# Shintani Cocycles on $\mathrm{GL}_n$

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**Abstract**

The aim of this paper is to define an $n-1$-cocycle $\sigma$ on $\mathrm{GL}_n(\mathbb{Q})$ with values in a certain space $\mathcal{D}$ of distributions on $\mathbb{A}_f^n \setminus \{0\}$. Here $\mathbb{A}_f$ denotes the ring of finite adèles of $\mathbb{Q}$, and the distributions take values in the Laurent series $\mathbb{C}((z_1, \ldots, z_n))$. This cocycle can be used to evaluate special values of Artin $L$-functions on number fields at negative integers. The construction generalizes that of Solomon [8] in the case $n = 2$.

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## 1 Introduction

Let $\mathbb{A}_f$ denote the ring of finite adèles of $\mathbb{Q}$. Let $\mathcal{D}$ be the space of distributions on $\mathbb{A}_f^n \setminus \{0\}$ with values in the Laurent series ring $\mathbb{C}((z_1, \ldots, z_n))$. In other words

$$\mathcal{D} = \mathrm{Hom}\Big(\mathcal{S}(\mathbb{A}_f^n \setminus \{0\}), \mathbb{C}((z_1, \ldots, z_n))\Big),$$

where $\mathcal{S}(\mathbb{A}_f^n \setminus \{0\})$ denotes the space of Bruhat-Schwartz functions $f : \mathbb{A}_f^n \setminus \{0\} \to \mathbb{C}$. The aim of this paper is to define an $n-1$-cocycle $\sigma$ on $\mathrm{GL}_n(\mathbb{Q})$ with values in $\mathcal{D}$. The construction generalizes that of Solomon [8] in the case $n = 2$. A similar but different cocycle was found by and Solomon and Hu [3] in the case $n = 2, 3$. For $n > 3$ the cocycle of Solomon and Hu is only defined on a Zariski-open subset of $\mathrm{GL}_n(\mathbb{Q})^n$. Following [8] and [3] we shall refer to the cocycle $\sigma$ as the *Shintani cocycle*.

Very briefly, to define the Shintani cocycle $\sigma$ we begin by choosing a non-zero vector $v \in \mathbb{Q}^n$. Given $\alpha_1, \ldots, \alpha_n \in \mathrm{GL}_n(\mathbb{Q})$, we shall define a cone $C$ in $\mathbb{Q}^n$. Roughly speaking, $C$ will be the set of linear combinations $\sum x_i \alpha_i v$ with the $x_i \in \mathbb{Q}$ positive. To be more precise, one must also include some of the faces of $C$. Then the cocycle is given by

$$\sigma(\alpha_1, \ldots, \alpha_n)(\varphi) = \sum_{v \in C} \varphi(v) \exp\left(\sum_{i=1}^n v_i z_i\right), \qquad \varphi \in \mathcal{S}(\mathbb{A}_f^n \setminus \{0\}).$$

Solomon and Hu showed how to make sense of the right hand side of this formula as an element of $\mathbb{C}((z_1, \ldots, z_n))$; we shall explain their method in §3 below. The difficulty tackled in this paper, is to define the cone $C$ correctly in the case where the vectors $\alpha_i v$ are not in general position. These problems are solved in §4. In §5, we describe the case $n = 2$ in detail, and show how $\sigma$ is related to Solomon's cocycle.

## 2 Special values of $L$-functions

Motivation for studying Shintani-cocycles lies in their relation with special values of $L$-functions. We shall spend a few moments discussing this connection. Since these matters are the subject of other papers ([2, 6, 7, 8, 9]), the discussion here is deliberately informal, and is independent of the rest of the paper.

Given a totally real number field $k$ of degree $n$, one can obtain formulae for the values of abelian $L$-functions of $k$ at negative integers by substituting units of $k$ (regarded as $n \times n$ integer matrices) into the $n-1$-cocycle on $\mathrm{GL}_n(\mathbb{Q})$. Similar formulae have been obtained by Sczech [4, 5] and Stevens [10].

**1. $L$-functions of $\mathbb{Q}$.** As a first example, we consider the case $n = 1$. In this case we have a (homogeneous) 0-cocycle $\sigma$ on $\mathrm{GL}_1(\mathbb{Q})$. Evaluating $\sigma$ at the identity element, we obtain the following distribution

$$\sigma(1)(\varphi) = \sum_{v \in \mathbb{Q},\, v>0} \varphi(v) \exp(zv) \in \mathbb{C}((z)), \quad \varphi \in \mathcal{S}(\mathbb{A}_f \setminus \{0\}).$$

If we substitute for $\phi$ a Dirichlet character $\chi : \hat{\mathbb{Z}} \to \mathbb{C}$, extended to be zero on $\mathbb{A}_f \setminus \hat{\mathbb{Z}}$, then this gives

$$\sigma(1)(\chi) = \sum_{n=1}^{\infty} \chi(n) \exp(nz).$$

Differentiating with respect to $z$, we obtain formally (following Euler):

$$L(\chi, 1-r) = \left(\frac{\partial}{\partial z}\right)^{r-1} \Big(\sigma(1)(\chi)\Big)\Big|_{z=0}, \qquad r \in \mathbb{N}.$$

To make sense of this equation, let $f \in \mathbb{N}$ be a conductor of $\chi$. We can group the terms into finitely many geometric progressions as follows:

$$\begin{aligned} \sigma(1)(\chi) &= \sum_{n=1}^{f} \chi(n) \exp(nx) \Big(1 + \exp(fz) + \exp(2fz) + \cdots\Big) \\ &= \sum_{n=1}^{f} \chi(n) \frac{\exp(nz)}{1 - \exp(fz)}. \end{aligned}$$

The ratio of exponentials can be expanded in terms of Bernoulli polynomials $B_m$ as follows.

$$\sigma(1)(\chi) = -\sum_{m=0}^{\infty} \left( \sum_{n=1}^{f} \chi(n) B_{m+1}\left(\frac{n}{f}\right)\right) \frac{(fz)^m}{(m+1)!}.$$

This gives the usual expression for $L(\chi, 1-r)$ (see for example §2.3 of [2]):

$$L(\chi, 1-r) = -\frac{f^{r-1}}{r} \sum_{n=1}^{f} \chi(n) B_r \left(\frac{n}{f}\right).$$

**2.** ***L*-functions of real quadratic fields.** Now suppose $k$ is a real quadratic field with ring of integers $\mathfrak{o}$. For simplicity, we shall assume that $k$ has narrow class number 1, i.e. every non-zero ideal of $\mathfrak{o}$ has a totally positive generator. Indeed, if $u$ denotes a generator for the group of totally positive units in $\mathfrak{o}$, then each ideal has a unique generator in the following cone:

$$C = \{x + yu : x, y \in \mathbb{Q},\ x \geq 0,\ y > 0\}.$$

We may therefore express the abelian $L$-functions of $k$ as follows:

$$L(\chi, s) = \sum_{a \in C \cap \mathfrak{o}} \chi(a) N(a)^{-s}.$$

By choosing an integral basis $\{b_1, b_2\}$, we can regard $k^\times$ as a subgroup of $\mathrm{GL}_2(\mathbb{Q})$ and $\mathfrak{o}^\times$ as a subgroup of $\mathrm{GL}_2(\mathbb{Z})$. The special values of $L(\chi, s)$ are encoded in the restriction of $\sigma$ to $\mathfrak{o}^\times$. We shall also identify $\mathbb{A}_f^2$ with the ring $\mathbb{A}_f \otimes k$ of finite adeles of $k$. An element $z \in \mathrm{Hom}_\mathbb{Q}(k, \mathbb{C})$ can be decomposed in terms of the dual basis $\{b_1^*, b_2^*\}$ as follows:

$$z = z_1 b_1^* + z_2 b_2^*, \qquad z_1, z_2 \in \mathbb{C}.$$

Evaluating our 1-cocycle on the fundamental unit $u$, we obtain the following the distribution:

$$\sigma(1, u)(\varphi) = \sum_{a \in C} \varphi(a) \exp(z \cdot a).$$

Although the right hand side actually converges for $z$ in a certain cone in $\mathrm{Hom}_\mathbb{Q}(k, \mathbb{C})$, we shall in fact interpret it as an element of $\mathbb{C}((z_1, z_2))$ by the method of Solomon and Hu.

Again, substituting a Dirichlet character $\chi$ for $\varphi$, we obtain:

$$\sigma(1, u)(\chi) = \sum_{a \in C \cap \mathfrak{o}} \chi(a) \exp(z \cdot a).$$

On the other hand, we can also write $z$ in the form

$$z = t_1 \tau_1 + t_2 \tau_2, \quad t_1, t_2 \in \mathbb{C},$$

where $\tau_1, \tau_2 : k \to \mathbb{R}$ are the two field embeddings. With this notation we have

$$N(a)^r = \left( \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} \right)^r \exp(z \cdot a) \Big|_{z=0}.$$

This gives (formally at least) the following:

$$L(\chi, -r) = \left( \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} \right)^r \sigma(1, u)(\chi) \Big|_{z=0}.$$

To make sense of this formula, we let $f \in \mathbb{N}$ be a (not necessarily minimal) conductor of $\chi$. We can group the terms of $\sigma(1, u)(\chi)$ into finitely many products of geometric progressions as before:

$$\begin{aligned} \sigma(1, u)(\chi) &= \sum_{a \in \mathcal{P} \cap \mathfrak{o}} \chi(a) \exp(a \cdot z) \sum_{r,s=0}^{\infty} \exp(f(r + su) \cdot z) \\ &= \sum_{a \in \mathcal{P} \cap \mathfrak{o}} \frac{\chi(a) \exp(z \cdot a)}{(1 - \exp(z \cdot f))(1 - \exp(z \cdot fu))}. \end{aligned}$$

Here $\mathcal{P}$ denotes the half-open parallelogram:

$$\mathcal{P} = \{x + yu : x, y \in \mathbb{Q},\ 0 \le x < f,\ 0 < y \le f\}.$$

When we expand the ratio of exponentials out as a power series in $z$, instead of values of Bernoulli polynomials, the coefficients will instead be generalized Dedekind sums. The expansion of such power series in terms of Dedekind sums and their generalization is described in [1, 8]. In particular, the reciprocity laws satisfied by these sums are shown to be consequences of the cocycle relation satisfied by $\sigma$. More precisely, we can expand out as follows:

$$\sigma(1,u)(\chi) \quad = \quad \sum_{m_1,m_2=0}^{\infty} S(m_1,m_2,\chi)\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}.$$

The coefficients $S(m_1, m_2, \chi)$ can be expressed in terms of Dedekind sums.

Using the $r$-th symmetric power of the transition matrix from $\{\tau_1, \tau_2\}$ to $\{b_1^*, b_2^*\}$, we obtain a formula for $L(\chi, -r)$ in terms the numbers $S(\chi, m_1, m_2)$ with $m_1 + m_2 = 2r$.

**3. Totally Real Fields.** Let $k$ be a totally real algebraic number field with $[k : \mathbb{Q}] = n$. Let $H_\infty$ denotes the narrow class group of $k$, i.e. the group of fractional ideals, modulo the principal ideals gererated by totally positive elements. It was shown by Shintani [6] (see also §2.7 of [2]) that there are finitely many cones $C_1, \dots, C_N$ such that any abelian $L$-function of $k$ can be expressed in the form

$$L(\chi, s) = \sum_{[\mathfrak{b}]\in H_\infty} \sum_{i=1}^{N} \sum_{a\in C_i\cap\mathfrak{b}^{-1}} \chi(a\mathfrak{b})N(a\mathfrak{b})^{-s}.$$

By choosing an integral basis, we can regard $\mathfrak{o}^\times$ as a subgroup of $\mathrm{GL}_n(\mathbb{Z})$. The method described above may be used to express the special values $L(s, -r)$ in terms of the restriction of $\sigma$ to $\mathfrak{o}^\times$.

As an example, suppose $k$ has narrow class number 1. Let $\{u_1, \dots, u_{n-1}\}$ be a basis for the group of totally positive units. Then we have, for a certain differential operator $\partial$:

$$L(\chi, -r) = \partial^r \left( \sum_{\xi\in S_{n-1}} \mathrm{sign}(\xi)\cdot\tilde{\sigma}(u_{\xi(1)}, \dots, u_{\xi(n-1)}) \right)(\chi)\Bigg|_{z=0},$$

where $\tilde{\sigma}$ is the corresponding inhomogeneous cocycle:

$$\tilde{\sigma}(\alpha_1, \dots, \alpha_{n-1}) = \sigma(1, \alpha_1, \alpha_1\alpha_2, \dots, \alpha_1\cdots\alpha_{n-1}).$$

Note that if $k$ has a complex place, then its $L$-functions are zero at negative integers. This can be seen from the functional equation.

# 3 Notation and Background Material

**1. The module of cones.** Let $v_1, \dots, v_r \in \mathbb{R}^n$ be linearly independent vectors. By the *open cone* of $v_1, \dots, v_r$, we shall mean the set

$$C^o(v_1, \dots, v_r) = \left\{ \sum \lambda_i v_i : \lambda_1, \dots, \lambda_r > 0 \right\}.$$

The *closed cone* $\bar{C}(v_1,\dots,v_r)$ is defined similarly but with the inequalities $>$ replaced by $\geq$. We shall write $\partial C$ for the points of the closed cone which are not in the open cone. A cone will be called *rational* if the vectors $v_1,\dots,v_r$ are in $\mathbb{Q}^n$. Let $\mathcal{K}^o_{\mathbb{Q}}$ be the abelian group of functions $\mathbb{R}^n \setminus \{0\} \to \mathbb{Z}$ generated by the characteristic functions of rational open cones. We shall also write $\mathcal{K}_{\mathbb{Q}}$ for the group of functions $\mathbb{R}^n \to \mathbb{Z}$ whose restrictions to $\mathbb{R}^n \setminus \{0\}$ are in $\mathcal{K}^o_{\mathbb{Q}}$.

We shall regard $\mathcal{K}$ (resp. $\mathcal{K}_{\mathbb{Q}}$) as a left $\mathrm{GL}_n(\mathbb{R})$- (resp. $\mathrm{GL}_n(\mathbb{Q})$-) module with the action given by

$$(\alpha * c)(v) = \mathrm{sign}(\det \alpha) \cdot c(\alpha^{-1} v).$$

The constant functions $\mathbb{R}^n \setminus \{0\} \to \mathbb{Z}$ are in $\mathcal{K}_{\mathbb{Q}}$, and form a submodule which we shall denote $\mathbb{Z}(-)$. The quotient $\mathcal{K}_{\mathbb{Q}}/\mathbb{Z}(-)$ will be written $\mathcal{L}_{\mathbb{Q}}$.

**2. The Solomon - Hu pairing.** In [3], Solomon and Hu introduced a pairing

$$\mathcal{L}_{\mathbb{Q}} \times \mathcal{S}(\mathbb{A}_f^n \setminus \{0\}) \to \mathbb{C}((z_1,\dots,z_n)).$$

This is defined in several steps.

Step 1. Let $\mathbb{Z}\{\mathbb{Q}^n\}$ be the space of all functions $\mathbb{Q}^n \to \mathbb{Z}$ and let $\mathbb{Z}[\mathbb{Q}^n]$ be the group ring of the group $\mathbb{Q}^n$, i.e. the elements of $\mathbb{Z}\{\mathbb{Q}^n\}$ of finite support. One defines a map $\Phi : \mathbb{Z}[\mathbb{Q}^n] \to \mathbb{C}((z_1,\dots,z_n))$ as follows:

$$\Phi(A) = \sum_{w \in \mathbb{Q}^n} A(w) \exp(w \cdot z).$$

Here $w \cdot z$ denotes the dot product $w_1 z_1 + \ldots + w_n z_n$.

Step 2. The group ring $\mathbb{Z}[\mathbb{Q}^n]$ acts on $\mathbb{Z}\{\mathbb{Q}^n\}$, $\mathbb{Z}[\mathbb{Q}^n]$ and $\mathbb{C}((z_1,\dots,z_n))$ and the map $\Phi$ is compatible with these actions. Define

$$\mathbb{Z}\{\mathbb{Q}^n\}^{(q)} = \left\{ B \in \mathbb{Z}\{\mathbb{Q}^n\} : \exists A \in \mathbb{Z}[\mathbb{Q}^n] \setminus \{0\} \text{ such that } AB \in \mathbb{Z}[\mathbb{Q}^n] \right\}.$$

As $\mathbb{Z}[\mathbb{Q}^n]$ is an integral domain, it follows that $\mathbb{Z}\{\mathbb{Q}^n\}^{(q)}$ is an additive subgroup of $\mathbb{Z}\{\mathbb{Q}^n\}$. Furthermore the map $\Phi : \mathbb{Z}[\mathbb{Q}^n] \to \mathbb{C}((z_1,\dots,z_n))$ extends uniquely to $\mathbb{Z}\{\mathbb{Q}^n\}^{(q)}$ in a way which is compatible with the actions of $\mathbb{Z}[\mathbb{Q}^n]$.

Step 3. We next define a pairing $\mathcal{K}_{\mathbb{Q}} \times \mathcal{S}(\mathbb{A}_f^n) \to \mathbb{C}((z_1,\dots,z_n))$. Given $c \in \mathcal{K}_{\mathbb{Q}}$ and $\varphi \in \mathcal{S}(\mathbb{A}_f^n)$, we define a function $c \cdot \varphi : \mathbb{Q}^n \to \mathbb{C}$ by

$$(c \cdot \varphi)(v) = \begin{cases} c(v)\phi(v) & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

It turns out that $c \cdot \varphi$ is in $\mathbb{Z}\{\mathbb{Q}^n\}^{(q)}$, and we define

$$\langle c, \varphi \rangle = \Phi(c \cdot \varphi).$$

We shall describe this more explicitly. Let $v_1,\dots,v_r \in \mathbb{Q}^n$ be linearly independent and let $c$ be the characteristic function of the open cone of $v_1,\dots,v_r$. Given $\varphi \in \mathcal{S}(\mathbb{A}_f^n)$, there is a lattice $L \subset \mathbb{Q}^n$ such that $\varphi$ is invariant under translation by $L$. By multiplying the vectors $v_1,\dots,v_n$ by natural numbers if necessary, we may assume $v_1,\dots,v_n \in L$. Let

$$\mathcal{P} = \{x_1 v_1 + \ldots + x_n v_n : x_1,\dots,x_n \in (0,1]\}.$$

We have $(1-[v_1])\dots(1-[v_r])(c\cdot\varphi) = p\cdot\varphi$, where $p$ is the characteristic function of the parallelotope $\mathcal{P}$. As $p\cdot\varphi$ has finite support, it follows that $c\cdot\phi$ is in $\mathbb{Z}\{\mathbb{Q}^n\}^{(q)}$, and the pairing is given by:

$$\langle c,\varphi\rangle = \frac{1}{1-\exp(v_1\cdot z)}\cdots\frac{1}{1-\exp(v_r\cdot z)}\sum_{w\in\mathcal{P}\cap\mathbb{Q}^n}\varphi(w)\exp(w\cdot z).$$

Step 4. Let $c$ be the constant function on $\mathbb{Q}^n\setminus\{0\}$ with value 1 and let $\varphi\in\mathcal{S}(\mathbb{A}_f^n)$. Then for any non-zero vector $v\in\mathbb{Q}^n$ such that $\varphi$ is periodic modulo $v$, we have

$$\Big((1-[v])(c\cdot\varphi)\Big)(w) = \begin{cases}\varphi(0) & \text{if } w=-v\\ -\varphi(0) & \text{if } w=0\\ 0 & \text{otherwise.}\end{cases}$$

It follows that the constant functions are orthogonal to the subspace $\mathcal{S}(\mathbb{A}_f^n\setminus\{0\})$. Hence the pairing factors through to give $\mathcal{L}_{\mathbb{Q}}\times\mathcal{S}(\mathbb{A}_f^n\setminus\{0\})\to\mathbb{C}((z_1,\dots,z_n))$.

**3. The Result.** To obtain a cocycle with values in $\mathcal{D}$, it is sufficient, using the pairing defined above, to construct one with values in $\mathcal{L}_{\mathbb{Q}}$. In the case $n=2$, Solomon's cocycle $s$ (with values in $\mathcal{L}_{\mathbb{Q}}$) is defined as follows. For $\alpha,\beta\in \mathrm{GL}_2(\mathbb{Q})$,

$$s(\alpha,\beta)(v) = \begin{cases}\operatorname{sign}\det(\alpha e_1,\beta e_1) & \text{if } v\in C^o(\alpha e_1,\beta e_1),\\ \frac{1}{2}\operatorname{sign}\det(\alpha e_1,\beta e_1) & \text{if } v\in\partial C(\alpha e_1,\beta e_1),\\ 0 & \text{otherwise.}\end{cases}$$

Here $e_1=\begin{pmatrix}1\\0\end{pmatrix}$. One could of course replace $e_1$ by any other non-zero vector to obtain a cohomologous cocycle. In the case that $\alpha e_1,\beta e_1,\gamma e_1$ are in general position, it is easy to see that $s$ satisfies the cocycle relation:

$$s(\beta,\gamma)-s(\alpha,\gamma)+s(\alpha,\beta)=0 \text{ modulo constant functions.}$$

If $\alpha e_1,\beta e_1$ are not linearly independent then the above definition makes no sense and we instead define $s(\alpha,\beta)=0$. With this completed definition the cocycle relation remains true modulo the kernel of the Solomon-Hu pairing.

Naively one would expect to generalize the cocycle $s$ above by defining for $\alpha_1,\dots,\alpha_n\in \mathrm{GL}_n(\mathbb{Q})$:

$$s(\alpha_1,\dots,\alpha_n)(v) = \begin{cases}\operatorname{sign}\det(\alpha_1e_1,\dots,\alpha_ne_1) & \text{if } v\in C^o(\alpha_1e_1,\dots,\alpha_ne_1),\\ 0 & \text{if } v\notin\bar{C}(\alpha_1e_1,\dots,\alpha_ne_1),\end{cases}$$

Indeed as long as $v,\alpha_1e_1,\dots,\alpha_ne_1$ are in general position, the above definition makes sense and a similar cocycle relation is satisfied. The difficulties are (a) how to define $s$ when $\alpha_1e_1,\dots,\alpha_ne_1$ are linearly dependent, and (b) how to define $s$ when $v\in\partial C(\alpha_1e_1,\dots,\alpha_ne_1)$ without losing the cocycle relation. Both these problems are solved by the same method in this paper.

# 4 Definition of the Shintani Cocycle

**1. A relation between signs of determinants** By an *ordered field* we shall mean a (commutative) field $\mathbb{F}$ equipped with a total ordering $>$ satisfying the condition:

- $\forall x, y, z \in \mathbb{F}$ if $x > y$ then $x + z > y + z$;
- $\forall x, y, z \in \mathbb{F}$ if $x > y$ and $z > 0$ then $xz > yz$.

Fix an ordered field $\mathbb{F}$ and define for $x \in \mathbb{F}^\times$:

$$\operatorname{sign}(x) = \begin{cases} 1 & x > 0, \\ -1 & x < 0. \end{cases}$$

A set of vectors in $\mathbb{F}^n$ will be said to be *in general position* if every subset with no more than $n$ elements is linearly independent. Given vectors $v_0, \ldots, v_n \in \mathbb{F}^n$ in general position, there are non-zero scalars $\lambda_0, \ldots, \lambda_n \in \mathbb{F}$ such that $\lambda_0 v_0 + \ldots + \lambda_n v_n = 0$. We define

$$d(v_0, \ldots, v_n) = \begin{cases} (-1)^i \operatorname{sign}\det(v_1, \ldots, \hat{v}_i, \ldots, v_n) & \text{if } \lambda_0, \ldots, \lambda_n \text{ all have the same sign,} \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 1** *(i) $d(v_0, \ldots, v_n)$ is well defined (i.e. independent of $i$).*

*(ii) For any permutation $\xi$ we have $d(v_{\xi(0)}, \ldots, v_{\xi(n)}) = \operatorname{sign}(\xi) d(v_0, \ldots, v_n)$.*

*(iii) If $\lambda_0, \ldots, \lambda_n \in \mathbb{F}^{>0}$ then $d(\lambda_0 v_0, \ldots, \lambda_n v_n) = d(v_0, \ldots, v_n)$.*

*(iv) For any $\alpha \in \mathrm{GL}_n(\mathbb{F})$ we have $d(\alpha v_0, \ldots, \alpha v_n) = \operatorname{sign}(\det \alpha) \cdot d(v_0, \ldots, v_n)$.*

*(v) Let $\mathbb{F}'$ be another ordered field and let $\iota : \mathbb{F} \hookrightarrow \mathbb{F}'$ be an order-preserving field homomorphism. Then for $v_0, \ldots, v_n$ in general position in $\mathbb{F}^n$ we have $d(\iota v_0, \ldots, \iota v_n) = d(v_0, \ldots, v_n)$.*

*(vi) If $v_1, \ldots, v_{n+2} \in \mathbb{F}^n$ are in general position then*

$$\sum_{i=0}^{n+2} (-1)^i d(v_1, \ldots, \hat{v}_i, \ldots, v_{n+2}) = 0.$$

*Proof.* (i) The case that $d(v_0, \ldots, v_n) = 0$ is clearly well defined, so assume $v_0 = -\sum_{j=1}^n \lambda_j v_j$ with $\lambda_j > 0$. We have by elementary properties of determinants:

$$\begin{aligned} (-1)^i \operatorname{sign}\det(v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n) &= (-1)^i \operatorname{sign}\det(-\lambda_i v_i, v_1, \ldots, \hat{v}_i, \ldots, v_n) \\ &= (-1)^{i-1} \operatorname{sign}\det(v_i, v_1, \ldots, \hat{v}_i, \ldots, v_n) \\ &= \operatorname{sign}\det(v_1, \ldots, v_n). \end{aligned}$$

(ii) This follows from (i) in the case that $\xi$ is an adjacent transposition $(i \quad i+1)$. The general case follows since the adjacent transpositions generate the group of all permutations.

Parts (iii), (iv) and (v) follow immediately from the definition.

(vi) By (ii), (iii) and (iv) we may reduce the general case to the case that $v_1, \ldots, v_n$ are the standard basis elements $e_1, \ldots, e_n$ in $\mathbb{F}^n$ and $v_{n+1}$ and $v_{n+2}$ are of the form

$$v_{n+1} = \begin{pmatrix} \left.\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}\right\} r \\ -1 \\ \vdots \\ -1 \end{pmatrix}; \quad v_{n+2} = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{pmatrix},$$

with $x_1 > \ldots > x_r$ and $x_{r+1} < \ldots < x_n$. For simplicity we assume $0 < r < n$; the cases $r = 0$ and $r = n$ may be handled similarly.

Case 1. Assume $i \leq r$. If we have

$$v_{n+2} = \sum_{j=1}^{n+1} \lambda_j v_j,$$

then this implies

$$\lambda_j = \begin{cases} x_j - x_i & \text{if } j \leq r,\ j \neq i, \\ x_j + x_i & \text{if } r < j \leq n. \\ x_i & \text{if } j = n+1. \end{cases}$$

For all the coefficients $\lambda_j$ to be negative we require $i = 1$, $x_1 < 0$ and $x_1 + x_n < 0$. From this we may deduce that

$$\sum_{i=1}^{r} (-1)^i d(v_1, \ldots, \hat{v}_i, \ldots, v_n) = \begin{cases} 1 & \text{if } x_1 < 0 \text{ and } x_1 + x_n < 0, \\ 0 & \text{otherwise.} \end{cases}$$

A similar calculation shows that

$$\sum_{i=r+1}^{n} (-1)^i d(v_1, \ldots, \hat{v}_i, \ldots, v_n) = \begin{cases} -1 & \text{if } x_n > 0 \text{ and } x_1 + x_n < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally we have $d(v_1, \ldots, v_{n+1}) = 0$ and

$$(-1)^{n+1} d(v_1, \ldots, v_n, \ldots, v_{n+2}) = \begin{cases} -1 & \text{if } x_1 < 0 \text{ and } x_n < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Adding everything up we obtain the result. □

**2. A relation between cone functions** Given a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{F}^n$ we define a function $c(v_1, \ldots, v_n) : \mathbb{F}^n \to \mathbb{Z}$ by

$$c(v_1, \ldots, v_n)(w) = \begin{cases} \operatorname{sign} \det(v_1, \ldots, v_n) & \text{if } v_1^*(w), \ldots, v_n^*(w) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus up to a sign, $c(v_1, \ldots, v_n)$ is the characteristic function of the open cone of $v_1, \ldots, v_n$. This may be expressed in terms of the function $d$:

$$c(v_1, \ldots, v_n)(w) = (-1)^n d(v_1, \ldots, v_n, -w). \tag{1}$$

The functions $c$ and $d$ satisfy the following cocycle relation.

**Proposition 2** *If $v_0, \ldots, v_n, w \in \mathbb{F}^n$ are in general position then*

$$\sum_{i=0}^{n} (-1)^i c(v_0, \ldots, \hat{v}_i, \ldots, v_n)(w) = d(v_0, \ldots, v_n).$$

*Proof.* This follows immediately from Proposition 1 (vi) and (1). $\square$

**3.** Consider the local field $\mathbb{F} = \mathbb{R}((\epsilon_1)) \ldots ((\epsilon_n))$. Every element of $\mathbb{F}$ may be expressed as

$$\sum_{\mathbf{r} \in \mathbb{Z}^n} a_{\mathbf{r}} \epsilon^{\mathbf{r}},$$

where $\epsilon^{\mathbf{r}} = \epsilon_1^{r_1} \cdots \epsilon_n^{r_n}$. The coefficients $a_{\mathbf{r}}$ are in $\mathbb{R}$. We shall order the multi-indices $\mathbf{r} \in \mathbb{Z}^n$ lexicographically, so $\mathbf{r} < \mathbf{s}$ if and only if there is an $i \in \{1, 2, \ldots, n\}$ such that

$$r_i < s_i \text{ and } \forall j > i,\ r_j = s_j.$$

Using this ordering we may define the leading term of a non-zero element of $\mathbb{F}$ to be the non-zero monomial $a_{\mathbf{r}} \epsilon^{\mathbf{r}}$ for which $\mathbf{r}$ is smallest. An element of $\mathbb{F}^\times$ will be said to be positive (resp. negative) if its leading term $a_{\mathbf{r}} \epsilon^{\mathbf{r}}$ has positive (resp. negative) coefficient. For $f, g \in \mathbb{F}$ we define $f > g$ if and only if $f - g$ is positive. Under this ordering $\epsilon_1$ is positive but smaller than every positive real number. For $i = 1, \ldots, n-1$, the element $\epsilon_{i+1}$ is positive but smaller than every power of $\epsilon_i$.

We shall also use the field $\mathbb{F}' = \mathbb{R}((\epsilon_0)) \ldots ((\epsilon_n))$, ordered in an analogous way. We have $n+1$ order-preserving field embeddings $\iota_i : \mathbb{F} \hookrightarrow \mathbb{F}'$ defined by

$$(\iota_i f)(\epsilon_0, \ldots, \epsilon_n) = f(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_n).$$

**4.** Define for $i = 1, \ldots, n$:

$$b(\epsilon_i) = \begin{pmatrix} 1 \\ \epsilon_i \\ \vdots \\ \epsilon_i^{n-1} \end{pmatrix}.$$

We shall regard $b(\epsilon_i)$ as an element of $\mathbb{F}^n$.

**Lemma 1** *For any $\alpha_0, \ldots, \alpha_n \in \mathrm{GL}_n(\mathbb{R})$ and any $w \in \mathbb{R}^n \backslash \{0\}$, the set $\{\alpha_0 b(\epsilon_0), \ldots, \alpha_n b(\epsilon_n), w\}$ is in general position in $\mathbb{F}'^n$.*

*Proof.* Regarding $b$ as a function $\mathbb{R} \to \mathbb{R}^n$, we note that the values of $\alpha_i b(\epsilon_i)$ span $\mathbb{R}^n$. We may therefore choose $\epsilon_1, \ldots, \epsilon_n \in \mathbb{R}$ so that $\{\alpha_1 b(\epsilon_1), \ldots, \alpha_n b(\epsilon_n)\}$ is a basis of $\mathbb{R}^n$. Hence $\det(\alpha_1 b(\epsilon_1), \ldots, \alpha_n b(\epsilon_n))$ is a non-zero function of $\epsilon_1, \ldots, \epsilon_n$, so is a non-zero element of $\mathbb{F}$. It follows that $\{\alpha_1 b(\epsilon_1), \ldots, \alpha_n b(\epsilon_n)\}$ is a basis of $\mathbb{F}^n$. A similar argument shows that for any $j$, $\{w, \alpha_i b(\epsilon_i) : i \neq j\}$ is also a basis of $\mathbb{F}^n$. $\square$

**5.** We now define our cocycle. Let $\mathcal{F}$ denote the space of all functions $\varphi : \mathbb{R}^n \setminus \{0\} \to \mathbb{Z}$. We let $\mathrm{GL}_n(\mathbb{R})$ act on $\mathcal{F}$ by:

$$(\alpha * \varphi)(w) = \operatorname{sign}\det \alpha \cdot \varphi(\alpha^{-1} w), \quad \alpha \in \mathrm{GL}_n(\mathbb{R}),\ \varphi \in \mathcal{F},\ w \in \mathbb{R}^n \setminus \{0\}.$$

The constant functions in $\mathcal{F}$ form a submodule, which we shall denote $\mathbb{Z}(-)$. We shall write $\mathcal{M}$ for the quotient. We shall describe a cocycle $\sigma \in H^{n-1}(\mathrm{GL}_n(\mathbb{R}), \mathcal{M})$.

For $\alpha_1, \ldots, \alpha_n \in \mathrm{GL}_n(\mathbb{R})$ and $w \in \mathbb{R}^n \setminus \{0\}$, we define

$$\sigma(\alpha_1, \ldots, \alpha_n)(w) = c(\alpha_1 b(\epsilon_1), \ldots, \alpha_n b(\epsilon_n))(w).$$

**Proposition 3** *(i) For $\alpha_0, \ldots, \alpha_n \in \mathrm{GL}_n(\mathbb{R})$ and $w \in \mathbb{R}^n \setminus \{0\}$ we have*

$$\sum_{i=0}^{n} (-1)^i \sigma(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_n)(w) = d(\alpha_0 b(\epsilon_0), \ldots, \alpha_n b(\epsilon_n)).$$

*(ii) For $\beta, \alpha_1, \ldots, \alpha_n \in \mathrm{GL}_n(\mathbb{R})$ we have*

$$\sigma(\beta\alpha_1, \ldots, \beta\alpha_n) = \beta * \sigma(\alpha_1, \ldots, \alpha_n).$$

*Proof.* (i) We have by definition

$$\sigma(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_n)(w) = c(\alpha_0 b(\epsilon_1), \ldots, \alpha_{i-1} b(\epsilon_i), \alpha_{i+1} b(\epsilon_{i+1}), \ldots, \alpha_n b(\epsilon_n))(w).$$

Applying the the order-preserving map $\iota_i : \mathbb{F} \to \mathbb{F}'$ we have

$$\sigma(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_n)(w) = c(\alpha_0 b(\epsilon_0), \ldots, \alpha_{i-1} b(\epsilon_{i-1}), \alpha_{i+1} b(\epsilon_{i+1}), \ldots, \alpha_n b(\epsilon_n))(w).$$

The result now follows from Lemma 1 and Proposition 2.

(ii) This follows from Propsition 1 (iv) and (1). □

The proposition shows that $\sigma$ represents an element of $H^{n-1}(\mathrm{GL}_n(\mathbb{R}), \mathcal{M})$. The short exact sequence

$$0 \to \mathbb{Z}(-) \to \mathcal{F} \to \mathcal{M} \to 0,$$

gives rise to a connecting homomorphism $\partial : H^{n-1}(\mathrm{GL}_n, \mathcal{M}) \to H^n(\mathrm{GL}_n, \mathbb{Z}(-))$. The proposition also shows that $\partial\sigma$ is given by the $n$-cocycle

$$\tau(\alpha_0, \ldots, \alpha_n) = d(\alpha_0 b(\epsilon_0), \ldots, \alpha_n b(\epsilon_n)).$$

**Aside.** In this context it is worth recording the following long exact sequence:

$$\ldots \to H^r(\mathrm{GL}_n(\mathbb{R}), \mathbb{Z}(-)) \to H^r(\mathrm{GL}_{n-1}(\mathbb{R}), \mathbb{Z}(-)) \to H^r(\mathrm{GL}_n(\mathbb{R}), \mathcal{M}) \to H^{r+1}(\mathrm{GL}_n(\mathbb{R}), \mathbb{Z}(-)) \to \ldots.$$

*Proof.* We need only show that $H^r(\mathrm{GL}_{n-1}(\mathbb{R}), \mathbb{Z}(-))$ is canonically isomorphic to $H^r(\mathrm{GL}_n(\mathbb{R}), \mathcal{F})$. Consider the *mirabolic* subgroup:

$$P = \{\alpha \in \mathrm{GL}_n(\mathbb{R}) : \alpha e_1 = e_1\}.$$

We have $\mathcal{F} = \mathrm{ind}_P^{\mathrm{GL}_n(\mathbb{R})} \mathbb{Z}(-)$. Hence by Shapiro's Lemma,

$$H^r(\mathrm{GL}_n(\mathbb{R}), \mathcal{F}) = H^r(P, \mathbb{Z}(-)).$$

The group extension

$$1 \to \mathbb{R}^{n-1} \to P \to \mathrm{GL}_{n-1}(\mathbb{R}) \to 1,$$

gives rise to the spectral sequence

$$H^p(\mathrm{GL}_{n-1}(\mathbb{R}), H^q(\mathbb{R}^{n-1}, \mathbb{Z}(-))) \Rightarrow H^{p+q}(P, \mathbb{Z}(-)).$$

The result now follows since

$$H^q(\mathbb{R}^{n-1}, \mathbb{Z}(-)) = \begin{cases} \mathbb{Z}(-) & \text{if} \quad q = 0, \\ 0 & \quad\quad q > 0. \end{cases}$$

□

**6.** We finally show that the values of $\sigma$ are actually in the module of cones.

**Theorem 1** *(i) For $\alpha_1, \ldots, \alpha_n \in \mathrm{GL}_n(\mathbb{R})$ we have $\sigma(\alpha_1, \ldots, \alpha_n) \in \mathcal{L}_{\mathbb{R}}$.*
*(ii) For $\alpha_1, \ldots, \alpha_n \in \mathrm{GL}_n(\mathbb{Q})$ we have $\sigma(\alpha_1, \ldots, \alpha_n) \in \mathcal{L}_{\mathbb{Q}}$.*

*Proof.* Note that $\mathcal{K}_{\mathbb{R}}$ is closed under pointwise multiplication of functions. Hence to prove the first part of the proposition, it is sufficient to show that, for any linear form $\phi : \mathbb{F}^n \to \mathbb{F}$, the set

$$S = \{w \in \mathbb{R}^n : \phi(w) > 0\}$$

is a finite disjoint union of open cones. The restriction $\phi : \mathbb{R}^n \to \mathbb{F}$ is $\mathbb{R}$-linear. We may write $\phi$ as

$$\phi = \sum_{\mathbf{r}} \phi_{\mathbf{r}} \epsilon^{\mathbf{r}},$$

with $\phi_{\mathbf{r}} : \mathbb{R}^n \to \mathbb{R}$ linear forms. In this sum $\mathbf{r}$ runs over the multipowers of the $\epsilon_i$. We may therefore decompose $S$ into disjoint subsets:

$$S = \bigcup_{\mathbf{r}} S_{\mathbf{r}},$$

where

$$S_{\mathbf{r}} = \{w \in \mathbb{R}^n : \phi_{\mathbf{r}}(w) > 0 \text{ and for all } \mathbf{s} < \mathbf{r},\ \phi_{\mathbf{s}}(w) = 0\}$$

Each non-empty $S_{\mathbf{r}}$ is an open half-subspace, and is hence a finite disjoint union of cones. It remains to show that only finitely many $S_{\mathbf{r}}$ are non-empty. If $S_{\mathbf{r}}$ and $S_{\mathbf{s}}$ are both non-empty and $\mathbf{s} < \mathbf{r}$ then $S_{\mathbf{r}}$ is contained in the boundary of the closure of $S_{\mathbf{s}}$ and is therefore of strictly smaller dimension.

This proves the first part of the proposition. Now assume $\alpha_1, \ldots, \alpha_n \in \mathrm{GL}_n(\mathbb{Q})$. It follows that the basis vectors $\alpha_i b(\epsilon_i)$ are in $\mathbb{Q}(\epsilon)$. From this it follows that $\phi_{\mathbf{r}} : \mathbb{Q}^n \to \mathbb{Q}$. Hence the sets $S_{\mathbf{r}}$ may be decomposed into rational cones. □

**Remark 1** *One could define a $K$-cone for any subfield $K$ of $\mathbb{R}$ and obtain a generalization of the above proposition.*

# 5 Comparison with previous results

We shall describe $\sigma$ in the case $n = 2$ and then give a coboundary relating it to Solomon's cocycle $s$.

**1.** As $\sigma$ is homogeneous, we need only calculate $\sigma(1, \alpha)$ for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. Let $w = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$. Recall that to calculate $\sigma(1, \alpha)(w)$, we express $w$ in the form

$$w = x' \begin{pmatrix} 1 \\ \epsilon_1 \end{pmatrix} + y' \alpha \begin{pmatrix} 1 \\ \epsilon_2 \end{pmatrix}, \quad x', y' \in \mathbb{R}((\epsilon_1))((\epsilon_2)).$$

To simplify notation consider the matrix

$$M = \left( \begin{pmatrix} 1 \\ \epsilon_1 \end{pmatrix}, \alpha \begin{pmatrix} 1 \\ \epsilon_2 \end{pmatrix} \right) = \begin{pmatrix} 1 & a + b\epsilon_2 \\ \epsilon_1 & c + d\epsilon_2 \end{pmatrix}.$$

We have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = M^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The cocycle is given by the formula:

$$\sigma(1, \alpha) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \mathrm{sign}(\det M) & \text{if } x' \text{ and } y' \text{ are both positive in } \mathbb{R}((\epsilon_1))((\epsilon_2)), \\ 0 & \text{otherwise.} \end{cases}$$

After solving these inequalities we obtain:

**Proposition 4** *(i) Let* $\alpha = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$.

- *If* $a > 0$ *and* $c > 0$ *then* $\sigma(1, \alpha) = 0$.
- *If* $a > 0$ *and* $c < 0$ *then* $\sigma(1, \alpha) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} -1 & \text{if } x > 0 \text{ and } y = 0, \\ 0 & \text{otherwise.} \end{cases}$
- *If* $a < 0$ *and* $c > 0$ *then* $\sigma(1, \alpha) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} 1 & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$
- *If* $a < 0$ *and* $c < 0$ *then* $\sigma(1, \alpha) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} 1 & \text{if } y > 0 \text{ or if } y = 0 \text{ and } x < 0 \\ 0 & \text{otherwise.} \end{cases}$

*(ii) Let* $\alpha = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$.

- *If* $a > 0$ *and* $c > 0$ *then* $\sigma(1, \alpha) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} 1 & \text{if } y > 0 \text{ and } cx - by > 0, \\ 0 & \text{otherwise.} \end{cases}$
- *If* $a > 0$ *and* $c < 0$ *then* $\sigma(1, \alpha) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} -1 & \text{if } y \leq 0 \text{ and } cx - by < 0, \\ 0 & \text{otherwise.} \end{cases}$

- *If $a<0$ and $c>0$ then* $\sigma(1,\alpha)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} 1 & \text{if } y>0 \text{ and } cx-by\le 0, \\ 0 & \text{otherwise.} \end{cases}$
- *If $a<0$ and $c<0$ then* $\sigma(1,\alpha)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} -1 & \text{if } y\le 0 \text{ and } cx-by\le 0, \\ 0 & \text{otherwise.} \end{cases}$

*Proof.* To give an impression of how to do this calculation, we shall prove (i) in the case $a,c<0$. The other cases are left to the reader. We have $M=\begin{pmatrix} 1 & a+b\epsilon_2 \\ \epsilon_1 & c\epsilon_2 \end{pmatrix}$. Hence $\det M = c\epsilon_2 - a\epsilon_1 - b\epsilon_1\epsilon_2$. The leading term of $\det M$ is $-a\epsilon_1$, which is positive. Therefore $\det M>0$. Furthermore

$$x' = \frac{1}{\det M}\left(-ay+(cx-by)\epsilon_2\right), \quad y' = \frac{1}{\det M}\left(-\epsilon_1 x + y\right).$$

For $y'$ to be positive we require either $y>0$ or $y=0$ and $x<0$. In both of these cases $x'$ is also positive. □

**2.** In [8] Solomon obtained a cocycle on $\mathrm{PGL}_2$ rather than on $\mathrm{GL}_2$; however the values of the cocycle were in $\frac{1}{2}\mathcal{L}$ rather than in $\mathcal{L}$. This cocycle $s\in Z^1(\mathrm{PGL}_2(\mathbb{R}),\frac{1}{2}\mathcal{L})$ is defined as follows:

$$s(\alpha,\beta)(w) = \begin{cases} \operatorname{sign}\det(\alpha e_1,\beta e_1) & \text{if } \{\alpha e_1,\beta e_1\} \text{ is a basis of } \mathbb{R}^2 \text{ and } w\in C^o(\alpha e_1,\beta e_1), \\ \frac{1}{2}\operatorname{sign}\det(\alpha e_1,\beta e_1) & \text{if } \{\alpha e_1,\beta e_1\} \text{ is a basis of } \mathbb{R}^2 \text{ and } w\in \partial C(\alpha e_1,\beta e_1), \\ 0 & \text{otherwise.} \end{cases}$$

This is related to $\sigma$ by the coboundary:

$$(\sigma - s)(\alpha,\beta) = \alpha * \tau - \beta * \tau,$$

where

$$\tau\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \frac{1}{2} & \text{if } y=0 \text{ and } x>0, \\ 0 & \text{otherwise.} \end{cases}$$

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