Constraint Algorithm for Extremals in Optimal Control Problems

María Barbero-Liñán * Miguel C. Muñoz-Lecanda†

Departamento de Matemática Aplicada IV Edificio C-3, Campus Norte UPC. C/ Jordi Girona 1. E-08034 Barcelona. Spain

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Abstract

A geometric method is described to characterize the different kinds of extremals in optimal control theory. This comes from the use of a presymplectic constraint algorithm starting from the necessary conditions given by Pontryagin's Maximum Principle. Apart from the design of this general algorithm useful for any optimal control problem, it is showed how it works to split the set of extremals and, in particular, to characterize the strict abnormality. An example of strict abnormal extremal for a particular control-affine system is also given.

Key words: Pontryagin's Maximum Principle, extremals, optimal control problems, abnormality, strict abnormality, presymplectic.

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1 Introduction

A difficult problem in optimal control is to obtain extremals, that is, curves candidates to be optimal solutions. The usual way to deal with that is through successive differentiations of some necessary conditions for optimality, see for instance [3, 6, 19]. Here we not only give a method to split the set of extremals for any optimal control problem, but also explain geometrically the meaning of the successive differentiations.

There are different kinds of extremals: normal, abnormal and strictly abnormal. The abnormal extremals have been partially ignored for years. In the nineties, the

^{*}e-mail: mbarbero@ma4.upc.edu

[†]e-mail: matmcml@ma4.upc.edu

papers by R. Montgomery, W. Liu and H. J. Sussmann [26, 23] showed up the importance of analyzing abnormal extremals, because they can be optimal. Therefore, the search for abnormal and also strict abnormal extremals has become an appealing issue for the last fourteen years, as is backed by [2, 4, 6, 8, 11, 12, 32, 33]. The main attraction of abnormality is its exclusive dependence on the geometry of the control system and for the strict abnormality is the fact that the strict abnormal extremals are not normal.

The essential result to describe the general method of this paper and to have techniques to solve optimal control problems is Pontryagin's Maximum Principle, despite only providing necessary conditions for optimality. Although the natural geometric framework of Pontryagin's Maximum Principle is the symplectic one [18, 22, 30], to our purpose the presymplectic formalism will be more useful [13, 25]. Then, we have an implicit equation including some compatibility conditions, that must be satisfied in order to have solution, besides the dynamical Hamilton's equations. The former is a necessary condition of the maximization of the hamiltonian over the controls according to the classic Maximum Principle [20, 29]. Hence, in the presymplectic framework a weaker version of Pontryagin's Maximum Principle is stated. Instead of the above classic necessary condition, we have an implicit differential equation that sets up a constraint algorithm in the sense given in [10, 15, 16, 17]. This presymplectic algorithm comes from the Dirac-Bergmann theory of constraints developed in the fifties for quantum field theory. This algorithm has been already adapted and used to study singular optimal control problems [13] and to study optimal control problems with nonholonomic constraints [21].

The aim of this paper, according to the previous optimal control formulation, is to give a precise and geometric description of how to use the constraint algorithm to determine where the dynamics of normal extremals takes place and also the dynamics of abnormal ones. We also obtain sufficient conditions to have both kinds of extremals. These conditions elucidate how to determine the strict abnormality. This adaptation of the algorithm to the study of the extremals is mostly developed in §3, under the assumption of the control set being open and the differentiability with respect to the controls whenever is needed.

The importance and the generality of the theory elaborated can be highlighted by the revisit of the characterization of abnormal extremals in some known examples such as subRiemannian geometry and single-input control-affine systems. Using the algorithm and distinguishing different cases that come up, it may be checked that some of the situations correspond with the results obtained by A. A. Agrachev, Y. Sachkov, I. Zelenko, W. Liu and H.J. Sussmann [3, 4, 23]. Our method collects all their results on the existence of abnormal extremals. So the described procedure allows us to study geometric and generically the extremals for any control system and obtain the dynamics of the abnormal extremals in a natural and understandable way.

The organization of the paper is as follows: in Section 2 after a brief review of

some notions in optimal control theory, we state the optimal control problem and Pontryagin's Maximum Principle in the suitable framework for this paper, that is, in the presymplectic one. Section 3 concentrates on the new material, so it is devoted to describe the geometric process used to characterize extremals in optimal control problems with fixed time. After studying the fixed time problem, we explain how the algorithm works for the free time case in Section 4. Finally, in Section 5, we find a strict abnormal extremal for a control-affine system using the presymplectic constraint algorithm.

In the sequel, all the manifolds are real, second countable and \mathcal{C}^{∞} . The maps are assumed to be \mathcal{C}^{∞} . Sum over repeated indices is understood.

2 Presymplectic optimal control problems

A control system is defined by a set of differential equations depending on parameters. More precisely, let M be a smooth manifold, dim M=m, U be an open set of \mathbb{R}^k called the control set with $k \leq m$. A vector field X along the projection $\pi \colon M \times U \to M$ is a map $X \colon M \times U \to TM$ such that the following diagram is commutative

$$\begin{array}{c|c} & TM \\ X & \tau_M \\ M \times U \xrightarrow{\pi} M \end{array}$$

where τ_M is the natural projection of the tangent bundle. We denote the set of these vector fields as $\mathfrak{X}(\pi)$. A control system is an element of $\mathfrak{X}(\pi)$.

Let $I \subset \mathbb{R}$, a curve $(\gamma, u) : I \to M \times U$ is an integral curve of X if

$$\dot{\gamma} = X \circ (\gamma, u), \text{ that is, } \dot{\gamma}(t) = X(\gamma(t), u(t)).$$
 (2.1)

Now, we can introduce a cost function $F: M \times U \to \mathbb{R}$ and the functional

$$\mathcal{S}[\gamma, u] = \int_{I} F(\gamma, u) \, dt$$

defined on curves (γ, u) with a compact interval as domain. We are interested in the following problem:

Problem 2.1. (Optimal Control Problem, OCP)

Given the elements M, U, X, F, $I = [a, b], x_a, x_b \in M$. Find (γ, u) such that

- (1) the endpoint conditions are satisfied $\gamma(a) = x_a$, $\gamma(b) = x_b$,
- (2) $\dot{\gamma}(t) = X(\gamma(t), u(t)), t \in I$, and
- (3) $S[\gamma, u]$ is minimum over all curves on $M \times U$ satisfying (1) and (2).

A solution (γ, u) to this problem is called *optimal curve*. The mappings $(\gamma, u): I \to M \times U$ are piecewise differentiable and the vector field X along π and the cost function $F: M \times U \to \mathbb{R}$ are differentiable enough.

2.1 Pontryagin's Maximum Principle

As was said in §1, we state Pontryagin's Maximum Principle from a presymplectic viewpoint [13, 14, 21, 25]. In this approach, the main elements are:

- The presymplectic manifold $(T^*M \times U, \Omega)$, where Ω is the closed 2-form on $T^*M \times U$ given by the pull-back through $\pi_1 : T^*M \times U \to T^*M$ of the canonical 2-form on T^*M .
- A presymplectic Hamiltonian system $(T^*M \times U, \Omega, H)$, where $H: T^*M \times U \to \mathbb{R}$ is the Pontryagin's Hamiltonian function given by

$$H(\lambda, u) = \langle \lambda, X(x, u) \rangle + p_0 F(x, u) = H_X(\lambda, u) + p_0 F(x, u),$$

with
$$\lambda \in T_x^*M$$
, $p_0 \in \{-1,0\}$ and the notation $H_X(\lambda,u) = \langle \lambda, X(x,u) \rangle$.

Observe that the kernel of Ω contains the π_1 -vertical vector fields, that is, π_1 -projectable vector fields $Z \in \mathfrak{X}(T^*M \times U)$ such that $(\pi_1)_*Z = 0$. Thus, Ω is degenerate. For details in presymplectic formalism see [10, 15, 16, 17, 25, 27].

Theorem 2.2. (Pontryagin's Maximum Principle, presymplectic form) Let $U \subset \mathbb{R}^k$ be an open set and $(\gamma, u) \colon [a, b] \to M \times U$ be a solution of the optimal control problem 2.1 with endpoint conditions x_a , x_b . Then there exist $\lambda \colon [a, b] \to T^*M$ along γ (i.e. the natural projection of λ to M is γ), and a constant $p_0 \in \{-1, 0\}$ such that:

1. (λ, u) is an integral curve of a Hamiltonian vector field X_H that satisfies

$$i_{X_H}\Omega = dH$$
, that is, $i_{(\dot{\lambda}(t),\dot{u}(t))}\Omega = dH(\lambda(t),u(t));$ (2.2)

- 2. (a) $H(\lambda(t), u(t))$ is constant everywhere in $t \in [a, b]$;
 - (b) $(p_0, \lambda(t)) \neq 0$ for each $t \in [a, b]$.

As Ω is degenerate, (2.2) does not have solution in the whole manifold $T^*M \times U$. As explained in § 3, it may have a solution if we restrict the equation to the submanifold defined implicitly by

$$S = \{ \beta \in T^*M \times U \mid i_v \, dH = 0, \quad \text{for } v \in \ker \Omega_\beta \},$$

and locally,
$$S = \{ \beta \in T^*M \times U \mid \frac{\partial H}{\partial u_l}(\beta) = 0, \quad l = 1, \dots, k \}.$$

Remark 2.3. Observe that this is a necessary condition for the Hamiltonian to have an extremum over the controls as long as U is an open set. In the classic Pontryagin's Maximum Principle [29], the Hamiltonian is equal to the maximum of the Hamiltonian over the controls. Therefore, Theorem 2.2 is a weaker version of the classic Maximum Principle.

The necessary conditions 1-2 of Theorem 2.2 determine different kinds of extremals.

Definition 2.4. A curve (γ, u) : $[a, b] \to M \times U$ is

- 1. an extremal for OCP if there exist $\lambda \colon [a,b] \to T^*M$ and a constant $p_0 \in \{-1,0\}$ such that (λ,u) satisfies the necessary conditions of Pontryagin's Maximum Principle;
- 2. a **normal extremal for** OCP if it is an extremal and $p_0 = -1$, that is, the Hamiltonian is $H^{[-1]} = H_X F$;
- 3. an abnormal extremal for OCP if it is an extremal and $p_0 = 0$, that is, the Hamiltonian is $H^{[0]} = H_X$;
- 4. a strictly abnormal extremal for OCP if it is not a normal extremal, but it is an abnormal extremal;

The curve (λ, u) : $[a, b] \to T^*M \times U$ is called **biextremal for** OCP.

Pontryagin's Maximum Principle lifts optimal solutions to the cotangent bundle. The uniqueness of the lifts is not guaranteed, that is, some extremals could be lifted in two different ways: normal and abnormal.

3 Characterization of extremals

Here we take advantage of the necessary conditions in Theorem 2.2 to determine where the different kinds of extremals above defined are contained. We are specially interested in strict abnormal extremals and abnormal extremals as a consequence of [23, 26]. A meaningful and constructive procedure in presymplectic manifolds in order to find a solution to Problem 3.1 is the constraint algorithm [10, 15, 16, 17, 27].

Problem 3.1. Given a presymplectic system (M, Ω, H) , find (N, X) such that

- (a) N is a submanifold of M,
- (b) $X \in \mathfrak{X}(M)$ is tangent to N and verifies $i_X \Omega = dH$ on N,
- (c) N is maximal among all the submanifolds satisfying (a) and (b).

As mentioned in § 2.1 the presymplectic equation (2.2), $i_{X_H}\Omega=\mathrm{d}H$, has solution in the primary constraint submanifold $N_0=\{x\in M\,|\,\exists\,v_x\in T_xM\,,\quad i_{v_x}\Omega=\mathrm{d}_xH\}$, or equivalently, $N_0=\{x\in M\,|\,(\mathrm{L}_ZH)_x=0\,,\,\,\forall\,\,Z\in\ker\Omega\}$, where L_Z is the Lie derivative with respect to Z. See [16, 27] for details on these equivalence.

Locally, $X_H = A^i \partial/\partial x^i + B_j \partial/\partial p_j + C_l \partial/\partial u_l$ with $A^i = \partial H/\partial p_i$ and $B_j = -\partial H/\partial x^j$ because of the presymplectic equation (2.2). Moreover, as $\partial/\partial u_l \in \ker \Omega$,

$$N_0 = \{(\lambda, u) \in T^*M \times U \mid \frac{\partial H}{\partial u_l} = \lambda_j \frac{\partial X^j}{\partial u_l} + p_0 \frac{\partial F}{\partial u_l} = 0, \quad l = 1, \dots, k\}.$$
 (3.3)

The solution on N_0 is not necessarily unique. Indeed, if X_0 is a solution, then $X_0 + \ker \Omega$ is the set of all the solutions. We may consider X_0 as a vector field defined on the whole M because N_0 is closed and we assume that N_0 is a submanifold of M.

Take the pair $(N_0, X_0 + \ker \Omega)$, rewritten as (N_0, X^{N_0}) . Observe that we are looking for an element in X^{N_0} tangent to N_0 . Then,

$$N_1 = \{ x \in N_0 \mid \exists X \in X^{N_0}, \quad X(x) \in T_x N_0 \},$$

locally

$$N_{1} = \{(\lambda, u) \in N_{0} \mid 0 = X_{H} \left(\frac{\partial H}{\partial u_{l}}\right) = \frac{\partial H}{\partial p_{i}} \frac{\partial^{2} H}{\partial x^{i}} \frac{\partial u_{l}}{\partial u_{l}} - \frac{\partial H}{\partial x^{j}} \frac{\partial^{2} H}{\partial p_{j}} \frac{\partial u_{l}}{\partial u_{l}} + C_{r} \frac{\partial^{2} H}{\partial u_{r}} \frac{\partial u_{l}}{\partial u_{l}}, \quad l = 1, \dots, k\}.$$

$$(3.4)$$

If the matrix $(\partial H/\partial u_r \partial u_l)_{rl}$, multiplying C_r , is not invertible, the OCP is singular [13], otherwise it is regular.

This step stabilizes the constraints in N_0 providing a new pair (N_1, X^{N_1}) where X^{N_1} is the set of the vector fields solution and tangent to N_0 . Inductively, we arrive at (N_i, X^{N_i}) where we assume that N_i is a submanifold of M and we define $N_{i+1} = \{x \in N_i \mid \exists X \in X^{N_i}, \quad X(x) \in T_x N_i\}$, obtaining the sequence

$$M \supseteq N_0 \supseteq N_1 \supseteq \ldots \supseteq N_i \supseteq N_{i+1} \supseteq \ldots$$

and the corresponding $X^{N_{i+1}}$. Let

$$N_f = \bigcap_{i>0} N_i, \quad X^{N_f} = \bigcap_{i>0} X^{N_i},$$

if (N_f, X^{N_f}) is a nontrivial pair, it is the solution to Problem 3.1. If at one step $N_i = N_{i+1}$, the algorithm finishes with $N_f = N_i$.

Note that each step of the algorithm can reduce the set of points of M where there exists solution, that is, $N_i \subsetneq N_{i-1}$, and can also reduce the degrees of freedom of the set of vector fields solution, $X^{N_i} \subsetneq X^{N_{i-1}}$. In terms of control systems, the desirable objectives are to restrict the problem to a smaller submanifold of $T^*M \times U$ and to determine the input controls. Observe that, generally, a step of the algorithm can

provide us new constraints and the determination of some controls at the same time. Hence, either a unique vector field is found or the new constraints must be stabilized or the set must be split in submanifolds. At the final step, we have either a unique or nonunique vector field and a submanifold that could be an empty or discrete set.

Remark 3.2. Observe that we do not miss any extremal using the constraint algorithm, in contrast to what happens in subRiemannian geometry in [23], where using a less geometric approach they miss the constant extremals.

Now, let us focus again on optimal control problems where there are two distinct Hamiltonians depending on the value of the constant p_0 . Thus, from (3.3) it is deduced that the constraint algorithm must be run twice, one for each Hamiltonian, as is explained in §3.1, 3.2.

3.1 Characterization of abnormality

First, we characterize a subset of $T^*M \times U$ where the abnormal biextremals are, if they exist. In this situation $p_0 = 0$ and the corresponding Pontryagin's Hamiltonian is $H^{[0]} = H_X$. Then the primary constraint submanifold (3.3) becomes

$$N_0^{[0]} = \{(\lambda, u) \in T^*M \times U \mid \lambda_j \frac{\partial X^j}{\partial u_l} = 0, \quad l = 1, \dots, k\},$$
 (3.5)

the submanifold (3.4) is

$$N_1^{[0]} = \{(\lambda, u) \in N_0^{[0]} \mid \lambda_j \left(X^i \frac{\partial^2 X^j}{\partial x^i \partial u_l} - \frac{\partial X^j}{\partial x^i} \frac{\partial X^i}{\partial u_l} + C_r \frac{\partial^2 X^j}{\partial u_r \partial u_l} \right) = 0, \quad l = 1, \dots, k\},$$

and the algorithm continues.

Once we have the final constraint submanifold $N_f^{[0]}$ for abnormality, we have to delete the biextremals in the zero fiber because these biextremals do not satisfy the necessary condition (2.b) of Pontryagin's Maximum Principle 2.2. For the sake of simplicity and clarity, we denote this actual final constraint submanifold with the same name $N_f^{[0]}$.

Proposition 3.3. If $N_f^{[0]} \neq \emptyset$, that is, $(\lambda, u) \in N_f^{[0]}$ with $\lambda \neq 0$, then $(\gamma, u) = (\pi_M \times \mathrm{Id})(\lambda, u)$ is an abnormal extremal.

3.2 Characterization of normality

Analogous to § 3.1, for $p_0 = -1$, Pontryagin's Hamiltonian is $H^{[-1]} = H_X - F$.

Then the primary constraint submanifold (3.3) becomes

$$N_0^{[-1]} = \{(\lambda, u) \in T^*M \times U \mid \lambda_j \frac{\partial X^j}{\partial u_l} - \frac{\partial F}{\partial u_l} = 0, \quad l = 1, \dots, k\}, \tag{3.6}$$

the submanifold (3.4) is

$$N_1^{[-1]} = \{ (\lambda, u) \in N_0^{[-1]} \mid \lambda_j \left(X^i \frac{\partial^2 X^j}{\partial x^i \partial u_l} - \frac{\partial X^j}{\partial x^i} \frac{\partial X^i}{\partial u_l} \right) - X^i \frac{\partial^2 F}{\partial x^i \partial u_l} + C_r \left(\lambda_j \frac{\partial^2 X^j}{\partial u_r \partial u_l} - \frac{\partial^2 F}{\partial u_r \partial u_l} \right) = 0, \quad l = 1, \dots, k \},$$

In general, the determination of the controls for normal extremals depends on the given cost function, unless it is quadratic or linear or independent on the controls. To a better understanding of all this process we address the reader to the examples in \S 4, 5.

It can be observed that Hamilton's equations for \dot{x}^i are the same for both Hamiltonian functions, for $p_0 = 0$ and $p_0 = -1$, since the cost function does not depend on the momenta p's. Hamilton's equations for \dot{p}_i are equal for cost functions not depending on x^i . For instance, if the cost function is constant, as in the case of time-optimal.

The final constraint submanifolds $N_f^{[0]}$ and $N_f^{[-1]}$ restrict the set of points where the biextremals of the Optimal Control Problem 2.1 are. But, even in the case that Hamilton's equations are the same, $N_f^{[0]}$ and $N_f^{[-1]}$ could be different. Then the integral curves of the same vector field in $T^*M \times U$ along the same extremal in M may be different depending on where the initial conditions for the momenta are taken. In other words, there may exist abnormal extremals being normal and viceversa. For a deeper study about how the extremals are we need to project the biextremals on the base manifold $M \times U$ using $\rho_1 = \pi_M \times \mathrm{Id} \colon T^*M \times U \to M \times U$.

Summarizing all the above comments, we have the following propositions.

Proposition 3.4. If there exists $(\lambda, u) \in N_f^{[-1]}$, then $(\gamma, u) = (\pi_M \times \mathrm{Id})(\lambda, u)$ is a normal extremal.

Proposition 3.5. Let (γ, u) be an abnormal extremal. If there exists a covector λ along γ such that $(\lambda, u) \in N_f^{[-1]}$, then (γ, u) is also a normal extremal.

Let (γ, u) be a normal extremal. If there exists a covector λ along γ such that $(\lambda, u) \in N_f^{[0]}$, then (γ, u) is also an abnormal extremal.

Proposition 3.6. If there exist $(\lambda^{[0]}, u^{[0]}) \in N_f^{[0]}$ with $\lambda^{[0]} \neq 0$ and $(\lambda^{[-1]}, u^{[-1]}) \in N_f^{[-1]}$ such that $\pi_M(\lambda^{[0]}) = \pi_M(\lambda^{[-1]}) = \gamma$, then γ is an abnormal extremal and also a normal extremal.

Remark 3.7. In this last proposition we do not consider the control as a part of the extremal, because it may happen that different controls give the same extremals in M depending on the control system. So we project onto M the biextremals to compare them. Under some assumptions about the control systems, such as control-affinity

with independent control vector fields, different controls give different extremals. If so happens, we will project the biextremals onto $M \times U$ through ρ_1 to compare them.

Remark 3.8. The union of both final constraint submanifods do not cover exactly the set of extremals in Definition 2.4, because the condition (2.a) in Theorem 2.2 is not included in the final constraint submanifolds. See § 4 to get a better understanding.

3.3 Characterization of strict abnormality

The normal and abnormal extremals in Definition 2.4 do not constitute a disjoint partition of the set of extremals as propositions 3.5, 3.6 shows. While in § 3.1 we do not care about the cost function, in § 3.2 the cost function takes part in the process. To characterize strict abnormal extremals the cost function is fundamental because these extremals are abnormal, but not normal. The only way to guarantee that an extremal is not normal is to use the cost function.

As a consequence of the final constraint submanifolds obtained from the algorithm for abnormality and normality, the strict abnormality can be studied. The adjective strict denotes that the extremal only admits one kind of lifts to the cotangent bundle. To find strict abnormal extremals we have to project the final constraint submanifolds to M. In the intersection there are the extremals admitting two different kinds of lifts: with $p_0 = 0$ and with $p_0 = -1$. This explanation makes evident the presence of the cost function to study strict abnormality because the final constraint submanifold for normality is used.

To sum up, all the biextremals in $N_f^{[0]}$ and $N_f^{[-1]}$ are projected through $\rho=\pi_M\circ\pi_1\colon T^*M\times U\to M$ due to Remark 3.7.

Proposition 3.9. Let
$$P = \rho(N_f^{[0]}) \cap \rho(N_f^{[-1]})$$
.

- (i) If $P = \emptyset$ and $\rho(N_f^{[0]}) \neq \emptyset$, then all the abnormal extremals are strict.
- (ii) If $P = \emptyset$ and $\rho(N_f^{[-1]}) \neq \emptyset$, then all the normal extremals are strict normal.
- (iii) If $P \neq \emptyset$ and $\rho(N_f^{[0]}) = P$, then there are no strict abnormal extremals.
- (iv) If $P \neq \emptyset$ and $\rho(N_f^{[0]}) \neq P$, then there are locally abnormal extremals.
- (v) If $P \neq \emptyset$ and $\rho(N_f^{[0]}) = \rho(N_f^{[-1]}) = P$, then all the abnormal extremals are also normal and viceversa.

In item (iv), it is said that there are strict abnormal extremals, but only locally since the extremal could have pieces in P. So at some points the extremal can be locally normal.

4 Free-time optimal control problem

Once the theory has been introduced let us deal with the particular case of the free time OCP. In this case the interval of definition of the extremals is another unknown of the problem.

Problem 4.1. (Free-time Optimal Control Problem, FOCP)

Given M, U, X, F, x_a , $x_b \in M$ (as in § 2). Find (γ, u) and $I = [a, b] \subset \mathbb{R}$ such that

- (1) $\gamma(a) = x_a, \ \gamma(b) = x_b,$
- (2) $\dot{\gamma}(t) = X(\gamma(t), u(t)), t \in I, and$
- (3) $S[\gamma, u]$ is minimum over all curves on $M \times U$ satisfying (1) and (2).

Pontryagin's Maximum Principle is the same as Theorem 2.2, but replacing (2.a) by

$$(2.a')$$
 $H(\lambda(t), u(t))$ is zero everywhere $t \in I$.

Thus the presymplectic equation (2.2) must be restricted to the submanifold defined by the condition

$$H = H_X + p_0 F = 0.$$

Hence, it must also be stabilized in the algorithm. Due to the properties of hamiltonian systems [1], the condition H=0 is trivially stabilized along integral curves of the hamiltonian vector field. Thus its tangency condition does not add any new constraint to the submanifolds of the algorithm. The same happens with H= constant, but this is not a suitable constraint for a submanifold, that is why it was not included in the primary constraint submanifold for fixed-time OCP \S 3. In contrast to Remark 3.8, the final constraint submanifolds we find here recover exactly the whole set of extremals since all the necessary conditions of Theorem 2.2 are taking into account. The trivial stabilization of H=0 makes possible to run the algorithm putting aside, then the same final constraint submanifolds as in \S 3.1, 3.2 are obtained. Those submanifolds are renamed, respectively, as $N_{ff}^{[0]}$ and $N_{ff}^{[-1]}$ since the actual final constraint submanifolds are obtained by considering the vanishing of the Hamiltonian:

$$N_f^{[0]} = N_{ff}^{[0]} \cap \{(\lambda, u) \in T^*M \times U \mid H_X = 0\},$$

$$N_f^{[-1]} = N_{ff}^{[-1]} \cap \{(\lambda, u) \in T^*M \times U \mid H_X - F = 0\}.$$

Due to condition (2.b) in Theorem 2.2, the zero fiber must be deleted from $N_f^{[0]}$.

Proposition 4.2. Given a free-time optimal control problem:

- 1. If $N_f^{[0]}$ has only zero covectors, there are no abnormal extremals.
- 2. If $N_f^{[0]}$ has nonzero covectors and $N_{ff}^{[0]} \subset \{(\lambda, u) \in T^*M \times U \mid H_X = 0\}$, then every abnormal extremal is strict and there are no normal extremals as long as F does not vanish.

5 Example

There are some classical optimal control problems where the classification of extremals has been described with different tools and approaches: geodesics in Riemannian geometry [23], shortest-paths in subRiemannian geometry [2, 23] and OCPs with control-affine systems [3, 4, 32, 33]. All of them can be studied in a unified way by direct application of the method we have proposed in this paper. Here we are going to use the algorithm for a particular example.

5.1 Control-affine mechanical system

Following the described method we find a strict abnormal extremal for a control-affine system on TQ, that, in fact, models an affine connection control mechanical system. See more details about these systems in [9].

Let $M=TQ=T\mathbb{R}^3$ (i.e. $Q=\mathbb{R}^3$), U be an open set in \mathbb{R}^2 containing the zero. In local natural coordinates (x,y,z,v_x,v_y,v_z) for TQ, the drift vector field of the control-system is

$$Z = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} ,$$

and the input vector fields are $Y_1 = \frac{\partial}{\partial v_x}$ and $Y_2 = (1-x)\frac{\partial}{\partial v_y} + x^2\frac{\partial}{\partial v_z}$. So the control system is given by $Z + u_1Y_1 + u_2Y_2$. The endpoint conditions on TQ are $v_a = (2,0,0,0,v_y^0,4(1-v_y^0)), \ v_b = (2,1,0,0,2(1-v_y^0),4v_y^0-4)$ with $v_y^0 \neq 1$. The cost function is $F = \frac{u_1^2 + u_2^2}{2}$. Hence Pontryagin's Hamiltonian is

$$H(\lambda, u_1, u_2) = p_x v_x + p_y v_y + p_z v_z + u_1 q_x + u_2 (1 - x) q_y + u_2 x^2 q_z + p_0 \frac{u_1^2 + u_2^2}{2}$$

with Hamilton's equations for abnormality and normality

$$\begin{array}{lll} \dot{x} = v_x & \dot{v_x} = u_1 & \dot{p}_x = q_y u_2 - 2q_z u_2 x & \dot{q}_x = -p_x \\ \dot{y} = v_y & \dot{v_y} = u_2 (1-x) & \dot{p}_y = 0 & \dot{q}_y = -p_y \\ \dot{z} = v_z & \dot{v}_z = u_2 x^2 & \dot{p}_z = 0 & \dot{q}_z = -p_z \end{array}$$

and Hamiltonian vector field $X_H = \sum_{i \in \{x,y,z\}} (A^i \partial/\partial i + B^i \partial/\partial v_i + C_i \partial/\partial p_i + D_i \partial/\partial q_i) + E_1 \partial/\partial u_1 + E_2 \partial/\partial u_2$, where A^i, B^i, C_i, D_i are determined by Hamilton's equations.

The constraint algorithm for abnormality gives us

$$\begin{array}{lll} N_0^{[0]} & = & \{(\lambda,u) \in T^*TQ \times U & | & \partial H^{[0]}/\partial u_k(\lambda,u) = H_{Y_k}(\lambda) = 0 \;,\; k = 1,2\} \\ & = & \{(\lambda,u) \in T^*TQ \times U & | & q_x = 0,\; \mathbf{q_y}(\mathbf{1-x}) + \mathbf{q_z}\mathbf{x^2} = \mathbf{0}\} \\ N_1^{[0]} & = & \{(\lambda,u) \in N_0^{[0]} & | & H_{[Z,Y_k]}(\lambda) = 0 \;\; \text{for} \;\; k = 1,2\} \\ & = & \{(\lambda,u) \in N_0^{[0]} & | & p_x = 0,\; (-1+x)p_y - x^2p_z - v_xq_y + 2xv_xq_z = 0\} \\ N_2^{[0]} & = & \{(\lambda,u) \in N_1^{[0]} & | & (H_{[Z,[Z,Y_k]]+u_l[Y_l,[Z,Y_k]]})(\lambda) = 0 \;\; \text{for} \;\; k = 1,2\} \\ & = & \{(\lambda,u) \in N_1^{[0]} & | & (-\mathbf{q_y} + 2\mathbf{x}\mathbf{q_z})\mathbf{u_2} = \mathbf{0}, \\ & & & (-q_y + 2xq_z)u_1 = -(2p_yv_x - 4xv_xp_z + 2v_x^2q_z)\} \;. \end{array}$$

In order to satisfy the endpoint conditions, not to have the zero covector and to have a strict abnormal extremal, the subset defined by $x(x-1)q_z u_2 = 0$, coming from the above bold equations, must be deleted from the constraint submanifold. Then

$$\begin{array}{lll} N_2^{[0]} & = & \left\{ (\lambda,u) \in T^*TQ \times U - \left\{ x(x-1)q_zu_2 = 0 \right\} \, | \, q_x = 0, \, \, -q_y + 4q_z = 0, \, \, p_x = 0, \\ & p_y - 4p_z = 0, \, \, x = 2, \, \, v_x = 0 \right\} \\ N_3^{[0]} & = & \left\{ (\lambda,u) \in N_2^{[0]} \, | \, v_x = 0, \, \, u_1 = 0 \right\} \\ N_4^{[0]} & = & \left\{ (\lambda,u) \in N_2^{[0]} \, | \, u_1 = 0, \, \, E_1 = 0 \right\} = N_5^{[0]} = N_f^{[0]}. \end{array}$$

By restriction to the final constraint submanifold and integrating Hamilton's equations on [0,1] we have $\lambda(t)=(0,4p_z^0,p_z^0,0,-4p_z^0t+4q_z^0,-p_z^0t+q_z^0)$ and

$$\gamma(t) = (2, -u_2 \frac{t^2}{2} + v_y^0 t, 2u_2 t^2 + 4(1 - v_y^0)t, 0, -u_2 t + v_y^0, 4u_2 t + 4(1 - v_y^0))$$

with $u_2 = 2(v_y^0 - 1)$.

The constraint algorithm for normality gives us

$$\begin{array}{lcl} N_0^{[-1]} & = & \{(\lambda,u) \in T^*TQ \times U & | & \partial H^{[-1]}/\partial u_k(\lambda,u) = 0 \;,\; k=1,2\} \\ & = & \{(\lambda,u) \in T^*TQ \times U & | & q_x-u_1=0 \;, \\ & & q_y(1-x) + q_z x^2 - u_2 = 0\} = N_1^{[-1]} = N_f^{[-1]}. \end{array}$$

If we substitute the curve γ in Hamilton's equations, we have $u_1=0$ and $u_2=2(1-v_y^0)$, then for the primary constraint submanifold $q_x=0$ and $u_2=q_y-4q_z$. Due to Hamilton's equations $p_x=0$ and $0=\dot{p}_x=u_2^2$. This last equality is only possible if $v_y^0=1$, but that was not the hypothesis. Thus there does not exist a covector with $p_0=-1$ along γ , that is, γ is a strict abnormal extremal whenever $v_y^0\neq 1$.

6 Conclusion and outlook

In this paper we have given a geometric method to study different kinds of extremals in a wide range of optimal control problems with an open control set. This can also be applied in the case of closed control set following ideas in [24]. This method is based on the suitable reinterpretation of the so-called *presymplectic algorithm* in other fields. The dependence on the cost function makes difficult to give general characterizations of normal and strict abnormal extremals since each problem must be studied by itself. However, the abnormal extremals only depend on the geometry of the control system, so some general results can be deduced. See [7] for an approach to a related problem for single-input control-affine systems.

One line of future research is to apply this general algorithm in the study of optimal control problems with affine connection control systems, which model the motion of different types of mechanical systems such as rigid bodies, nonholonomic systems and robotic arms [9]. Eventually, we will focus on particular problems for

mechanical systems, as for instance time-optimal problems and control-quadratic cost function.

Apart from having sufficient conditions to determine where the extremals are, it may be interesting to prove the density and the optimality of them, similar to the work done in [23] for control-linear systems with two input vector fields. Here the cost function takes part in, even in determining the optimality of abnormal extremals. There are some results that do not contribute to the optimism in relation to the existence of strict abnormal minimizers, at least in a generic sense. For instance, in [11, 12] it is proved the existence of an open and dense subset in the set of control systems where every nontrivial strict abnormal extremal is not a minimizer for control-quadratic cost functions and control-affine systems.

Another meaningful research issue is to establish connections between the controllability of the system [9, 28, 31] and the final constraint submanifold obtained for abnormality, since both notions are independent of the functional to be minimized. In fact, we are already working on some properties of controllability, similar to results in [5], that can also be justified using the algorithm here described.

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