Topological Free Entropy Dimension in Unital C* algebras

II : Orthogonal Sum of Unital C*-algebras

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Abstract: In the paper, we obtain a formula for topological free entropy dimension in the orthogonal sum (or direct sum) of unital C* algebras. As an application, we compute the topological free entropy dimension of any family of self-adjoint generators of a finite dimensional C* algebra.

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1. Introduction

The theory of free probability and free entropy was developed by Voiculescu from 1980s. It played a crucial role in the recent study of finite von Neumann algebras (see [**1**], [**3**], [**4**], [**5**], [**6**], [**7**], [**8**], [**9**], [**12**], [**16**], [**17**], [**25**], [**26**], [**27**]). An analogue of free entropy dimension in C* algebra context, the notion of topological free entropy dimension of of n-tuples of elements in a unital C* algebra, was introduced by Voiculescu in [**28**], where some basic properties of free entropy dimension are discussed.

We start our investigation of the properties of topological free entropy dimension in [**13**], where we computed the topological free entropy dimension of a self-adjoint element in a unital C* algebra. Some estimation of topological free entropy dimension in an infinite dimensional, unital, simple C* algebra with a unique trace was also obtained in the same paper. In this article, we will continue our investigation on the properties of topological free entropy dimension.

First, we compute the topological free entropy dimension in an $n \times n$ complex matrix algebra $\mathcal{M}_n(\mathbb{C})$ as follows (see **Theorem 3.1**):

$$\delta_{top}(x_1, \ldots, x_m) = 1 - \frac{1}{n^2},$$

where $x_1, \ldots, x_m$ is any family of self-adjoint generators of $\mathcal{M}_n(\mathbb{C})$ and $\delta_{top}(x_1, \ldots, x_m)$ is the Voiculescu's topological free entropy dimension of $x_1, \ldots, x_m$.

In [**28**], Voiculescu asked the question whether the equality

$$\chi_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \max\{\chi_{top}(x_1, \ldots, x_n), \chi_{top}(y_1, \ldots, y_n)\},$$

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holds when $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are self-adjoint elements in a unital C* algebras $\mathcal{A}$, or $\mathcal{B}$ respectively, and χ_{top} is the topological free entropy defined in [**28**]. Motivated by his question, in the paper we consider the topological free entropy dimension in the orthogonal sum of unital C* algebras. More specifically, we prove the following result.

Theorem 4.2: Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C* algebras and $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is a family of self-adjoint elements that generate $\mathcal{A} \bigoplus \mathcal{B}$ as a C*-algebra. Assume

$$s = \delta_{top}(x_1, \ldots, x_n) \qquad \text{and} \qquad t = \delta_{top}(y_1, \ldots, y_n).$$

(1) If $s \geq 1$ or $t \geq 1$, then

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \max\{\delta_{top}(x_1, \ldots, x_n), \delta_{top}(y_1, \ldots, y_n)\}$$

(2) If $s < 1$, $t < 1$ and both families $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ are stable (see Definition 4.1) , then (i)

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \frac{st - 1}{s + t - 2};$$

and (ii) the family of elements $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is also stable.

Combining the preceding two results, Theorem 3.1 and Theorem 4.2, we obtain the topological free entropy dimension of any family of self-adjoint generators in a finite dimensional C* algebra (see **Theorem 5.1**): Suppose that $\mathcal{A}$ is a finite dimensional C* algebra and $dim_{\mathbb{C}}\mathcal{A}$ is the complex dimension of $\mathcal{A}$. Then

$$\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{dim_{\mathbb{C}}\mathcal{A}},$$

where $x_1, \ldots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$.

The organization of the paper is as follows. In section 2, we recall Voiculescu's definition of topological free entropy dimension. The computation of topological free entropy dimension in an $n \times n$ complex matrix algebra is carried out in section 3. In section 4, we prove a formula of the topological free entropy dimension in the orthogonal sum of the unital C* algebras. In section 5, we calculate the topological free entropy dimension in any finite dimensional C* algebra.

In this article, we only discuss unital C* algebras which have the approximation property (see Definition 5.3 in [**13**]).

2. Definitions and preliminary

In this section, we will recall Voiculescu's definition of the topological free entropy dimension of n-tuples of elements in a unital C* algebra.

2.1. A Covering of a set in a metric space. Suppose (X, d) is a metric space and K is a subset of X. A family of balls in X is called a covering of K if the union of these balls covers K and the centers of these balls are in K.

2.2. Covering numbers in complex matrix algebra $(\mathcal{M}_k(\mathbb{C}))^n$**.** Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}Tr$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}(k)$ denote the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k^{s.a}(\mathbb{C})$ be the subalgebra of $\mathcal{M}_k(\mathbb{C})$ consisting of all self-adjoint matrices of $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ be the direct sum (or orthogonal sum) of n copies of $\mathcal{M}_k^{s.a}(\mathbb{C})$. Let $\|\cdot\|$ be an operator norm on $\mathcal{M}_k(\mathbb{C})^n$ defined by

$$\|(A_1, \ldots, A_n)\| = \max\{\|A_1\|, \ldots, \|A_n\|\}$$

for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$. Let $\|\cdot\|_{Tr}$ denote the usual trace norm induced by Tr on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \ldots, A_n)\|_{Tr} = \sqrt{Tr(A_1^*A_1) + \ldots + Tr(A_n^*A_n)}$$

for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$. Let $\|\cdot\|_2$ denote the trace norm induced by τ_k on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \ldots, A_n)\|_2 = \sqrt{\tau_k(A_1^*A_1) + \ldots + \tau_k(A_n^*A_n)}$$

for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-$\|\cdot\|$-ball $Ball(B_1, \ldots, B_n; \omega, \|\cdot\|)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\| < \omega.$$

Definition 2.1. *Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_\infty(\Sigma, \omega)$ to be the minimal number of ω-$\|\cdot\|$-balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.*

For every $\omega > 0$, we define the ω-$\|\cdot\|_2$-ball $Ball(B_1, \ldots, B_n; \omega, \|\cdot\|_2)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\|_2 < \omega.$$

Definition 2.2. *Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_2(\Sigma, \omega)$ to be the minimal number of ω-$\|\cdot\|_2$-balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.*

The following lemma is obvious.

Lemma 2.1. *Suppose K is a subset of $\mathcal{M}_k(\mathbb{C})^n$, equipped with a distance d. Suppose that $\{B_\lambda\}_{\lambda\in\Lambda}$ is a family of balls of radius ω so that*

$$K \subseteq \cup_{\lambda\in\Lambda} B_\lambda.$$

Then

Covering number of K by balls of radius $2\omega \le$ Cardinality of Λ.

2.3. Noncommutative polynomials. In this article, we always assume that $\mathcal{A}$ is a unital C*-algebra. Let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be self-adjoint elements in $\mathcal{A}$. Let $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$ be the unital noncommutative polynomials in the indeterminates $X_1, \ldots, X_n, Y_1, \ldots, Y_m$. Let $\{P_r\}_{r=1}^\infty$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$ with rational complex coefficients. (Here "rational complex coefficients" means that the real and imaginary parts of all coefficients of P_r are rational numbers).

Remark 2.1. *We alsways assume that* $1 \in \mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$.

2.4. Voiculescu's Norm-microstates Space. For all integers $r, k \geq 1$, real numbers $R, \epsilon > 0$ and noncommutative polynomials $P_1, \ldots, P_r$, we define

$$\Gamma_R^{(top)}(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

to be the subset of $(\mathcal{M}_k^{s.a}(\mathbb{C}))^{n+m}$ consisting of all these

$$(A_1, \ldots, A_n, B_1, \ldots, B_m) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^{n+m}$$

satisfying

$$\max\{\|A_1\|, \ldots, \|A_n\|, \|B_1\|, \ldots, \|B_m\|\} \leq R$$

and

$$|\|P_j(A_1, \ldots, A_n, B_1, \ldots, B_m)\| - \|P_j(x_1, \ldots, x_n, y_1, \ldots, y_m)\|| \leq \epsilon, \qquad \forall\ 1 \leq j \leq r.$$

Define the norm-microstates space of $x_1, \ldots, x_n$ in the presence of $y_1, \ldots, y_m$, denoted by

$$\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$$

as the projection of $\Gamma_R^{(top)}(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$ onto the space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ via the mapping

$$(A_1, \ldots, A_n, B_1, \ldots, B_m) \to (A_1, \ldots, A_n).$$

2.5. Voiculescu's topological free entropy dimension. Define

$$\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega)$$

to be the covering number of the set $\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$ by ω-$\|\cdot\|$-balls in the metric space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ equipped with operator norm.

Definition 2.3. *Define*

$$\begin{aligned} &\delta_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m; \omega) \\ &= \sup_{R>0} \inf_{\epsilon>0, r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega))}{-k^2 \log \omega} \end{aligned}$$

The topological entropy dimension *of $x_1, \ldots, x_n$ in the presence of $y_1, \ldots, y_m$ is defined by*

$$\delta_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m) = \limsup_{\omega\to 0^+} \delta_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m; \omega)$$

Remark 2.2. *Let $R > \max\{\|x_1\|, \ldots, \|x_n\|, \|y_1\|, \ldots, \|y_m\|\}$ be some positive number. By definition, we know*

$$\begin{aligned} &\delta_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m) \\ &= \limsup_{\omega\to 0^+} \inf_{\epsilon>0, r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega))}{-k^2 \log \omega} \end{aligned}$$

Remark 2.3. *Apparently, $\delta_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m)$ does not depend on the order of the sequence $\{P_r\}_{r=1}^\infty$.*

2.6. C* algebra ultraproduct and von Neumann algebra ultraproduct. Suppose $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^{\infty}$ is a sequence of complex matrix algebras where k_m goes to infinity as m goes to infinity. Let γ be a free ultrafilter in $\beta(\mathbb{N}) \setminus \mathbb{N}$. We can introduce a unital C* algebra $\prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C})$ as follows:

$$\prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C}) = \{(Y_m)_{m=1}^{\infty} \mid \forall\, m \geq 1,\ Y_m \in \mathcal{M}_{k_m}(\mathbb{C}) \ \text{ and } \ \sup_{m \geq 1} \|Y_m\| < \infty\}.$$

We can also introduce norm closed two sided ideals $\mathcal{I}_\infty$ and $\mathcal{I}_2$ as follows.

$$\mathcal{I}_\infty = \{(Y_m)_{m=1}^{\infty} \in \prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C}) \mid \lim_{m \to \gamma} \|Y_m\| = 0\}$$

$$\mathcal{I}_2 = \{(Y_m)_{m=1}^{\infty} \in \prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C}) \mid \lim_{m \to \gamma} \|Y_m\|_2 = 0\}$$

Definition 2.4. *The C* algebra ultraproduct of $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^{\infty}$ along the ultrfilter γ, denoted by $\prod_{m=1}^{\gamma} \mathcal{M}_{k_m}(\mathbb{C})$, is defined to be the quotient algebra of $\prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C})$ by the ideal $\mathcal{I}_\infty$. The image of $(Y_m)_{m=1}^{\infty} \in \prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C})$ in the quotient algebra is denoted by $[(Y_m)_m]$.*

Definition 2.5. *The von Neumann algebra ultraproduct of $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^{\infty}$ along the ultrfilter γ, also denoted by $\prod_{m=1}^{\gamma} \mathcal{M}_{k_m}(\mathbb{C})$ if no confusion arises, is defined to be the quotient algebra of $\prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C})$ by the ideal $\mathcal{I}_2$. The image of $(Y_m)_{m=1}^{\infty} \in \prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C})$ in the quotient algebra is denoted by $[(Y_m)_m]$.*

Remark 2.4. *The von Neumann algebra ultraproduct $\prod_{m=1}^{\gamma} \mathcal{M}_{k_m}(\mathbb{C})$ is a finite factor (see* [**18**]*).*

3. Topological free entropy dimension in $\mathcal{M}_n(\mathbb{C})$

In this section, we are going to calculate the topological free entropy dimension of a family of self-adjoint generators of $\mathcal{M}_n(\mathbb{C})$.

3.1. Upper-bound.

Proposition 3.1. *Let n be a positive integer and $\mathcal{M}_n(\mathbb{C})$ be the $n \times n$ matrix algebra over the complex numbers. Let $x_1, \ldots, x_m$ be a family of self-adjoint matrices that generate $\mathcal{M}_n(\mathbb{C})$. Then*

$$\delta_{top}(x_1, \ldots, x_m) \leq 1 - \frac{1}{n^2}.$$

Proof. Since $\mathcal{M}_n(\mathbb{C})$ is a unital C* algebra with a unique tracial state, by Theorem 5.1 in [**13**], we know that

$$\delta_{top}(x_1, \ldots, x_m) \leq \kappa\delta(x_1, \ldots, x_m),$$

where $\kappa\delta(x_1, \ldots, x_m)$ is the Voiculescu's free dimension capacity in [**28**]. By [**14**] or Proposition 1 in [**12**], we have

$$\kappa\delta(x_1, \ldots, x_m) \leq 1 - \frac{1}{n^2}.$$

Therefore,

$$\delta_{top}(x_1,\ldots,x_m)\le 1-\frac{1}{n^2}.$$

□

3.2. Some lammas. In this subsection, we let n,t be some positive integers and $k=nt$. Let

$$A=\begin{pmatrix} 1\cdot I_t & 0 & \cdots & 0\\ 0 & 2\cdot I_t & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & n\cdot I_t\end{pmatrix}\quad \text{and} \quad W=\begin{pmatrix} 0 & 1\cdot I_t & 0 & \cdots & 0\\ 0 & 0 & 1\cdot I_t & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots & \cdots\\ 1\cdot I_t & 0 & 0 & \cdots & 0\end{pmatrix}$$

be in $\mathcal{M}_k(\mathbb{C})$, where I_t is the identity matrix of $\mathcal{M}_t(\mathbb{C})$.

Lemma 3.1. *Let $\delta>0$. Suppose $\|U_1AU_1^*-U_2AU_2^*\|_2\le\delta$ and $\|U_1WU_1^*-U_2WU_2^*\|_2\le\delta$ for some unitary matrices U_1 and U_2 in $\mathcal{U}(k)$. Then there are a unitary matrix V_1 in $\mathcal{M}_t(\mathbb{C})$ and*

$$V=\begin{pmatrix} V_1 & 0 & \cdots & 0\\ 0 & V_1 & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & V_1\end{pmatrix}\in\mathcal{U}(k)$$

so that

$$\|U_1-U_2V\|_2\le 14n^2\delta.$$

Proof. Assume that

$$U_2^*U_1=\begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1,n}\\ U_{21} & U_{22} & \cdots & U_{2,n}\\ \cdots & \cdots & \cdots & \cdots\\ U_{n,1} & U_{n,2} & \cdots & U_{n,n}\end{pmatrix}\in\mathcal{U}(k)$$

where each $U_{i,j}$ is a $t\times t$ matrix for all $1\le i,j\le n$.

Let

$$S=\begin{pmatrix} U_{11} & 0 & \cdots & 0\\ 0 & U_{22} & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & U_{n,n}\end{pmatrix}.$$

It is easy to see that $\|S\|_2 \le 1$ and

$$\begin{aligned}\delta^2 &\ge \|U_1AU_1^* - U_2AU_2^*\|_2^2 = \frac{1}{k}Tr((U_2^*U_1A - AU_2^*U_1)(U_2^*U_1A - AU_2^*U_1)^*)\\ &= \frac{1}{k}\sum_{1\le i\ne j\le m} Tr(|i-j|^2 U_{ij}U_{ij}^*)\\ &\ge \frac{1}{k}\sum_{1\le i\ne j\le m} Tr(U_{ij}U_{ij}^*).\end{aligned}$$

Hence

$$\|U_1 - U_2S\|_2 = \|U_2^*U_1 - S\|_2 = \sqrt{\frac{1}{k}\sum_{1\le i\ne j\le m} Tr(U_{ij}U_{ij}^*)} \le \delta. \tag{3.2.1}$$

Thus,

$$\left\|\begin{pmatrix} U_{22} & 0 & \cdots & 0\\ 0 & U_{33} & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & U_{11}\end{pmatrix} - \begin{pmatrix} U_{11} & 0 & \cdots & 0\\ 0 & U_{22} & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & U_{n,n}\end{pmatrix}\right\|_2$$

$$= \|W^*SW - S\|_2 = \|SW - WS\|_2 = \|U_2^*U_1W - WU_2^*U_1 - (U_2^*U_1 - S)W + W(U_2^*U_1 - S)\|_2 \le 3\delta.$$

It follows that

$$\frac{1}{\sqrt{k}}\sqrt{Tr((U_{j,j} - U_{j+1,j+1})(U_{j,j} - U_{j+1,j+1})^*)} \le 3\delta, \quad \forall\ 1 \le j \le n-1. \tag{3.2.2}$$

Let

$$X = \begin{pmatrix} U_{11} & 0 & \cdots & 0\\ 0 & U_{11} & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & U_{11}\end{pmatrix}.$$

By inequality (3.2.2), we have

$$\begin{aligned}\|S - X\|_2 &\le \frac{1}{\sqrt{k}}\sqrt{\sum_{i=2}^{n} Tr((U_{11} - U_{ii})(U_{11} - U_{ii})^*)} \le \frac{1}{\sqrt{k}}\sum_{i=2}^{n}\sqrt{Tr((U_{11} - U_{ii})(U_{11} - U_{ii})^*)}\\ &\le \frac{1}{\sqrt{k}}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\sqrt{Tr((U_{j,j} - U_{j+1,j+1})(U_{j,j} - U_{j+1,j+1})^*)} < 3n^2\delta. \end{aligned}\tag{3.2.3}$$

Let $U_{11}=V_1H$ be the polar decomposition of U_{11} in $\mathcal{M}_t(\mathbb{C})$ and

$$V=\begin{pmatrix} V_1 & 0 & \cdots & 0\\ 0 & V_1 & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & V_1\end{pmatrix}.$$

Note $\|H\|=\|U_{11}\|\le\|S\|\le 1$. We have

$$\begin{aligned}\|X-V\|_2 &= \left\|\begin{pmatrix} H & 0 & \cdots & 0\\ 0 & H & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & H\end{pmatrix}-I\right\|_2\\ &\le \left\|\begin{pmatrix} H^2 & 0 & \cdots & 0\\ 0 & H^2 & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & H^2\end{pmatrix}-I\right\|_2\\ &= \|X^*X-I\|_2\le 2\|S-X\|_2+\|S^*S-I\|_2\le 6n^2\delta+2\delta,\end{aligned}$$

where the last inequality follows from inequalities (3.2.1) and (3.2.3). It follows that

$$\|U_1-U_2V\|_2\le\|U_1-U_2S\|_2+\|S-X\|_2+\|X-V\|_2\le 3\delta+3n^2\delta+6n^2\delta+2\delta\le 14n^2\delta.$$

□

Lemma 3.2. *Let $k=nt$ and*

$$\mathcal{N}_1=\left\{\begin{pmatrix} V_1 & 0 & \cdots & 0\\ 0 & V_1 & \cdots & 0\\ \cdots & \cdots & \ddots & \cdots\\ 0 & 0 & \cdots & V_1\end{pmatrix}\in\mathcal{U}(k)\mid V_1\in\mathcal{U}(t)\right\}\subset\mathcal{M}_k(\mathbb{C}).$$

For every $U\in\mathcal{U}(k)$, let

$$\Sigma(U)=\{U_1\in\mathcal{U}(k)\mid \exists \text{ a unitary matrix } V \text{ in } \mathcal{N}_1 \text{ such that } \|U_1-UV\|_2\le 14n^2\delta\ \}.$$

Then

$$\mu(\Sigma(U))\le(C_1\cdot 30n^2\delta)^{k^2}\cdot\left(\frac{C}{\delta}\right)^{t^2},$$

where μ is the normalized Haar measure on $\mathcal{U}(k)$ and C,C_1 are constants independent of t,δ.

PROOF. By computing the covering number of $\mathcal{N}_1$ by δ-$\|\cdot\|_2$-balls in $\mathcal{U}(k)$, we know

$$\nu_2(\mathcal{N}_1,\delta) \le \left(\frac{C}{\delta}\right)^{t^2},$$

where C is a constant independent of t,δ. Thus, by Lemma 2.1, the covering number of the set $\Sigma(U)$ by the $30n^2\delta$-$\|\cdot\|_2$-balls in $\mathcal{U}(k)$ is bounded by

$$\nu_2(\Sigma(U), 30n^2\delta) \le \nu_2(\mathcal{N}_1,\delta) \le \left(\frac{C}{\delta}\right)^{t^2}.$$

But the ball of radius $30n^2\delta$ in $\mathcal{U}(k)$ has the volume bounded by

$$\mu(\text{ball of radius } 30n^2\delta \text{ in } \mathcal{U}(k)) \le (C_1\cdot 30n^2\delta)^{k^2},$$

where C_1 is a universal constant. Thus

$$\mu(\Sigma(U)) \le (C_1\cdot 30n^2\delta)^{k^2}\cdot\left(\frac{C}{\delta}\right)^{t^2}.$$

□

Lemma 3.3. *Let A, W and*

$$\mathcal{N}_1 = \left\{ \begin{pmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & V_1 \end{pmatrix} \in \mathcal{U}(k) \mid V_1 \in \mathcal{U}(t) \right\} \subset \mathcal{M}_k(\mathbb{C}).$$

be defined as above. Let

$$\Omega(A,W) = \{(U^*AU,\ \frac{1}{2}U^*(W+W^*)U,\ \frac{1}{2\sqrt{-1}}U^*(W-W^*)U) \mid U\in\mathcal{U}(k)\}.$$

Then, for each $\delta>0$,

$$\nu_2(\Omega(A,W),\frac{1}{4}\delta) \ge (C_1\cdot 30n^2\delta)^{-k^2}\cdot\left(\frac{C}{\delta}\right)^{-t^2},$$

where C_1, C are some universal constants independent of t,δ.

PROOF. For every $U\in\mathcal{U}(k)$, define

$$\Sigma(U) = \{U_1\in\mathcal{U}(k) \mid \exists \text{ a unitary matrix } V\in\mathcal{N}_1, \text{ such that } \|U_1 - UV\|_2 \le 14n^2\delta\}.$$

By preceding lemma, we have

$$\mu(\Sigma(U)) \le (C_1\cdot 30n^2\delta)^{k^2}\cdot\left(\frac{C}{\delta}\right)^{t^2}.$$

A "parking" (or exhausting) argument will show the existence of a family of unitary elements $\{U_i\}_{i=1}^N \subset \mathcal{U}(k)$ such that

$$N \geq (C_1 \cdot 30n^2\delta)^{-k^2} \cdot \left(\frac{C}{\delta}\right)^{-t^2}$$

and

$$U_i \text{ is not contained in } \cup_{j=1}^{i-1} \Sigma(U_j), \qquad \forall\, i = 1, \ldots, N.$$

From the definition of each $\Sigma(U_j)$, it follows that

$$\|U_i - U_j V\|_2 \geq 14n^2\delta, \qquad \forall \text{ unitary matrix } V \in \mathcal{N}_1, \forall\, 1 \leq j < i \leq N.$$

By Lemma 3.1, we know that

$$\|U_i A U_i^* - U_j A U_j^*\|_2 > \delta \qquad \text{or} \qquad \|U_i W U_i^* - U_j W U_j^*\|_2 > \delta,$$

which implies that

$$\nu_2(\Omega(A, W), \frac{1}{4}\delta) \geq N \geq (C_1 \cdot 30n^2\delta)^{-k^2} \cdot \left(\frac{C}{\delta}\right)^{-t^2}.$$

□

3.3. Lower-bound. Suppose $x_1, \ldots, x_m$ is a family of self-adjoint elements that generate $\mathcal{M}_n(\mathbb{C})$. Let $\{e_{st}\}_{s,t=1}^n$ be a canonical system of matrix units in $\mathcal{M}_n(\mathbb{C})$. We might assume that

$$x_i = \sum_{s,t=1}^n x_{st}^{(i)} \cdot e_{st}, \qquad \forall\, 1 \leq i \leq m,$$

for some $\{x_{st}^{(i)}\}_{1 \leq s,t \leq n, 1 \leq i \leq m} \subset \mathbb{C}$. Let

$$a = \sum_{i=1}^n i \cdot e_{ii} \qquad \text{and} \qquad w = \sum_{i=1}^{n-1} e_{i,i+1} + e_{n,1}.$$

Note that $\mathcal{M}_n(\mathbb{C})$ is a finite dimensional C* algebra. It is easy to see that there exist noncommutative polynomials $P_1(x_1, \ldots, x_m)$ and $P_2(x_1, \ldots, x_m)$ such that

$$a = P_1(x_1, \ldots, x_m) \qquad \text{and} \qquad w = P_2(x_1, \ldots, x_m).$$

The proof of Lemma 5.1 in [**13**] can be easily adapted to prove the following Lemma 3.4.

Lemma 3.4. *We have*

$$\delta_{top}(a, \frac{w + w^*}{2}, \frac{w - w^*}{2\sqrt{-1}} : x_1, \ldots, x_m) \leq \delta_{top}(x_1, \ldots, x_m).$$

Lemma 3.5. *We have*

$$\delta_{top}(a, \frac{w + w^*}{2}, \frac{w - w^*}{2\sqrt{-1}} : x_1, \ldots, x_m) \geq 1 - \frac{1}{n^2}.$$

PROOF. Let t be positive integer and $k=nt$. Note that

$$\begin{aligned} x_i &= \sum_{s,t=1}^{n} x_{st}^{(i)} \cdot e_{st}, \qquad \forall\ 1 \le i \le m \\ a &= \sum_{i=1}^{n} i \cdot e_{ii} \\ w &= \sum_{i=1}^{n-1} e_{i,i+1} + e_{n,1}. \end{aligned}$$

We let

$$\begin{aligned} X_i &= \left(\sum_{s,t=1}^{n} x_{st}^{(i)} \cdot e_{st} \right) \otimes I_t, \qquad \forall\ 1 \le i \le m \\ A &= \left(\sum_{i=1}^{n} i \cdot e_{ii} \right) \otimes I_t \\ W &= \left(\sum_{i=1}^{n-1} e_{i,i+1} + e_{n,1} \right) \otimes I_t \end{aligned}$$

be matrices in $\mathcal{M}_k(\mathbb{C})$. It is not hard to see that, for every $t \in \mathbb{N}$ and $k = nt$,

$$(A, \frac{W+W^*}{2}, \frac{W-W^*}{2\sqrt{-1}}) \in \Gamma_R^{(top)}(a, \frac{w+w^*}{2}, \frac{w-w^*}{2\sqrt{-1}} : x_1, \ldots, x_m; k, \epsilon, P_1, \ldots, P_r)$$

when $R > \max\{\|a\|, \|x_1\|, \ldots, \|x_m\|, 1\}$, $\epsilon > 0$ and $r \ge 1$. Therefore,

$$\Omega(A, W) \subset \Gamma_R^{(top)}(a, \frac{w+w^*}{2}, \frac{w-w^*}{2\sqrt{-1}} : x_1, \ldots, x_m; k, \epsilon, P_1, \ldots, P_r),$$

where $\Omega(A,W)$ is defined in Lemma 3.3. Letting $\delta = 4\omega$, by lemma 3.3, we have

$$\nu_2(\Gamma_R^{(top)}(a, \frac{w+w^*}{2}, \frac{w-w^*}{2\sqrt{-1}} : x_1, \ldots, x_m; k, \epsilon, P_1, \ldots, P_r), \omega) \ge (C_1 \cdot 120 n^2 \omega)^{-k^2} \cdot \left(\frac{4C}{\omega} \right)^{-t^2},$$

where C_1, C are some constants independent of t, ω. By the definitions of the operator norm and the trace norm on $(\mathcal{M}_k(\mathbb{C}))^3$, we get

$$\begin{aligned} &\nu_\infty(\Gamma_R^{(top)}(a, \frac{w+w^*}{2}, \frac{w-w^*}{2\sqrt{-1}} : x_1, \ldots, x_m; k, \epsilon, P_1, \ldots, P_r), \frac{\omega}{\sqrt{3}}) \\ &\qquad \ge \nu_2(\Gamma_R^{(top)}(a, \frac{w+w^*}{2}, \frac{w-w^*}{2\sqrt{-1}} : x_1, \ldots, x_m; k, \epsilon, P_1, \ldots, P_r), \omega) \\ &\qquad \ge (C_1 \cdot 120 n^2 \omega)^{-k^2} \cdot \left(\frac{4C}{\omega} \right)^{-t^2}. \end{aligned} \tag{3.3.1}$$

It quickly induces that

$$\delta_{top}(a, \frac{w+w^*}{2}, \frac{w-w^*}{2\sqrt{-1}} : x_1, \ldots, x_m) \geq 1 - \frac{1}{n^2}.$$

□

Combining Lemma 3.4 and Lemma 3.5, we have the following result.

Proposition 3.2. *Suppose $x_1, \ldots, x_m$ is a family of self-adjoint generators of $\mathcal{M}_n(\mathbb{C})$. Then*

$$\delta_{top}(x_1, \ldots, x_m) \geq 1 - \frac{1}{n^2}.$$

3.4. Conclusion. By Proposition 3.1 and Proposition 3.2, we obtain the following result.

Theorem 3.1. *Suppose $x_1, \ldots, x_m$ is a family of self-adjoint generators of $\mathcal{M}_n(\mathbb{C})$. Then*

$$\delta_{top}(x_1, \ldots, x_m) = 1 - \frac{1}{n^2}.$$

4. Topological free entropy dimension in orthogonal sum of C* algebras

In this section, we assume that $\mathcal{A}$, $\mathcal{B}$ are two unital C* algebras and $\mathcal{A} \bigoplus \mathcal{B}$ is the orthogonal sum, or direct sum, of $\mathcal{A}$ and $\mathcal{B}$. We assume that the self-adjoint elements $x_1 \oplus y_1, \cdots, x_n \oplus y_n$ generate $\mathcal{A} \bigoplus \mathcal{B}$ as a C* algebra. Thus $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, are the families of self-adjoint generators of $\mathcal{A}$, or $\mathcal{B}$ respectively.

4.1. Upper-bound of topological free entropy dimension in orthogonal sum of C* algebras. Let $R > \max\{\|x_1\|, \ldots, \|x_n\|, \|y_1\|, \ldots, \|y_n\|\}$ be a positive number. By the definition of topological free entropy dimension, we have the following.

Lemma 4.1. *For each*

$$\alpha > \delta_{top}(x_1, \ldots, x_n) \qquad and \qquad \beta > \delta_{top}(y_1, \ldots, y_n),$$

(i) there is some $\frac{1}{10} > \omega_0 > 0$ so that, if $0 < \omega < \omega_0$,

$$\inf_{r\in\mathbb{N}} \limsup_{k_1\to\infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; k_1, \frac{1}{r}, P_1, \ldots, P_r), \omega))}{-k_1^2 \log \omega} < \alpha;$$

$$\inf_{r\in\mathbb{N}} \limsup_{k_2\to\infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(y_1, \ldots, y_n; k_2, \frac{1}{r}, P_1, \ldots, P_r), \omega))}{-k_2^2 \log \omega} < \beta.$$

(ii) Thus, for each $0 < \omega < \omega_0$, there is $r(\omega) \in \mathbb{N}$ satisfying

$$\limsup_{k_1\to\infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; k_1, \frac{1}{r(\omega)}, P_1, \ldots, P_{r(\omega)}), \omega))}{-k_1^2 \log \omega} < \alpha;$$

$$\limsup_{k_2\to\infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(y_1, \ldots, y_n; k_2, \frac{1}{r(\omega)}, P_1, \ldots, P_{r(\omega)}), \omega))}{-k_2^2 \log \omega} < \beta.$$

(iii) Therefore, for each $0<\omega<\omega_0$ and $r(\omega)\in\mathbb{N}$, there is some $K(r(\omega))\in\mathbb{N}$ satisfying

$$\log(\nu_\infty(\Gamma_R^{(top)}(x_1,\ldots,x_n;k_1,\frac{1}{r(\omega)},P_1,\ldots,P_{r(\omega)}),\omega))<-\alpha k_1^2\log\omega,\quad \forall\ k_1\geq K(r(\omega));$$

$$\log(\nu_\infty(\Gamma_R^{(top)}(y_1,\ldots,y_n;k_2,\frac{1}{r(\omega)},P_1,\ldots,P_{r(\omega)}),\omega))<-\beta k_2^2\log\omega,\quad \forall\ k_2\geq K(r(\omega)).$$

Lemma 4.2. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C^* algebras and $x_1\oplus y_1,\ldots,x_n\oplus y_n$ is a family of self-adjoint elements that generate $\mathcal{A}\bigoplus\mathcal{B}$. Let $R>\max\{\|x_1\|,\ldots,\|x_n\|,\|y_1\|,\ldots,\|y_n\|\}$ be a positive number. For any $\omega>0$, $r_0\in\mathbb{N}$, there is some $t>0$ so that the following holds: $\forall\ r>t$, $\forall\ k\geq 2$, if*

$$(X_1,\ldots,X_n)\in\Gamma_R^{(top)}(x_1\oplus y_1,\ldots,x_n\oplus y_n;k,\frac{1}{r},P_1,\ldots,P_r),$$

then there are $k_1,k_2\in\mathbb{N}$,

$$(A_1,\ldots,A_n)\in\Gamma_R^{(top)}(x_1,\ldots,x_n;k_1,\frac{1}{r_0},P_1,\ldots,P_{r_0}),$$

$$(B_1,\ldots,B_n)\in\Gamma_R^{(top)}(y_1,\ldots,y_n;k_2,\frac{1}{r_0},P_1,\ldots,P_{r_0})$$

and $U\in\mathcal{U}(k)$ so that (i) $k_1+k_2=k$; and (ii)

$$\left\|(X_1,\ldots,X_n)-U^*(\begin{pmatrix}A_1&0\\0&B_1\end{pmatrix},\ldots,\begin{pmatrix}A_n&0\\0&B_n\end{pmatrix})U\right\|\leq\omega.$$

Proof. We will prove the result by using the contradiction. Assume, to the contrary, the result of the lemma does not hold, i.e. there are some $\omega_0>0$, $r_0\geq 1$, two strictly increasing sequences $\{r_m\}_{m=1}^\infty$ and $\{k_m\}_{m=1}^\infty$, and

$$(X_1^{(m)},\ldots,X_n^{(m)})\in\Gamma_R^{(top)}(x_1\oplus y_1,\ldots,x_n\oplus y_n;k_m,\frac{1}{r_m},P_1,\ldots,P_{r_m})$$

satisfying

$$\left\|(X_1^{(m)},\ldots,X_n^{(m)})-U^*(\begin{pmatrix}A_1^{(m)}&0\\0&B_1^{(m)}\end{pmatrix},\ldots,\begin{pmatrix}A_n^{(m)}&0\\0&B_n^{(m)}\end{pmatrix})U\right\|>\omega. \tag{4.1.1}$$

for all

$$(A_1^{(m)},\ldots,A_n^{(m)})\in\Gamma_R^{(top)}(x_1,\ldots,x_n;s_{1,m},\frac{1}{r_0},P_1,\ldots,P_{r_0}),$$

$$(B_1^{(m)},\ldots,B_n^{(m)})\in\Gamma_R^{(top)}(y_1,\ldots,y_n;s_{2,m},\frac{1}{r_0},P_1,\ldots,P_{r_0})$$

and all $U\in\mathcal{U}(k)$ where $s_{1,m}+s_{2,m}=k_m$.

Let γ be a free ultra-filter in $\beta(\mathbb{N})\setminus\mathbb{N}$. Let $\prod_{m=1}^\gamma\mathcal{M}_{k_m}(\mathbb{C})$ be the C* algebra ultra-product of $(\mathcal{M}_{k_m}(\mathbb{C}))_{m=1}^\infty$ along the ultra-filter γ, i.e. $\prod_{m=1}^\gamma\mathcal{M}_{k_m}(\mathbb{C})$ is the quotient algebra of the C* algebra $\prod_m\mathcal{M}_{k_m}(\mathbb{C})$ by $\mathcal{I}_\infty$, the 0-ideal of the norm $\|\cdot\|_\gamma$, where $\|(Y_m)_{m=1}^\infty\|_\gamma=\lim_{m\to\gamma}\|Y_m\|$ for each $(Y_m)_{m=1}^\infty$ in $\prod_m\mathcal{M}_{k_m}(\mathbb{C})$.

By mapping $x_i \oplus y_i$ to $[(X_i^{(m)})_{m=1}^\infty]$ in $\prod_{m=1}^\gamma \mathcal{M}_{k_m}(\mathbb{C})$ for each $1 \le i \le n$, we obtain a unital $*$-isomorphism ψ from the C* algebra $\mathcal{A} \bigoplus \mathcal{B}$ onto the C* subalgebra generated by $[(X_1^{(m)})_{m=1}^\infty]$, $\ldots$, $[(X_n^{(m)})_{m=1}^\infty]$ in $\prod_{m=1}^\gamma \mathcal{M}_{k_m}(\mathbb{C})$. Thus $\psi(I_\mathcal{A} \oplus 0)$ and $\psi(0 \oplus I_\mathcal{B})$ are two projections in $\prod_{m=1}^\gamma \mathcal{M}_{k_m}(\mathbb{C})$ satisfying

$$\psi(I_\mathcal{A} \oplus 0) + \psi(0 \oplus I_\mathcal{B}) = I_{\prod_{m=1}^\gamma \mathcal{M}_{k_m}(\mathbb{C})}.$$

Without loss of generality, we can assume that there is a sequence of projections $\{P_m\}_{m=1}^\infty$ with $P_m \in \mathcal{M}_{k_m}(\mathbb{C})$ such that

$$[(P_m)_{m=1}^\infty] = \psi(I_\mathcal{A} \oplus 0) \qquad \text{and} \qquad [(I_{k_m} - P_m)_{m=1}^\infty] = \psi(0 \oplus I_\mathcal{B}),$$

where I_{k_m} is the identity matrix of $\mathcal{M}_{k_m}(\mathbb{C})$. For each P_m in $\mathcal{M}_{k_m}(\mathbb{C})$, there are positive integers $s_{1,m}, s_{2,m}$, with $s_{1,m} + s_{2,m} = k_m$, and a unitary matrix U_m in $\mathcal{U}(k_m)$ so that

$$P_m = U_m^* \begin{pmatrix} I_{s_{1,m}} & 0 \\ 0 & 0 \end{pmatrix} U_m \qquad \text{and} \qquad I_{k_m} - P_m = U_m^* \begin{pmatrix} 0 & 0 \\ 0 & I_{s_{2,m}} \end{pmatrix} U_m,$$

where $I_{s_{1,m}}$ are $I_{s_{2,m}}$ the identity matrices of $\mathcal{M}_{s_{1,m}}(\mathbb{C})$, or $\mathcal{M}_{s_{2,m}}(\mathbb{C})$ respectively.

Note

$$x_i \oplus 0 = (I_\mathcal{A} \oplus 0)(x_i \oplus y_i)(I_\mathcal{A} \oplus 0) \in \mathcal{A} \bigoplus 0.$$

Thus

$$\begin{aligned} \psi(x_i \oplus 0) &= [(P_m)_{m=1}^\infty][(X_i^{(m)})_{m=1}^\infty][(P_m)_{m=1}^\infty] \\ &= [(P_m X_i^{(m)} P_m)_{m=1}^\infty] \\ &= [(U_m^* \begin{pmatrix} I_{s_{1,m}} & 0 \\ 0 & 0 \end{pmatrix} U_m X_i^{(m)} U_m^* \begin{pmatrix} I_{s_{1,m}} & 0 \\ 0 & 0 \end{pmatrix} U_m)_{m=1}^\infty]. \end{aligned}$$

Similarly,

$$\begin{aligned} \psi(0 \oplus y_i) &= [(I_{k_m} - P_m)_{m=1}^\infty][(X_i^{(m)})_{m=1}^\infty][(I_{k_m} - P_m)_{m=1}^\infty] \\ &= [((I_{k_m} - P_m) X_i^{(m)} (I_{k_m} - P_m))_{m=1}^\infty] \\ &= [(U_m^* \begin{pmatrix} 0 & 0 \\ 0 & I_{s_{2,m}} \end{pmatrix} U_m X_i^{(m)} U_m^* \begin{pmatrix} 0 & 0 \\ 0 & I_{s_{2,m}} \end{pmatrix} U_m)_{m=1}^\infty]. \end{aligned}$$

Let

$$\begin{pmatrix} A_i^{(m)} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{s_{1,m}} & 0 \\ 0 & 0 \end{pmatrix} U_m X_i^{(m)} U_m^* \begin{pmatrix} I_{s_{1,m}} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for } i = 1, \ldots, n$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & B_i^{(m)} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I_{s_{2,m}} \end{pmatrix} U_m X_i^{(m)} U_m^* \begin{pmatrix} 0 & 0 \\ 0 & I_{s_{2,m}} \end{pmatrix}, \quad \text{for } i = 1, \ldots, n$$

where $A_1^{(m)}, \ldots, A_n^{(m)}$ are in $\mathcal{M}_{s_{1,m}}(\mathbb{C})$ and $B_1^{(m)}, \ldots, B_n^{(m)}$ are in $\mathcal{M}_{s_{2,m}}(\mathbb{C})$. Then,

$$\begin{aligned}
[(\begin{pmatrix} A_i^{(m)} & 0 \\ 0 & 0 \end{pmatrix})_{m=1}^{\infty}] &= [(U_m)_{m=1}^{\infty}]\psi(x_i \oplus 0)[(U_m^*)_{m=1}^{\infty}] \ \text{ for } i = 1, \ldots, n \\
[(\begin{pmatrix} 0 & 0 \\ 0 & B_i^{(m)} \end{pmatrix})_{m=1}^{\infty}] &= [(U_m)_{m=1}^{\infty}]\psi(0 \oplus y_i)[(U_m^*)_{m=1}^{\infty}] \ \text{ for } i = 1, \ldots, n.
\end{aligned}$$

Therefore, when m is large enough, we have

$$\begin{aligned}
(A_1^{(m)}, \ldots, A_n^{(m)}) \in \Gamma_R^{(top)}(x_1, \ldots, x_n; s_{1,m}, \frac{1}{r_0}, P_1, \ldots, P_{r_0}), \\
(B_1^{(m)}, \ldots, B_n^{(m)}) \in \Gamma_R^{(top)}(y_1, \ldots, y_n; s_{2,m}, \frac{1}{r_0}, P_1, \ldots, P_{r_0}),
\end{aligned}$$

On the other hand,

$$\begin{aligned}
([(X_1^{(m)})_{m=1}^{\infty}], \ldots, [(X_1^{(m)})_{m=1}^{\infty}]) &= (\psi(x_1 \oplus y_1), \ldots, \psi(x_n \oplus y_n)) \\
&= (\psi(x_1 \oplus 0) + \psi(0 \oplus y_1), \ldots, \psi(x_n \oplus 0) + \psi(0 \oplus y_n)) \\
&= [(U_m^*)_{m=1}^{\infty}] \Bigg([(\begin{pmatrix} A_1^{(m)} & 0 \\ 0 & 0 \end{pmatrix})_{m=1}^{\infty}] + [(\begin{pmatrix} 0 & 0 \\ 0 & B_1^{(m)} \end{pmatrix})_{m=1}^{\infty}], \ldots, \\
&\qquad [(\begin{pmatrix} A_n^{(m)} & 0 \\ 0 & 0 \end{pmatrix})_{m=1}^{\infty}] + [(\begin{pmatrix} 0 & 0 \\ 0 & B_n^{(m)} \end{pmatrix})_{m=1}^{\infty}] \Bigg) [(U_m)_{m=1}^{\infty}] \\
&= [(U_m^*)_{m=1}^{\infty}] \Bigg([(\begin{pmatrix} A_1^{(m)} & 0 \\ 0 & B_1^{(m)} \end{pmatrix})_{m=1}^{\infty}], \ldots, [(\begin{pmatrix} A_n^{(m)} & 0 \\ 0 & B_n^{(m)} \end{pmatrix})_{m=1}^{\infty}] \Bigg) [(U_m)_{m=1}^{\infty}]
\end{aligned}$$

which is against the inequality (4.1.1). This completes the proof.

□

Lemma 4.3. *Let* $\alpha, \beta > 0$ *and*

$$f(s) = \alpha s^2 + \beta(1-s)^2 + 1 - s^2 - (1-s)^2, \qquad \textit{for } 0 \le s \le 1.$$

Then

$$\max_{0 \le s \le 1} f(s) = \begin{cases} \dfrac{\alpha\beta - 1}{\alpha + \beta - 2} & \textit{if } \alpha < 1, \beta < 1 \\ \max\{\alpha, \beta\} & \textit{otherwise.} \end{cases}$$

Proof. Note that

$$f(s) = (\alpha + \beta - 2)s^2 - 2(\beta - 1)s + \beta.$$

Thus, if $\alpha + \beta \neq 2$, then f has an extreme point at

$$s_0 = \frac{\beta - 1}{\alpha + \beta - 2},$$

with

$$f(s_0) = \frac{\alpha\beta - 1}{\alpha + \beta - 2}.$$

Case one: If $\alpha + \beta > 2$, we know

$$\frac{\alpha\beta - 1}{\alpha + \beta - 2} = \alpha - \frac{\alpha^2 - 2\alpha + 1}{\alpha + \beta - 2} \le \alpha = f(1); \quad \text{similarly} \quad \frac{\alpha\beta - 1}{\alpha + \beta - 2} \le \beta = f(0).$$

Thus

$$\max_{0 \le s \le 1} f(s) = \max\{\alpha, \beta\} \quad \text{if} \quad \alpha + \beta > 2.$$

Case two: If $\alpha + \beta - 2 < 0$ and f achieves its absolute maximum in the interval $(0, 1)$, then $0 < s_0 < 1$. This is equivalent to

$$\alpha < 1 \qquad \text{and} \qquad \beta < 1.$$

Thus

$$f(s_0) = \frac{\alpha\beta - 1}{\alpha + \beta - 2} = \alpha - \frac{\alpha^2 - 2\alpha + 1}{\alpha + \beta - 2} \ge \alpha = f(1), \quad \text{and} \quad f(s_0) = \frac{\alpha\beta - 1}{\alpha + \beta - 2} \ge \beta = f(0).$$

It follows that

$$\max_{0 \le s \le 1} f(s) = \begin{cases} \dfrac{\alpha\beta - 1}{\alpha + \beta - 2} & \text{if } \alpha < 1, \beta < 1 \\ \max\{\alpha, \beta\} & \text{if } \alpha + \beta < 2,\ \alpha \ge 1 \text{ or } \alpha + \beta < 2,\ \beta \ge 1. \end{cases}$$

Case three: If $\alpha + \beta - 2 = 0$, it is easy to check that

$$\max_{0 \le s \le 1} f(s) = \max\{\alpha, \beta\}.$$

As a summary, we obtain

$$\max_{0 \le s \le 1} f(s) = \begin{cases} \dfrac{\alpha\beta - 1}{\alpha + \beta - 2} & \text{if } \alpha < 1, \beta < 1 \\ \max\{\alpha, \beta\} & \text{otherwise.} \end{cases}$$

$\square$

Proposition 4.1. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C^* algebras and $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is a family of self-adjoint elements that generate $\mathcal{A} \bigoplus \mathcal{B}$. If*

$$\alpha > \delta_{top}(x_1, \ldots, x_n) \qquad \textit{and} \qquad \beta > \delta_{top}(y_1, \ldots, y_n),$$

then

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) \le \begin{cases} \dfrac{\alpha\beta - 1}{\alpha + \beta - 2} & \textit{if } \alpha < 1, \beta < 1 \\ \max\{\alpha, \beta\} & \textit{otherwise.} \end{cases}$$

PROOF. Let $R > \max\{\|x_1 \oplus y_1\|, \ldots, \|x_n \oplus y_n\|\}$ be a positive number. By Lemma 4.1, there is some $\omega_0 > 0$ so that the following hold: for any $0 < \omega < \omega_0$, there are $r(\omega) \in \mathbb{N}$ and $K(r(\omega)) \in \mathbb{N}$ satisfying

$$\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; k_1, \frac{1}{r(\omega)}, P_1, \ldots, P_{r(\omega)}), \omega) < \left(\frac{1}{\omega}\right)^{\alpha k_1^2}, \quad \forall\, k_1 \geq K(r(\omega)); \tag{4.1.2}$$

$$\nu_\infty(\Gamma_R^{(top)}(y_1, \ldots, y_n; k_2, \frac{1}{r(\omega)}, P_1, \ldots, P_{r(\omega)}), \omega) < \left(\frac{1}{\omega}\right)^{\beta k_2^2}, \quad \forall\, k_2 \geq K(r(\omega)). \tag{4.1.3}$$

On the other hand, for each $0 < \omega < \omega_0$ and $r(\omega) \in \mathbb{N}$, it follows from Lemma 4.2 that there is some $t \in \mathbb{N}$ so that $\forall\, r > t$, $\forall\, k \geq 1$, if

$$(X_1, \ldots, X_n) \in \Gamma_R^{(top)}(x_1 \oplus y_1, \ldots, x_n \oplus y_n; k, \frac{1}{r}, P_1, \ldots, P_r),$$

then there are

$$(A_1, \ldots, A_n) \in \Gamma_R^{(top)}(x_1, \ldots, x_n; k_1, \frac{1}{r_\omega}, P_1, \ldots, P_{r_\omega}),$$
$$(B_1, \ldots, B_n) \in \Gamma_R^{(top)}(y_1, \ldots, y_n; k_2, \frac{1}{r_\omega}, P_1, \ldots, P_{r_\omega})$$

and $U \in \mathcal{U}(k)$ so that (i) $k_1 + k_2 = k$; and (ii)

$$\left\|(X_1, \ldots, X_n) - U^*(\begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix})U\right\| < \omega.$$

Moreover, we can further assume that $U \in \mathcal{U}(k)/(\mathcal{U}(k_1) \bigoplus \mathcal{U}(k_2))$.

Now it is a standard argument to show that for $r > t$,

$$\begin{aligned}
&\nu_\infty(\Gamma_R^{(top)}(x_1 \oplus y_1, \ldots, x_n \oplus y_n; k, \frac{1}{r}, P_1, \ldots, P_r), 3\omega) \\
&\leq \sum_{k_1+k_2=k} \left(\left(\frac{C_2}{\omega}\right)^{k^2-k_1^2-k_2^2} \cdot \ \nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; k_1, \frac{1}{r_\omega}, P_1, \ldots, P_{r_\omega}), \omega) \right. \\
&\qquad \left. \cdot\ \nu_\infty(\Gamma_R^{(top)}(y_1, \ldots, y_n; k_2, \frac{1}{r_\omega}, P_1, \ldots, P_{r_\omega}), \omega) \right),
\end{aligned} \tag{4.1.4}$$

where C_2 is some constant independent of k, ω. But

$$\begin{aligned}
(4.1.4) = &\left(\sum_{k_1=1}^{K(r(\omega))} + \sum_{k_1=K(r(\omega))+1}^{k-K(r(\omega))-1} + \sum_{k_1=k-K(r(\omega))}^{k} \right) \left(\left(\frac{C_2}{\omega}\right)^{k^2-k_1^2-(k-k_1)^2} \right. \\
&\cdot\ \nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; k_1, \frac{1}{r_\omega}, P_1, \ldots, P_{r_\omega}), \omega) \\
&\left. \cdot\ \nu_\infty(\Gamma_R^{(top)}(y_1, \ldots, y_n; k - k_1, \frac{1}{r_\omega}, P_1, \ldots, P_{r_\omega}), \omega) \right).
\end{aligned} \tag{4.1.5}$$

Let

$$M_\omega = \max_{1\le k_1\le K(r(\omega))} \nu_\infty(\Gamma_R^{(top)}(x_1,\ldots,x_n;k_1,\frac{1}{r_\omega},P_1,\ldots,P_{r_\omega}),\omega)$$
$$N_\omega = \max_{1\le k_2\le K(r(\omega))} \nu_\infty(\Gamma_R^{(top)}(y_1,\ldots,y_n;k_2,\frac{1}{r_\omega},P_1,\ldots,P_{r_\omega}),\omega)$$

By (4.1.2) and (4.1.3), we get that if $k > 2K(r(\omega))$ then

$$\begin{aligned}
&\nu_\infty(\Gamma_R^{(top)}(x_1\oplus y_1,\ldots,x_n\oplus y_n;k,\frac{1}{r},P_1,\ldots,P_r),3\omega)\\
&\quad\le K(r(\omega))M_\omega\left(\frac{C_2}{\omega}\right)^{k^2-(k-K(r(\omega)))^2}\left(\frac{1}{\omega}\right)^{\beta k^2}+K(r(\omega))N_\omega\left(\frac{C_2}{\omega}\right)^{k^2-(k-K(r(\omega)))^2}\left(\frac{1}{\omega}\right)^{\alpha k^2}\\
&\qquad+\sum_{k_1=K(r(\omega))+1}^{k-K(r(\omega))-1}\left(\frac{C_2}{\omega}\right)^{k^2-k_1^2-(k-k_1)^2}\left(\frac{1}{\omega}\right)^{\alpha k_1^2}\left(\frac{1}{\omega}\right)^{\beta(k-k_1)^2}.
\end{aligned}\tag{4.1.6}$$

Let

$$f(s)=\alpha s^2+\beta(1-t)^2+1-s^2-(1-s)^2,\qquad \text{for } 0\le s\le 1.$$

And

$$L(\alpha,\beta)=\max_{0\le s\le 1} f(s).$$

Then

$$\begin{aligned}
(4.1.6)\le&\left[K(r(\omega))(M_\omega+N_\omega)\left(\frac{C_2}{\omega}\right)^{k^2-(k-K(r(\omega)))^2}+kC_2^{k^2}\right]\\
&\cdot\left\{\left(\frac{1}{\omega}\right)^{\beta k^2}+\left(\frac{1}{\omega}\right)^{\alpha k^2}+\left(\frac{1}{\omega}\right)^{L(\alpha,\beta)k^2}\right\}.
\end{aligned}\tag{4.1.7}$$

Note that

$$\lim_{k\to\infty}\frac{\log\left[K(r(\omega))(M_\omega+N_\omega)\left(\frac{C_2}{\omega}\right)^{k^2-(k-K(r(\omega)))^2}+kC_2^{k^2}\right]}{k^2}=\log C_2;$$

and

$$L(\alpha,\beta)\ge\max\{\alpha,\beta\}.$$

We obtain,

$$\limsup_{k\to\infty}\frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1\oplus y_1,\ldots,x_n\oplus y_n;k,\frac{1}{r},P_1,\ldots,P_r),3\omega))}{k^2}\le\log C_2+L(\alpha,\beta)\log\left(\frac{1}{\omega}\right).$$

It induces that

$$\begin{aligned}
&\delta_{top}(x_1\oplus y_1,\ldots,x_n\oplus y_n)\\
&\quad=\limsup_{\omega\to 0^+}\inf_{r\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1\oplus y_1,\ldots,x_n\oplus y_n;k,\frac{1}{r},P_1,\ldots,P_r),\omega))}{-k^2\log\omega}\\
&\quad\le L(\alpha,\beta)=\begin{cases}\dfrac{\alpha\beta-1}{\alpha+\beta-2} & \text{if } \alpha<1,\beta<1\\ \max\{\alpha,\beta\} & \text{otherwise,}\end{cases}
\end{aligned}$$

where the last equation is from Lemma 4.3. □

Proposition 4.2. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C^* algebras and $x_1\oplus y_1,\ldots,x_n\oplus y_n$ is a family of self-adjoint elements that generate $\mathcal{A}\bigoplus\mathcal{B}$. If*

$$s=\delta_{top}(x_1,\ldots,x_n)\qquad \text{and}\qquad t=\delta_{top}(y_1,\ldots,y_n),$$

then

$$\delta_{top}(x_1\oplus y_1,\ldots,x_n\oplus y_n)\le\begin{cases}\dfrac{st-1}{s+t-2} & \text{if } s<1,\ t<1;\\ \max\{s,t\} & \text{otherwise.}\end{cases}$$

Proof. It follows directly from the preceding lemma.

□

4.2. One of topological free entropy dimensions ≥ 1.

Lemma 4.4. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C^* algebras and $x_1\oplus y_1,\ldots,x_n\oplus y_n$ is a family of self-adjoint elements that generate $\mathcal{A}\bigoplus\mathcal{B}$. Then*

$$\delta_{top}(x_1\oplus y_1,\ldots,x_n\oplus y_n)\ge\max\{\delta_{top}(x_1,\ldots,x_n),\delta_{top}(y_1,\ldots,y_n)\}.$$

Proof. Let $R>\max\{\|x_1\oplus y_1\|,\ldots,\|x_n\oplus y_n\|\}$ be a positive number. For any $r\ge 1$, $\epsilon>0$, $k>k_1$, and any

$$(A_1,\ldots,A_n)\in\Gamma_R^{(top)}(x_1,\ldots,x_n;k_1,\epsilon,P_1,\ldots,P_r),$$
$$(B_1,\ldots,B_n)\in\Gamma_R^{(top)}(y_1,\ldots,y_n;k-k_1,\epsilon,P_1,\ldots,P_r)$$

we have

$$(\begin{pmatrix}A_1 & 0\\ 0 & B_1\end{pmatrix},\ldots,\begin{pmatrix}A_n & 0\\ 0 & B_n\end{pmatrix})\in\Gamma_R^{(top)}(x_1\oplus y_1,\ldots,x_n\oplus y_n;k,\epsilon,P_1,\ldots,P_r).$$

Thus,

$$\nu_\infty(\Gamma_R^{(top)}(x_1\oplus y_1,\ldots,x_n\oplus y_n;k,\epsilon,P_1,\ldots,P_r),\omega)\ge\nu_\infty(\Gamma_R^{(top)}(y_1,\ldots,y_n;k-k_1,\epsilon,P_1,\ldots,P_r),2\omega).$$

It follows that

$$\delta_{top}(x_1\oplus y_1,\ldots,x_n\oplus y_n)\ge\delta_{top}(y_1,\ldots,y_n).$$

Similarly,

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) \geq \delta_{top}(x_1, \ldots, x_n).$$

Hence we have proved the result of the lemma. □

Theorem 4.1. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C^* algebras and $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is a family of self-adjoint elements that generates $\mathcal{A} \bigoplus \mathcal{B}$. If one of $\delta_{top}(x_1, \ldots, x_n)$ and $\delta_{top}(y_1, \ldots, y_n)$ is larger than or equal to 1, then*

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \max\{\delta_{top}(x_1, \ldots, x_n), \delta_{top}(y_1, \ldots, y_n)\}$$

Proof. The result follows directly from Proposition 4.2 and Lemma 4.4.

□

4.3. Both of topological free entropy dimensions < 1. We start this subsection with the following definition.

Definition 4.1. *Suppose that $\mathcal{A}$ is a unital C^* algebra and $x_1, \ldots, x_n$ is a family of self-adjoint elements in $\mathcal{A}$. The family of elements $x_1, \ldots, x_n$ is called stable if for any $\alpha < \delta_{top}(x_1, \ldots, x_n)$ there are positive numbers $C_3 > 0$ and $\omega_0 > 0$, $r_0 \in \mathbb{N}$, $k_0 \in \mathbb{N}$ so that*

$$\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; q \cdot k_0, \frac{1}{r}, P_1, \ldots, P_r), \omega) \geq C_3^{(q \cdot k_0)^2} \left(\frac{1}{\omega}\right)^{\alpha \cdot (q \cdot k_0)^2}, \forall\ 0 < \omega < \omega_0, r > r_0, q \in \mathbb{N}.$$

Example 4.1.

(1) *From the inequality (3.3.1), it follows that any family of self-adjoint generators $x_1, \ldots, x_n$ of $\mathcal{M}_n(\mathbb{C})$ is stable.*

(2) *A self-adjoint element x in a unital C^* algebra is stable (see [**12**]).*

(3) *Suppose that $\mathcal{K}$ is the algebra of all compact operators in a separable Hilbert space H and unital C^* algebra $\mathcal{A}$ is the unitization of $\mathcal{K}$. Then any family of self-adjoint generators $x_1, \ldots, x_n$ of $\mathcal{A}$ is stable since $\delta_{top}(x_1, \ldots, x_n) = 0$ (see Theorem 5.6 in [**13**]).*

Notation 4.1. *Suppose that $A \in \mathcal{M}_{k_1}(\mathbb{C})$ and $B \in \mathcal{M}_{k_2}(\mathbb{C})$. We denote the element*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{M}_{k_1+k_2}(\mathbb{C})$$

by $A \oplus B$.

Notation 4.2. *Suppose that $\Gamma_1 \subset (M_{k_1}(\mathbb{C}))^n$ and $\Gamma_2 \subset (\mathcal{M}_{k_2}(\mathbb{C}))^n$. We denote the set*

$$\left\{ \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} \mid (A_1, \ldots, A_n) \in \Gamma_1,\ (B_1, \ldots, B_n) \in \Gamma_2 \right\}$$

in $(\mathcal{M}_{k_1+k_2}(\mathbb{C}))^n$ by $\Gamma_1 \oplus \Gamma_2$.

The main goal of this subsection is to prove the following result.

Proposition 4.3. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C^* algebras and $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is a family of self-adjoint elements that generates $\mathcal{A} \bigoplus \mathcal{B}$ as a C^* algebra. Assume*

$$s = \delta_{top}(x_1, \ldots, x_n) < 1 \qquad \text{and} \qquad t = \delta_{top}(y_1, \ldots, y_n) < 1.$$

If both families $\{x_1, \ldots, x_n\}$ *and* $\{y_1, \ldots, y_n\}$ *are stable, then*

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \frac{st-1}{s+t-2}.$$

Moreover, the family of elements $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ *is also stable.*

Remark 4.1. *The difficulty to prove the preceding result lies in the fact that* $I_{\mathcal{A}} \oplus 0$ *might not be in the* $*$*-algebra generated by* $x_1 \oplus y_1, \ldots, x_n \oplus y_n$*.*

The proof of Proposition 4.3 will be postponed after we prove some lemmas firstly. Recall the definition of the packing number of a set in a metric space as follows.

Definition 4.2. *Suppose that* X *is a metric space with a metric distance* d*. The packing number of a set* K *by* δ*-balls in* X*, denoted by* $Pack_d(K, \delta)$*, is the maximal cardinality of the subsets* F *in* K *satisfying for all* a, b *in* F *either* $a = b$ *or* $d(a, b) \geq \delta$*.*

The following result follows easily from the definition of packing number.

Lemma 4.5. *For any subset* K *of* $((\mathcal{M}_k(\mathbb{C}))^n, \|\cdot\|)$*, we have*

$$Pack_\infty(K, \delta) \geq \nu_\infty(K, 2\delta) \geq Pack_\infty(K, 4\delta),$$

where $Pack_\infty(K, \delta)$ *is the packing number of the set* K *by* δ*-*$\|\ \|$*-balls in* $(\mathcal{M}_k(\mathbb{C}))^n$*.*

Lemma 4.6. *Let*

$$\Gamma_1 \subset (\mathcal{M}_{k_1}(\mathbb{C}))^n \qquad\qquad \Gamma_2 \subset (\mathcal{M}_{k_2}(\mathbb{C}))^n.$$

Then, for $\delta > 0$*,*

$$Pack_\infty(\Gamma_1 \oplus \Gamma_2, \delta) \geq \nu_\infty(\Gamma_1, 2\delta) \cdot \nu_\infty(\Gamma_2, 2\delta),$$

where $\Gamma_1 \oplus \Gamma_2$ *is as in Notation 4.2.*

Proof. By Lemma 4.5, there exists a family of elements $\{(A_1^\lambda, \ldots, A_n^\lambda)\}_{\lambda\in\Lambda}$, or $\{(B_1^\sigma, \ldots, B_n^\sigma)\}_{\sigma\in\Sigma}$, in Γ_1, or in Γ_2 respectively, such that

$$\|(A_1^{\lambda_1}, \ldots, A_n^{\lambda_1}) - (A_1^{\lambda_2}, \ldots, A_n^{\lambda_2})\| \geq \delta, \qquad \forall\ \lambda_1 \neq \lambda_2 \in \Lambda$$
$$\|(B_1^{\sigma_1}, \ldots, B_n^{\sigma_1}) - (B_1^{\sigma_2}, \ldots, B_n^{\sigma_2})\| \geq \delta, \qquad \forall\ \sigma_1 \neq \sigma_2 \in \Sigma;$$

and

$$Card(\Lambda) \geq \nu_\infty(\Gamma_1, 2\delta), \qquad Card(\Sigma) \geq \nu_\infty(\Gamma_2, 2\delta),$$

where $Card(\Lambda)$, or $Card(\Sigma)$, is the cardinality of the set Λ, or Σ respectively. Thus, if $\lambda_1 \neq \lambda_2$ or $\sigma_1 \neq \sigma_2$,

$$\begin{aligned}
&\left\| \left(\begin{pmatrix} A_1^{\lambda_1} & 0 \\ 0 & B_1^{\sigma_1} \end{pmatrix}, \ldots, \begin{pmatrix} A_n^{\lambda_1} & 0 \\ 0 & B_n^{\sigma_1} \end{pmatrix} \right) - \left(\begin{pmatrix} A_1^{\lambda_2} & 0 \\ 0 & B_1^{\sigma_2} \end{pmatrix}, \ldots, \begin{pmatrix} A_n^{\lambda_2} & 0 \\ 0 & B_n^{\sigma_2} \end{pmatrix} \right) \right\| \\
&= \max\{\|(A_1^{\lambda_1}, \ldots, A_n^{\lambda_1}) - (A_1^{\lambda_2}, \ldots, A_n^{\lambda_2})\|, \|(B_1^{\sigma_1}, \ldots, B_n^{\sigma_1}) - (B_1^{\sigma_2}, \ldots, B_n^{\sigma_2})\|\} \\
&\geq \delta.
\end{aligned}$$

Hence

$$Pack_\infty(\Gamma_1 \oplus \Gamma_2, \delta) \geq \nu_\infty(\Gamma_1, 2\delta) \cdot \nu_\infty(\Gamma_2, 2\delta).$$

□

Definition 4.3. *Let k_1, k_2, s be some positive integers such that $k_1 \geq 2s$, $k_2 \geq 2s$. Define $\Omega(k_1, k_2, s)$ be the collection of all these $k_1 \times k_2$ matrices T satisfying $\|T\| \leq 2$ and $rank(T) \leq 2s$, where $rank(T)$ is the rank of the matrix T.*

Sublemma 4.3.1. *Let k, k_1, k_2, s be some positive integers such that $k_1 \geq 2s$, $k_2 \geq 2s$ and $k = k_1 + k_2$. Let ι be the embedding of $\mathcal{M}_{k_1,k_2}(\mathbb{C})$ into $\mathcal{M}_k(\mathbb{C})$ by the mapping*

$$\iota : \quad A \to \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

for any A in $\mathcal{M}_{k_1,k_2}(\mathbb{C})$. For any $\delta > 0$, we have

$$\nu_2(\iota(\Omega(k_1, k_2, s)), \delta) \leq \left(\frac{C_4}{\delta}\right)^{4s(k_1+k_2)+2s},$$

where C_4 is a constant independent of k_1, k_2 and s.

Proof. For any T in $\Omega(k_1, k_2, s)$, by Definition 4.3 we have $\|T\| \leq 2$ and $rank(T) \leq 2s$. Thus by polar decomposition, there are partial isometry V_1 in $\mathcal{M}_{k_1,k_2}(\mathbb{C})$, a unitary matrix V_2 in $\mathcal{M}_{k_2}(\mathbb{C})$ and a family of numbers $0 \leq \lambda_1, \ldots, \lambda_{2s} \leq 2$ such that,

$$\begin{aligned} T =& V_1 \; diag(\lambda_1, \ldots, \lambda_{2s}, 0, \ldots, 0) \; V_2^* \\ =& (V_1(I_{2s} \oplus 0 \cdot I_{k_2-2s})) \; diag(\lambda_1, \ldots, \lambda_{2s}, 0, \ldots, 0) \; (V_2(I_{2s} \oplus 0 \cdot I_{k_2-2s}))^*. \end{aligned}$$

Now it is a standard argument (for example see [**23**]) to show that

$$\nu_2(\iota(\Omega(k_1, k_2, s)), \delta) \leq \left(\frac{C_4}{\delta}\right)^{4s(k_1+k_2)+2s},$$

where C_4 is a constant independent of k_1, k_2 and s. □

Let s_1, s_2, s_3 be positive integers so that $s_1 \geq s_3, s_2 \geq s_3$.

Definition 4.4. *Define $R(s_1, s_3)$ be the collection of all these self-adjoint matrices Q in $\mathcal{M}_{s_1+s_3}(\mathbb{C})$ satisfying: there are some unitary matrix U_1 in $\mathcal{M}_{s_1+s_3}(\mathbb{C})$ and real numbers*

$$\lambda_1, \ldots, \lambda_{s_1}, \ldots, \lambda_{s_1+s_3}$$

such that (i)

$$Q = U_1^* diag(\lambda_1, \ldots, \lambda_{s_1}, \ldots, \lambda_{s_1+s_3}) U_1;$$

and (ii)

$$\lambda_i \geq 2, \qquad \forall \; 1 \leq i \leq s_1.$$

Define $Q(s_2, s_3)$ be the collection of all these self-adjoint matrices Q in $\mathcal{M}_{s_2+s_3}(\mathbb{C})$ satisfying: there are some unitary matrix U_2 in $\mathcal{M}_{s_2+s_3}(\mathbb{C})$ and real numbers

$$\mu_1, \ldots, \mu_{s_2}, \ldots, \mu_{s_2+s_3}$$

such that (i)

$$Q = U_2^* diag(\mu_1, \ldots, \mu_{s_2}, \ldots, \mu_{s_2+s_3}) U_2;$$

and (ii)

$$|\lambda_i| \le 1, \qquad \forall\ 1 \le i \le s_2.$$

Sublemma 4.3.2. *Let $\delta > 0$ be a positive number. Let s_1, s_2, s_3 be positive integers so that $s_1 \ge s_3, s_2 \ge s_3$. Let $k_1 = s_1 + s_3$, $k_2 = s_2 + s_3$ and $k = k_1 + k_2$. Suppose X is a $k_1 \times k_2$ complex matrix such that, (i) $\|X\| \le 1$; and (ii) for some Q_1 in $R(s_1, s_3)$ and Q_2 in $Q(s_2, s_3)$,*

$$\frac{Tr((Q_1X - XQ_2)^*(Q_1X - XQ_2))}{k} \le \delta.$$

Then, there is some T in $\Omega(k_1, k_2, s_3)$ (as defined in Definition 4.3) such that

$$\frac{Tr((X - T)^*(X - T))}{k} \le \delta.$$

Proof. By the definitions of $R(s_1, s_3)$ and $Q(s_2, s_3)$, we know there are some unitary matrix U_1 in $\mathcal{U}(k_1)$, U_2 in $\mathcal{U}(k_2)$, and families of real numbers $\lambda_1, \ldots, \lambda_{k_1}$ and $\mu_1, \ldots, \mu_{k_2}$ such that (i)

$$Q_1 = U_1^* diag(\lambda_1, \ldots, \lambda_{k_1}) U_1$$
$$Q_2 = U_2^* diag(\mu_1, \ldots, \mu_{k_2}) U_2;$$

and (ii)

$$\lambda_i \ge 2,\ |\mu_j| \le 1, \quad \forall\ 1 \le i \le s_1,\ 1 \le j \le s_2.$$

Let

$$U_1 X U_2^* = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \in\ \mathcal{M}_{k_1,k_2}(\mathbb{C}),$$

where $Y_{11} \in \mathcal{M}_{s_1,s_2}(\mathbb{C})$, $Y_{12} \in \mathcal{M}_{s_1,s_3}(\mathbb{C})$, $Y_{21} \in \mathcal{M}_{s_3,s_2}(\mathbb{C})$ and $Y_{22} \in \mathcal{M}_{s_3,s_3}(\mathbb{C})$.

From the facts that

$$\frac{Tr((Q_1X - XQ_2)^*(Q_1X - XQ_2))}{k} \le \delta,$$

and

$$\lambda_i \ge 2,\ |\mu_j| \le 1, \quad \forall\ 1 \le i \le s_1,\ 1 \le j \le s_2,$$

we know that

$$\frac{Tr(Y_{11}^* Y_{11})}{k} \le \delta.$$

Let

$$T_1 = \begin{pmatrix} 0 & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

Then $\|T_1\| \le 2\|X\| \le 2$, $rank(T_1) \le 2s_3$, and

$$\frac{Tr((X - U_1^* T U_2)^*(X - U_1^* T U_2))}{k} = \frac{Tr(Y_{11}^* Y_{11})}{k} \le \delta.$$

Let $T = U_1^* T_1 U_2$ and we finished the proof of the sublemma.

□

Lemma 4.7. *Let s_1, s_2, s_3 be positive integers so that $s_1 \geq s_3, s_2 \geq s_3$, and $R(s_1, s_3), Q(s_2, s_3)$ be defined in Definition 4.4. Let $k_1 = s_1 + s_3$, $k_2 = s_2 + s_3$ and $k = k_1 + k_2$. Then there exists a family of unitary matrices $\{U_\gamma\}_{\gamma \in \mathcal{I}}$ in $\mathcal{M}_k(\mathbb{C})$ so that (i) when $\gamma_1 \neq \gamma_2 \in \mathcal{I}$,*

$$\|U_{\gamma_1}^*(Q_1 \oplus Q_2)U_{\gamma_1} - U_{\gamma_2}^*(\tilde{Q}_1 \oplus \tilde{Q}_2)U_{\gamma_2}\| \geq \delta, \quad \forall\ Q_1, \tilde{Q}_1 \in R(s_1, s_3),\ \ Q_2, \tilde{Q}_2 \in Q(s_2, s_3);$$

and (ii)

$$Card(\mathcal{I}) \geq (C_6 \cdot 130\delta)^{-k^2} \cdot \left(\frac{C_5}{\delta}\right)^{-(s_1^2 + s_2^2 + 8s_3 + 12(k_1 s_3 + k_2 s_3))},$$

where C_5, C_6 are some constants independent of k, s_1, s_2, s_3; and $Q_1 \oplus Q_2$, $\tilde{Q}_1 \oplus \tilde{Q}_2$ are defined as in Notation 4.1.

As we will see, Lemma 4.7 is a consequence of the Sublemma 4.3.3, Sublemma 4.3.4 and Sublemma 4.3.5, which we will prove first. Following the notations as before, let s_1, s_2, s_3, $k_1 = s_1 + s_3$, $k_2 = s_2 + s_3$, and $k = k_1 + k_2$ be as above.

Definition 4.5. *Define $\mathcal{S}(s_1, s_2, s_3)$ to be the collection of all these matrices*

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \in \mathcal{M}_k(\mathbb{C}),$$

where $S_{ij} \in \mathcal{M}_{k_i, k_j}(\mathbb{C})$ for $1 \leq i, j \leq 2$, satisfying (i) $\|S_{i,j}\| \leq 2$ for $1 \leq i, j \leq 2$; (ii) $S_{12} \in \Omega(k_1, k_2, s_3)$ and $S_{21} \in \Omega(k_2, k_1, s_3)$, where $\Omega(k_1, k_2, s_3)$ and $\Omega(k_2, k_1, s_3)$ are defined in Definition 4.3.

Sublemma 4.3.3. *Suppose that $\delta > 0$ and U_1, U_2 are unitary matrices in $\mathcal{M}_k(\mathbb{C})$ so that the following holds: there are some $Q_1, \tilde{Q}_1 \in R(s_1, s_3)$, and $Q_2, \tilde{Q}_2 \in Q(s_2, s_3)$ such that*

$$\|U_1^*(Q_1 \oplus Q_2)U_1 - U_2^*(\tilde{Q}_1 \oplus \tilde{Q}_2)U_2\| \leq \delta,$$

where $R(s_1, s_3)$, $Q(s_2, s_3)$ are defined in Definition 4.4 and $Q_1 \oplus Q_2$, $\tilde{Q}_1 \oplus \tilde{Q}_2$ are as in Notation 4.1. Then, there is some S in $\mathcal{S}(s_1, s_2, s_3)$ such that

$$\|U_1 - U_2 S\|_2 \leq 2\delta.$$

Proof. Let

$$U_2 U_1^* = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

where U_{ij} is $k_i \times k_j$ complex matrix for $1 \leq i, j \leq 2$. By the conditions on U_1, U_2, we know that $\|U_{12}\| \leq 1$ and

$$\frac{Tr((U_{12}Q_2 - \tilde{Q}_1 U_{12})^*(U_{12}Q_2 - \tilde{Q}_1 U_{12}))}{k} \leq \delta^2.$$

By Sublemma 4.3.2, we know that there is some T_{12} in $\Omega(k_1, k_2, s_3)$ so that

$$\frac{Tr((U_{12} - T_{12})^*(U_{12} - T_{12}))}{k} \leq \delta^2.$$

Similarly, there is some T_{21} in $\Omega(k_2, k_1, s_3)$ so that

$$\frac{Tr((U_{21} - T_{21})^*(U_{21} - T_{21}))}{k} \le \delta^2.$$

Let

$$S = \begin{pmatrix} U_{11} & T_{12} \\ T_{21} & U_{22} \end{pmatrix}$$

be in $\mathcal{M}_k(\mathbb{C})$. Now it is not hard to check that S is in $\mathcal{S}(s_1, s_2, s_3)$ and

$$\|U_1 - U_2 S\|_2 \le 2\delta.$$

□

Sublemma 4.3.4. *For any $\delta > 0$, let*

$$\mathcal{S}_\delta(s_1, s_2, s_3) = \{S \in \mathcal{S}(s_1, s_2, s_3) \mid \exists\ U \in \mathcal{U}(k) \ \text{such that}\ \|U - S\|_2 \le 2\delta\}.$$

We have

$$\nu_2(\mathcal{S}_\delta(s_1, s_2, s_3), 64\delta) \le \left(\frac{C_5}{\delta}\right)^{s_1^2 + s_2^2 + 8s_3 + 12(k_1+k_2)s_3},$$

where C_5 is some constant independent of k, s_1, s_2, s_3.

Proof. Assume

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

is in $\mathcal{S}_\delta(s_1, s_2, s_3)$, where S_{ij} is $k_i \times k_j$ complex matrix for $1 \le i, j \le 2$.

Assume that

$$S_{11} = H_{11} W_{11}$$

is the polar decompositions of elements S_{11} in $\mathcal{M}_{k_1}(\mathbb{C})$, where W_{11} is unitary matrix in $\mathcal{M}_{k_1}(\mathbb{C})$ and H_{11} is a positive matrix in $\mathcal{M}_{k_1}(\mathbb{C})$. From the fact that $\|U - S\|_2 \le 2\delta$, it follows that

$$\begin{aligned}(16\delta)^2 &\ge (\|(S-U)S^*\|_2 + \|U(S-U)^*\|_2)^2 \ge \|SS^* - I_k\|_2^2 \\ &\ge \frac{Tr((H_{11}^2 - (I_{k_1} - S_{12}S_{12}^*))^2)}{k}.\end{aligned}$$

Let

$$2 \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{k_1} \ge 0,$$

be the eigenvalues of H_{11} in $\mathcal{M}_{k_1}(\mathbb{C})$ arranged in the decreasing order. Note that S_{12} is in $\Omega(k_1, k_2, s_3)$. By the Definition 4.3, S_{12} is a $k_1 \times k_2$ complex matrix satisfying $\|S_{12}\| \le 2$ and $rank(S_{12}) \le 2s_3$. We can assume that

$$4 \ge \mu_1 \ge \mu_2 \ge \cdots, \ge \mu_{2s_3} \ge 0 \ge \cdots \ge 0$$

are eigenvalues of $S_{12}S_{12}^*$ in $\mathcal{M}_{k_1}(\mathbb{C})$ arranged in the decreasing order. By Lemma 4.1 in [**25**], we have

$$k(16\delta)^2 \ge Tr((H_{11}^2 - (I_{k_1} - S_{12}S_{12}^*))^2) \ge \sum_{i=1}^{k_1 - 2s_3} |\lambda_i^2 - 1|^2 + \sum_{i=k_1-2s_3+1}^{k_1} |\lambda_i^2 + \mu_i - 1|^2 \ge \sum_{i=1}^{k_1-2s_3} |\lambda_i - 1|^2.$$

Thus, there is some

$$U_{11} \in \mathcal{U}(k_1)/(\mathcal{U}(k_1-2s_3) \oplus I_{2s_3})$$

such that

$$\frac{Tr(H_{11} - U_{11}^* diag(1,1,\ldots,1,\lambda_{k_1-2s_3+1},\ldots,\lambda_{k_1})U_{11})^2}{k} = \frac{\sum_{i=1}^{k_1-2s_3}|\lambda_i - 1|^2}{k} \le (16\delta)^2. \quad (4.3.1)$$

Similarly, assume that

$$S_{22} = H_{22}W_{22}$$

is the polar decomposition of S_{22} in $\mathcal{M}_{k_2}(\mathbb{C})$ where W_{22} is a unitary matrix and H_{22} is a positive matrix in $\mathcal{M}_{k_2}(\mathbb{C})$. Then there are some

$$U_{22} \in \mathcal{U}(k_2)/(\mathcal{U}(k_2-2s_3) \oplus I_{2s_3})$$

and some $0 \le \sigma_{k_2-2s_3+1},\ldots,\sigma_{k_2} \le 2$ such that

$$\frac{Tr(H_{22} - U_{22}^* diag(1,1,\ldots,1,\sigma_{k_2-2s_3+1},\ldots,\sigma_{k_2})U_{22})^2}{k} \le (16\delta)^2. \quad (4.3.2)$$

Define the mapping ρ from the space

$$\begin{aligned}\mathcal{X} = (\ \mathcal{U}(k_1), \frac{\|\cdot\|_{Tr}}{\sqrt{k}}) \ \times\ (\mathcal{U}(k_2), \frac{\|\cdot\|_{Tr}}{\sqrt{k}}) \ \times\ (\mathcal{U}(k_1)/(\mathcal{U}(k_1-2s_3)\oplus I_{2s_3}), \frac{\|\cdot\|_{Tr}}{\sqrt{k}}) \\ \times\ (\mathcal{U}(k_2)/(\mathcal{U}(k_2-2s_3)\oplus I_{2s_3}), \frac{\|\cdot\|_{Tr}}{\sqrt{k}}) \ \times\ (\Omega(k_1,k_2,s_3), \frac{\|\cdot\|_{Tr}}{\sqrt{k}}) \ \times\ (\Omega(k_2,k_1,s_3), \frac{\|\cdot\|_{Tr}}{\sqrt{k}}) \\ \times\ \{(\lambda_{k_1-2s_3+1},\ldots,\lambda_{k_1}) \mid 0 \le \lambda_j \le 2, \forall\ k_1-2s_3+1 \le j \le k_1\} \\ \times\ \{(\sigma_{k_2-2s_3+1},\ldots,\sigma_{k_2}) \mid 0 \le \sigma_j \le 2, \forall\ k_2-2s_3+1 \le j \le k_2\}\end{aligned}$$

into $\mathcal{S}$ by sending

$$(W_{11}, W_{22}, U_{11}, U_{22}, S_{12}, S_{21}, (\lambda_{k_1-2s_3+1},\ldots,\lambda_{k_1}), (\sigma_{k_2-2s_3+1},\ldots,\sigma_{k_2})) \in \mathcal{X}$$

to

$$\tilde{S} = \begin{pmatrix} U_{11}^*\tilde{H}_{11}U_{11}W_{11} & S_{12} \\ S_{21} & U_{22}^*\tilde{H}_{22}U_{22}W_{22} \end{pmatrix} \in \mathcal{M}_k(\mathbb{C}),$$

where

$$\tilde{H}_1 = diag(1,1,\ldots,1,\lambda_{k_1-2s_3+1},\ldots,\lambda_{k_1}) \qquad \tilde{H}_2 = diag(1,1,\ldots,1,\sigma_{k_2-2s_3+1},\ldots,\sigma_{k_2}).$$

By inequalities (4.3.1) and (4.3.2), we know for any $S \in \mathcal{S}_\delta$, there is some $x \in \mathcal{X}$ satisfying

$$\|S - \rho(x)\|_2 \le 16\sqrt{2}\delta.$$

Computing the covering number of $\rho(\mathcal{X})$ by combining with Sublemma 4.3.1 and Lemma 2.1, we get

$$\begin{aligned}\nu_2(\mathcal{S}_\delta(s_1,s_2,s_3),60\delta) &\leq \left(\frac{C_5}{\delta}\right)^{k_1^2+k_2^2+k_1^2-(k_1-2s_3)^2+k_2^2-(k_2-2s_3)^2+8(k_1+k_2)s_3+4s_3+4s_3} \\ &\leq \left(\frac{C_5}{\delta}\right)^{k_1^2+k_2^2+8s_3+12(k_1s_3+k_2s_3)},\end{aligned}$$

where C_5 is some constant independent of k, s_1, s_2, s_3.

□

Sublemma 4.3.5. *For every $U \in \mathcal{U}(k)$, let*

$$\Sigma(U) = \{W \in \mathcal{U}(k) \mid \exists\ S \in \mathcal{S}(s_1,s_2,s_3) \text{ such that } \|W - US\|_2 \leq 2\delta\}.$$

Then the volume of $\Sigma(U)$ is bounded by the following:

$$\mu(\Sigma(U)) \leq (C_6 \cdot 130\delta)^{k^2} \cdot \left(\frac{C_5}{\delta}\right)^{s_1^2+s_2^2+8s_3+12(k_1s_3+k_2s_3)},$$

where μ is the normalized Haar measure on the unitary group $\mathcal{U}(k)$ and C_5, C_6 are some constants independent of k, s_1, s_2, s_3.

Proof. For any $\delta > 0$, let

$$\mathcal{S}_\delta(s_1,s_2,s_3) = \{S \in \mathcal{S}(s_1,s_2,s_3) \mid \exists\ U \in \mathcal{U}(k) \text{ such that } \|U - S\|_2 \leq 2\delta\}.$$

It follows from the preceding sublemma that

$$\nu_2(\mathcal{S}_\delta(s_1,s_2,s_3),60\delta) \leq \left(\frac{C_5}{\delta}\right)^{s_1^2+s_2^2+8s_3+12(k_1s_3+k_2s_3)}$$

where C_5 is a constant independent of s_1, s_2, s_3. Thus, by Lemma 2.1, the covering number of the set $\Sigma(U)$ by the 130δ-$\|\cdot\|_2$-balls in $\mathcal{M}_k(\mathbb{C})$ is bounded by

$$\nu_2(\Sigma(U),130\delta) \leq \left(\frac{C_5}{\delta}\right)^{s_1^2+s_2^2+8s_3+12(k_1s_3+k_2s_3)}.$$

But the ball of radius 130δ in $\mathcal{U}(k)$ has a volume bounded by

$$\mu(\text{ball of radius } 130\delta \text{ in } \mathcal{U}(k)) \leq (C_6 \cdot 130\delta)^{k^2},$$

where C_6 is a universal constant. Thus

$$\mu(\Sigma(U)) \leq (C_6 \cdot 130\delta)^{k^2} \cdot \left(\frac{C_5}{\delta}\right)^{s_1^2+s_2^2+8s_3+12(k_1s_3+k_2s_3)}.$$

□

Proof of Lemma 4.7: For every $U \in \mathcal{U}(k)$, define

$$\Sigma(U) = \{W \in \mathcal{U}(k) \mid \exists S \in \mathcal{S}(s_1, s_2, s_3), \text{ such that } \|W - US\|_2 \le 2\delta\}.$$

By previous lemma, we have

$$\mu(\Sigma(U)) \le (C_6 \cdot 130\delta)^{k^2} \cdot \left(\frac{C_5}{\delta}\right)^{s_1^2+s_2^2+8s_3+12(k_1s_3+k_2s_3)}.$$

A "parking" (or exhausting) argument will show the existence of a family of unitary elements $\{U_i\}_{i=1}^N \subset \mathcal{U}(k)$ such that

$$N \ge (C_6 \cdot 130\delta)^{-k^2} \cdot \left(\frac{C_5}{\delta}\right)^{-(s_1^2+s_2^2+8s_3+12(k_1s_3+k_2s_3))}.$$

and

$$U_i \text{ is not contained in } \cup_{j=1}^{i-1} \Sigma(U_j).$$

Hence

$$\|U_i - U_jS\|_2 \ge 2\delta, \qquad \forall\ S \in \mathcal{S}(s_1, s_2, s_3), \text{ with } \forall 1 \le j < i \le N.$$

By Sublemma 4.3.3, we know that for all $1 \le j < i \le N$

$$\|U_i^*(Q_1 \oplus Q_2)U_i - U_j^*(\tilde{Q}_1 \oplus \tilde{Q}_2)U_j\| \ge \delta, \ \ \forall\ Q_1, \tilde{Q}_1 \in R(s_1, s_3),\ Q_2, \tilde{Q}_2 \in Q(s_2, s_3);$$

i.e. there exists a family of unitary matrices $\{U_\gamma\}_{\gamma\in\mathcal{I}}$ in $\mathcal{M}_k(\mathbb{C})$ so that (i) when $\gamma_1 \ne \gamma_2 \in \mathcal{I}$,

$$\|U_{\gamma_1}^*(Q_1 \oplus Q_2)U_{\gamma_1} - U_{\gamma_2}^*(\tilde{Q}_1 \oplus \tilde{Q}_2)U_{\gamma_2}\| \ge \delta, \ \ \forall\ Q_1, \tilde{Q}_1 \in R(s_1, s_3),\ Q_2, \tilde{Q}_2 \in Q(s_2, s_3);$$

and (ii)

$$Card(\mathcal{I}) \ge (C_6 \cdot 130\delta)^{-k^2} \cdot \left(\frac{C_5}{\delta}\right)^{-(s_1^2+s_2^2+8s_3+12(k_1s_3+k_2s_3))},$$

where C_5, C_6 are some constants independent of k, s_1, s_2, s_3 and $R(s_1, s_3), Q(s_2, s_3)$ are defined in Definition 4.4. □

Lemma 4.8. *Let $k_1, m \ge 2$ be some positive integers. Suppose that Q is a self-adjoint element in $\mathcal{M}_{k_1}^{s.a.}(\mathbb{C})$ such that*

$$\|Q - 3I_{k_1}\| < \frac{2}{m^3},$$

where I_{k_1} is the identity matrix of $\mathcal{M}_{k_1}(\mathbb{C})$. Then Q is in $R(k_1 - \frac{4k_1}{m^4}, \frac{4k_1}{m^4})$, where $R(k_1 - \frac{4k_1}{m^4}, \frac{4k_1}{m^4})$ is defined in Definition 4.4.

Proof. Suppose that $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{k_1}$ are the eigenvalues of Q. Let

$$T_1 = \{i \in \mathbb{N}\ |1 \le i \le k_1 \text{ and } |\lambda_i - 3| \le \frac{1}{m}\}$$

and

$$T_2 = \{1, 2, \ldots, k_1\} \setminus T_1.$$

By Lemma 4.1 in [**25**], we have

$$k_1\left(\frac{2}{m^3}\right)^2 \geq Tr((Q-3I_{k_1})^2) \geq \sum_{i\ \in\{1,\ldots,k_1\}\setminus T_1} |\lambda_i - 3|^2 \geq \left(\frac{1}{m}\right)^2\ card(T_2),$$

where $card(T_2)$ is the cardinality of the set T_2. Thus

$$card(T_2) \leq \frac{4k_1}{m^4}.$$

Hence, by Definition 4.3, we have Q is in $R(k_1 - \frac{4k_1}{m^4}, \frac{4k_1}{m^4})$.

□

Similarly, we have the following result.

Lemma 4.9. *Let $k_2, m \geq 2$ be some positive integers. Suppose that Q is a self-adjoint element in $\mathcal{M}_{k_2}^{s.a.}(\mathbb{C})$ such that*

$$\|Q\| < \frac{2}{m^3}.$$

Then Q is in $Q(k_2 - \frac{4k_2}{m^4}, \frac{4k_2}{m^4})$, where $Q(k_2 - \frac{4k_2}{m^4}, \frac{4k_2}{m^4})$ is defined in Definition 4.4.

Lemma 4.10. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C^* algebras and $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is a family of self-adjoint elements that generates $\mathcal{A} \bigoplus \mathcal{B}$. For $m \geq 2$, choose*

$$z_m = P_m(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$$

to be a self-adjoint element in $\mathcal{A} \bigoplus \mathcal{B}$, where $P_m(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$ is a noncommutative polynomial of $x_1 \oplus y_1, \ldots, x_n \oplus y_n$, satisfying

$$\|z_m - 3I_{\mathcal{A}} \oplus 0\| \leq \frac{1}{m^3}.$$

Then

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n, z_m) \leq \delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n).$$

PROOF. The result can be proved in the similar fashion as the one of Lemma 5.1 in [**13**]. □

Now we are ready to present the proof of Proposition 4.3.

Proof of Proposition 4.3: Let $R > \max\{4, \|x_1 \oplus y_1\|, \ldots, \|x_n \oplus y_n\|\}$ be a positive number. Since both families of $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are stable, if

$$\alpha < \delta_{top}(x_1, \ldots, x_n)$$
$$\beta < \delta_{top}(y_1, \ldots, y_n),$$

then there are some constants $C_7 > 0$ and $\omega_0 > 0$, $r_0 \geq 1$, $k_1, k_2 \geq 1$ so that

$$\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; q \cdot k_1, \frac{1}{r}, P_1, \ldots, P_r), \omega) \geq C_7^{q \cdot k_1}\left(\frac{1}{\omega}\right)^{\alpha \cdot q \cdot k_1}, \forall \omega < \omega_0, r > r_0, q \in \mathbb{N}, \quad (4.3.3)$$

$$\nu_\infty(\Gamma_R^{(top)}(y_1, \ldots, y_n; q \cdot k_2, \frac{1}{r}, P_1, \ldots, P_r), \omega) \geq C_7^{q \cdot k_2}\left(\frac{1}{\omega}\right)^{\beta \cdot q \cdot k_2}, \forall \omega < \omega_0, r > r_0, q \in \mathbb{N}. \quad (4.3.4)$$

For such $\alpha, \beta > 0$, define

$$f(a) = \alpha a^2 + \beta(1-a)^2 + 1 - a^2 - (1-a)^2, \quad \forall\, 0 \le a \le 1.$$

Since $\alpha < 1$ and $\beta < 1$, we know

$$\max_{0\le a\le 1} f(a) = \frac{\alpha\beta - 1}{\alpha+\beta-1}.$$

For any $\gamma > 0$, let b, c be some positive integers such that

$$f\left(\frac{bk_1}{bk_1+ck_2}\right) > \frac{\alpha\beta-1}{\alpha+\beta-1} - \gamma.$$

For $m \ge 2$, choose

$$z_m = Q_m(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$$

to be a self-adjoint element in $\mathcal{A} \bigoplus \mathcal{B}$, where $Q_m(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$ is a self-adjoint noncommutative polynomial of $x_1 \oplus y_1, \ldots, x_n \oplus y_n$, satisfying

$$\|z_m - 3I_{\mathcal{A}} \oplus 0\| \le \frac{1}{m^3},$$

i.e.

$$\|Q_m(x_1, \ldots, x_m) - 3I_{\mathcal{A}}\| \le \frac{1}{m^3};$$
$$\|Q_m(y_1, \ldots, y_m)\| \le \frac{1}{m^3}.$$

For any given $r \ge 1$ and $\epsilon > 0$, by the definition of topological free entropy dimension, there exist $r' \ge r$ and $\epsilon' < \epsilon$ such that the following hold: $\forall\, q \in \mathbb{N}$, if

$$(A_1, \ldots, A_n) \in \Gamma_R^{(top)}(x_1, \ldots, x_n; qbk_1, \epsilon', P_1, \ldots, P_{r'}) = \Gamma_1$$
$$(B_1, \ldots, B_n) \in \Gamma_R^{(top)}(y_1, \ldots, y_n; qck_2, \epsilon', P_1, \ldots, P_{r'}) = \Gamma_2,$$

then

$$(A_1 \oplus B_1, \ldots, A_n \oplus B_n, Q_m(A_1 \oplus B_1, \ldots, A_n \oplus B_n)) \in \Gamma_R^{(top)}(x_1 \oplus y_1, \ldots, x_n \oplus y_n, z_m; k, \epsilon, P_1, \ldots, P_r),$$

where $k = qbk_1 + qck_2$.

Let

$$\Omega(\Gamma_1, \Gamma_2) = \{U^*(A_1 \oplus B_1, \ldots, A_n \oplus B_n, Q_m(A_1 \oplus B_1, \ldots, A_n \oplus B_n))U \mid U \in \mathcal{U}(k),\ (A_1, \ldots, A_n) \in \Gamma_1,\ (B_1, \ldots, B_n) \in \Gamma_2\}.$$

By Lemma 4.6, there is a family of elements $\{(A_1^\lambda, \ldots, A_n^\lambda)\}_{\lambda\in\Lambda}$, or $\{(B_1^\sigma, \ldots, B_n^\sigma)\}_{\sigma\in\Sigma}$, in Γ_1, or Γ_2 respectively, so that

$$\|(A_1^\lambda \oplus B_1^\sigma, \ldots, A_n^\lambda \oplus B_n^\sigma) - (A_1^{\lambda'} \oplus B_1^{\sigma'}, \ldots, A_n^{\lambda'} \oplus B_n^{\sigma'})\| > \omega,\ \forall\, (\lambda, \sigma) \ne (\lambda', \sigma') \in \Lambda \times \Sigma; \quad (4.3.5)$$

and

$$Card(\Lambda) Card(\Sigma) \ge \nu_\infty(\Gamma_1, 2\omega) \cdot \nu_\infty(\Gamma_2, 2\omega).$$

Note, for any $(\lambda,\sigma)\in\Lambda\times\Sigma$, we have

$$\|Q_m(A_1^\lambda,\ldots,A_n^\lambda)-3I_{qbk_1}\|\le\|Q_m(x_1,\ldots,x_n)-3I_{\mathcal{A}}\|+\epsilon<\frac{2}{m^3}$$
$$\|Q_m(B_1^\sigma,\ldots,B_1^\sigma)\|\le\|Q_m(y_1,\ldots,y_n)\|+\epsilon<\frac{2}{m^3}$$

By Lemma 4.8 and Lemma 4.9, we have that

$$Q_m(A_1^\lambda,\ldots,A_n^\lambda)\ \in\ R(qbk_1-\frac{4k_1}{m^4},\frac{4k_1}{m^4})\subseteq R(qbk_1-\frac{4k}{m^4},\frac{4k}{m^4}) \tag{4.3.6}$$

and

$$Q_m(B_1^\sigma,\ldots,B_1^\sigma)\ \in\ R(qbk_1-\frac{4k_2}{m^4},\frac{4k_2}{m^4})\subseteq Q(qck_2-\frac{4k}{m^4},\frac{4k}{m^4}). \tag{4.3.7}$$

On the other hand, from Lemma 4.7, there exists a family of unitary matrices $\{U_\gamma\}_{\gamma\in\mathcal{I}}$ in $\mathcal{M}_k(\mathbb{C})$ so that (i) when $\gamma_1\ne\gamma_2\in\mathcal{I}$,

$$\begin{aligned}\|U_{\gamma_1}^*(Q_1\oplus Q_2)U_{\gamma_1}-U_{\gamma_2}^*(\tilde{Q}_1\oplus\tilde{Q}_2)U_{\gamma_2}\|\ge\omega,&\\ \forall\ Q_1,\tilde{Q}_1\in R(qbk_1-\frac{4k}{m^4},\frac{4k}{m^4}),\ Q_2,\tilde{Q}_2\in Q(qck_2-\frac{4k}{m^4},\frac{4k}{m^4});&\end{aligned} \tag{4.3.8}$$

and (ii)

$$\begin{aligned}Card(\mathcal{I})&\ge(C_6\cdot130\omega)^{-k^2}\cdot\left(\frac{C_5}{\omega}\right)^{-((qbk_1-\frac{4k}{m^4})^2+(qck_2-\frac{4k}{m^4})^2+8\frac{4k}{m^4}+12(k\frac{4k}{m^4}))}\\&\ge(C_6\cdot130\omega)^{-k^2}\cdot\left(\frac{C_5}{\omega}\right)^{-((qbk_1)^2+(qck_2)^2+\frac{72k^2}{m^4})}\end{aligned}$$

where C_5,C_6 are some constants independent of k,m.

Consider the family of matrices

$$\{U_\gamma^*(A_1^\lambda\oplus B_1^\sigma,\ldots,A_n^\lambda\oplus B_n^\sigma,Q_m(A_1^\lambda\oplus B_1^\sigma,\ldots,A_n^\lambda\oplus B_n^\sigma))U_\gamma\}_{\lambda\in\Lambda,\sigma\in\Sigma,\gamma\in\mathcal{I}}$$

in $\Omega(\Gamma_1,\Gamma_2)$. By (4.3.6), (4.3.7) and (4.3.8) we know that, if $\gamma_1\ne\gamma_2\in\mathcal{I}$, then for any (λ_1,σ_1) and (λ_2,σ_2) in $\Lambda\times\Sigma$,

$$\begin{aligned}&\|U_{\gamma_1}^*Q_m(A_1^{\lambda_1}\oplus B_1^{\sigma_1},\ldots,A_n^{\lambda_1}\oplus B_n^{\sigma_1})U_{\gamma_1}\\&\quad-U_{\gamma_2}^*Q_m(A_1^{\lambda_2}\oplus B_1^{\sigma_2},\ldots,A_n^{\lambda_2}\oplus B_n^{\sigma_2})U_{\gamma_2}\|\ge\omega.\end{aligned}$$

Combining with (4.3.5), we have

$$\begin{aligned}Pack_\infty(\Omega(\Gamma_1,\Gamma_2),\omega)&\ge Card(\Lambda)Card(\Sigma)Card(\mathcal{I})\\&\ge\nu_\infty(\Gamma_1,2\omega)\nu_\infty(\Gamma_2,2\omega)(C_6\cdot130\omega)^{-k^2}\cdot\left(\frac{C_5}{\omega}\right)^{-((qbk_1)^2+(qck_2)^2+\frac{72k^2}{m^4})}.\end{aligned}$$

By inequalities (4.3.3) and (4.3.4), when ω, ϵ are small, $\forall\ q \in \mathbb{N}$ we have

$$\begin{aligned}\nu_\infty(\Gamma_R^{(top)}&(x_1 \oplus y_1, \ldots, x_n \oplus y_n, z_m; k, \epsilon, P_1, \ldots, P_r)), \omega/2) \\ &\geq Pack_\infty(\Omega(\Gamma_1, \Gamma_2), \omega) \\ &\geq C_7^{(qbk_1)^2} \left(\frac{1}{\omega}\right)^{\alpha\cdot(qbk_1)^2} \cdot C_7^{(qck_2)^2} \left(\frac{1}{\omega}\right)^{\beta\cdot(qck_2)^2} \cdot (130C_6\omega)^{-k^2} \cdot \left(\frac{C_5}{\omega}\right)^{-(qbk_1)^2-(qck_2)^2-\frac{72k^2}{m^4}} \\ &\geq C_8^{k^2} \left(\frac{1}{\omega}\right)^{(\frac{\alpha\beta-1}{\alpha+\beta-2}-\gamma-\frac{72}{m^4})k^2}, \end{aligned} \tag{4.3.9}$$

where $k = qbk_1 + qck_2$ and C_8 is a constant independent of k, ω. Then, it induces that

$$\begin{aligned}\limsup_{\omega\to 0^+} \inf_{r\in\mathbb{N}} \limsup_{k\to\infty} &\frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1 \oplus y_1, \ldots, x_n \oplus y_n, z_m; k, \epsilon, P_1, \ldots, P_r)), \omega))}{-k^2 \log \omega} \\ &\geq \frac{\alpha\beta - 1}{\alpha + \beta - 2} - \gamma - \frac{72}{m^4}.\end{aligned}$$

Since γ, m are arbitrary, we obtain

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n, z_m) \geq \frac{\alpha\beta - 1}{\alpha + \beta - 2}.$$

Hence, by Lemma 4.10,

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) \geq \delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n, z_m) \geq \frac{st - 1}{s + t - 2},$$

where

$$s = \delta_{top}(x_1, \ldots, x_n) \quad \text{and} \quad t = \delta_{top}(y_1, \ldots, y_n).$$

(i) Combining with Proposition 4.2, we have that

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \frac{st - 1}{s + t - 2},$$

where

$$s = \delta_{top}(x_1, \ldots, x_n) \quad \text{and} \quad t = \delta_{top}(y_1, \ldots, y_n).$$

(ii) Moreover, by inequality (4.3.9), we know that $x_1 \oplus y_1, \ldots, x_n \oplus y_n, z_m$ is a stable family. Since z_m is a polynomial of $x_1 \oplus y_1, \ldots, x_n \oplus y_n$, we know that $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is also a stable family.

□

Remark 4.2. *If $I_\mathcal{A} \oplus 0$ is in the $*$-algebra generated by $x_1 \oplus y_1, \ldots, x_n \oplus y_n$, i.e. there is a non-commutative polynomial P such that $I_\mathcal{A} \oplus 0 = P(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$, then a much simpler proof can be provided by using Lemma 3.3 in* [**13**] *instead of Lemma 4.7 here.*

4.4. Conclusion. As a summary, we have the following result.

Theorem 4.2. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C^* algebras and $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is a family of self-adjoint elements that generates $\mathcal{A} \bigoplus \mathcal{B}$. Assume*

$$s = \delta_{top}(x_1, \ldots, x_n) \qquad \text{and} \qquad t = \delta_{top}(y_1, \ldots, y_n).$$

(i) If $s \geq 1$ or $t \geq 1$, then

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \max\{\delta_{top}(x_1, \ldots, x_n), \delta_{top}(y_1, \ldots, y_n)\}$$

(ii) If $s < 1$, $t < 1$ and both families $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ are stable, then

$$\delta_{top}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \frac{st - 1}{s + t - 2};$$

and the family of elements $x_1 \oplus y_1, \ldots, x_n \oplus y_n$ is also stable.

5. Topological free entropy dimension of finite dimensional C* algebras

In this section, we are going to compute the topological free entropy dimension of a family of self-adjoint generators of a finite dimensional C* algebra.

Theorem 5.1. *Suppose that $\mathcal{A}$ is a finite dimensional C^* algebra and $dim_{\mathbb{C}}\mathcal{A}$ is the complex dimension of $\mathcal{A}$. If $x_1, \ldots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$, then*

$$\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{dim_{\mathbb{C}}\mathcal{A}}.$$

Proof. It is well known that

$$\mathcal{A} \backsimeq \mathcal{M}_{n_1}(\mathbb{C}) \bigoplus \mathcal{M}_{n_2}(\mathbb{C}) \bigoplus \cdots \bigoplus \mathcal{M}_{n_m}(\mathbb{C}),$$

for a sequence of positive integers $n_1, \ldots, n_m$. By Theorem 3.1 and Theorem 4.2, we have

$$\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{n_1^2 + \cdots + n_m^2} = 1 - \frac{1}{dim_{\mathbb{C}}\mathcal{A}}.$$

□

Similarly, we have the following result.

Theorem 5.2. *Suppose that $\mathcal{K}$ is the algebra of all compact operators in a separable Hilbert space H. Suppose that $\mathcal{A}$ is the unitization of $\mathcal{K}$ and $\mathcal{B}$ is a finite dimensional C^* algebra. If $x_1, \ldots, x_m$ is a family of self-adjoint elements that generates $\mathcal{A} \bigoplus \mathcal{B}$ as a C^* algebra, then*

$$\delta_{top}(x_1, \ldots, x_m) = 1 - \frac{1}{dim_{\mathbb{C}}\mathcal{B} + 1}.$$

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