

# PRIME NUMBERS WITH BEATTY SEQUENCES

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## Abstract

A study of certain Hamiltonian systems has lead Y. Long to conjecture the existence of infinitely many primes of the form  $p = 2 \lfloor \alpha n \rfloor + 1$ , where  $1 < \alpha < 2$  is a fixed irrational number. An argument of P. Ribenboim coupled with classical results about the distribution of fractional parts of irrational multiples of primes in an arithmetic progression immediately imply that this conjecture holds in a much more precise asymptotic form. Motivated by this observation, we give an asymptotic formula for the number of primes  $p = q \lfloor \alpha n + \beta \rfloor + a$  with  $n \leq N$ , where  $\alpha, \beta$  are real numbers such that  $\alpha$  is positive and irrational of finite type (which is true for almost all  $\alpha$ ) and  $a, q$  are integers with  $0 \leq a < q \leq N^\kappa$  and  $\gcd(a, q) = 1$ , where  $\kappa > 0$  depends only on  $\alpha$ . We also prove a similar result for primes  $p = \lfloor \alpha n + \beta \rfloor$  such that  $p \equiv a \pmod{q}$ .

# 1 Introduction

For two fixed real numbers  $\alpha$  and  $\beta$ , the corresponding *non-homogeneous Beatty sequence* is the sequence of integers defined by

$$\mathcal{B}_{\alpha,\beta} = (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty}.$$

Beatty sequences appear in a variety of apparently unrelated mathematical settings, and because of their versatility, the arithmetic properties of these sequences have been extensively explored in the literature; see, for example, [1, 3, 4, 5, 7, 11, 12, 18, 19, 24] and the references contained therein.

In 2000, while investigating the Maslov-type index theory for Hamiltonian systems, Long [17] made the following conjecture:

CONJECTURE. *For every irrational number  $1 < \alpha < 2$ , there are infinitely many prime numbers of the form  $p = 2 \lfloor \alpha n \rfloor + 1$  for some  $n \in \mathbb{N}$ .*

Jia [9] has given a lower bound for the number of such primes  $p$  in the interval  $(x/2, x]$ . We remark that, using a simple modification to an argument given by Ribenboim [20, Chapter 4.V], one can show further that the number of such primes  $p \leq x$  is asymptotic to  $\alpha^{-1} \pi(x)$  as  $x \rightarrow \infty$ ; see also [16]. Moreover, Ribenboim's method also applies to the general problem of estimating

$$\mathcal{N}_{\alpha,\beta;q,a}(x) = \#\{n \leq x : p = q \lfloor \alpha n + \beta \rfloor + a \text{ is prime}\},$$

where  $\alpha, \beta$  are fixed real numbers such that  $\alpha$  is positive and irrational, and  $a, q$  are integers with  $0 \leq a < q$  and  $\gcd(a, q) = 1$ . In fact, if  $a$  and  $q$  are fixed, one easily derives the asymptotic formula

$$\mathcal{N}_{\alpha,\beta;q,a}(x) = (1 + o(1)) \frac{q}{\varphi(q)} \pi(x) \quad (x \rightarrow \infty),$$

where the function implied by  $o(\cdot)$  depends on  $\alpha, \beta$  and  $q$ , and  $\varphi(\cdot)$  is the *Euler function*. Motivated by this observation, we consider here the problem of finding uniform estimates for  $\mathcal{N}_{\alpha,\beta;q,a}(x)$  if  $q$  is allowed to grow with  $x$ . We also consider the same problem for the counting function

$$\mathcal{M}_{\alpha,\beta;q,a}(x) = \#\{n \leq x : p = \lfloor \alpha n + \beta \rfloor \text{ is prime, and } p \equiv a \pmod{q}\}.$$

In particular, in the case that  $\alpha$  is of *finite type* (which is true for *almost all*  $\alpha$  in sense of Lebesgue measure), our main results yield (by partial summation)

nontrivial results for both  $\mathcal{N}_{\alpha,\beta;q,a}(x)$  and  $\mathcal{M}_{\alpha,\beta;q,a}(x)$  even if  $q$  grows as a certain power of  $x$ .

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## 2 Notation

The notation  $\|x\|$  is used to denote the distance from the real number  $x$  to the nearest integer; that is,

$$\|x\| = \min_{n \in \mathbb{Z}} |x - n| \quad (x \in \mathbb{R}).$$

As usual, we denote by  $\lfloor x \rfloor$ ,  $\lceil x \rceil$ , and  $\{x\}$  the greatest integer  $\leq x$ , the least integer  $\geq x$ , and the fractional part of  $x$ , respectively.

We also put  $\mathbf{e}(x) = e^{2\pi i x}$  for all real numbers  $x$  and use  $\Lambda(\cdot)$  to denote the *von Mangoldt function*:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n \text{ is a power of a prime } p; \\ 0 & \text{otherwise.} \end{cases}$$

Throughout the paper, the implied constants in symbols  $O$ ,  $\ll$  and  $\gg$  may depend on the parameters  $\alpha$  and  $\beta$  but are absolute unless indicated otherwise. We recall that the notations  $A \ll B$ ,  $B \gg A$  and  $A = O(B)$  are all equivalent to the statement that  $|A| \leq c|B|$  for some constant  $c > 0$ .

### 3 Preliminaries

Recall that the *discrepancy*  $D(M)$  of a sequence of (not necessarily distinct) real numbers  $a_1, a_2, \dots, a_M \in [0, 1)$  is defined by

$$D(M) = \sup_{\mathcal{I} \subseteq [0,1)} \left| \frac{V(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|, \quad (1)$$

where the supremum is taken all subintervals  $\mathcal{I} = (c, d)$  of the interval  $[0, 1)$ ,  $V(\mathcal{I}, M)$  is the number of positive integers  $m \leq M$  such that  $a_m \in \mathcal{I}$ , and  $|\mathcal{I}| = d - c$  is the length of  $\mathcal{I}$ .

For an irrational number  $\gamma$ , we define its *type*  $\tau$  by the relation

$$\tau = \sup \left\{ \varrho \in \mathbb{R} : \liminf_{\substack{n \rightarrow \infty \\ (n \in \mathbb{N})}} n^\varrho \|\gamma n\| = 0 \right\}.$$

Using *Dirichlet's approximation theorem*, it is easily seen that  $\tau \geq 1$  for every irrational number  $\gamma$ . The celebrated theorems of Khinchin [10] and of Roth [21, 22] assert that  $\tau = 1$  for almost all real (in the sense of the Lebesgue measure) and all irrational algebraic numbers  $\gamma$ , respectively; see also [6, 23].

For every irrational number  $\gamma$ , it is well known that the sequence of fractional parts  $\{\gamma\}, \{2\gamma\}, \{3\gamma\}, \dots$ , is *uniformly distributed modulo 1* (for instance, see [14, Example 2.1, Chapter 1]). If  $\gamma$  is of finite type, this statement can be made more precise. Let  $D_{\gamma, \delta}(M)$  denote the discrepancy of the sequence of fractional parts  $(\{\gamma m + \delta\})_{m=1}^M$ . By [14, Theorem 3.2, Chapter 2] we have:

**Lemma 3.1.** *Let  $\gamma$  be a fixed irrational number of finite type  $\tau < \infty$ . Then, for all  $\delta \in \mathbb{R}$  the following bound holds:*

$$D_{\gamma, \delta}(M) \leq M^{-1/\tau + o(1)} \quad (M \rightarrow \infty),$$

where the function implied by  $o(\cdot)$  depends only on  $\gamma$ .

The following elementary result characterizes the set of numbers that occur in a Beatty sequence  $\mathcal{B}_{\alpha, \beta}$  in the case that  $\alpha > 1$ :

**Lemma 3.2.** *Let  $\alpha, \beta \in \mathbb{R}$  with  $\alpha > 1$ . Then, an integer  $m$  has the form  $m = \lfloor \alpha n + \beta \rfloor$  for some integer  $n$  if and only if*

$$0 < \{\alpha^{-1}(m - \beta + 1)\} \leq \alpha^{-1}.$$

*The value of  $n$  is determined uniquely by  $m$ .*

*Proof.* It is easy to see that an integer  $m$  has the form  $m = \lfloor \alpha n + \beta \rfloor$  for some integer  $n$  if and only if the inequalities

$$\frac{m - \beta}{\alpha} \leq n < \frac{m - \beta + 1}{\alpha}$$

hold, and since  $\alpha > 1$  the value of  $n$  is determined uniquely.  $\square$

We also need the following statement, which is a simplified and weakened version of a theorem of Balog and Perelli [2] (see also [15]):

**Lemma 3.3.** *For an arbitrary real number  $\vartheta$  and coprime integers  $a, q$  with  $0 \leq a < q$ , if  $|\vartheta - b/d| \leq 1/L$  and  $\gcd(b, d) = 1$ , then the bound*

$$\sum_{\substack{n \leq L \\ n \equiv a \pmod{q}}} \Lambda(n) \mathbf{e}(\vartheta n) \ll \left( \frac{L}{d^{1/2}} + d^{1/2} L^{1/2} + L^{4/5} \right) (\log L)^3$$

holds, where the implied constant is absolute.

Finally, we use the *Siegel–Walfisz theorem* (see, for example, the book [8] by Huxley), which asserts that for any fixed constant  $B > 0$  and uniformly for integers  $L \geq 3$  and  $0 \leq a < q \leq (\log L)^B$  with  $\gcd(a, q) = 1$ , one has

$$\sum_{\substack{n \leq L \\ n \equiv a \pmod{q}}} \Lambda(n) = \frac{L}{\varphi(q)} + O\left(L \exp\left(-C_B \sqrt{\log L}\right)\right), \quad (2)$$

where  $C_B > 0$  is an absolute constant that depends only on  $B$ .

## 4 Bounds on exponential sums

The following result may be well known but does not seem to be recorded in the literature. Thus, we present it here with a complete proof.

**Theorem 4.1.** *Let  $\gamma$  be a fixed irrational number of finite type  $\tau < \infty$ . Then, for every real number  $0 < \varepsilon < 1/(8\tau)$ , there is a number  $\eta > 0$  such that the bound*

$$\left| \sum_{m \leq M} \Lambda(qm + a) \mathbf{e}(\gamma km) \right| \leq M^{1-\eta}$$

holds for all integers  $1 \leq k \leq M^\varepsilon$  and  $0 \leq a < q \leq M^{\varepsilon/4}$  with  $\gcd(a, q) = 1$  provided that  $M$  is sufficiently large.

*Proof.* Fix a constant  $\varrho$  such that

$$1 \leq \tau < \varrho < \frac{1}{8\varepsilon} \quad (3)$$

Since  $\gamma$  is of type  $\tau$ , for some constant  $c > 0$  we have

$$\|\gamma d\| > cd^{-\varrho} \quad (d \geq 1). \quad (4)$$

Let  $k, a, q$  be integers with the properties stated in the proposition, and write

$$\sum_{m \leq M} \Lambda(qm + a) \mathbf{e}(\gamma km) = \mathbf{e}(-\vartheta a) \sum_{\substack{n \leq L \\ n \equiv a \pmod{q}}} \Lambda(n) \mathbf{e}(\vartheta n), \quad (5)$$

where  $\vartheta = \gamma k/q$  and  $L = qM + a$ . Let  $b/d$  be the convergent in the continued fraction expansion of  $\vartheta$  which has the largest denominator  $d$  not exceeding  $L^{1-\varepsilon}$ ; then,

$$\left| \frac{\gamma k}{q} - \frac{b}{d} \right| \leq \frac{1}{dL^{1-\varepsilon}}. \quad (6)$$

Multiplying by  $qd$ , we get from (4):

$$\frac{q}{L^{1-\varepsilon}} \geq |\gamma kd - bq| \geq \|\gamma kd\| > c(kd)^{-\varrho}.$$

Thus, since  $k \leq L^\varepsilon$  and  $q \leq L^{\varepsilon/4} \leq L^\varepsilon$ , we see that under the condition (3) the bound

$$d \geq CL^{(1-2\varepsilon)/\varrho-\varepsilon} \geq CL^{1/(4\varrho)} \quad (7)$$

holds, where  $C = c^{1/\varrho}$  and  $L$  is sufficiently large.

Inserting (7) into (6) and using (3) again, we conclude that

$$\left| \frac{\gamma k}{q} - \frac{b}{d} \right| \leq \frac{1}{CL^{1+1/(4\varrho)-\varepsilon}} \leq \frac{1}{L}$$

if  $L$  is sufficiently large. We are therefore in a position to apply Lemma 3.3; taking into account (3), (7), and the fact that  $d \leq L^{1-\varepsilon}$ , it follows that the bound

$$\sum_{\substack{n \leq L \\ n \equiv a \pmod{q}}} \Lambda(n) \mathbf{e}(\vartheta n) \ll (L^{1-1/(8\varrho)} + L^{1-\varepsilon/2}) (\log L)^3 \leq L^{1-\varepsilon/3}$$

holds for all sufficiently large  $L$ . Since  $L \ll qM \leq M^{1+\varepsilon/4}$ , the result now follows from simple calculations after inserting this estimate into (5).  $\square$

Using similar arguments, we have:

**Theorem 4.2.** *Let  $\gamma$  be a fixed irrational number of finite type  $\tau < \infty$ . Then, for every real number  $0 < \varepsilon < 1/(8\tau)$ , there is a number  $\eta > 0$  such that the bound*

$$\left| \sum_{\substack{m \leq M \\ m \equiv a \pmod{q}}} \Lambda(m) \mathbf{e}(\gamma km) \right| \leq M^{1-\eta}$$

*holds for all integers  $1 \leq k \leq M^\varepsilon$  and  $0 \leq a < q \leq M^{\varepsilon/4}$  with  $\gcd(a, q) = 1$  provided that  $M$  is sufficiently large.*

## 5 Main Results

**Theorem 5.1.** *Let  $\alpha$  and  $\beta$  be a fixed real numbers with  $\alpha$  positive, irrational, and of finite type. Then there is a positive constant  $\kappa > 0$  such that for all integers  $0 \leq a < q \leq N^\kappa$  with  $\gcd(a, q) = 1$ , we have*

$$\sum_{n \leq N} \Lambda(q \lfloor \alpha n + \beta \rfloor + a) = \alpha^{-1} \sum_{m \leq \lfloor \alpha N + \beta \rfloor} \Lambda(qm + a) + O(N^{1-\kappa})$$

*where the implied constant depends only on  $\alpha$  and  $\beta$ .*

*Proof.* Suppose first that  $\alpha > 1$ . It is obvious that if  $\alpha$  is of finite type, then so is  $\alpha^{-1}$ . We choose

$$0 < \varepsilon < \frac{1}{16\tau},$$

where  $1 \leq \tau < \infty$  is the type of  $\alpha^{-1}$ .

First, let us suppose that  $\alpha > 1$ . Put  $\gamma = \alpha^{-1}$ ,  $\delta = \alpha^{-1}(1 - \beta)$ , and  $M = \lfloor \alpha N + \beta \rfloor$ . By Lemma 3.2, it follows that

$$\begin{aligned} S_{\alpha, \beta; q, a}(N) &= \sum_{n \leq N} \Lambda(q \lfloor \alpha n + \beta \rfloor + a) \\ &= \sum_{\substack{m \leq M \\ 0 < \{\gamma m + \delta\} \leq \gamma}} \Lambda(qm + a) + O(1) \\ &= \sum_{m \leq M} \Lambda(qm + a) \psi(\gamma m + \delta) + O(1), \end{aligned} \tag{8}$$

where  $\psi(x)$  is the periodic function with period one for which

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x \leq \gamma; \\ 0 & \text{if } \gamma < x \leq 1. \end{cases}$$

By a classical result of Vinogradov (see [26, Chapter I, Lemma 12]) it is known that for any  $\Delta$  such that

$$0 < \Delta < \frac{1}{8} \quad \text{and} \quad \Delta \leq \frac{1}{2} \min\{\gamma, 1 - \gamma\},$$

there is a real-valued function  $\psi_\Delta(x)$  with the following properties:

- (i)  $\psi_\Delta(x)$  is periodic with period one;
- (ii)  $0 \leq \psi_\Delta(x) \leq 1$  for all  $x \in \mathbb{R}$ ;
- (iii)  $\psi_\Delta(x) = \psi(x)$  if  $\Delta \leq x \leq \gamma - \Delta$  or if  $\gamma + \Delta \leq x \leq 1 - \Delta$ ;
- (iv)  $\psi_\Delta(x)$  can be represented as a Fourier series

$$\psi_\Delta(x) = \gamma + \sum_{k=1}^{\infty} (g_k \mathbf{e}(kx) + h_k \mathbf{e}(-kx)),$$

where the coefficients satisfy the uniform bound

$$\max\{|g_k|, |h_k|\} \ll \min\{k^{-1}, k^{-2}\Delta^{-1}\} \quad (k \geq 1). \quad (9)$$

Therefore, from (8) we deduce that

$$S_{\alpha, \beta; q, a}(N) = \sum_{m \leq M} \Lambda(qm + a) \psi_\Delta(\gamma m + \delta) + O(1 + V(\mathcal{I}, M) \log N), \quad (10)$$

where  $V(\mathcal{I}, M)$  denotes the number of positive integers  $m \leq M$  such that

$$\{\gamma m + \delta\} \in \mathcal{I} = [0, \Delta) \cup (\gamma - \Delta, \gamma + \Delta) \cup (1 - \Delta, 1).$$

Since  $|\mathcal{I}| \ll \Delta$ , it follows from the definition (1) and Lemma 3.1, that

$$V(\mathcal{I}, M) \ll \Delta N + N^{1-\varepsilon}, \quad (11)$$

where the implied constant depends only on  $\alpha$ .



To estimate the sum in (10), we use the Fourier expansion for  $\psi_\Delta(\gamma m + \delta)$  and change the order of summation, obtaining

$$\begin{aligned} & \sum_{m \leq M} \Lambda(qm + a) \psi_\Delta(\gamma m + \delta) \\ &= \gamma \sum_{m \leq M} \Lambda(qm + a) + \sum_{k=1}^{\infty} g_k \mathbf{e}(\delta k) \sum_{m \leq M} \Lambda(qm + a) \mathbf{e}(\gamma km) \\ & \quad + \sum_{k=1}^{\infty} h_k \mathbf{e}(-\delta k) \sum_{m \leq M} \Lambda(qm + a) \mathbf{e}(-\gamma km). \end{aligned} \quad (12)$$

By Theorem 4.1 and the bound (9), we see that for  $0 \leq a < q \leq M^{\varepsilon/4}$ , we have

$$\sum_{k \leq M^\varepsilon} g_k \mathbf{e}(\delta k) \sum_{m \leq M} \Lambda(qm + a) \mathbf{e}(\gamma km) \ll M^{1-\eta} \sum_{k \leq M^\varepsilon} k^{-1} \ll M^{1-\eta/2}, \quad (13)$$

for some  $\eta > 0$  that depends only on  $\alpha$ . Similarly,

$$\sum_{k \leq M^\varepsilon} h_k \mathbf{e}(-\delta k) \sum_{m \leq M} \Lambda(qm + a) \mathbf{e}(-\gamma km) \ll M^{1-\eta/2}. \quad (14)$$

On the other hand, using the trivial bound

$$\left| \sum_{m \leq M} \Lambda(qm + a) \mathbf{e}(\gamma km) \right| \leq \sum_{n \leq N} \Lambda(n) \ll N,$$

we have

$$\sum_{k > M^\varepsilon} g_k \mathbf{e}(\delta k) \sum_{m \leq M} \Lambda(qm + a) \mathbf{e}(\gamma km) \ll N \sum_{k > M^\varepsilon} k^{-2} \Delta^{-1} \ll N^{1-\varepsilon} \Delta^{-1}, \quad (15)$$

and

$$\sum_{k > M^\varepsilon} h_k \mathbf{e}(-\delta k) \sum_{m \leq M} \Lambda(qm + a) \mathbf{e}(-\gamma km) \ll N^{1-\varepsilon} \Delta^{-1}. \quad (16)$$

Inserting the bounds, (13), (14), (15) and (16) into (12), we obtain

$$\begin{aligned} & \sum_{m \leq M} \Lambda(qm + a) \psi_\Delta(\gamma m + \delta) \\ &= \gamma \sum_{m \leq M} \Lambda(qm + a) + O(M^{1-\eta/2} + N^{1-\varepsilon} \Delta^{-1}), \end{aligned} \quad (17)$$

where the constant implied by  $O(\cdot)$  depends only on  $\alpha$  and  $\beta$ .

Substituting (11) and (17) in (10) and choosing  $\Delta = N^{-\varepsilon/4}$ , it follows that

$$S_{\alpha,\beta;q,a}(N) = \gamma \sum_{m \leq M} \Lambda(qm + a) + O(N^{1-\kappa}), \quad (18)$$

for some  $\kappa$  which depends only on  $\alpha$ . This concludes the proof in the case that  $\alpha > 1$ .

If  $\alpha < 1$ , we put  $t = \lceil \alpha^{-1} \rceil$  and write

$$\sum_{n \leq N} \Lambda(q \lfloor \alpha n + \beta \rfloor + a) = \sum_{j=0}^{t-1} \sum_{n \leq (N-j)/t} \Lambda(q \lfloor \alpha t n + \alpha j + \beta \rfloor + a).$$

Applying the preceding argument with the irrational number  $\alpha t > 1$ , we conclude the proof.  $\square$

In particular, using the Siegel–Walfisz theorem (2) to estimate the sum in (18) for “small”  $a$  and  $q$ , we obtain:

**Corollary 5.2.** *Under the conditions of Theorem 5.1, for any constant  $B > 0$  and uniformly for all integers  $N \geq 3$  and  $0 \leq a < q \leq (\log N)^B$  with  $\gcd(a, q) = 1$ , we have*

$$\sum_{n \leq N} \Lambda(q \lfloor \alpha n + \beta \rfloor + a) = \frac{q}{\varphi(q)} N + O\left(N \exp\left(-C\sqrt{\log N}\right)\right)$$

for some constant  $C > 0$  that depends only on  $\alpha$ ,  $\beta$  and  $B$ .

In the special case that  $(a, q) = (0, 1)$  or  $(1, 2)$  (the latter case corresponding to primes in the Long conjecture), we can use a well known bound on the error term in the Prime Number Theorem (proved independently by Korobov [13] and Vinogradov [25]) to achieve the following sharper result:

**Corollary 5.3.** *Suppose that  $(a, q) = (0, 1)$  or  $(a, q) = (1, 2)$ . Then, under the conditions of Theorem 5.1, for any constant  $B > 0$  and uniformly for all integers  $N \geq 3$ , we have*

$$\sum_{n \leq N} \Lambda(q \lfloor \alpha n + \beta \rfloor + a) = qN + O\left(N \exp\left(-c(\log N)^{3/5}(\log \log N)^{-1/5}\right)\right)$$

for some absolute constant  $c > 0$ .

Finally, using Lemma 4.2 in place of Lemma 4.1, we obtain the following analogues of Theorem 5.1 and its two corollaries:

**Theorem 5.4.** *Let  $\alpha$  and  $\beta$  be a fixed real numbers with  $\alpha$  positive, irrational, and of finite type. Then there is a positive constant  $\kappa > 0$  such that for all integers  $0 \leq a < q \leq N^\kappa$  with  $\gcd(a, q) = 1$ , we have*

$$\sum_{\substack{n \leq N \\ \lfloor \alpha n + \beta \rfloor \equiv a \pmod{q}}} \Lambda(\lfloor \alpha n + \beta \rfloor) = \alpha^{-1} \sum_{\substack{m \leq \lfloor \alpha N + \beta \rfloor \\ m \equiv a \pmod{q}}} \Lambda(m) + O(N^{1-\kappa})$$

where the implied constant depends only on  $\alpha$  and  $\beta$ .

**Corollary 5.5.** *Under the conditions of Theorem 5.4, for any constant  $B > 0$  and uniformly for all integers  $N \geq 3$  and  $0 \leq a < q \leq (\log N)^B$  with  $\gcd(a, q) = 1$ , we have*

$$\sum_{\substack{n \leq N \\ \lfloor \alpha n + \beta \rfloor \equiv a \pmod{q}}} \Lambda(\lfloor \alpha n + \beta \rfloor) = \frac{N}{\varphi(q)} + O\left(N \exp\left(-C\sqrt{\log N}\right)\right)$$

for some constant  $C > 0$  that depends only on  $\alpha$ ,  $\beta$  and  $B$ .

**Corollary 5.6.** *Suppose that  $(a, q) = (0, 1)$  or  $(a, q) = (1, 2)$ . Then, under the conditions of Theorem 5.4, for any constant  $B > 0$  and uniformly for all integers  $N \geq 3$ , we have*

$$\sum_{\substack{n \leq N \\ \lfloor \alpha n + \beta \rfloor \equiv a \pmod{q}}} \Lambda(\lfloor \alpha n + \beta \rfloor) = N + O\left(N \exp\left(-c(\log N)^{3/5}(\log \log N)^{-1/5}\right)\right)$$

for some absolute constant  $c > 0$ .

## References

- [1] A. G. Abercrombie, ‘Beatty sequences and multiplicative number theory’, *Acta Arith.* **70** (1995), 195–207.
- [2] A. Balog and A. Perelli, ‘Exponential sums over primes in an arithmetic progression’, *Proc. Amer. Math. Soc.* **93** (1985), 578–582.

- [3] W. Banks and I. E. Shparlinski, ‘Non-residues and primitive roots in Beatty sequences’, *Bull. Austral. Math. Soc.* **73** (2006), 433–443.
- [4] W. Banks and I. E. Shparlinski, ‘Short character sums with Beatty sequences’, *Math. Res. Lett.* **13** (2006), 539–547.
- [5] A. V. Begunts, ‘An analogue of the Dirichlet divisor problem’, *Moscow Univ. Math. Bull.* **59** (2004), no. 6, 37–41.
- [6] Y. Bugeaud, *Approximation by algebraic numbers*. Cambridge University Press, Cambridge, 2004.
- [7] A. S. Fraenkel and R. Holzman, ‘Gap problems for integer part and fractional part sequences’, *J. Number Theory* **50** (1995), 66–86.
- [8] M. N. Huxley, *The distribution of prime numbers. Large sieves and zero-density theorems*. Clarendon Press, Oxford, 1972.
- [9] C. Jia, ‘On a conjecture of Yiming Long’, *Acta Arith.* **122** (2006), no. 1, 57–61.
- [10] A. Y. Khinchin, ‘Zur metrischen Theorie der diophantischen Approximationen’, *Math. Z.* **24** (1926), no. 4, 706–714.
- [11] T. Komatsu, ‘A certain power series associated with a Beatty sequence’, *Acta Arith.* **76** (1996), 109–129.
- [12] T. Komatsu, ‘The fractional part of  $n\vartheta + \varphi$  and Beatty sequences’, *J. Théor. Nombres Bordeaux* **7** (1995), 387–406.
- [13] H. M. Korobov, ‘Estimates of trigonometric sums and their applications’ (Russian), *Uspehi Mat. Nauk* **13** (1958), 185–192.
- [14] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience, New York-London-Sydney, 1974.
- [15] A. F. Lavrik, ‘Analytic method of estimates of trigonometric sums by the primes of an arithmetic progression’, (Russian) *Dokl. Akad. Nauk SSSR* **248** (1979), no. 5, 1059–1063.
- [16] D. Leitman and D. Wolke, ‘Primzahlen der Gestalt  $[f(n)]$ ’, *Math. Z.* **145** (1975), 81–92.

- [17] Y. Long, ‘Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics’, *Adv. Math.* **154** (2000), no. 1, 76–131.
- [18] G. S. Lü and W. G. Zhai, ‘The divisor problem for the Beatty sequences’, *Acta Math. Sinica* **47** (2004), 1213–1216 (in Chinese).
- [19] K. O’Byrant, ‘A generating function technique for Beatty sequences and other step sequences’, *J. Number Theory* **94** (2002), 299–319.
- [20] P. Ribenboim, *The new book of prime number records*. Springer-Verlag, New York, 1996.
- [21] K. F. Roth, ‘Rational approximations to algebraic numbers’, *Mathematika* **2** (1955), 1–20.
- [22] K. F. Roth, ‘Corrigendum to “Rational approximations to algebraic numbers”’, *Mathematika* **2** (1955), 168.
- [23] W. M. Schmidt, *Diophantine approximation*. Springer-Verlag, Berlin, 1980.
- [24] R. Tijdeman, ‘Exact covers of balanced sequences and Fraenkel’s conjecture’, *Algebraic number theory and Diophantine analysis (Graz, 1998)*, 467–483, de Gruyter, Berlin, 2000.
- [25] I. M. Vinogradov, ‘A new estimate for  $\zeta(1 + it)$ ’ (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **22** (1958), 161–164.
- [26] I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*. Dover Publications, New York, 2004.