phi/F)$ is uniquely determined by the F -isomorphism class of Φ ; hence, Φ is subject to the alternative in (i) unless F is real closed.

Corollary 3.10. *Let E be a quasilocal field, and suppose that $M \in \text{Gal}(E)$ has the property that $\rho_{E/L}$ is surjective, where L is the fixed field of the Fitting subgroup of $G(M/E)$. Then $N(R/E) = N(R/E)_{\text{Ab}}$, for each $R \in I(M/E)$.*

Proof. In view of Theorem 1.1 (i), one may consider the special case where $M \neq L$. It is easily seen that the exponent of $N(L/E)/N(M/E)$ divides $[M:L]$. Observe also that, by Fitting's theorem, $G(M/L)$ is normal in $G(M/E)$ and $G(M/L) \in \text{Nil}$. Hence, by Galois theory, the field $M \cap L(p) := M_p$ lies in $\text{Gal}(E) \cap I(L(p)/L)$, and by Lemma 2.1, $p \nmid [M:M_p]$, for any $p \in \bar{P}$, $p \mid [M:L]$. This shows that p does not divide the exponents of $N(M/M_p)$ and $N(M/E)/N(M_p/E)$. At the same time, by Proposition 2.3 (iii), M_p/E and p satisfy the conditions of Theorem 3.1 (i). Since $p \nmid [M:M_p]$, one deduces from Galois theory that $M_{\text{Ab},p} \subseteq M_p$, so the obtained results imply p does not divide the exponent of $N(M/E)/N(M/E)_{\text{Ab}}$ either. Thus it follows that $N(R/E) = N(R/E)_{\text{Ab}}$, as claimed.

It follows from Corollary 3.5 and Remark 3.9 (ii) that the conditions of Theorem 1.1 (i) guarantee the surjectivity of $\rho_{E/R}$, for all $R \in I(M/E)$. This means that Theorem 1.1 (i) is a special case of Corollary 3.10. Remark 3.9 (ii) shows that the conclusion of Theorem 1.1 (i) is not necessarily true without the assumption that $M \in \text{Gal}(E)$. We prove in Section 6 that the scope of Corollary 3.10 is larger than that of Theorem 1.1 (i).

4. On the relative Brauer group of function field extensions of arbitrary fields, associated with norm equations

The results of this Section form the technical basis for the proof of Theorem 1.2 (iii) and (iv). The main one is obtained by the method of proving Proposition 2.6, using at crucial points (2.3) (ii), Theorem 1.1 (i), the regularity and other known properties of function fields of Brauer-Severi varieties and of their transfers over finite Galois extensions (see [14, Theorem 1] with its proof). It illustrates the fact that the applications of p -quasilocal fields to the study of Brauer groups do not restrict to stable fields with Henselian

valuations (see Section 6) and to the corestriction mapping. In what follows, we use the abbreviation tr-degree of the transcendency degree of an arbitrary field extension.

Proposition 4.1. *Let E and M_1, \dots, M_s be fields, and for $j = 1, \dots, s$, suppose that $M_j \in \text{Gal}(E)$, $[M_j : E] = n_j$, $N_{M_j/E} = N_{M_j/E}(\widehat{X}_j)$ is a norm form of M_j/E in an n_j -tuple $\widehat{X}_j = (X_{j,1}, \dots, X_{j,n_j})$ of algebraically independent variables over E , $c_j \in E^*$, and $\Lambda(M_j/E; c_j)$ is the fraction field of the quotient ring $E[\widehat{X}_j]/(N_{M_j/E} - c_j)$. Then the field $\Lambda = \widetilde{\otimes}_{j=1}^s \Lambda(M_j/E; c_j)$, where $\widetilde{\otimes} = \widetilde{\otimes}_E$, has the following properties:*

- (i) Λ is regular over E and $(M \otimes_E \Lambda)/M$ is rational of tr-degree $\sum_{j=1}^s (n_j - 1)$;
- (ii) $\text{Br}(\Lambda/E)$ equals the group $B_c(M/E) = \langle [(L_j/E, \sigma_{L_j/E}, c_j)] \rangle$, $j = 1, \dots, s$, where L_j runs across the set of cyclic extensions of E in M_j , and $\sigma_{L_j/E}$ is a generator of $G(L_j/E)$. In particular, if $c_j \in N(M_j/E)$, $j = 1, \dots, s$, then $\text{Br}(\Lambda/E) = \{0\}$.

Proof. (i): For each index j , $M_j[\widehat{X}_j]/(N_{M_j/E} - c_j)$ is a domain, its fraction field Θ_j is rational over M_j of tr-degree $n_j - 1$, and $G(M_j/E)$ acts on Θ_j as a group of automorphisms. Therefore, Θ_j/M_j is regular and $\Lambda(M_j/E; c_j)$ is the fixed field of $G(M_j/E)$ in Θ_j . In view of [21, Ch. VII, Proposition 20], this implies $\Theta_j \in \text{Gal}(\Lambda(M_j/E; c_j))$, $G(\Theta_j/\Lambda(M_j/E; c_j)) \cong G(M_j/E)$ and $M_j \cap \Lambda(M_j/E; c_j) = E$. It is now clear that $\Lambda(M_j/E; c_j)/E$ is regular and $\Theta_j \cong_E M_j \otimes_E \Lambda(M_j/E; c_j)$, so Proposition 4.1 (i) can easily be proved by induction on s .

(ii) If $s \geq 2$, then Λ is E -isomorphic to $\widetilde{\otimes}_{j=1}^{s-1} \Lambda'_j$, where $\widetilde{\otimes} = \widetilde{\otimes}_{\Lambda(M_s/E; c_s)}$ and $\Lambda'_j = \Lambda(M_j/E; c_j) \otimes_E \Lambda(M_s/E; c_s)$, $j = 1, \dots, s-1$. Thus our proof reduces to the case of $s = 1$.

To simplify notation, put $M = M_1$, $c = c_1$, $n = n_1$ and $X_u = X_{1,u}$, for $u = 1, \dots, n$. The extensions of E considered in the rest of our proof are assumed to lie in $I(\overline{\Theta}/E)$, for some algebraically closed extension $\overline{\Theta}$ of E of countable tr-degree. In what follows, $B = \{\xi_u : u = 1, \dots, n\}$ denotes the basis of M/E with respect to which $N_{M/E}$ is defined, $Y = \sum_{j=1}^n \xi_j X_j$, and \bar{f} stands for the image in $M\Lambda(M/E; c)$ of any polynomial $f \in M[X_1, \dots, X_n]$. Recall that $M\Omega \in \text{Gal}(\Omega)$ and $G(M\Omega/\Omega) \cong G(M/E)$, for any $\Omega \in I(\overline{\Theta}/E)$ with $\Omega \cap M = E$. In addition, then $I(M/E)$ is mapped bijectively on $I(M\Omega/\Omega)$, by the rule $D \rightarrow D\Omega$. Moreover, $D \in \text{Gal}(E)$ if and only if $D\Omega \in \text{Gal}(\Omega)$; when this is the case, one may identify when necessary $G(D/D_0)$ with $G(D\Omega/D_0\Omega)$, for any $D_0 \in I(D/E)$. Note that the norm map $N_{D_0\Omega}^{D\Omega}$ extends $N_{D_0}^D$. Let $R \in I(M/E)$, $[R : E] = r$, ψ_1, \dots, ψ_r the E -embeddings of R into M , and for each index $k \leq r$, let $\tilde{\psi}_k$ be an automorphism of M extending ψ_k , and $\rho_k = \prod_{\sigma \in G(M/R)} (\sigma \tilde{\psi}_k^{-1})(\bar{Y})$. It is easily seen that $\bar{\rho}_1, \dots, \bar{\rho}_{r-1}$ are algebraically independent over R , $\prod_{k=1}^r \bar{\rho}_k = c$ and $\bar{\rho}_k \in R\Lambda(M/E; c)$, $k = 1, \dots, r$. Hence, the above observations and the transitivity of norms in towers of finite extensions lead to the conclusion that $R\Lambda(M/E; c)$ is R -isomorphic to $\widetilde{\otimes}_R$ of the fields $\Lambda(\widetilde{M}/\widetilde{R}; c)$ and $\Lambda(\widetilde{M}/\widetilde{R}; \bar{\rho}_k)$, $k = 1, \dots, r-1$, where $\widetilde{R} = R(\bar{\rho}_1, \dots, \bar{\rho}_{r-1})$ and $\widetilde{M} = M\widetilde{R}$. This enables us to complement Proposition 4.1 (ii) as follows:

(4.1) The sum of groups $\text{Br}(\Lambda(\widetilde{M}/\widetilde{R}; c)/\widetilde{R})$ and $\text{Br}(\Lambda(\widetilde{M}/\widetilde{R}; \rho_k)/\widetilde{R})$, $k = 1, \dots, r-1$, is direct and equal to $\text{Br}(R\Lambda(M/E; c)/\widetilde{R})$. Also, $\text{Br}(R\Lambda(M/E; c)/R) = \text{Br}(\Lambda(M/R; c)/R)$.

Note that $c \in N(M\Lambda(M/E; c)/\Lambda(M/E; c))$, since $N_{\Lambda(M/E; c)}^{M\Lambda(M/E; c)}$ extends N_E^M . In view of [27, Sect. 15.1, Proposition b], this implies that $B_c(M/E) \subseteq \text{Br}((M\Lambda(M/E; c))/\Lambda(M/E; c))$. We prove the converse implication. By [14, Theorem 1], there is $B(M/E) \in$

$I(\overline{\Theta}/E)$ which is E -isomorphic to $\tilde{\otimes}_E$ of function fields of Brauer-Severi E -varieties, such that $\text{Br}(B(M/E)/E) = B_c(M/E)$. This ensures that E is algebraically closed in $B(M/E)$. Therefore, considering $MB(M/E)/B(M/E)$ instead of M/E , one obtains that Proposition 4.1 (ii) will follow, if we show that $\text{Br}((M\Lambda(M/E; c))/\Lambda(M/E; c)) = \{0\}$ in case $B_c(M/E) = \{0\}$. Applying Theorem 1.2 (i) and the concluding statement of Theorem 1.2 (ii) as in the proof of Proposition 2.6 in [10] (with E instead of E_0), and using Theorem 1.1, one sees further that it suffices to prove the final assertion of Proposition 4.1 (ii). Suppose first that $[M:E] = p$, for some $p \in \overline{P}$. We show that $\text{Br}(\Lambda(M/E; c)/E) = \{0\}$ by proving the following statement:

(4.2) If $c \in N(M/E)$, then $\Lambda(M/E; c)/E$ is rational of tr-degree $p - 1$.

It is sufficient to establish (4.2) in the case of $c = 1$. Fix a generator σ of $G(M/E)$ and put $Y_i = \sum_{i=1}^p \sigma^{i-1}(\xi_i)X_i$. As $N_{\Lambda(M/E; 1)}^{M\Lambda(M/E; 1)}(\overline{Y}_1) = 1$, Hilbert's Theorem 90 yields $\overline{Y}_1 = \overline{Z}\sigma(\overline{Z})^{-1}$, for some $Z \in M[X_1, \dots, X_p]$. Clearly, $\lambda\overline{Z}\sigma(\lambda\overline{Z})^{-1} = \overline{Y}_1$, for each $\lambda \in \Lambda(M/E; c)^*$. Denote by $W_1(\lambda), \dots, W_p(\lambda)$ the coordinates of $\lambda\overline{Z}$ in $\Lambda(M/E; 1)$ with respect to B . Observe that λ can be fixed so that $W_p(\lambda) \in E$. Since $\overline{Y}_1, \dots, \overline{Y}_{p-1}$ form a tr-basis of $M\Lambda(M/E; c)/M$, the choice of λ guarantees that $W_1(\lambda), \dots, W_{p-1}(\lambda)$ are algebraically independent over E . Note finally that the equality $\overline{Y}_1 = \overline{Z}\sigma(\overline{Z})^{-1}$ implies that $\overline{X}_i \in E(W_1(\lambda), \dots, W_{p-1}(\lambda))$, for $i = 1, \dots, p$, so it follows that $\Lambda(M/E; c) = E(W_1(\lambda), \dots, W_{p-1}(\lambda))$, proving (4.2).

Assume now that $[M:E] = p^k$, for some $p \in P(E)$ and $k \in \mathbb{N}$, $k \geq 2$. Proceeding by induction on k and using the fact that finite extensions of E are quasilocal, one obtains that it suffices to prove Proposition 4.1 (ii) when $c \in N(M/E)$, under the hypothesis that its conclusion holds in general, for every Galois extension Φ/Φ_0 of degree $p^u < p^k$, where $u \in \mathbb{N}$. This ensures the validity of (4.1) for M'_0/R_0 whenever $M'_0 \in \text{Gal}(E)$, $E \subseteq R_0 \subseteq M'_0 \subseteq M$ and $[M'_0:R_0] < p^k$. It is well-known (see e.g. [21, Ch. I, Sect. 6; Ch. VIII]) that $\text{Gal}(E) \cap I(M/E)$ contains fields R_1, \dots, R_k , such that $[R_j:E] = p^j$, $j = 1, \dots, k$, and $R_{j-1} \subset R_j$, for $j \geq 2$. Put $H_j = G(M/R_j)$, $y_j = (\prod_{h_j \in H_j} h_j(\overline{Y}))$, $S_j = \{\tau_j(y_j) : \tau_j \in G(R_j/E)\}$ and $\tilde{R}_j = R_j(S_j)$, for $j = 1, \dots, k$. It is easy to see that $G(R_j/E)$ is an automorphism group of \tilde{R}_j whose fixed field, say N_j , is E -isomorphic to $\Lambda(R_j/E; c)$ and satisfies the equality $N_j(S_j) = \tilde{R}_j$ (S_j will be viewed as a standard generating set of \tilde{R}_j/R_j in $\Lambda(M/E; c)$). To prove that $\text{Br}(\Lambda(M/E; c)/E) = \{0\}$ we show (setting $B_0 = \Phi_0 = E$) the existence of a tower of field extensions $T_1, B_1, \Phi_1, \dots, T_{k-1}, B_{k-1}, \Phi_{k-1}, T_k$ of N_1 satisfying the following conditions, for each index $j \geq 1$:

(4.3) (i) T_j/Φ_{j-1} is rational of tr-degree p^j and contains N_j as a subfield; more precisely, T_j is a transfer over $R_j\Phi_{j-1}$ to Φ_{j-1} of a rational function field $T'_j = \Phi_{j-1}(Z_{j,1})$, such that the compositum of R_j and the transfer $T'_{j,(j-1)}$ of T'_j over $(R_j\Phi_{j-1})/(R_{j-1}\Phi_{j-1})$ includes the set $S_{j,(j-1)} = \{\tau'_j(y_j), \tau'_j \in G(R_j/R_{j-1}), \tau'_j \neq 1\}$. Specifically, the union $\{Z_{j,1}\} \cup S_{j,(j-1)}$ is a tr-basis of $(R_jT'_{j,(j-1)})/(R_j\Phi_{j-1})$.

(ii) B_j is an $(R_jT_j)/T_j$ -transfer of an extension B'_j of R_jT_j isomorphic to a tensor compositum over R_jT_j of function fields of Brauer-Severi (R_jT_j) -varieties, and such that $\text{Br}(B'_j/R_jT_j)$ equals the sum Γ_j of the images of the groups $\text{Br}(\Lambda(M\tilde{R}_j/\tilde{R}_j; s_j))$, $s_j \in S_j$, under $\rho_{\tilde{R}_j/(R_jT_j)}$. In particular, T_j is algebraically closed in B_j .

- (iii) $\text{Br}(B_j/T_j)$ equals the image of Γ_j under $\text{Cor}_{(R_j T_j)/T_j}$, and $\Gamma_j \cap \text{Br}(\Phi_{j-1}) = \{0\}$.
- (iv) Φ_j is p -quasilocal, $B_j \subseteq \Phi_j$, B_j is algebraically closed in Φ_j and $\text{Br}(\Phi_j/B_j) = \{0\}$.

The first part of (4.3) (iii) and the second half of (4.3) (ii) are implied by the first part of (4.3) (ii) and the following lemma.

Lemma 4.2. *With assumptions being as in Proposition 2.8, suppose that $M \in \text{Gal}(F)$. Then $M\Lambda$ is $\tilde{\otimes}_M$ of function fields of Brauer-Severi M -varieties and Λ' embeds canonically in $M\Lambda$ as an M -subalgebra. Moreover, E is algebraically closed in Λ and $\text{Br}(\Lambda/E)$ coincides with the images of $\text{Br}(\Lambda'/M)$ and $\text{Br}(M\Lambda/M)$ under $\text{Cor}_{M/E}$.*

Proof. The assertion about $\text{Br}(\Lambda/E)$ has been deduced from Lemma 2.4 in [10, Sect. 2]. The other conclusions of the lemma follow from its assumptions and Galois theory (see the beginning of [30, Sect. 3], for more details).

Statement (4.3) (ii) and the properties of cyclic algebras described in [27, Sect. 15.1] guarantee that $S_j \subset N(I_j B_j/R_j B_j)$, for every $I_j \in I(M/R_j)$ cyclic over R_j . As B_j is separably closed in Φ_j , this means that $S_j \subset N(I_j \Phi_j/R_j \Phi_j)$, so it follows from Galois theory and the p -quasilocal property of Φ_j (apply (2.3) (ii)) that $S_j \subset N(M \Phi_j/R_j \Phi_j)$, for each admissible j . These facts enable one to deduce (4.3) (i) from the following lemma.

Lemma 4.3. *Assume that E is a field, R, L and M lie in $\text{Gal}(E)$, $R \subset L \subseteq M$, $[L:R] = p$ and $[M:E] = p^k$, for some $p \in P(E)$, $k \in \mathbb{N}$. Suppose also that the inductive hypothesis holds and fix an element $\rho \in N(M/R)$ so that $c = N_E^R(\rho)$. Then $\Lambda(L/R; \rho) \in I(\Omega'/R)$, for a rational extension Ω' of R satisfying the following conditions:*

- (i) Ω' is rational over $\Lambda(L/R; \rho)$ and has tr-degree p over R ; also, Ω' is the L/R -transfer of a rational extension Ω_1 of L in one indeterminate;
- (ii) The L/E -transfer Ω of Ω_1 is rational over E of tr-degree $[L:E]$;
- (iii) The R -algebra $\tilde{\otimes}_R \Lambda(L/R; \tau(\rho))$, where τ runs across $G(R/E)$, is isomorphic to a field $\Lambda_R \in I(R\Omega/R)$, such that $R\Omega/\Lambda_R$ is rational of tr-degree $[R:E]$.

Proof. It is clearly sufficient to consider the special case where $\rho = 1$. Identifying $G(L/R)$ with $G(L\Lambda(L/R; 1)/\Lambda(L/R; 1))$, fix a generator σ of $G(L/R)$ and take elements $y_i \in L\Lambda(L/R; 1)$: $i = 1, \dots, p$, so that $\prod_{i=1}^p y_i = 1$, $\{y_1, \dots, y_{p-1}\}$ is a tr-basis of $\Lambda(L/R; 1)/L$ and $y_i = \sigma^{i-1}(y_1)$, for each index i . Note also that $L\Lambda(L/R; 1)$ is a subfield of a rational function field $L' = L(z_1, \dots, z_p)$ with generators subject to the relations $z_i/z_{i+1}^{-1} = y_i$, $i = 1, \dots, p-1$. This guarantees the existence of an E -automorphism $\tilde{\sigma}$ of L' extending σ , and such that $\tilde{\sigma}(z_p) = z_1$ and $\tilde{\sigma}(z_i) = z_{i+1}$, $i < p$. Therefore, $G(L/R)$ can be viewed as a group of automorphisms of L' whose fixed field Ω' includes $\Lambda(L/R; 1)$. Since, by Hilbert's Theorem 90, $y_p = w_p \sigma(w_p)^{-1}$, for some $w_p \in L\Lambda(L/R; 1)$, it follows that $z_p = z w_p$, for some $z \in \Omega'$ transcendental over $\Lambda(M_0/E; 1)$. Thus it becomes clear that $\Omega' = \Lambda(L/R; 1)(z)$ and $L' = L\Lambda(M_0/E; 1)(z)$. Putting $\Omega_1 = L(z_1)$ and applying [30, Lemmas 3.1 and 3.2], one completes the proof of (i) and (ii). Statement (iii) is implied by (i), (ii) and the definition of the transfer map.

The latter assertion of (4.3) (iii) follows from the former one and our next lemma.

Lemma 4.4. *In the setting of Lemma 4.3, let Λ_R be the field $\tilde{\otimes}_{\tau \in G(R/E)} \Lambda(M/R; \tau(\rho))$, where $\tilde{\otimes} = \tilde{\otimes}_R$. For each $\tau \in G(R/E)$, identify $\Lambda(L/R; \tau(\rho))$ with its canonical R -*

isomorphic copy in $\Lambda(M/R; \tau(\rho))$, and denote by $S_{\tau(\rho)}$ the standard generating set of $L\Lambda(L/R; \tau(\rho))/L$ in $\Lambda(M/R; \tau)$. Also, let $\Sigma_{L\Lambda_R}$ be the sum of the images of $\text{Br}(\Lambda(M(S_{\tau(\rho)})/L(S_{\tau(\rho)}); s_{\tau(\rho)})$ under $\rho_{L(S_{\tau(\rho)})/L, \Lambda_R}$, where τ runs across $G(R/E)$ and, for each τ , $s_{\tau(\rho)}$ runs across $S_{\tau(\rho)}$. Then the image Δ_E of $\Sigma_{L\Lambda_R}$ under $\text{Cor}_{(L\Omega)/\Omega} \circ \rho_{L\Lambda_R/L\Omega}$ intersects trivially with $\text{Br}(E)$.

Proof. The assertion is obvious in the case where E is finite, since then $\text{Br}(E) = \{0\}$, by Wedderburn's theorems. Suppose further that E is infinite, denote by Δ_L the image of $\Sigma_{L\Lambda_R}$ under $\rho_{(L\Lambda_R)/(L\Omega)}$, let $G(L/E) = \{\varphi_1, \dots, \varphi_l\}$, $\varphi_1 = 1$, $[L:E] = l$, and take a tr-basis Z_1, \dots, Z_l of $L\Omega/\Omega$ so that $L(Z_1) = \Omega_1$ and $Z_j = \varphi_j(Z_1)$, for each index j . Statement (4.1) and the inductive hypothesis on M/E imply that $\Delta_L \subseteq \text{Br}(M\Omega/L\Omega)$ and $\Delta_E \subseteq \text{Br}(M\Omega/\Omega)$. In addition, it follows from Lemma 4.3 (ii) that $\text{Br}(E) \cap \Delta_E \subseteq \text{Br}(M/E)$. Denote by A_L^l the l -dimensional affine L -space, and for each $d \in \mathbb{N}$, let $A_{L/E}^d = \{(a_1^d, \dots, a_l^d) : a_i \in L^* \text{ and } a_i = \varphi_i(a_1), i = 1, \dots, l\}$. Observing that L is algebraic over $E(Z_1^d, \dots, Z_l^d)$, and using (4.1), the infinity of E and the definition of the corestriction mapping, one proves the following:

(4.4) (i) The sets A_E^l and $A_{L/E}^d$: $d \in \mathbb{N}$, are dense in A_L^l , in the sense of Zariski;

(ii) $A_{L/E}^l$ possesses a subset $U \neq \emptyset$, Zariski-open in A_L^l and such that each specialization of (Z_1, \dots, Z_l) into U induces group homomorphisms $\pi_L: (\text{Br}(M/L) + \Delta_L) \rightarrow \text{Br}(L)$ and $\pi_E: (\text{Br}(M/E) + \Delta_E) \rightarrow \text{Br}(E)$ satisfying the equality $\text{Cor}_{L/E} \circ \pi_L = \pi_E \circ \text{Cor}_{(L\Omega)/\Omega}$ and acting as the identity on $\text{Br}(M/L)$ and $\text{Br}(M/E)$, respectively.

Statement (4.4) (i) and [27, Sect. 15.1, Proposition b] ensure the existence of many specializations for which $\Delta_L \subseteq \text{Ker}(\pi_L)$. This implies that $\Delta_E \subseteq \text{Ker}(\pi_E)$ and $\text{Br}(E) \cap \Delta_E = \{0\}$, so Lemma 4.4 is proved.

We are now in a position to complete the proof of (4.3) and Proposition 4.1. The existence of B_j^l satisfying the conditions in the first part of (4.3) (ii) is obtained by applying [14, Theorem 1]. Statement (4.3) (iv) follows from Lemma 2.4, so (4.3) holds and $\text{Br}(T_k/E) = \{0\}$. Since, by (4.3) (i), $N_k \in I(T_k/E)$ and N_k is E -isomorphic to $\Lambda(M/E; c)$, this proves Proposition 4.1 (ii) in case $M \subseteq E(p)$.

For the proof in general, take G_p and E_p as in Lemma 3.6. By Lemma 2.2, $c \in N(M/E_p)$, so (4.1) and the established special case of our assertion imply that $\text{Br}(E_p\Lambda(M/E; c)/E_p) = \{0\}$. It is now easy to see that $\text{Br}(\Lambda(M/E; c)/E) \cap \text{Br}(E)_p = \{0\}$, for every $p \in \overline{P}$ (see [27, Sect. 13.4]), which completes the proof of Proposition 4.1 (ii).

Remark 4.5. Propositions 2.6 and 4.1 imply that if E is a field, $M \in \text{Gal}(E)$ and $R \in I(M/E)$, then $\text{Cor}_{R/E}$ maps $\text{Br}(\Lambda(M/R; c)/R)$ into $\text{Br}(\Lambda(M/E; c)/E)$, for each $c \in E^*$. Moreover, it follows from the RC-formula that $\text{Br}(\Lambda(M/E; c)/E)$ equals the image of $\text{Br}(\Lambda(M/R; c)/R)$ under $\text{Cor}_{R/E}$, provided that $\text{g.c.d.}([R:E], [M:R]) = 1$.

Proposition 4.6. *Let E be a field, Ω an algebraically closed extension of E_{sep} , s a positive integer, Σ, Λ and $M_1, \Lambda_1, \dots, M_s, \Lambda_s$ lie in $I(\Omega/E)$ so that $\Lambda = \Lambda_1 \dots \Lambda_s$, $\Lambda \cap \Sigma_{\text{sep}} = E$ and $\Lambda \cong \widetilde{\otimes}_{j=1}^s \Lambda_j$, where $\widetilde{\otimes} = \widetilde{\otimes}_E$ and \cong is an E -isomorphism. For each j , assume that $M_j \in \text{Gal}(E)$, put $\Sigma_j = \Sigma \cap M$ and let Λ_j/E be of one of the following types:*

(i) $\Lambda_j = \Lambda(M_j/E; c_j)$, for some $c_j \in E^*$;

(ii) Λ_j/E is an M_j/E transfer of \otimes_{M_j} of function fields of Brauer-Severi M_j -varieties, such that $\text{Br}(\Lambda_j/M_j)$ is a submodule of $\text{Br}(M_j)$ over the integral group ring $\mathbb{Z}[G(M_j/E)]$. Suppose also that $\tilde{\Delta}_j = \text{Br}(\Lambda_j\Sigma_j/\Sigma_j)$ in case (i), and let $\tilde{\Delta}_j$ be the image of $\text{Br}(\Lambda_j M_j/M_j)$ under $\text{Cor}_{M_j/\Sigma_j}$, otherwise. Then E and Σ are algebraically closed in Λ and $\Lambda\Sigma$, respectively, and $\text{Br}(\Lambda\Sigma/\Sigma)$ equals the sum of the images Δ_j of $\tilde{\Delta}_j$ under $\rho_{\Sigma_j/\Sigma}$, for $j = 1, \dots, s$.

Proof. It is clear from Galois theory and the condition $\Lambda \cap \Sigma_{\text{sep}} = E$ that if $F \in I(\Sigma/E)$, then F is algebraically closed in ΛF . For the rest of our proof, suppose first that $s = 1$. It follows from Galois theory and the definitions of Λ and of the transfer map that Λ/E and $\Lambda\Sigma/\Sigma$ are of one and the same type relative to M_1/E and $M_1\Sigma/\Sigma$, respectively. In addition, if Λ/E is of type (ii), then direct calculations show that the mappings of $\text{Br}(M_1)$ into $\text{Br}(\Sigma)$ defined by the rules $\text{Cor}_{M_1\Sigma/\Sigma} \circ \rho_{M_1/M_1\Sigma}$ and $\rho_{\Sigma_1/\Sigma} \circ \text{Cor}_{M_1/\Sigma_1}$ coincide. Applying now Proposition 4.1 and (4.1) in case Λ/E is of type (i), and using Lemma 4.2, otherwise, one proves that $\text{Br}(\Lambda\Sigma/\Sigma) = \Delta_1$, as claimed. It remains to be seen that the concluding assertion of Proposition 4.6 holds in the case of $s \geq 2$. Consider the fields Λ , $\tilde{E} = \Lambda_1 \dots \Lambda_{s-1}$, $\tilde{\Sigma} = \Sigma\tilde{E}$ and $M_s\tilde{E}$ instead of Λ , E , Σ and M_1, \dots, M_s , respectively. It is not difficult to deduce from Galois theory and well-known properties of tensor products and of function fields of Brauer-Severi varieties (see [27, Sects. 9.2 and 9.4] and the proof of [14, Theorem 1]) that $\text{Br}(\Lambda\tilde{\Sigma}/\tilde{\Sigma})$ is determined in accordance with Proposition 4.1, for $s = 1$, and equals the image of $\text{Br}(\Lambda_s\Sigma_s/\Sigma_s)$ under $\rho_{\Sigma_s/\tilde{\Sigma}}$. Proceeding now by induction on s and applying the inductive hypothesis to $\text{Br}(\tilde{\Sigma}/\Sigma)$, we complete our proof.

5. Proof of Theorem 1.2

Let E_0 be an arbitrary field. Theorem 1.2 will be proved by constructing E as a union of a certain tower of fields E_n : $n \in \mathbb{N}$, such that $E_0 \subset E_1$ and E_{n-1} is algebraically closed in E_n , for every index n . It should be emphasized that the proof of Theorem 1.2 (i) and of the latter part of Theorem 1.2 (ii) in the special case where $\chi = \text{Fin}$ does not use Proposition 2.8 in full generality and is independent of Propositions 2.6, 4.1 and 4.6 (i) (but relies on Lemma 4.2 and Propositions 3.4 (c), 4.6 (ii)). In order to ensure generally that our construction has the desired properties we also need the following lemmas.

Lemma 5.1. *Let E and M be fields, and χ an abelian closed class, such that $\text{Nil} \subseteq \chi$, $M \in \text{Gal}(E)$, $G(M/E) \notin \chi$ and $G(L/E) \in \chi$, for $L \in \text{Gal}(E) \cap I(M/E)$, $L \neq M$. Then:*

(i) *$G(M/E)$ is simple or has a unique minimal normal subgroup G_0 ; in the former case, $G(M/E) \notin \text{Sol}$;*

(ii) *If $G(M/E)$ is not simple, then $G_0 \in \text{Ab}$ if and only if $G(M/E) \in \text{Sol}$; in this case, $\chi = \text{Nil}$;*

(iii) *If $G(M/E) \in \text{Sol}$, then $G_0 \subseteq [G(M/E), G(M/E)]$ and $G_0 \in \text{Syl}_p(M/E)$, for some $p \in \bar{P}$ not dividing the order of $G(M/E)/G_0$.*

Proof. Suppose for a moment that $G(M/E)$ has normal proper subgroups H_1 and H_2 , such that $H_1 \cap H_2 = \{1\}$. Then Galois theory and our assumptions ensure that $G(M/E)/H_j \in \chi$, for $j = 1, 2$. Hence, by the choice of χ , it contains $G(M/E)/H_1 \times G(M/E)/H_2$, and since $G(M/E)$ embeds canonically into $G(M/E)/H_1 \times G(M/E)/H_2$, this requires that $G(M/E) \in \chi$, a contradiction proving Lemma 5.1 (i). In the rest of

the proof, we may assume that $G(M/E)$ has a unique minimal normal subgroup G_0 . It is well-known that if $G(M/E) \in \text{Sol}$, then $G_0 \in \text{Ab}$ and G_0 is of exponent $p \in \overline{P}$. Conversely, if $G_0 \in \text{Ab}$, then χ could not be abelian closed (by Galois theory and the assumptions on M/E), so the conditions on χ guarantee that it equals Nil. Since Sol is closed under the formation of group extensions, this proves Lemma 5.1 (ii). Suppose finally that $G(M/E) \in \text{Sol}$ and fix a group $G_p \in \text{Syl}_p(M/E)$. As $G(M/E) \notin \text{Nil}$ and Nil is a saturated group formation (in the sense of [34]), G_0 is not included in the Frattini subgroup of $G(M/E)$. Hence, $G(M/E) = G_0H$, for some maximal subgroup H of $G(M/E)$. In view of Lemma 5.1 (ii), this means that $G_0 \cap H = \{1\}$ and $H \cong G(M/E)/G_0$. In particular, $H \in \text{Nil}$, which implies that G_p is normal in $G(M/E)$. The centre $Z(G_p)$ of G_p is characteristic in G_p , so the obtained result shows that $Z(G_p)$ is normal in $G(M/E)$. As $G_0 \cap Z(G_p) \neq \{1\}$ (see [21, Ch. I, Sect. 6]), the minimality of G_0 implies that $G_0 \subseteq Z(G_p)$. It is now easily seen that the group $H_p = G_p \cap H$ is normal in $G(M/E)$. Since $H_p \cap G_0 = \{1\}$, Lemma 5.1 (i) yields $H_p = \{1\}$. Note finally that if $G_0 \not\subseteq [G(M/E), G(M/E)]$, then G_0 must be a direct summand in $G(M/E)$. This, however, means that $G(M/E) \cong G_0 \times G(M/E)/G_0$, which contradicts the assumption that $G(M/E) \notin \text{Nil}$, and so proves Lemma 5.1 (iii).

Lemma 5.2. *Let Φ , L and M be fields, such that $L, M \in \text{Gal}(\Phi)$, and let $p \in \overline{P}$ be a divisor of $[M:L \cap M]$. Suppose that R_p and Y_p are the fixed fields of some groups $\tilde{H}_p \in \text{Syl}_p(L/L \cap M)$ and $H_p \in \text{Syl}_p(M/L \cap M)$, respectively, and $\delta_p \neq 0$ be an element of $\text{Ker}(Y_p/L \cap M)$. Then $\rho_{Y_p/(R_p Y_p)}(\delta_p) \notin \text{Im}(R_p/R_p Y_p)$.*

Proof. It is easily verified that $[R_p Y_p: L \cap M] = [R_p: L \cap M][Y_p: L \cap M]$. This implies that $p \nmid [R_p Y_p: Y_p]$, whence $\rho_{Y_p/(R_p Y_p)}(\delta_p) := \bar{\delta}_p \neq 0$. Also, it follows that $\text{Cor}_{(R_p Y_p)/R_p}(\bar{\delta}_p) = \rho_{(L \cap M)/R_p}(\text{Cor}_{E_p/(L \cap M)}(\delta_p)) = 0$. On the other hand, if $\bar{\delta}_p = \rho_{R_p/(R_p Y_p)}(\tilde{\delta}_p)$, for some $\tilde{\delta}_p \in \text{Br}(R_p)$, then the RC-formula yields $\text{Cor}_{(R_p Y_p)/R_p}(\bar{\delta}_p) = \tilde{\delta}_p^{m_p} \neq 0$, where $m_p = [Y_p: L \cap M]$. The obtained contradiction proves our assertion.

Let E be a field and $M \in \text{Gal}(E)$. Before stating our next lemma, we denote by $N(M/E)_{\text{cyc}}$ the intersection of the norm groups of cyclic extensions of E in M . By Proposition 4.1, $N(M/E)_{\text{cyc}} = \{c \in E^*: \text{Br}(\Lambda(M/E; c)/E) = \{0\}\}$.

Lemma 5.3. *Assume that E , M and χ satisfy the conditions of Lemma 5.1, put $M_0 = M \cap E_\chi$, take a divisor $p \in \overline{P}$ of $[M: M_0]$, and suppose that $H_p \in \text{Syl}_p(M/M_0)$, Y_p is the fixed field of H_p , and c is an element of $E^* \setminus N(M/Y_p)_{\text{cyc}}$. Let also M_1, \dots, M_s be fields lying in $\text{Gal}(E)$, for some $s \in \mathbb{N}$, and let $\Omega, \Lambda_1, \dots, \Lambda_s$ and Λ be extensions of E associated with M_1, \dots, M_s as in Proposition 4.6. Then $c \notin N(M\Lambda/Y_p\Lambda)_{\text{cyc}}$ in the following cases:*

- (i) $(M_1 \dots M_s) \cap M \subseteq M_0$ and $\chi \neq \text{Nil}$;
- (ii) $M_i \subseteq E_{\text{Nil}}$ and $\Lambda_i = \Lambda(M_i/E; c_i)$, for some $c_i \in E^*$, and $i = 1, \dots, s$;
- (iii) For each index j , Λ_j is of type (ii) (in the sense of Proposition 4.6) and $\text{Br}(M_j\Lambda_j/M_j) \subseteq \text{Ker}_{M_j/L_j}$, where $L_j \in \text{Gal}(E)$ is chosen so that $L_j \subseteq M_j \subseteq L_j(p_j)$, for some $p_j \in \overline{P}$. In this case, $\text{Br}(Y_p\Lambda/Y_p) = \{0\}$.

Proof. Arguing as in the concluding part of the proof of Proposition 4.6, one reduces our considerations to the case of $s = 1$. As E , M and χ satisfy the conditions of Lemma 5.1, we have $M \subseteq M_1$ or $M \cap M_1 \subseteq M_0$. Our first objective is to prove Lemma 5.3 (iii). Observe that if $M \subseteq M_1 \subseteq L_1(p_j)$, then $\text{Br}(Y_p\Lambda/Y_p) = \{0\}$. Indeed, it follows from Galois

theory and the assumptions on M/E that $M \subseteq L_1$ except, possibly, in the case of $\chi = \text{Nil}$ and $p_1 = p$. When $\chi = \text{Nil}$ and $p_1 = p$, we have $Y_p = M_0 \subseteq L_1$. It is therefore clear from Lemma 4.2 that $\text{Br}(Y_p\Lambda/Y_p) = \{0\}$, as required. The same assertion is implied in the case of $L_1 \cap M \neq M$ by Proposition 4.6, since then $\text{Cor}_{L_1/(L_1 \cap M)}$ is injective and $L_1 \cap M \subseteq M_0 \subset Y_p$. Hence, by Galois theory and [27, Sect. 15.1, Proposition b], $c \notin N(M\Lambda/Y_p\Lambda)_{\text{cyc}}$, so Lemma 5.3 (iii) is proved. Assume now that $M \cap M_1 \subseteq M_0$, $\tilde{H}_p \in \text{Syl}_p(M_0M_1/M_0)$ and R_p is the fixed field of \tilde{H}_p . It follows from (4.1) and Proposition 4.1 that if $\chi \neq \text{Nil}$, then $\text{Br}(R_pY_p\Lambda/R_pY_p)$ is the image of $\text{Br}(R_p\Lambda/R_p)$ under $\rho_{R_p/(R_pY_p)}$. When $\chi = \text{Nil}$, $M_1 \dots M_s \subseteq M_0$, which enables one to obtain similarly that $\text{Br}(K_p\Lambda/K_p) \cap \text{Br}(K_p)_p$ is the image of $\text{Br}(\Lambda/E) \cap \text{Br}(E)_p$ under ρ_{E/K_p} , for $K_p = R_p, R_pY_p$. On the other hand, Lemma 5.1 and Galois theory show that Lemma 3.6 applies to $(M/E, p)$, if $\chi = \text{Nil}$, and to $(M/M_0, p)$, otherwise. In view of Lemma 5.2, these observations prove Lemma 5.3.

Let now E_0 be an arbitrary field, T_0 a subgroup of $\text{Br}(E_0)$ and R_{Fin} a system of representatives of the isomorphism classes of finite groups. Replacing, if necessary, E_0 by its rational extensions of sufficiently large tr-degree, one easily reduces (e.g., from [27, Sect. 19.6]) our considerations to the special case where T is a divisible hull of T_0 . Note further that E_0 has a regular PQL-extension E'_0 , such that $\text{Br}(E'_0/E_0) = \{0\}$ and $\text{Br}(E'_0)$ is divisible (apply [10, Lemma 1.4], proved on the basis of (2.2)), so one may assume for the proof that $T_0 = T$. It is known [39] that each profinite group G is (continuously) isomorphic to $G(L(G)/K(G))$, for some rational extension $L(G)/E$ of countable tr-degree and a suitably chosen $K(G) \in I(L(G)/E)$. Since $\text{Br}(L(G)/E) = \text{Br}(K(G)/E) = \{0\}$, this applied to the case in which G is a topological product of the groups in R_{Fin} , allows us to assume for the proof of Theorem 1.2 that all $G \in \text{Fin}$ are realizable as Galois groups over E_0 . It follows from the choice of E_0 that it has Galois extensions $\Sigma_{0,1}$ and $\Sigma_{0,2}$ in $E_{0,\text{sep}}$, such that $\Sigma_{0,1} \cap \Sigma_{0,2} = E$ and each $G \in \text{Fin}$ is isomorphic to $G(Y_j/E)$, where $Y_j \in \text{Gal}(E)$ and $Y_j \subseteq \Sigma_{0,j}$, for $j = 1, 2$. Our objective is to prove the existence of a quasilocal extension E/E_0 with the properties required by Theorem 1.2. The field E will be obtained as a union $\cup_{n=1}^{\infty} E_n$ of an inductively defined tower of regular extensions of E_0 . Suppose that the field E_k has already been defined, for some integer $k \geq 0$, and denote by T_k the image of $T_0 = T$ under ρ_{E_0/E_k} . As T_k is divisible, it is a direct summand in $\text{Br}(E_k)$, i.e. $\text{Br}(E_k)$ possesses a subgroup T'_k , such that $T'_k + T_k = \text{Br}(E_k)$ and $T'_k \cap T_k = \{0\}$. Hence, by [14, Theorem 1], there is an extension Λ_k of E_k , such that $\text{Br}(\Lambda_k/E_k) = T'_k$ and Λ_k is presentable as $\tilde{\otimes}_{E_k}$ of function fields of Brauer-Severi E_k -varieties; in particular Λ_k/E_k is regular. Identifying $E_{k,\text{sep}}$ with its E_k -isomorphic copy in $\Lambda_{k,\text{sep}}$, put $\Sigma_{k,j} = \Sigma_{0,j}\Lambda_k$, for $j = 1, 2$. The regularity of Λ_k/E_k ensures that E_k is algebraically closed in Λ_k , so it follows from Galois theory that $\Sigma_{k,1}/\Lambda_k$ and $\Sigma_{k,2}/\Lambda_k$ have the same properties as $\Sigma_{0,1}/E_0$ and $\Sigma_{0,2}/E_0$. Denote by Z_k the extension of Λ_k defined as follows:

- (5.1) (α) If $\chi = \text{Nil}$, then $Z_k = \Sigma_{k,1} \cap \Lambda_{k,\chi'}$ in case $\chi' \neq \text{Nil}$, and $Z_k = \Lambda_k$, otherwise.
- (β) If $\chi \neq \text{Nil}$, then $Z_k = \Lambda_{k,\chi}(\Sigma_{k,1} \cap \Lambda_{k,\chi'})$ in case $\chi' \neq \chi$, and $Z_k = \Lambda_{k,\chi}$, otherwise.

Let now $M_k \in \text{Gal}(\Lambda_k)$ and $W(M_k)$ be a tensor compositum over M_k of function fields of Brauer-Severi M_k -varieties, such that $\text{Br}(W(M_k)/M_k) = \text{Ker}(M_k/\Lambda_k)$. Denote by $W(M_k/\Lambda_k)$ the M_k/Λ_k -transfer of $W(M_k)$ and put $W(M_k)' = M_kW(M_k/\Lambda_k)$. It is known that $\text{Br}(W(M_k)'/M_k) = \text{Br}(W(M_k)/M_k)$, and by Lemma 4.2, $\text{Br}(W(M_k/\Lambda_k)/\Lambda_k) = \{0\}$.

Denote by W_k the tensor compositum over Λ_k of the fields $W(M_k/\Lambda_k)$, taken over all $M_k \in \text{Gal}(\Lambda_k)$, $M_k \subseteq Z_k$. It is easily obtained from Galois theory and case (ii) of Proposition 4.6 that Λ_k is algebraically closed in W_k and $\text{Br}(W_k/\Lambda_k) = \{0\}$. For each $p \in \overline{P}$, let $Q(W_k)_p$ be the set of all pairs $\widetilde{W}_{k,p} = (W'_k, W'_{k,p}) \in \text{Gal}(W_k) \times \text{Gal}(W_k)$, for which $W'_{k,p} \in I(W'_k/W'_k)$. Replacing $\text{Ker}(\text{Cor}_{M_k/\Lambda_k})$ by $\text{Ker}(\text{Cor}_{W'_{k,p}/W'_k}) \cap \text{Br}(W'_{k,p})_p$, attach to each $\widetilde{W}_{k,p} \in Q(W_k)_p$ a field extension $W(\widetilde{W}_{k,p})/W'_{k,p}$ in the same way as $W(M_k)/M_k$ is associated with (Λ_k, M_k) , and let $W(\widetilde{W}_{k,p}; W'_{k,p}/W_k)$ be the $W'_{k,p}/W_k$ -transfer of $W(\widetilde{W}_{k,p})$. Consider the tensor compositum Θ_k over W_k of the fields $W(\widetilde{W}_{k,p}; W'_{k,p}/W_k)$: $p \in \overline{P}$, $\widetilde{W}_{k,p} \in Q(W_k)_p$. By Proposition 4.6 (ii), W_k is algebraically closed in Θ_k and $\text{Br}(\Theta_k/W_k) = \{0\}$. Observing further that Sol equals the intersection of abelian closed group classes, and applying Lemmas 2.2 and 5.3, one obtains the following result:

(5.2) If E'_k/E_k and $\widetilde{\chi}$ satisfy the conditions of Lemma 5.1, then $J_k \cap N(E'_k/(E'_k \cap E_{k,\widetilde{\chi}}))$ coincides with $J_k \cap N(E'_k \Theta_k/(E'_k \cap E_{k,\widetilde{\chi}}) \Theta_k)$ in the following cases:

- (i) $\widetilde{\chi} = \chi' \cup \text{Sol}$, $G(E'_k/(E'_k \cap E_{k,\chi'})) \notin \text{Sol}$ and $J_k = E'_k \cap E_{k,\chi'}$.
- (ii) $\widetilde{\chi} = \chi \cup \text{Sol}$, $G(E'_k/E_k) \in \chi'$, $J_k = E'_k \cap E_{k,\chi}$, $G(E'_k/J_k) \notin \text{Sol}$ and $E'_k \subset \Sigma_{0,2} E_k$.
- (iii) $\widetilde{\chi} = \chi = \text{Nil}$, $G(E'_k/E_k) \in \text{Sol}$, $J_k = E_k$, and in the case of $\chi' \neq \chi$, $E'_k \subset \Sigma_{0,2} E_k$.

Putting $\widetilde{\Theta}_k = \Theta_k$ and $E_{k+1} = \Theta_k$ in the cases of $\chi' = \text{Fin}$ and $\chi = \text{Fin}$, respectively, we continue the presentation of the inductive step in our construction with the definition of the extension $\widetilde{\Theta}_k/\Theta_k$ in the case where $\chi' \neq \text{Fin}$. Let $\text{Un}(\Theta_k) = \{\Theta'_k \in \text{Gal}(\Theta_k) : G(\Theta'_k/(\Theta'_k \cap \Theta_{k,\chi'})) \notin \text{Sol}\}$, and let Ω_k/Θ_k be a rational extension with a tr-basis $X = \{X_{\Theta'_k} : \Theta'_k \in \text{Un}(\Theta_k)\}$. Using notation as in Proposition 4.1, put $\widetilde{\Lambda}_{\Theta'_k} = \Lambda((\Theta''_k \Omega_k)/\Omega_k; X_{\Theta'_k})$, for each $\Theta'_k \in \text{Un}(\Theta_k)$, where $\Theta''_k = \Theta'_k \cap (\Theta_{k,\chi'} \Theta_{k,\text{Sol}})$. Denote by $\widetilde{\Theta}_k$ the tensor compositum over Ω_k of the fields $\widetilde{\Lambda}_{\Theta'_k}$, $\Theta'_k \in \text{Un}(\Theta_k)$. It is not difficult to see that Θ_k and Ω_k are algebraically closed in $\widetilde{\Theta}_k$. At the same time, one observes that $X_{\Theta'_k} \notin N(O_k \Omega_k/F_k \Omega_k)$, for any finite extension O_k of Θ_k in $\Theta_{k,\text{sep}}$ and any $F_k \in I(O_k/\Theta_k)$. Thus it follows that the conditions of Lemma 5.3 are fulfilled by $\Theta'_k \Omega_k/\Theta''_k \Omega_k$, $X_{\widetilde{\Theta}'_k}$ and any $p \in \overline{P}$ dividing $[\Theta'_k : \Theta''_k]$. Identifying $\Theta_{k,\text{sep}}$ with its Θ_k -isomorphic copy in $\widetilde{\Theta}_{k,\text{sep}}$, one deduces from Proposition 4.1 and Lemmas 3.6, 5.3 and 2.2 that $\text{Br}(\nabla_k \widetilde{\Theta}_k/\nabla_k) = \{0\}$, for every finite extension ∇_k of Θ_k in $\Theta_{k,\text{sep}}$, and also, that $X_{\Theta'_k} \in N(\Theta''_k \widetilde{\Theta}_k/\widetilde{\Theta}_k) \setminus N(\Theta'_k \widetilde{\Theta}_k/\Theta''_k \widetilde{\Theta}_k)$, for each $\Theta'_k \in \text{Un}(\Theta_k)$. Hence, by Lemma 2.2, $X_{\Theta'_k} \notin N(\Theta'_k \widetilde{\Theta}_k/\widetilde{\Theta}_k)$, $\Theta'_k \in \text{Un}(\Theta_k)$.

Assuming that $\Delta_k = \widetilde{\Theta}_k$, if $\chi' = \chi$ or $\chi' \subseteq \text{Sol}$, we give the definition of Δ_k under the hypothesis that $\chi \neq \chi' \neq \text{Sol}$. Then χ' is abelian closed and $\text{Sol} \subset \chi'$. Let $Z(\widetilde{\Theta}_k) = \widetilde{\Theta}_k \Sigma_{k,2}$, $\text{Un}(\widetilde{\Theta}_k)' = \{\widetilde{\Theta}'_k \in \text{Gal}(\widetilde{\Theta}_k) : \widetilde{\Theta}'_k \subseteq Z(\widetilde{\Theta}_k), G(\widetilde{\Theta}'_k/(\widetilde{\Theta}'_k \cap \widetilde{\Theta}_{k,\chi})) \notin \text{Sol}\}$, $\widetilde{\Theta}''_k = \widetilde{\Theta}'_k \cap (\widetilde{\Theta}_{k,\chi} \widetilde{\Theta}_{k,\text{Sol}})$, for each $\widetilde{\Theta}'_k \in \text{Un}(\widetilde{\Theta}_k)'$, and let $\widetilde{\Omega}_k/\widetilde{\Theta}_k$ be a rational extension with a tr-basis $\widetilde{X} = \{\widetilde{X}_{\widetilde{\Theta}'_k} : \widetilde{\Theta}'_k \in \text{Un}(\widetilde{\Theta}_k)'\}$. We define Δ_k to be the tensor compositum over $\widetilde{\Omega}_k$ of the fields $\widetilde{\Lambda}_{\widetilde{\Theta}'_k} = \Lambda((\widetilde{\Theta}''_k \widetilde{\Omega}_k)/\widetilde{\Omega}_k; \widetilde{X}_{\widetilde{\Theta}'_k})$, $\widetilde{\Theta}'_k \in \text{Un}(\widetilde{\Theta}_k)'$. It is easily seen that $\widetilde{\Theta}_k$ and $\widetilde{\Omega}_k$ are algebraically closed in Δ_k . Note also that $\widetilde{X}_{\widetilde{\Theta}'_k} \notin N(\widetilde{O}_k \widetilde{\Omega}_k/\widetilde{F}_k \widetilde{\Omega}_k)$ in case $\widetilde{O}_k/\widetilde{\Theta}_k$ is a finite extension and $\widetilde{F}_k \in I(\widetilde{O}_k/\widetilde{\Theta}_k)$. Hence, Lemma 5.3 applies to $\widetilde{\Theta}'_k \widetilde{\Omega}_k/\widetilde{\Theta}''_k \widetilde{\Omega}_k$, $\widetilde{X}_{\widetilde{\Theta}'_k}$

and any $p \in \overline{P}$ dividing $[\tilde{\Theta}'_k: \tilde{\Theta}''_k]$. Identifying $\tilde{\Theta}_{k,\text{sep}}$ and $\tilde{\Omega}_{k,\text{sep}}$ with their isomorphic copies in $\Delta_{k,\text{sep}}$ (over $\tilde{\Theta}_k$ and $\tilde{\Omega}_k$, respectively), one deduces from Proposition 4.1 that $\text{Br}(\tilde{\nabla}_k \Delta_k / \tilde{\nabla}_k \tilde{\Omega}_k) = \{0\}$ (and $\text{Br}(\tilde{\nabla}_k \Delta_k / \tilde{\nabla}_k) = \{0\}$), for each finite extension $\tilde{\nabla}_k$ of $\tilde{\Theta}_k$ in $\tilde{\Theta}_{k,\text{sep}}$. Using Galois theory and Lemmas 3.6, 5.3 and 2.2, one also proves the following:

$$(5.3) \text{ (i) } X_{\Theta'_k} \in N(\Theta''_k \Delta_k / \Delta_k) \setminus N(\Theta'_k \Delta_k / \Theta''_k \Delta_k), \Theta'_k \in \text{Un}(\Theta_k);$$

$$\text{(ii) } \tilde{X}_{\tilde{\Theta}'_k} \in N(\tilde{\Theta}''_k \Delta_k / \Delta_k) \setminus N(\tilde{\Theta}'_k \Delta_k / \tilde{\Theta}''_k \Delta_k), \tilde{\Theta}'_k \in \text{Un}(\tilde{\Theta}_k)'.$$

We are now in a position to finish the construction of E_{k+1} and to show that the field $E = \cup_{n=1}^{\infty} E_n$ is quasilocal and has the properties required by Theorem 1.2 (i) and (ii). Let $E_{k+1} = \Delta_k$, provided that $\chi \neq \text{Nil}$, and suppose further that $\chi = \text{Nil}$. Denote by $Z(\Delta_k)'$ the field $\Delta_{k,\text{Sol}}$, if $\chi' = \text{Nil}$, and put $Z(\Delta_k)' = (\Delta_k \Sigma_{k,2}) \cap \Delta_{k,\text{Sol}}$, otherwise. Fix a rational field extension $\hat{\Omega}_k / \Delta_k$ with a tr-basis $\hat{X} = \{\hat{X}_{\Delta'_k}\}$, indexed by all $\Delta'_k \in I(Z(\Delta_k)' / \Delta_k)$, for which Δ'_k / Δ_k satisfies the conditions of Lemma 5.1. Taking as E_{k+1} the tensor compositum over $\hat{\Omega}_k$ of the fields $\Lambda((\Delta'_k \cap \Delta_{k,\text{Nil}}) \cdot \hat{\Omega}_k / \hat{\Omega}_k; \hat{X}_{\Delta'_k})$, $\hat{X}_{\Delta'_k} \in \hat{X}$, we finish the construction of E . It is easily verified that E_{n+1} / E_n and E / E_n are regular, and that $E_n, \Lambda_n, W_n, \Theta_n, \tilde{\Theta}_n$ and Δ_n are algebraically closed in E_{n+1} and E , for all $n \in \mathbb{N}$. Hence, by Galois theory and the observation preceding statement (5.1), E has the following property:

$$(5.4) \text{ For each } G \in \text{Fin} \text{ and } j = 1, 2, \text{ there exists } M_j(G) \in \text{Gal}(E) \cap I(\Sigma_{0,j} E / E), \text{ such that } G(M_j(G) / E) \cong G.$$

Note also that $\rho_{E_n / E_{n+1}}$ maps T_n bijectively upon T_{n+1} , for every $n \in \mathbb{N}$, and each $L \in \text{Gal}(E)$ has a subfield $L_k \in \text{Gal}(\Lambda_k)$, such that $G(L_k / \Lambda_k) \cong G(L / E)$, for some $k \in \mathbb{N}$. When $L_k \subseteq Z_k$, this implies that $L_k \Lambda_n \subseteq Z_n$ and $\text{Ker}(\text{Cor}_{(L_k \Lambda_n) / \Lambda_n}) \subseteq \text{Br}(L_k E_n / L_k \Lambda_n)$, for $n > k$. Therefore, $\rho_{E_0 / E}$ maps $T_0 = T$ isomorphically on $\text{Br}(E)$, and whenever $L \in \text{Gal}(E)$ and $L \subseteq \cup_{n=1}^{\infty} Z_n := Z_{\infty}$, $\text{Cor}_{L / E}$ is injective. Similarly, it follows from Galois theory and the established properties of E_n, Λ_n, Θ_n and E_{n+1} that if E' and E'_p lie in $\text{Gal}(E)$ and $E' \subseteq E'_p \subseteq E'(p)$, for some $p \in \overline{P}$, then $\text{Br}(E'_p)_p \cap \text{Ker}(\text{Cor}_{E'_p / E'}) = \{0\}$. As $\text{Br}(E)$ is divisible, these observations show that E satisfies condition (c) of Proposition 3.4, for $\Omega = E_{\text{sep}}$ and each $p \in \overline{P}$. Thus it turns out that E is quasilocal and nonreal. Applying now Corollary 3.5, one also concludes that $\rho_{E / R}$ is surjective, for every finite extension R of E in Z_{∞} . Since, by (5.1), $E_{\chi} \subseteq Z_{\infty}$ in case $\chi \neq \text{Nil}$, the obtained results, combined with (1.1) (ii) and Theorem 1.1 (i), prove Theorem 1.2 (i) and (ii).

It remains to be seen that E has the properties required by Theorem 1.2 (iii) and (iv). Our argument goes along similar lines to those drawn in the proof of Theorem 1.2 (i) and (ii), so we present it omitting details. Let M be a field lying in $\text{Gal}(E)$. Suppose first that $G(M / (M \cap E_{\chi'})) \notin \text{Sol}$. Then $M = M_{\kappa} E$, for some $\kappa \in \mathbb{N}$, $M_{\kappa} \in \text{Un}(\Theta_{\kappa})$. Since Θ_{κ} is algebraically closed in E , this implies that $G(M_{\kappa} / \Theta_{\kappa}) \cong G(M / E)$. Let L_{κ} be a minimal subfield of M_{κ} lying in $\text{Un}(\Theta_k)$. Then it follows from the definition of E , (5.3) (i) and (5.2) (i) that $X_{L_{\kappa}}$ lies in $N(L \cap (E_{\chi'} E_{\text{Sol}}) / E) \setminus N(L / L \cap (E_{\chi'} E_{\text{Sol}}))$, where $L = L_{\kappa} E$. Hence, by Lemma 2.2, $X_{L_{\kappa}} \notin N(L / E)$. As $N(M / E) \subseteq N(L / E)$ and, by Lemma 3.2, $N(L \cap (E_{\chi'} E_{\text{Sol}}) / E) = N(L / E) N(M \cap (E_{\chi'} E_{\text{Sol}}) / E)$, this implies that $N(M / E) \neq N(M \cap (E_{\chi'} E_{\text{Sol}}) / E)$. When $\chi' \neq \text{Nil}$, the obtained result proves Theorem 1.2 (iii), since then $E_{\text{Sol}} \subseteq E_{\chi'}$. Similarly, one gets from (5.3) (ii) and (5.2) (ii) that

$N(M/E) \neq N(M \cap (E_\chi E_{\text{Sol}})/E)$ also in case $G(M/E) \in \chi'$, $G(M/(M \cap E_\chi)) \notin \text{Sol}$ and $M \subseteq \Sigma_{0,2}E$. Assume now that $\chi = \text{Nil}$, put $Z(E)' = \cup_{n=1} Z(\Delta_n)'$, and suppose that $M \subseteq Z(E)'$ and the pair $(M/E, \chi)$ is chosen as in Lemma 5.1. Then $G(M/E) \in \text{Sol}$ and our construction of E guarantees that $N(M/E) \neq N(M/E)_{\text{Nil}}$. As each $M' \in \text{Gal}(E)$, with $G(M'/E) \notin \text{Nil}$ and $M' \subset Z(E)'$, possesses a subfield satisfying the conditions of Lemma 5.1 with respect to E , this enables one to deduce from (1.1) (ii) and Theorem 1.1 (ii) that $N(M'/E) \neq N(M'/E)_{\text{Ab}}$. Thus Theorem 1.2 (iii) is proved. For the proof of Theorem 1.2 (iv), drop the assumption that $\chi = \text{Nil}$, let $\chi' \neq \chi$ and take fields $M_1(G)$ and $M_2(G)$ as in (5.4), for an arbitrary $G \in \chi' \setminus \chi$. As shown in the process of proving Theorem 1.2 (i)-(ii), then $\Sigma_{0,1}E \cap E_{\chi'} \subseteq Z_\infty$ and $N(M_1(G)/E) = N(M_1(G)/E)_{\text{Ab}}$. On the contrary, Theorem 1.2 (ii), the proof of Theorem 1.2 (iii) and the inclusion $\text{Ab} \subset \chi$ indicate that $N(M_2(G)/E) \neq N(M_2(G)/E)_\chi = N(M_2(G)/E)_{\text{Ab}}$, so Theorem 1.2 is proved.

Remark 5.4. With assumptions being as in Theorem 1.2, suppose that E_0 is infinite of cardinality $d \geq \max\{d_p: p \in \overline{P}\}$, where d_p is the dimension of the subgroup $\{t_p \in T: pt_p = 0\} \subseteq T$ as a vector space over \mathbb{F}_p , for each $p \in \overline{P}$. Analyzing the proofs of (4.3) (iv) and Theorem 1.2, one concludes that our construction of the quasilocal field E can be specified so that d equals the cardinality of E as well as the ranks of $G(E(p)/E)$ and of any Sylow pro- p -subgroup G_p of G_E as pro- p -groups, for each $p \in \overline{P}$. Applying [7, II, Lemma 3.3], one also sees that whenever E/E_0 , T and $\chi = \text{Fin}$ are related as in Theorem 1.2, d_p equals the dimension of the (continuous) cohomology group $H^2(G_p, \mathbb{F}_p)$ as an \mathbb{F}_p -vector space. Moreover, by the same lemma, then the dimension of $H^2(G(E(p)/E), \mathbb{F}_p)$ is equal to d_p or zero, depending on whether or not E_0 contains a primitive p -th root of unity.

6. Topological interpretation of Theorem 1.2 and applications to Brauer groups of fields

Let E be a field, $\text{char}(E) = q \geq 0$, $D(E)$ and $R(E)$ the divisible and the reduced parts of E^* , respectively. For each (nonempty group) formation $\Psi \subseteq \text{Fin}$, put $N_\Psi(E) = \cap_{M \in \text{Gal}(E)} N(M/E)_\Psi$. It is easily obtained that $E^*/N_\Psi(E)$ can be canonically viewed as a topological group (totally disconnected, see [28, Theorem 9]). Specifically, $N_\Psi(E) = D(E)$, if E is perfect and quasilocal, $\text{Br}(E)_p \neq \{0\}$, for every $p \in (\Pi(E) \setminus \{q\})$, and $\text{Met} \cap \text{Fin}_{\Pi(E)} \subseteq \Psi$, where $\text{Fin}_{\Pi(E)}$ is the class of all $G \in \text{Fin}$ of orders not divisible by any $p \in \overline{P} \setminus \Pi(E)$ (see [7, II, Sect. 2]). With the same assumptions on E , the equality $N_\Psi(E) = D(E)$ holds also when $P(E) \setminus \{q\} = \Pi(E) \setminus \{q\}$ and $\text{Ab} \cap \text{Fin}_{\Pi(E)} \subseteq \Psi$. Hence, by the isomorphism $R(E) \cong E^*/D(E)$, Ψ induces on $R(E)$ a structure T_Ψ of a topological group. Let Ψ_1 be another formation of this kind. It follows from Lemma 3.2 that the topologies T_Ψ and T_{Ψ_1} are equivalent ($T_\Psi \sim T_{\Psi_1}$) if and only if $N(M/E) = N(M/E)_{(\Psi \cap \Psi_1)}$ when $M \in \text{Gal}(E)$ and $M \subseteq (E_\Psi \cup E_{\Psi_1})$. Clearly, this occurs if and only if $T_\Psi \sim T_{(\Psi \cap \Psi_1)}$ and $T_{\Psi_1} \sim T_{(\Psi \cap \Psi_1)}$, so solving the equivalence problem for T_Ψ and T_{Ψ_1} reduces to the case in which $\Psi \subset \Psi_1$. Consider now the set Ω_{ac} of all pairs (χ, χ') of distinct subclasses of Fin satisfying the conditions of Theorem 1.2. A binary sequence $\bar{a} = a_{i,j}: (i,j) \in \Omega_{\text{ac}}$, is called multiplicative, if $a_{k,l}a_{l,m} = a_{k,m}$ for every (k,l) and (l,m) lying in Ω_{ac} . When $P(E) \setminus \{q\} = \Pi(E) \setminus \{q\}$, such a sequence $\bar{a}(E)$ can be canonically attached to E , putting $a_{i,j}(E) = 1$, if $T_i \sim T_j$, and $a_{i,j}(E) = 0$, otherwise. By Theorem 1.1 (i), $a_{i,j}(E) = 1$,

for all $(i, j) \in \Omega_{\text{ac}}$, provided that $\rho_{E/M}$ is surjective, for each $M \in \text{Gal}(E)$. This applies to the presently known, and conjecturally, to all perfect fields with LCFT in the sense of Neukirch-Perlis (see [8, Proposition 3.3 and Remark 3.4 (ii)]). Conversely, when \bar{a} is multiplicative and T is a divisible abelian torsion group, one obtains by modifying the proof of Theorem 1.2 that there exists a quasilocal perfect field F , such that:

(6.1) (i) $\text{Br}(F) \cong T$ and all $G \in \text{Fin}$ are realizable as Galois groups over F ; in particular, $\Pi(F) = P(F) = \bar{P}$.

(ii) $\bar{a} = \bar{a}(F)$ and $\rho_{F_i/M_{(i)}}$ is surjective when $a_{i,j} = 1$ and $M_{(i)} \in \text{Gal}(F_i) \cap I(F_j/E)$.

The groups $D(E)$ and $R(E)$ are related with group formations as above, also if E is SQL and almost perfect (in the sense of [7]).

Remark 6.1. Let R_{sim} be a system of representatives of the isomorphism classes of finite simple groups, $\Sigma_{\text{sim}} = \{S \subseteq R_{\text{sim}}: \text{Ab} \cap R_{\text{sim}} \subseteq S\}$, and for each $S \in \Sigma_{\text{sim}}$, let \tilde{S} be the class of those $G \in \text{Fin}$, whose simple quotients are isomorphic to groups from S . It follows from the Jordan-Hölder theorem that the correspondence $S \rightarrow \tilde{S}: S \in \Sigma_{\text{sim}}$, is injective and maps Σ_{sim} into the set of abelian closed classes.

Now we turn our attention to the Brauer groups of the basic types of PQL-fields. First we present two results which substantially generalize Theorem 1.2 (i) and (ii). They also complement Theorem 1.1 (i), [8, Corollary 6.2] and an observation made by M. Auslander (see [31, Ch. II, Sect. 3.1]) about the class of fields with trivial Brauer groups.

Proposition 6.2. *Let E_0 be a field, L_0 a Galois extension of E_0 in $E_{0,\text{sep}}$, S a nonempty set of profinite groups, and let T and T_0 satisfy the conditions of Theorem 1.2. Then there exists a regular field extension E/E_0 with the following properties:*

(i) E is quasilocal, $\text{Br}(E) \cong T$, $\rho_{E_0/E}$ maps T_0 injectively into $\text{Br}(E)$, and all $G \in S$ are realizable as Galois groups over E ;

(ii) A finite extension R of E in E_{sep} lies in $I(L_0E/E)$ if and only if $\rho_{E/R}$ is surjective; when $R \notin I(L_0E/E)$ and $p \in \bar{P}$ does not divide $[\tilde{R}: (R \cap L_0E)]$, where $\tilde{R} \subseteq E_{\text{sep}}$ is the normal closure of R over E , $\text{Br}(R)_p$ properly includes the image of $\text{Br}(E)_p$ under $\rho_{E/R}$.

Proof. The existence of E is proved constructively, and in this respect, our proof is very similar to the one of Theorem 1.2 (i) and (ii), in the special case where $\chi = \text{Fin}$. In the first place, it becomes clear that one may additionally assume that $T_0 = T$, all $G \in S$ are realizable as Galois groups over E_0 , and each $H \in \text{Ab}$ has an isomorphic copy $s_H \in S$. In this setting, E is obtained as a union $\cup_{n=0}^{\infty} \Lambda_n = \cup_{n=0}^{\infty} W_n = \cup_{n=0}^{\infty} \Theta_n = \cup_{n=0}^{\infty} E_n$ of inductively defined field towers, such that $E_n \subseteq \Lambda_n \subseteq W_n \subseteq \Theta_n \subseteq E_{n+1}$, for every index n . The extensions Λ_n/E_n and Θ_n/W_n are defined in exactly the same way as in the proof of Theorem 1.2 (whence they are regular), E_{n+1}/Θ_n are rational of tr-degree 2, and W_n is $\tilde{\otimes}_{\Lambda_n}$ of the fields $W(M_n/\Lambda_n)$, where M_n runs across $\text{Gal}(\Lambda_n) \cap I(L_0\Lambda_n/\Lambda_n)$. This ensures that E has the properties required by Proposition 6.2 (i) as well as the surjectivity of $\rho_{E/R}$, for every finite extension R of E in L_0E . In particular, the regularity of E/E_0 and the additional conditions satisfied by E_0 and S enable one to deduce from Galois theory that $C(\Phi(\pi)/\Phi)$ contains infinitely many elements of order π , for each $\Phi \in I(E/E_0)$ and $\pi \in \bar{P}$. It remains for us to prove the latter assertion of Proposition 6.2 (ii), so we assume

further that E , R and \tilde{R} satisfy its conditions. Fix some $p \in \bar{P}$ not dividing $[\tilde{R}:V]$, where $V = R \cap L_0E$, put $U_n = U \cap \Theta_{n,\text{sep}}$, for each index n and $U \in I(\tilde{R}/E)$, and define $\text{Im}_p(V/R)$ and $\text{Ker}_p(R/V)$ in accordance with the proof of Corollary 3.5. Using the RC-formula and Proposition 2.3 (ii), one obtains that it suffices to establish the inequality $\text{Im}_p(V/R) \neq \text{Br}(R)_p$ under the extra hypothesis that $I(R/E) \setminus \{R\} \subseteq I(L_0E/E)$. It follows from Proposition 2.3 (ii) and the choice of p that $\text{Br}(R)_p \cong \text{Im}_p(V/R) \oplus \text{Ker}_p(R/V)$ (see also [27, Sect. 13.4]), so our assertion is equivalent to the one that $\text{Ker}_p(R/V) \neq \{0\}$. Clearly, $\tilde{R} = \tilde{R}_kE$, for some integer $k \geq 0$ and $\tilde{R}_k \in \text{Gal}(\Theta_k)$. As E/Θ_k is regular, $G(\tilde{R}_k/\Theta_k)$ and $G(\tilde{R}/E)$ are canonically isomorphic, and for convenience, they will be further identified. Fix a tr-basis X_k, Y_k of E_{k+1}/Θ_k and a primitive element φ_k of \tilde{R}_k/Θ_k . Observe that E_{sep} contains as a subfield a cyclic extension Θ'_k of $\Theta_k(Y_k)$ of degree p , such that $\Theta'_k \cap \tilde{R}_k(Y_k) = \Theta_k(Y_k)$. This can be easily deduced from Galois theory and the fact that the maximal subgroup of $C(\Theta_k(Y_k)(p)/\Theta_k(Y_k))$ of exponent p is infinite. Regarding now the groups $\tilde{R}_k(X_k)/\tilde{R}_k(X_k)^{*p}$ and ${}_p\text{Br}(\tilde{R}_kE_{k+1})$ as modules over the group algebra $\mathbb{F}_p[G(\tilde{R}/E)]$, and considering the images of the element $X_k - \varphi_k$ under the action of $G(\tilde{R}/E)$, one obtains without difficulty that $\mathbb{F}_p[G(\tilde{R}/E)]$ is \mathbb{F}_p -isomorphic to submodules of $\tilde{R}_k(X_k)/\tilde{R}_k(X_k)^{*p}$ and ${}_p\text{Br}(\tilde{R}_kE_{k+1})$. Let $[R:V] = r$ and $\{\xi_j: j = 1, \dots, r\}$ be a system of representatives of the right co-sets of $G(\tilde{R}/R)$ in $G(\tilde{R}/V)$. The embeddability of $\mathbb{F}_p[G(\tilde{R}/E)]$ in ${}_p\text{Br}(\tilde{R}_kE_{k+1})$ indicates that ${}_p\text{Br}(\tilde{R}_kE_{k+1})$ contains an element δ , such that $[\sum_{j=1}^r (\xi_j - 1)]\delta$ lies in $\text{Ker}(\tilde{R}_kE_{k+1}/V_kE_{k+1}) \setminus \text{Ker}(\tilde{R}_kE_{k+1}/R_kE_{k+1})$. This implies that $\text{Ker}_p(R_kE_{k+1}/V_kE_{k+1}) \neq \{0\}$. At the same time, it follows from Galois theory and the regularity of E/Θ_k that $U_n = U_k\Theta_n$, $V_n = R \cap L_0\Theta_n$ and $G(\tilde{R}_n/\Theta_n) \cong G(\tilde{R}/E)$, for $n \geq k$ and $U \in I(\tilde{R}/E)$. Hence, by Proposition 4.6 (i), the extra hypothesis on R and the noted properties of the construction of E , $\text{Br}(R_n)_p \cap \text{Br}(R_{n+1}/R_n) \subseteq \text{Im}_p(V_n/R_n)$, for every index $n \geq k$. In view of the RC-formula and the choice of p , these observations show that $\rho_{R_k/R}$ maps $\text{Ker}_p(R_k/V_k)$ injectively into $\text{Ker}_p(R/V)$. In particular, it turns out that $\text{Ker}_p(R/V) \neq \{0\}$, so Proposition 6.2 is proved.

Our next result, applied to the formation Ψ of supersolvable groups $W \in \text{Fin}$, and to a set S of profinite groups containing an isomorphic copy of the symmetric group Sym_4 , proves the existence of a quasilocal field E , such that Sym_4 is realizable as a Galois group over E , and each $M \in \text{Gal}(E)$ with $G(M/E) \cong \text{Sym}_4$ satisfies both the condition of Corollary 3.10 and the one that $\rho_{E/\Phi}$ is not surjective, for any $\Phi \in \text{Gal}(E)$ including M . This indicates that the assertions of Theorem 1.2 (i), (ii) and (iv) cannot simultaneously be true, if χ and χ' are replaced by Ψ and Sol , respectively.

Proposition 6.3. *Let T and S satisfy the conditions of Proposition 6.2, and let $\Psi \subseteq \text{Fin}$ be a formation. Then there exists a quasilocal field E , such that $\text{Br}(E) \cong T$, every $G \in S$ is realizable as a Galois group over E , and a finite extension R of E in E_{sep} lies in $I(E_\Psi/E)$ if and only if $\rho_{E/R}$ is surjective.*

Proof. The field E can be obtained as an extension of an arbitrary fixed field E_0 , which is a union $E = \cup_{n=0}^\infty E_n$ of an inductively defined tower, such that E_n/E_{n-1} has the properties required by Proposition 6.2 with respect to $E_{n-1,\Psi}$, for each $n \in \mathbb{N}$.

Proposition 2.3 (ii) and the following result, combined with [15, Theorem 23.1], describe the isomorphism classes of Brauer groups of PQL-fields.

Proposition 6.4. *Let E_0 be a formally real field, $\text{Odd} \subset \text{Fin}$ the class of groups of odd orders, $\Psi \subseteq \text{Odd}$ a formation, and T an abelian torsion group, such that the p -components T_p : $p \in \overline{P} \setminus \{2\}$, are divisible, and T_2 of order 2. Suppose also that T_0 is a subgroup of $\text{Br}(E_0)$ embeddable in T and with $T_{0,2} = \{0\}$. Then there exists a field extension E/E_0 with the following properties:*

(i) *E is formally real and PQL, $E_0(2)$ contains as a subfield the algebraic closure of E_0 in E , $\text{Br}(E) \cong T$ and $T_0 \cap \text{Br}(E/E_0) = \{0\}$;*

(ii) *Finite extensions of E in E_{Odd} are p -quasilocal, for every $p \in \overline{P} \setminus \{2\}$, and all $G \in \text{Odd}$ are realizable as Galois groups over E ;*

(iii) *A finite extension R of E in E_{Odd} is included in E_Ψ if and only if $\text{Br}(R)_p \subseteq \text{Im}(E/R)$, for every $p \in \overline{P} \setminus \{2\}$; if $R \notin I(E_\Psi/E)$, then $\text{Br}(R)_{p'} \cap \text{Im}(E/R) \neq \text{Br}(R)_{p'}$, for any $p' \in \overline{P} \setminus \{2\}$, $p' \nmid [R: E_\Psi \cap R]$.*

Proof. We obtain E as a union $\cup_{n=1}^\infty E_n$ of a field tower defined inductively as in the proof of Proposition 6.3. Omitting the details, note that E is constructed by considering Odd , $\overline{P} \setminus \{2\}$ and the sums of the subgroups $T_{0,p}$: $p > 2$, $\text{Br}(E_n)_p$: $p > 2$, instead of Fin , \overline{P} , T_0 and $\text{Br}(E_n)$, respectively, extending at the end of the inductive step the obtained field by a real closure in its maximal 2-extension. The correctness of the construction is proved essentially as in the proof of Proposition 6.2; the Artin-Schreier theory ensures that E_n , $n \in \mathbb{N}$, and E are formally real, since the function field of each of the Brauer-Severi varieties V used for constructing E embeds over the field of definition $\Phi(V)$ of V into a field $R(V)$ that is rational over some extension $\Phi'(V)$ of $\Phi(V)$ of odd degree.

Remark 6.5. In the setting of Proposition 6.4, when $\Psi = \{1\}$, the proof of Theorem 1.2 enables one to modify the construction of E so as to satisfy the inequality $N(M/E) \neq N(M/E)_{\text{Ab}}$, for each $M \in \text{Gal}(E)$ with $G(M/E) \in \text{Odd} \setminus \text{Nil}$.

Our next result complements [8, Theorems 1.2, 1.3 and Proposition 3.6] as follows:

Corollary 6.6. *Let T satisfy the condition of some of Propositions 6.3 or 6.4, $\Psi = \text{Fin}$ in the former case, $\Psi = \text{Odd}$ in the latter one, and let $\text{Sup}(T) = \{p \in \overline{P}: T_p \neq \{0\}\}$. Then there is a strictly PQL-field F , such that $\text{Br}(F) \cong T$ and every $G \in \Psi$, for which the index $|G: [G, G]|$ has no divisor $p \in \overline{P} \setminus \text{Sup}(T)$, is realizable as a Galois group over F . When T is divisible and $\text{Sup}(T) = \overline{P}$, F can be chosen from the class of SQL-fields.*

Proof. Our latter conclusion follows at once from Theorem 1.2, and in case $\text{Sup}(T) = \overline{P}$, the former one is contained in Theorem 1.2 and Proposition 6.4. Assume now that $\text{Sup}(T) \neq \overline{P}$, put $\overline{\text{Sup}}(T) = \overline{P} \setminus \text{Sup}(T)$, consider a field F_0 with $\text{Br}(F_0) \cong T$, and let $T_0 = \{0\} \subset \text{Br}(F_0)$. Then one can take as F the union $\cup_{n=0}^\infty F'_n = \cup_{n=0}^\infty F''_n = \cup_{n=1}^\infty F_n$ of fields defined inductively so as to satisfy the following conditions, for each index $n \geq 0$:

(6.2) (i) $\text{Br}(F'_n) \cong T$ and F'_n/F_n has the properties required by Theorem 1.2 or Proposition 6.4. More precisely, ρ_{F_0/F'_n} is an isomorphism and, for every $p \in \overline{P} \setminus \{2\}$ and each finite extension R_n of F'_n in $F'_{n,\Psi}$, $\rho_{F'_n/R_n}$ maps $\text{Br}(F'_n)_p$ surjectively on $\text{Br}(R_n)_p$.

(ii) F''_n is the compositum of the fields $F'_n(p)$: $p \in \overline{\text{Sup}}(T)$.

(iii) $F_{n+1} = F_n''$, if T is divisible, and F_{n+1} is a real closure of F_n'' in $F_n''(2)$, otherwise.

When T and F_0 have the properties required by Theorem 1.2 (i)-(ii), for $\chi = \text{Fin}$, one may use only a countable iteration of (6.2) (ii), with (F_n, F_{n+1}) instead of (F_n', F_n'') , for any n .

Corollary 6.7. *Let E be a field and T an abelian torsion group with the properties required by Theorem 1.2 (i), Proposition 6.4 or Corollary 6.6. Assume E has a Henselian valuation v . Then the value group $v(E)$ of (E, v) is divisible and every $D \in d(E)$ is defectless with respect to v .*

Proof. Consider first a Henselian valued quasilocal field (F, w) , such that $w(F) = qw(F)$, $\text{Br}(F)_q \neq \{0\}$, $\text{char}(\widehat{F}) = q > 0$, and $q \in \Pi(\widehat{F})$, where \widehat{F} is the residue field of (F, w) . By [20, Theorem 3.16], F and \widehat{F} are nonreal fields. For each finite extension L/F , denote by $w(L)$ the value group, and by \widehat{L} the residue field of L with respect to its unique (up-to an equivalence) valuation w_L extending w . It is well-known that $w(F)$ is a subgroup of $w(L)$ of index $|w(L):w(F)| \leq [L:F]$ (cf. [21, Ch. XII, Proposition 12]), and that $w(L)$ is isomorphic to a totally ordered subgroup of $w(F)$. The equality $w(F) = qw(F)$ implies that q does not divide $|w(L):w(F)|$, and yields $w(L) = qw(L)$ as well. We show that L/F is defectless with respect to w (and w_L), provided that $L \subset F_{\text{sep}}$. The inequality $\text{cd}_q(G_{\widehat{F}}) > 0$ and [19, Theorem 2.8 (a)] guarantee the existence of an inertial finite extension Φ/F , such that $q \nmid [\Phi:F]$ and $q \in P(\widehat{\Phi})$. At the same time, it is easily seen that $\text{Br}(\Phi/F) \cap \text{Br}(F)_q = \{0\}$ (cf. [27, Sect. 13.4]), whence the nontriviality of $\text{Br}(F)_q$ is preserved by $\text{Br}(\Phi)_q$. Moreover, it follows from Ostrowski's theorem (cf. [12]) that our assertion holds if and only if finite extensions of Φ in Φ_{sep} are defectless with respect to v_{Φ} , so the preceding observations reduce its proof to the special case in which $q \in P(\widehat{F})$. Then the quasilocal property of F and the nontriviality of $\text{Br}(F)_q$ imply that $N(L_1/F) \neq F^*$, for every inertial cyclic extension L_1 of F in $F(q)$. Hence, by the lifting property of w , applied to a norm form of L_1/F with coefficients in the valuation ring of (F, w) , the assumption on $w(F)$ ensures that $N(\widehat{L}_1/\widehat{F}) \neq \widehat{F}^*$. In view of [27, Sect. 15.1, Proposition b] and [19, Theorem 2.8 (a)], this proves the existence of an inertial F -algebra $\Delta_q \in d(F)$ of index q . Observing that every extension of F embeddable in Δ_q is inertial, one obtains from the quasilocal property of F that there are no immediate cyclic extensions of F of degree q . Since, by Witt's theorem, $\text{Br}(F)_q$ is divisible, and by [42, Theorem 2] and Galois theory, $q \in P(\widehat{R})$, for every finite extension R/F , this observation makes it easy to deduce the claimed defectlessness of L/F (in case $L \subset F_{\text{sep}}$) from Ostrowski's theorem and well-known formulae about valuation prolongations. As shown in [36], the obtained result implies that every $\Delta \in d(F)$ is defectless. When $w(F)$ is divisible, this means that Δ is inertial over F . Also, it follows from the Ostrowski-Draxl theorem [12], that our conclusions remain valid, if (F, v) is a Henselian valued field, such that $\text{char}(\widehat{F}) = 0$. Thus the latter conclusion of Corollary 6.7 turns out to be a consequence of the former one.

Our objective now is to establish the divisibility of $v(E)$. In the first two cases, this follows directly from the former assertion of [8, (2.3)]. In the third one, one obtains from Galois theory that if E is nonreal, then simple groups $\Sigma \in \text{Fin} \setminus \text{Ab}$ are realizable as Galois groups over E , so our assertion again is implied by the noted part of [8, (2.3)]. Assume further that E is formally real, $\text{Br}(E) \cong T$, $\text{Sup}(T) \neq \overline{P}$ and E has the properties required

by Corollary 6.6. By [20, Theorem 3.16], then the residue field \widehat{E} of (E, v) is formally real, whence $\text{char}(\widehat{E}) = 0$. In addition, it is well-known that if $G \in \text{Fin}$, $p \in \overline{P}$ and ν is the order of G , then G embeds in $\text{Aut}(P(G))$, where $P(G) \in \text{Ab}$ is of exponent p and order p^ν . When $G \notin \text{Ab}$ and $p \nmid \nu$, this implies that G has an irreducible representation over \mathbb{F}_p of dimension ≥ 2 . Hence, by Galois theory and the assumptions on E , for each $p \in \overline{P} \setminus \{2\}$, there exists $M_p \in \text{Gal}(E) \cap I(E_{\text{Odd}}/E)$, such that $p \nmid [M_p: E]$ and $G(M_p(p)/M_p)$ is of rank ≥ 2 as a pro- p -group (one may put $M_p = E$, if $p \in \text{Sup}(T)$). Let now v_p be the prolongation of v on M_p . By the proof of [8, (2.3)], then $v_p(M_p) = pv_p(M_p) = 2v_p(M_p)$ (M_p is formally real), and since $v(E)$ is a subgroup of $v_p(M_p)$ of index $\leq [M_p: E]$, this means that $v(E) = pv(E)$, $p \in \overline{P}$, which completes our proof.

In conclusion, we use Theorem 1.2 (i) for describing, up-to an isomorphism, the abelian torsion groups that can be realized as reduced parts of Brauer groups of absolutely stable fields K possessing equicharacteristic Henselian valuations v , such that $v(K)$ are totally indivisible (i.e. with $v(K) \neq pv(K)$, for every $p \in \overline{P}$). This enables one to construct various new examples of Brauer groups that are not simply presentable (see [16, Lemma 1 and pages 492-493]). Before giving the description, note that the residue field \widehat{K} is quasilocal, and that a Henselian discrete valued field (L, w) is absolutely stable, provided that \widehat{L} is quasilocal and perfect (see [7, I, Proposition 2.3 and the beginning of Sect. 8]).

Proposition 6.8. *An abelian torsion group Θ is isomorphic to a maximal reduced subgroup of $\text{Br}(K)$, for an absolutely stable field $K = K(\Theta)$ with a Henselian valuation v such that $\text{char}(K) = \text{char}(\widehat{K})$ and $v(K)$ is totally indivisible if and only if the p -component Θ_p decomposes into a direct sum of cyclic groups of the same order p^{n_p} , for each $p \in \overline{P}$.*

Proof. The necessity of the conditions on Θ has been proved in [7, II, Sect. 3], so we show here only their sufficiency. Let $\widetilde{\Theta}$ be a divisible hull of Θ , $\Pi(\Theta) = \{p \in \overline{P}: n_p > 0\}$, $\overline{\mathbb{Q}}$ an algebraic closure of \mathbb{Q} , and $\varepsilon_n \in \overline{\mathbb{Q}}$ a primitive n -th root of unity, for each $n \in \mathbb{N}$. Denote by F_0 the extension of \mathbb{Q} generated by the set $\{\varepsilon_{p^{n_p}}: p \in \Pi(\Theta)\} \cup \{\varepsilon_{p^\nu}, \nu \in \mathbb{N}: p \in \overline{P} \setminus \Pi(\Theta)\}$. Take an extension E/F_0 in accordance with Theorem 1.2 so that $\text{Br}(E) \cong \widetilde{\Theta}$, and consider a Henselian discrete valued field $(K(\Theta), v)$ with a residue field F_0 -isomorphic to E . By Scharlau's generalization of Witt's decomposition theorem [31], then $\text{Br}(K(\Theta)) \cong \text{Br}(E) \oplus C_E$ (see also [38, (3.10)]), so it follows from (2.5) that the reduced part of $\text{Br}(K(\Theta))$ is isomorphic to Θ . Thus Proposition 6.8 is proved.

Note finally that the groups Θ singled out by Proposition 6.8 are those realizable as reduced parts of Brauer groups of Henselian discrete valued absolutely stable fields. The sufficiency of the conditions on Θ is shown by the proof of Proposition 6.8, and their necessity follows from [7, II, Lemma 3.2], [38, (3.10)] and the divisibility of $\text{Br}(F)_q$ and $C(F(q)/F)$, for any field F of characteristic $q \neq 0$ (Witt, see [11, Sect. 15]). On the other hand, each sequence $\{p^{h_p}: h_p \in (\mathbb{N} \cup \{\infty\}), p \in \overline{P}\}$ with $h_2 \neq \infty$ and $p^{h_p} := \infty, h_p = \infty$, equals the sequence $\{e_p(K): p \in \overline{P}\}$ of exponents of reduced parts of $\text{Br}(K)_p$, for some Henselian discrete valued stable field (K, v) (see the reference at the end of [10, Sect. 3]).

References

- [1] A.A. Albert, *Modern Higher Algebra*. Univ. of Chicago Press, XIV, Chicago, Ill., 1937.
- [2] E. Artin, J. Tate, *Class Field Theory*. Benjamin, New York-Amsterdam, 1968.
- [3] A. Blanchet, Function fields of generalized Brauer-Severi varieties. *Comm. Algebra* 19 (1991), No 1, 97-118.
- [4] L. Bröcker, Characterization of fans and hereditarily Pythagorean fields. *Math. Z.* 151 (1976), 149-173.
- [5] I.D. Chipchakov, On nilpotent Galois groups and the scope of the norm limitation theorem in one-dimensional abstract local class field theory. In: *Proc. of ICTAMI 05*, Alba Iulia, Romania, 15.9-18.9, 2005; *Acta Univ. Apulensis* 10 (2005), 149-167.
- [6] I.D. Chipchakov, Class field theory for strictly quasilocal fields with Henselian discrete valuations. *Manuscr. math.* 119 (2006), 383-394.
- [7] I.D. Chipchakov, On the residue fields of Henselian valued stable fields. I, *J. Algebra* 319 (2008), 16-49; II, *C.R. Acad. Bulg. Sci* 60 (2007), No 5, 471-478.
- [8] I.D. Chipchakov, One-dimensional abstract local class field theory. Preprint, arXiv:math/0506515v5 [math.RA].
- [9] I.D. Chipchakov, Norm groups and class fields of formally real quasi-local fields. Preprint, arXiv:math/0508019v1 [math.RA].
- [10] I.D. Chipchakov, Embeddings in quasilocal fields and computations in Brauer groups of arbitrary fields. *C.R. Acad. Bulg. Sci.* 61 (2008), 1229-1238.
- [11] P.K. Draxl, *Skew Fields*. Lond. Math. Soc. Lecture Notes, Cambridge etc., Cambridge Univ. Press, 1983.
- [12] P.K. Draxl, Ostrowski's theorem for Henselian valued skew fields. *J. Reine Angew Math.* 354 (1984), 213-218.
- [13] B. Fein, D.J. Saltman, M. Schacher, Embedding problems for finite dimensional division algebras. *J. Algebra* 167 (1994), No 3, 588-626.
- [14] B. Fein, M. Schacher, Relative Brauer groups, I. *J. Reine Angew. Math.* 321 (1981), 179-194.
- [15] L. Fuchs, *Infinite Abelian Groups*. V. I and II, Pure and Applied Mathematics, Academic Press, IX and XI, New York-London, 1970.
- [16] A.W. Hales, Abelian groups as Brauer and character groups. *Abelian group theory*, Proc. 3rd Conf., Oberwolfach/FRG 1985, 487-507 (1987).
- [17] K. Iwasawa, *Local Class Field Theory*. Iwanami Shoten, Japan, 1980 (Japanese: Russian transl. in Mir, Moscow, 1983).
- [18] G. Karpilovsky, *Topics in Field Theory*. North-Holland Math. Studies, 155, North Holland, Amsterdam etc., 1989.
- [19] B. Jacob and A. Wadsworth, Division algebras over Henselian fields. *J. Algebra* 128 (1990), 126-179.
- [20] T.Y. Lam, *Orderings, valuations and quadratic forms*. Conf. Board Math. Sci. Regional Conf. Ser. Math. No 52, Amer. Math. Soc., Providence, RI, 1983.
- [21] S. Lang, *Algebra*. Addison-Wesley Publ. Comp., Mass., 1965.
- [22] P. Mammone, A. Merkurjev, On the corestriction of the p^n -symbol. *Isr. J. Math.* 76 (1991), 73-80.

- [23] A.S. Merkurjev, Index reduction formula. *J. Ramanujan Math. Soc.* 12 (1997), 49-95.
- [24] A.S. Merkurjev, I.A. Panin, A.R. Wadsworth, Index reduction formulas for twisted flag varieties. I, *K-Theory* 10 (1996), 517-596; II, *K-Theory* 14 (1998), 101-196.
- [25] A.S. Merkurjev, A.A. Suslin, K -cohomology of Brauer-Severi varieties and the norm residue homomorphism. *Izv. Akad. Nauk SSSR* 46 (1982), 1011-1046 (Russian: Engl. transl. in *Math. USSR Izv.* 21 (1983), 307-340).
- [26] J. Neukirch, R. Perlis, Fields with local class field theory. *J. Algebra* 42 (1976), 531-536.
- [27] R. Pierce, *Associative Algebras*. Graduate Texts in Math. 88, Springer-Verlag, New York-Heidelberg-Berlin, 1982.
- [28] L.S. Pontrjagin, *Topological Groups*, 4-th Ed. Nauka, Moscow, 1984 (Russian).
- [29] L. Redei, *Algebra*, v. 1. Akademiai Kiado, Budapest, 1967.
- [30] D.J. Saltman, The Schur index and Moody's theorem. *K-Theory* 7 (1993), No 3, 309-332.
- [31] W. Scharlau, Über die Brauer-Gruppe eines Hensel-Körpers. *Abh. Math. Semin. Univ. Hamb.* 33 (1969), 243-249.
- [32] J.-P. Serre, *Cohomologie Galoisienne*. Lecture Notes in Math. 5, Springer-Verlag, Berlin-Heidelberg-New York, 1965.
- [33] J.-P. Serre, *Local Fields*. Graduate Texts in Mathematics 88, Springer-Verlag, Berlin-Heidelberg-New York, 1979.
- [34] L.A. Shemetkov, *Formations of Finite Groups*. Nauka, Moscow, 1978 (in Russian).
- [35] J.-P. Tignol, On the corestriction of central simple algebras. *Math. Z.* 194 (1987), No 2, 267-274.
- [36] I.L. Tomchin and V.I. Yanchevskij, On defects of valued division algebras. *Algebra i Analiz* 3 (1991), No 3, 147-164 (Russian: Engl. transl. in *St. Petersburg Math. J.* 3 (1992), No 3, 631-647).
- [37] M. Van den Bergh, A. Schofield, Division algebra coproducts of index n . *Trans. Amer. Math. Soc.* 341 (1994), 505-518.
- [38] A.R. Wadsworth, Valuation theory on finite dimensional division algebras. In: F.-V. Kuhlmann (ed.) et al., *Valuation Theory and its applications*, I. Proc. of international conference and workshop, Univ. of Saskatchewan, Saskatoon, Canada, 28.7-11.8, 1999.
- [39] W.C. Waterhouse, Profinite groups are Galois groups. *Proc. Amer. Math. Soc.* 42 (1974), 639-640.
- [40] E. Weiss, *Cohomology of Groups*. Pure and Applied Mathematics 34, Academic Press, New York-London, 1969.
- [41] G. Whaples, Generalized local class field theory. II. Existence Theorem. *Duke Math. J.* 21 (1954), 247-255.
- [42] G. Whaples, Algebraic extensions of arbitrary fields. *Duke Math. J.* 24 (1957), 201-204.