Vortex Images and q-Elementary Functions

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Abstract

In the present paper problem of vortex images in annular domain between two coaxial cylinders is solved by q-elementary functions. We show that all images are determined completely as poles of q-logarithmic function, where dimensionless parameter $q = r_2^2/r_1^2$ is determined by square ratio of the cylinder radiuses. The resulting solution for the complex potential is represented in terms of the Jackson q-exponential function. By composing pairs of q-exponents as the first Jacobi theta function and conformal mapping to rectangular domain we link our solution with result of Johnson and McDonald. We found that one vortex cannot remain at rest except at the geometric mean distance, but must orbit the cylinders with constant angular velocity which is related with q-geometric series. Vortices in two particular geometries in $q \to \infty$ limit are studied.

1 Introduction

The classical method of images introduced by W. Thomson in 1845 becomes powerful method for solving boundary value problems in electrostatics and hydrodynamics [1],[2]. The method has been successfully applied to simple geometries as spheres, cylinders and half-spaces, where explicit formulas have been given. Unfortunately for complex body shapes the image principle becomes extremely difficult even to find approximate solution. This is why the image problem for which solution can be found in an exact form, like merging cylinders [3] for example, becomes a member of very exclusive family . In the present paper we consider exact solution of planar vortex problem in annular domain between two coaxial cylinders by method of images. This problem has many interesting applications. One of them is related to hydrodynamic interaction in which the modification of ambient flow by cylinders is carried out by obtaining the effect of single cylinder on the flow and then applying the boundary conditions on each cylinder to determine the unknown coefficients that appear in the series expansions. [4] solved the diffraction problem of water waves by multiple cylinders placed at the free surface.

Another application is related with inviscid two-dimensional fluid dynamics experiments with magnetized electron columns confined in a cylindrical trap[19]. The flow vorticity is proportional to the electron density and the electric potential is analogous to the two dimensional stream function. Thus, the electrons mimic ideal two dimensional fluid equations and by creating electron columns with the appropriate density, one can model fluid flows. It allows to model real problems of vortex interaction with topography [9]. Motion of a vortex in the neighborhood of a cosmic string [20] and influency on this the cylindrically compactificated extra space dimensions is another class of possible cosmological applications.

The problem of one vortex and one cylinder is connected with the Circle Theorem of Milne-Thomson [1] which can be rewritten for the complex velocity of the flow $\bar{V}(z) = u_1 - iu_2$ in the form

$$\bar{V}(z) = \bar{v}(z) - \frac{r_1^2}{z^2}\, v\left(\frac{r_1^2}{z}\right) \tag{1}$$

where $v(z)$ is complex velocity of the flow in unbounded domain, and the second term in (1) represents the correction to the complex velocity by the cylinder of radius r_1 placed at the origin. For a vortex at z_0, of strength κ and circulation, $\Gamma = -2\pi\kappa$, (1) can be written explicitly as

$$\bar{V}(z) = \frac{i\kappa}{z - z_0} - \frac{i\kappa}{z - \frac{r_1^2}{\bar{z}_0}} + \frac{i\kappa}{z} \tag{2}$$

where the second term represents a vortex of strength $-\kappa$ at the inverse point of z_0, $\frac{r_1^2}{\bar{z}_0}$, with respect to the cylinder. Henceforth, we shall call the vortices at inverse points and at the centres of cylinders (or at the infinity) "vortex images" or simply "images". Therefore, in (2), there are two images; one positive image at the centre of the cylinder and another negative image at the inverse point. In fact, images are used to replace the circle in the infinite 2-D plane.

Another application is the case of a vortex at point z_0 inside a cylindrical domain with radius r_2, C : $|z| < r_2$,

$$\bar{V}(z) = \frac{i\kappa}{z - z_0} - \frac{i\kappa}{z - \frac{r_2^2}{\bar{z}_0}} \tag{3}$$

where the vortex image is located at point $r_2^2/\bar{z}_0$ outside C. The solution (3) can be obtained from the circle theorem by first using the mapping $z = 1/\omega$ and also from the Laurent series expansion of the solution.

The above two examples are limiting cases of the problem of a point vortex in annular domain between two coaxial cylinders with inner radius r_1 and outer radius r_2. We find in this case that the solution is given as the infinite set of images in two cylinders,

$$\bar{V}(z) = \sum_{n=-\infty}^{\infty} \left[\frac{i\kappa}{z - z_0 q^n} - \frac{i\kappa}{z - \frac{r_1^2}{\bar{z}_0} q^n} \right]. \tag{4}$$

As we show in this paper these images are determined completely in terms of q-logarithmic and q-exponential functions. Mathematical study of these functions is connected with some applications in the number theory, for calculation of the Euler's constant γ by generalization of a classical formula due to Ramanujan and Vacca [13], irrationality test [12], [13], the Stieltjes transform of a positive discrete measure [11] and construction of the Pade approximations [11]. Another class of applications related with physics is quantum groups and their representations [16]. The physical systems with such symmetries started from the quantum integrable systems and then extended to the several q-deformed physical systems like the quantum linear harmonic oscillator, generalized coherent states in quantum optics, composite particle with the Chern-Simons flux - the anyons. It was observed in general that physical systems with a fundamental length scale have a symmetry of a quantum group [5]. In nuclear physics the deformation parameter is related to the time scale of strong interactions [6], while in solid state physics with the lattice spacing [7]. Despite of this particular progress, the direct interpretation of the deformation parameter in these cases is sometimes incomplete or even nonexistent. In the present paper we study the classical problem of N vortices placed in the annular region between two coaxial cylinders with radii $r_1 < r_2$, and we see that the natural dimensionless parameter $q = r_2^2/r_1^2$ plays the role of the fundamental length and the infinite set of images produces the one dimensional q- lattice.

2 N Vortices in Annular Domain

We consider the problem of N point vortices in annular domain $D : \{r_1 \leq |z| \leq r_2\}$, where $z_1, ..., z_N$ are positions of vortices with strengths $\kappa_1, ..., \kappa_N$ respectively. The region is bounded by two concentric circles: $C_1 : z\bar{z} = r_1^2$ and $C_2 : z\bar{z} = r_2^2$

The complex velocity is given by the Laurent series

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} + \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_{n+1}}{z^{n+1}} \tag{5}$$

which has to satisfy the boundary conditions at both cylinders,

$$\left[\bar{V}(z)z + V(\bar{z})\right]|_{C_k} = 0,\ k = 1, 2. \tag{6}$$

The conditions (6) imply that no fluid can penetrate any of the circular walls of the domain. If $\mathbf{n}$ denotes the normal to the boundary, the boundary condition is that normal velocity must be zero, $\mathbf{u} \cdot \mathbf{v} = u_n = 0$.

To find the unknown coefficients we have to determine the boundary conditions,

$$\Gamma = \bar{V}(z)z + V(\bar{z})\bar{z} = \sum_{k=1}^{N} \frac{i\kappa_k z}{z - z_k} + \sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+1}} + C.C. \tag{7}$$

where $C.C.$ stands for the complex conjugate. Since on boundary C_1: $z\bar{z} = {r_1}^2$ and $|z_k| > |z|$, we can rewrite equation (7) as follows;

$$\begin{aligned}\Gamma_{|C_1} &= \sum_{k=1}^{N}(-i\kappa_k)\sum_{n=0}^{\infty}\left(\frac{z}{z_k}\right)^{n+1} + \sum_{n=0}^{\infty}a_n z^{n+1} + \sum_{n=0}^{\infty}\frac{b_{n+2}}{z^{n+1}} + C.C.\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=1}^{N}\frac{-i\kappa_k}{z_k^{n+1}} + a_n + \bar{b}_{n+2}\frac{1}{r_1^{2(n+1)}}\right]z^{n+1} + C.C. = 0. \end{aligned} \tag{8}$$

This implies the following algebraic system

$$\sum_{k=1}^{N}\frac{-i\kappa_k}{z_k^{n+1}} + a_n + \bar{b}_{n+2}\frac{1}{r_1^{2(n+1)}} = 0, \quad (n = 0, 1, 2...) \quad \texttt{and} \quad b_1 = 0. \tag{9}$$

Since on boundary C_2: $z\bar{z} = {r_2}^2$ and $|z_k| < |z|$, we can rewrite equation (7) as follows;

$$\begin{aligned}\Gamma_{|C_2} &= \sum_{k=1}^{N}(i\kappa_k)\sum_{n=0}^{\infty}\left(\frac{z_k}{z}\right)^{n} + \sum_{n=0}^{\infty}a_n z^{n+1} + \sum_{n=0}^{\infty}\frac{b_{n+2}}{z^{n+1}} + C.C.\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=1}^{N}\frac{-i\kappa_k\bar{z}_k^{n+1}}{r_2^{2(n+1)}} + a_n + \bar{b}_{n+2}\frac{1}{r_2^{2(n+1)}}\right]z^{n+1} + C.C. = 0. \end{aligned} \tag{10}$$

This implies another algebraic system

$$\sum_{k=1}^{N}\frac{-i\kappa_k\bar{z}_k^{n+1}}{r_2^{2(n+1)}} + a_n + \bar{b}_{n+2}\frac{1}{r_2^{2(n+1)}} = 0, \quad (n = 0, 1, 2, ...). \tag{11}$$

We have two algebraic systems (9) and (11). By substracting (11) from (9), we eliminate a_n

$$\bar{b}_{n+2}\left[\frac{1}{r_2^{2(n+1)}} - \frac{1}{r_1^{2(n+1)}}\right] + \sum_{k=1}^{N}(-i\kappa_k)\left[\frac{\bar{z}_k^{n+1}}{r_2^{2(n+1)}} - \frac{1}{z_k^{n+1}}\right] = 0. \tag{12}$$

If $q \equiv r_2^2/r_1^2$ is used we find

$$b_{n+2} = \sum_{k=1}^{N}\left(\frac{-i\kappa_k}{\bar{z}_k^{n+1}}\,\frac{r_2^{2(n+1)} - |z_k|^{2(n+1)}}{q^{n+1}-1}\right) \tag{13}$$

and from (9) we determine a_n,

$$a_n = \sum_{k=1}^{N}\frac{i\kappa_k}{z_k^{n+1}} - \bar{b}_{n+2}\frac{1}{r_1^{2(n+1)}}, \tag{14}$$

or

$$a_n = \sum_{k=1}^{N}\frac{-i\kappa_k}{z_k^{n+1}}\,\frac{r_1^{2(n+1)} - |z_k|^{2(n+1)}}{r_1^{2(n+1)}(q^{n+1}-1)}. \tag{15}$$

The Taylor series part of (5) gives the following,

$$\begin{aligned}\sum_{n=0}^{\infty}a_n z^n &= \sum_{n=0}^{\infty}\sum_{k=1}^{N}\frac{-i\kappa_k}{z_k^{n+1}}\,\frac{z^n}{q^{n+1}-1} - \sum_{n=0}^{\infty}\sum_{k=1}^{N}\frac{(-i\kappa_k)\bar{z}_k^{(n+1)}z^n}{r_1^{2(n+1)}(q^{n+1}-1)}\\ &= \sum_{k=1}^{N}\frac{-i\kappa_k}{z(q-1)}\sum_{n=0}^{\infty}\frac{q-1}{q^{n+1}-1}\left(\frac{z}{z_k}\right)^{n+1} + \sum_{k=1}^{N}\frac{i\kappa_k}{z(q-1)}\sum_{n=0}^{\infty}\frac{q-1}{q^{n+1}-1}\left(\frac{z\bar{z}_k}{r_1^2}\right)^{n+1}\\ &= \sum_{k=1}^{N}\frac{-i\kappa_k}{z(q-1)}\sum_{n=0}^{\infty}\frac{1}{[n+1]}\left(\frac{z}{z_k}\right)^{n+1} + \sum_{k=1}^{N}\frac{i\kappa_k}{z(q-1)}\sum_{n=0}^{\infty}\frac{1}{[n+1]}\left(\frac{z\bar{z}_k}{r_1^2}\right)^{n+1}\\ &= \sum_{k=1}^{N}\frac{i\kappa_k}{z(q-1)}Ln_q\left(1-\frac{z}{z_k}\right) - \sum_{k=1}^{N}\frac{i\kappa_k}{z(q-1)}Ln_q\left(1-\frac{z\bar{z}_k}{r_1^2}\right) \end{aligned} \tag{16}$$

and the Laurent part gives,

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{b_{n+2}}{z^{n+2}} &= \sum_{k=1}^{N}\left(\frac{1}{z}\sum_{n=0}^{\infty}\frac{-i\kappa_k}{\bar{z}_k^{n+1}}\frac{r_2^{2(n+1)}}{q^{n+1}-1}\frac{1}{z^{n+1}}\right)+\sum_{k=1}^{N}\left(\frac{1}{z}\sum_{n=0}^{\infty}\frac{i\kappa_k z_k^{n+1}}{q^{n+1}-1}\frac{1}{z^{n+1}}\right) \\
&= \sum_{k=1}^{N}\frac{-i\kappa_k}{z(q-1)}\sum_{n=0}^{\infty}\frac{q-1}{q^{n+1}-1}\left(\frac{r_2^2}{z\bar{z}_k}\right)^{n+1}+\sum_{k=1}^{N}\frac{i\kappa_k}{z(q-1)}\sum_{n=0}^{\infty}\frac{q-1}{q^{n+1}-1}\left(\frac{z_k}{z}\right)^{n+1} \\
&= \sum_{k=1}^{N}\frac{-i\kappa_k}{z(q-1)}\sum_{n=0}^{\infty}\frac{1}{[n+1]}\left(\frac{r_2^2}{z\bar{z}_k}\right)^{n+1}+\sum_{k=1}^{N}\frac{i\kappa_k}{z(q-1)}\sum_{n=0}^{\infty}\frac{1}{[n+1]}\left(\frac{z_k}{z}\right)^{n+1} \\
&= \sum_{k=1}^{N}\frac{i\kappa_k}{z(q-1)}Ln_q\left(1-\frac{r_2^2}{z\bar{z}_k}\right)-\sum_{k=1}^{N}\frac{i\kappa_k}{z(q-1)}Ln_q\left(1-\frac{z_k}{z}\right) \qquad (17)
\end{aligned}$$

where $[n]=\frac{q^n-1}{q-1}$ and $Ln_q(1-x)\equiv-\sum_{n=1}^{\infty}\frac{x^n}{[n]},\ |x|<q,\ q>1.$

3 Logarithmic and q-Exponential Functions

By analogy with ordinary logarithmic function : $|x|\leq 1$, $x\neq -1$

$$\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n} \qquad (18)$$

q-logarithmic function is defined as

$$Ln_q(1-x)\equiv-\sum_{n=1}^{\infty}\frac{x^n}{[n]},\ |x|<q,\ q>1. \qquad (19)$$

where q-number

$$[n]\equiv 1+q+q^2+...+q^{n-1}=\frac{q^n-1}{q-1} \qquad (20)$$

for any positive integer n. In the limiting case $q\to 1$

$$\lim_{q\to 1}[n]=n \qquad (21)$$

$$\lim_{q\to 1}Ln_q(1-x)=\ln(1-x) \qquad (22)$$

Our definition relates with the q-Logarithmic function of Borwein [11]

$$L_q(x)=\sum_{n=1}^{\infty}\frac{x^n}{[n]} \qquad (23)$$

by

$$Ln_q(1-x)=-L_q(x) \qquad (24)$$

and with that of Sondow and Zudilin [13]

$$\ln_q(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{q^n-1},\ |x|<q,\ q>1 \qquad (25)$$

by

$$Ln_q(1+x)=(q-1)\ln_q(1+x) \qquad (26)$$

The q-derivative is defined as

$$D_qf(x)=\frac{f(qx)-f(x)}{(q-1)x}. \qquad (27)$$

Taking q-derivative of x^n and the q-logarithmic function, we get

$$D_q x^n = [n] x^{n-1} \tag{28}$$

and

$$D_q Ln_q(1-x) = -\frac{1}{1-x} \tag{29}$$

For our problem, the following representation of q-Logarithmic function [11], [13] is crucial in the complex domain: let q be real, $q > 1$, then for $0 < |z| < q$ the following identity hold

$$Ln_q(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{[n]} = (q-1) \sum_{n=1}^{\infty} \frac{z}{q^n + z}. \tag{30}$$

The proof is given by the following chain of transformations

$$\sum_{n=1}^{\infty} \frac{z}{q^n + z} = z \sum_{n=1}^{\infty} \frac{1}{q^n} \frac{1}{1 + z\, q^{-n}} = z \sum_{n=1}^{\infty} \frac{1}{q^n} \sum_{k=1}^{\infty} \frac{(-z)^{k-1}}{q^{n(k-1)}} =$$

$$\sum_{k=1}^{\infty} (-1)^{k-1} z^k \sum_{n=1}^{\infty} \frac{1}{q^{nk}} = \sum_{k=1}^{\infty} (-1)^{k-1} z^k \frac{1}{q^k} \frac{1}{1-q^{-k}} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^k}{q^k - 1} =$$

$$= \frac{1}{q-1} Ln_q(1+z) \tag{31}$$

Next we will need Jackson's q-exponential functions defined as [11]

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \tag{32}$$

$$E_q^*(z) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{z^n}{[n]!} \tag{33}$$

where $[n]! \equiv [1][2]...[n]$ and q is real. As $q \to 1$, both functions reduce to the ordinary exponential. Application of q-derivative gives

$$D_q E_q(x) = E_q(x), \quad D_q E_q^*(x) = E_q^*(qx) \tag{34}$$

The function $E_q(z)$ is entire in z if $|q| > 1$, while it has radius of convergence $|1-q|^{-1}$ if $|q| < 1$. The function $E_q^*(z)$ is entire for $|q| < 1$ and converges for $|z| < 1$ if $|q| > 1$. These two functions are related by

$$E_q(z) = E_{1/q}^*(z) \tag{35}$$

and

$$E_q(-z) E_q^*(z) = 1 \tag{36}$$

Particularly important for us is the infinite product representation [11]

$$E_q^*(z) = \prod_{k=0}^{\infty} \left(1 + z q^k (1-q)\right), \quad |q| < 1 \tag{37}$$

and the related identity obtained by setting $q \to 1/q$ and shifting the argument

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{q^k}\right) = \frac{1}{1-z} E_q \left(\frac{-z}{1-q^{-1}}\right) \tag{38}$$

which is entire for $|q| > 1$.

3.1 Vortex Images and Poles of q-Logarithm

Substituting equations (16) and (17) in (5) we get the following,

$$\bar{V}(z) = \sum_{k=1}^{N} i\kappa_k \left[\frac{1}{z-z_k} + \frac{1}{z(q-1)}\left[Ln_q\left(1-\frac{z}{z_k}\right) - Ln_q\left(1-\frac{z\bar{z}_k}{r_1^2}\right) + Ln_q\left(1-\frac{r_2^2}{z\bar{z}_k}\right) - Ln_q\left(1-\frac{z_k}{z}\right)\right]\right]. \quad (39)$$

It can be written in a more compact form in terms of q-derivatives with different basis q:

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{i\kappa_k}{z-z_k} - \frac{i\kappa_k}{z_k}\left([\alpha]_q D_{q^\alpha} Ln_q\left(1-\frac{z}{z_k}\right) - \frac{z_k^2}{z^2}[1-\alpha]_q D_{q^{1-\alpha}} Ln_q\left(1-\frac{z_k}{z}\right)\right) \quad (40)$$

where the real q-number $[\alpha]_q = (q^\alpha - 1)/(q-1)$, $\alpha = \alpha(k) \equiv \log_q(|z_k|^2/r_1^2)$. Parameter α is restricted by $0 < \alpha < 1$. Particular values of this parameter correspond to different positions of a vortex: 1) for $\alpha = 1$ the vortex is on the outer cylinder $|z_0| = r_2$, 2) for $\alpha = 0$ the vortex is on the inner cylinder $|z_0| = r_1$ 3) for $\alpha = 1/2$ the vortex is at the geometric mean distance $|z_0| = \sqrt{r_1 r_2}$ (see Section 6 equation (84)) 4) for $\alpha = m/n$, where $m < n$ are positive integers, the vortex is at the generalized mean distance $|z_0|^n = r_1^{n-m} r_2^m$.

Expanding q-log according to (30) we have

$$\begin{aligned} \bar{V}(z) &= \sum_{k=1}^{N} \frac{i\kappa_k}{z-z_k} + \sum_{k=1}^{N}\sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^n} - \sum_{k=1}^{N}\sum_{n=1}^{\infty} \frac{i\kappa_k}{z - q^n \frac{r_1^2}{\bar{z}_k}} \\ &+ \sum_{k=1}^{N}\sum_{n=1}^{\infty}\left[\frac{i\kappa_k}{z} - \frac{i\kappa_k}{z - q^{-n}\frac{r_2^2}{\bar{z}_k}}\right] - \sum_{k=1}^{N}\sum_{n=1}^{\infty}\left[\frac{i\kappa_k}{z} - \frac{i\kappa_k}{z - q^{-n} z_k}\right] \end{aligned} \quad (41)$$

or

$$\begin{aligned} \bar{V}(z) &= \sum_{k=1}^{N} \frac{i\kappa_k}{z-z_k} + \sum_{k=1}^{N}\left[\sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^n} + \sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^{-n}}\right], \\ &- \sum_{k=1}^{N}\left[\sum_{n=0}^{\infty} \frac{i\kappa_k}{z - \frac{r_1^2}{\bar{z}_k} q^{-n}} + \sum_{n=0}^{\infty} \frac{i\kappa_k}{z - \frac{r_2^2}{\bar{z}_k} q^n}\right]. \end{aligned} \quad (42)$$

Equation (42) for complex velocity has countable infinite number of pole singularities. These singularities can be interpreted as vortex images in two cylindrical surfaces. For simplicity let us consider only one vortex at position z_0, $r_1 < |z_0| < r_2$. Then the set of images in the cylinder C_1 we denote $z_I^{(1)}, z_I^{(2)}, ...$ and in the cylinder C_2 as $z_{II}^{(1)}, z_{II}^{(2)}, ...$, where

$$z_I^{(1)} = \frac{r_1^2}{\bar{z}_0} \qquad z_{II}^{(1)} = \frac{r_2^2}{\bar{z}_0} \quad (43)$$

$$z_I^{(2)} = \frac{r_1^2}{\bar{z}_{II}^{(1)}} = \frac{z_0}{q} \qquad z_{II}^{(2)} = \frac{r_2^2}{\bar{z}_I^{(1)}} = q z_0 \quad (44)$$

$$z_I^{(3)} = \frac{r_1^2}{\bar{z}_{II}^{(2)}} = \frac{r_1^2}{\bar{z}_0}\frac{1}{q} \qquad z_{II}^{(3)} = \frac{r_2^2}{\bar{z}_I^{(2)}} = \frac{r_2^2}{\bar{z}_0} q \quad (45)$$

$$z_I^{(4)} = \frac{r_1^2}{\bar{z}_{II}^{(3)}} = \frac{z_0}{q^2} \qquad z_{II}^{(4)} = \frac{r_2^2}{\bar{z}_I^{(3)}} = z_0 q^2 \quad (46)$$

$$z_I^{(5)} = \frac{r_1^2}{\bar{z}_{II}^{(4)}} = \frac{r_1^2}{\bar{z}_0}\frac{1}{q^2} \qquad z_{II}^{(5)} = \frac{r_2^2}{\bar{z}_I^{(4)}} = \frac{r_2^2}{\bar{z}_0} q^2 \quad (47)$$

$$\dots\dots\dots\dots\dots \quad (48)$$

Combining together and taking into account alternating signs (the negative for the first image and the positive for the next one - the image of the image) we have two sets of consecutive images

$$z_0,\ z_I^{(1,-)},\ z_{II}^{(2,+)},\ z_I^{(3,-)},\ z_{II}^{(4,+)},\ z_I^{(5,-)}, ... \quad (49)$$

and

$$z_0,\ z_{II}^{(1,-)},\ z_I^{(2,+)},\ z_{II}^{(3,-)},\ z_I^{(4,+)},\ z_{II}^{(5,-)},.... \tag{50}$$

This shows that the set of vortex images is completely determined by simple pole singularities of the q-logarithmic function. In the above representation (42) by identity $r_2^2/q^n = r_1^2/q^{n-1}$ we can combine sums so that, we have

$$\bar{V}(z) = \sum_{k=1}^{N}\left[\sum_{n=-\infty}^{\infty}\frac{i\kappa_k}{z - z_k q^n}\right] - \sum_{k=1}^{N}\left[\sum_{n=0}^{\infty}\frac{i\kappa_k}{z - \frac{r_1^2}{\bar{z}_k}q^{-n}} + \sum_{n=1}^{\infty}\frac{i\kappa_k}{z - \frac{r_1^2}{\bar{z}_k}q^{n}}\right] \tag{51}$$

$$= \sum_{k=1}^{N} i\kappa_k \sum_{n=-\infty}^{\infty}\left[\frac{1}{z - z_k q^n} - \frac{1}{z - \frac{r_1^2}{\bar{z}_k}q^n}\right]. \tag{52}$$

4 Complex Potential and q-Exponential function

In this section we derive the complex potential of the flow according to the relation

$$\bar{V}(z) = F'(z) \tag{53}$$

in terms of Jackson q-exponential function (32). To construct $F(z)$ we use new function defined in [11] as

$$f_q(z) \equiv \prod_{n=1}^{\infty}\left(1 - \frac{z}{q^n}\right) = \frac{E_q\left(\frac{zq}{1-q}\right)}{1-z} \tag{54}$$

where $|q| > 1$ and observe

$$\frac{f_q'(z)}{f_q(z)} = \frac{d}{dz}\ln f_q(z) = -\sum_{n=1}^{\infty}\frac{q^{-n}}{1 - zq^{-n}} = \sum_{n=1}^{\infty}\frac{1}{z - q^n}. \tag{55}$$

Using (30) we have the relation with q-logarithmic function

$$\frac{Ln_q(1-z)}{(q-1)z} = \frac{d}{dz}\ln f_q(z) = \frac{d}{dz}\ln\frac{E_q\left(\frac{qz}{1-q}\right)}{1-z}. \tag{56}$$

This expression can be simplified if we use q-derivative of the exponential function (34) by rescaling the argument

$$E_q\left(\frac{qz}{1-q}\right) = (1-z)E_q\left(\frac{z}{1-q}\right) \tag{57}$$

so that

$$\frac{Ln_q(1-\alpha z)}{(q-1)z} = \frac{d}{dz}\ln\, E_q\left(\frac{\alpha z}{1-q}\right). \tag{58}$$

By similar arguments we find also

$$\frac{Ln_q(1-\frac{\alpha}{z})}{(q-1)z} = -\frac{d}{dz}\ln\, E_q\left(\frac{\alpha}{(1-q)z}\right). \tag{59}$$

If now we apply these formulas to (39) we get

$$\bar{V}(z) = \sum_{k=1}^{N} i\kappa_k \frac{d}{dz}\ln\left[(z - z_k)\frac{E_q\left(\frac{z}{(1-q)z_k}\right)E_q\left(\frac{z_k}{(1-q)z}\right)}{E_q\left(\frac{z\bar{z}_k}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z\bar{z}_k}\right)}\right]. \tag{60}$$

Finally this implies complex potential in the form

$$F(z) = \sum_{k=1}^{N} i\kappa_k\left[\ln(z - z_k) + \ln\frac{E_q\left(\frac{z}{(1-q)z_k}\right)E_q\left(\frac{z_k}{(1-q)z}\right)}{E_q\left(\frac{z\bar{z}_k}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z\bar{z}_k}\right)}\right]. \tag{61}$$

The first term in the bracket corresponds to the vortex at position z_k while the second term describes its images. All these images completely are determined by zeroes of q-exponential functions. For the zeros of the q-analogue of exponential function and asymptotic formulas for varying parameter q see [15].

5 Conformal Mapping and Elliptic Functions

To compare our solution (61) with Johnson & Mc Donald (2004) we rewrite it in terms of elliptic functions. For comparison purposes we fix the radius $r_2 = 1$ so that $q = \frac{r_2^2}{r_1^2} = \frac{1}{r_1^2} \equiv \frac{1}{\tilde{q}^2}$, where we introduced new parameter $\tilde{q} < 1$. Then according to (35) $E_q(z) = E^*_{\tilde{q}^2}(z)$ so that we find complex potential in terms of the second Jackson q-exponent

$$F(z) = \sum_{k=1}^{N} i\kappa_k \ln \left[(z - z_k) \frac{E^*_{\tilde{q}^2}(\frac{\tilde{q}^2 z}{(\tilde{q}^2-1)z_k}) E^*_{\tilde{q}^2}(\frac{\tilde{q}^2 z_k}{(\tilde{q}^2-1)z})}{E^*_{\tilde{q}^2}(\frac{z\bar{z}_k}{(\tilde{q}^2-1)}) E^*_{\tilde{q}^2}(\frac{\tilde{q}^2}{(\tilde{q}^2-1)z\bar{z}_k})} \right]. \quad (62)$$

Using representation of q-exponentials as an infinite products (37) we have

$$E^*_{\tilde{q}^2}\left(\frac{z}{\tilde{q}^2 - 1}\right) = \prod_{n=0}^{\infty} (1 - \tilde{q}^{2n} z) \equiv (1 - z)^{\infty}_{\tilde{q}^2} \quad (63)$$

$$E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2 z}{\tilde{q}^2 - 1}\right) = \prod_{n=1}^{\infty} (1 - \tilde{q}^{2n} z) = \frac{(1 - z)^{\infty}_{\tilde{q}^2}}{1 - z}. \quad (64)$$

The first Jacobi theta function is defined as an infinite product [17]

$$\Theta_1(x; \tilde{q}) = 2\, G\, \tilde{q}^{\frac{1}{4}}\, \sin x \prod_{n=1}^{\infty} \left(1 - \tilde{q}^{2n} e^{2ix}\right) \left(1 - \tilde{q}^{2n} e^{-2ix}\right) \quad (65)$$

where

$$G \equiv \prod_{n=1}^{\infty} (1 - \tilde{q}^{2n}) \quad (66)$$

($\tilde{q} < 1$) or

$$\Theta_1(x; \tilde{q}) = \frac{G\, \tilde{q}^{\frac{1}{4}}}{2 \sin x} \left(1 - e^{2ix}\right)^{\infty}_{\tilde{q}^2} \left(1 - e^{-2ix}\right)^{\infty}_{\tilde{q}^2}. \quad (67)$$

This theta function is composed from two q-exponentials

$$\Theta_1(x; \tilde{q}) = 2\, G\, \tilde{q}^{\frac{1}{4}}\, \sin x\, E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2 e^{2ix}}{\tilde{q}^2 - 1}\right) E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2 e^{-2ix}}{\tilde{q}^2 - 1}\right). \quad (68)$$

Then complex potential becomes

$$F(z) = \sum_{k=1}^{N} i\kappa_k \ln \left[z \frac{\left(1 - \frac{1}{z\bar{z}_k}\right)}{\left(1 - \frac{z}{z_k}\right)} \frac{\left(1 - \frac{z}{z_k}\right)^{\infty}_{\tilde{q}^2} \left(1 - \frac{z_k}{z}\right)^{\infty}_{\tilde{q}^2}}{(1 - z\bar{z}_k)^{\infty}_{\tilde{q}^2} \left(1 - \frac{1}{z\bar{z}_k}\right)^{\infty}_{\tilde{q}^2}} \right]. \quad (69)$$

If we denote

$$\frac{z}{z_k} = e^{2iu_k}, \quad z\bar{z}_k = e^{2iv_k}, \quad z_k\bar{z}_k = \frac{e^{2iv_k}}{e^{2iu_k}} \quad (70)$$

where $k = 1, \cdots, N$, we obtain,

$$F(z) = \sum_{k=1}^{N} i\kappa_k \ln \left[z_k\, e^{2iu_k} \frac{(1 - e^{-2iv_k})}{(1 - e^{2iu_k})} \frac{\left(1 - e^{2iu_k}\right)^{\infty}_{\tilde{q}^2} \left(1 - e^{-2iu_k}\right)^{\infty}_{\tilde{q}^2}}{\left(1 - e^{2iv_k}\right)^{\infty}_{\tilde{q}^2} \left(1 - e^{-2iv_k}\right)^{\infty}_{\tilde{q}^2}} \right]. \quad (71)$$

By using (67) we have

$$F(z) = \sum_{k=1}^{N} i\kappa_k \left[\ln \left[\frac{\Theta_1(u_k, \tilde{q})}{\Theta_1(v_k, \tilde{q})} \right] + \ln \left[-\left(\frac{z_k}{\bar{z}_k}\right)^{1/2} \right] \right]. \quad (72)$$

In terms of coordinates $\tau \equiv -\ln z$, $\tau_k \equiv -\ln z_k$ and $\bar{\tau}_k \equiv -\ln \bar{z}_k$, conformally mapping the annulus in the z plane to a rectangle in the τ plane, finally we find

$$F(z) = \sum_{k=1}^{N} i\kappa_k \ln \left[\frac{\Theta_1(i\frac{\tau - \tau_k}{2}, \tilde{q})}{\Theta_1(i\frac{\tau + \bar{\tau}_k}{2}, \tilde{q})} \right] + F_0 \quad (73)$$

where F_0 is a real constant and branch for logarithm is chosen such that $\ln(-1) = i\pi$,

$$F_0 = -\frac{i}{2}\sum_{k=1}^{N}\kappa_k(\tau_k - \bar{\tau}_k) - \pi\sum_{k=1}^{N}\kappa_k. \tag{74}$$

For the stream function, we have

$$\Psi = \frac{F - \bar{F}}{2i} = \sum_{k=1}^{N}\kappa_k \ln\left|\frac{\Theta_1(\frac{i}{2}(\tau - \tau_k), \tilde{q})}{\Theta_1(\frac{i}{2}(\tau + \bar{\tau}_k), \tilde{q})}\right|. \tag{75}$$

This coincides with the result of Johnson & Mc Donald (2004).

6 Motion of a Point Vortex in Annular Domain

We use the above formulas to determine the motion of a single vortex in annular domain. Complex velocity at the vortex position is determined by,

$$\dot{z}_0 = \dot{x}_0 + i\dot{y}_0 = V_0(\bar{z})|_{z=z_0} \tag{76}$$

where in the complex velocity

$$\begin{aligned}\bar{V}_0(z) &= \frac{i\kappa}{z(q-1)}\left[Ln_q\left(1 - \frac{z}{z_0}\right) - Ln_q\left(1 - \frac{z_0}{z}\right) + Ln_q\left(1 - \frac{r_2^2}{z\bar{z}_0}\right) - Ln_q\left(1 - \frac{z\bar{z}_0}{r_1^2}\right)\right] \\ &= \sum_{n=\pm 1}^{\pm\infty}\frac{i\kappa}{z - z_0 q^n} - \sum_{n=\pm 1}^{\pm\infty}\frac{i\kappa}{z - \frac{r_1^2}{\bar{z}_0}q^n}.\end{aligned}$$

contribution of the vortex itself is excluded. If we take into account that q-harmonic series [14]

$$H(q) \equiv \sum_{n=1}^{\infty}\frac{1}{[n]} = -Ln_q 0 \tag{77}$$

converges for $q > 1$ then at $z = z_0$ the first two terms cancel each other and we get the following equation of motion

$$\dot{z}_0 = \frac{i\kappa}{\bar{z}_0(q-1)}\left[Ln_q\left(1 - \frac{|z_0|^2}{r_1^2}\right) - Ln_q\left(1 - \frac{r_2^2}{|z_0|^2}\right)\right]. \tag{78}$$

The last equation gives,

$$\bar{z}_0\dot{z}_0 + z_0\dot{\bar{z}}_0 = \frac{d}{dt}|z_0|^2 = 0 \to |z_0| = \text{const}. \tag{79}$$

This implies that the distance of the vortex from the origin is a constant of motion. Then only the argument of $z_0 = |z_0|e^{i\varphi(t)}$, is a time dependent function,

$$\varphi(t) = \omega t + \varphi_0 \tag{80}$$

where constant frequency ω dependens on modulus $|z_0|$,

$$\omega = \frac{\kappa}{|z_0|^2(q-1)}\left[Ln_q\left(1 - \frac{|z_0|^2}{r_1^2}\right) - Ln_q\left(1 - \frac{r_2^2}{|z_0|^2}\right)\right]. \tag{81}$$

So we find that the vortex uniformly rotates around the origin,

$$z_0(t) = |z_0|e^{i\omega t + i\varphi_0} = z_0(0)e^{i\omega t}. \tag{82}$$

with frequency depending on the vortex strength, the initial position and geometry of the annular domain. This reflects the fact that the motion of vortex results from interaction with an infinite set of its images in the cylinders. The frequency (81) vanishes when

$$Ln_q\left(1 - \frac{|z_0|^2}{r_1^2}\right) = Ln_q\left(1 - \frac{r_2^2}{|z_0|^2}\right) \tag{83}$$

or $|z_0|^4 = r_1^2 r_2^2$. It means that at the geometric mean distance

$$|z_0| = \sqrt{r_1 r_2} \tag{84}$$

vortex is at the rest. This equation has simple geometrical meaning that any two intersection points of cylinders with a ray (say $r_1 e^{i\alpha}$ and $r_2 e^{i\alpha}$) are images of each other in a cylinder with radius $\sqrt{r_1 r_2}$. At this distance angular velocity changes the sign and when the vortex approaches the cylinders, ω grows in modulus; $\omega \to \kappa/(2r_1^2\epsilon)$ when $|z_0| \to r_1 e^{\epsilon} \approx r_1(1+\epsilon)$, $0 < \epsilon << 1$, and $\omega \to -\kappa/(2r_2^2\epsilon)$ when $|z_0| \to r_2 e^{-\epsilon} \approx r_2(1-\epsilon)$, $0 < \epsilon << 1$.

Here we like to indicate an intriguing relation of our solution with number theory. The frequency (81) is combination of two opposite sign frequencies, $\omega = \omega_1 + \omega_2$, made from the q-logarithm functions

$$\omega_1 = \frac{\kappa}{|z_0|^2(q-1)} Ln_q\left(1 - \frac{|z_0|^2}{r_1^2}\right) = \frac{-\kappa}{|z_0|^2(q-1)} \sum_{n=1}^{\infty} \frac{\left(\frac{|z_0|^2}{r_1^2}\right)^n}{[n]} \tag{85}$$

$$\omega_2 = -\frac{\kappa}{|z_0|^2(q-1)} Ln_q\left(1 - \frac{r_2^2}{|z_0|^2}\right) = \frac{\kappa}{|z_0|^2(q-1)} \sum_{n=1}^{\infty} \frac{\left(\frac{|r_2^2}{|z_0|^2}\right)^n}{[n]} \tag{86}$$

The last representation shows that every frequency is infinite superposition of frequencies coming from every vortex image. Moreover for $|z_0| = r_1$ the frequency $\omega_1 = H(q)$, while for $|z_0| = r_2$ the frequency $\omega_2 = -H(q)$ (for simplicity we took coefficients equal one), where $H(q)$ is the q-harmonic series. Contribution of N images in the frequency at these limiting cases are given by q-harmonic numbers [21]

$$\omega^{(N)} = H_N(q) = \sum_{n=1}^{N} \frac{1}{[n]_q} \tag{87}$$

The frequencies (85) and (86) are compensating each other at the geometric mean distance (84). In the annular region $r_1 < |z_0| < \sqrt{r_1 r_2}$, $\omega_1 > |\omega_2|$ and resulting $\omega > 0$, while in the region $\sqrt{r_1 r_2} < |z_0| < r_2$, $\omega_1 < |\omega_2|$ and resulting $\omega < 0$. If we consider geometry with parameter $q \geq 2$ and the unit strength vortex at the distance such that all arguments are non-zero rational, then problem of rationality of the frequency of motion is related with problem of rationality of q-logarithms. Starting from early result of Erdos it was proved that the last one is irrational [12]. We can expect influence of this irrationality on the character of multiple vortex dynamics.

7 The One Vortex Problem in $q \to \infty$ Limiting Cases

Usually in applications of q-calculus the limit $q \to 1$ corresponds to reduction of the q-elementary functions to the standard elementary functions. However in our problem this limit corresponds to $r_1 = r_2$ and the region reduces to the circle. More interesting is the limit when $q \to \infty$. To study this limit we need corresponding limits of q-elementary functions. When the q-logarithm is expanded we get the following

$$Ln_q(1+z) = (q-1)\sum_{n=1}^{\infty} \frac{z}{q^n + z} = (1 - \frac{1}{q}) \sum_{n=1}^{\infty} \frac{z}{q^{n-1}}(1 - \frac{z}{q^n} + ...). \tag{88}$$

This implies that

$$\lim_{q\to\infty} Ln_q(1+z) = z. \tag{89}$$

Using the $q-$ derivative of $Ln_q(1-z)$ function (29)

$$\frac{Ln_q(1-qz) - Ln_q(1-z)}{(q-1)z} = D_q Ln_q(1-z) = -\frac{1}{1-z} \tag{90}$$

and (89) we get another limit

$$\lim_{q\to\infty} \frac{Ln_q(1-qz)}{q-1} = \frac{z}{z-1}. \tag{91}$$

For the large $q >> 1$ we have

$$[n] = \frac{q^n - 1}{q-1} \approx q^{n-1} \tag{92}$$

$$[n]! = [1] \cdot [2] \cdot [3]...[n] \approx q \cdot q^2 \cdot q^3...q^{n-1} = q^{n(n-1)/2} \tag{93}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \approx \sum_{n=0}^{\infty} \frac{z^n}{q^{n(n-1)/2}} = 1 + z + \frac{z^2}{q} + ..., \tag{94}$$

so that

$$\lim_{q\to\infty} E_q(z) = 1 + z. \tag{95}$$

Applying the q-derivative

$$\frac{E_q(qz) - E_q(z)}{(q-1)z} = D_q E_q(z) = E_q(z), \tag{96}$$

we have identities

$$E_q\left(\frac{qz}{q-1}\right) = (1+z)E_q\left(\frac{z}{q-1}\right) \tag{97}$$

$$E_q\left(\frac{z}{q}\right) = \frac{E_q(z)}{1 + \frac{q-1}{q}z}. \tag{98}$$

In the limit $q \to \infty$ they imply that

$$\lim_{q\to\infty} E_q\left(\frac{z}{q}\right) = 1 \tag{99}$$

$$\lim_{q\to\infty} E_q\left(\frac{qz}{q-1}\right) = 1 + z. \tag{100}$$

Since $q = r_2^2/r_1^2$, for the limit $q \to \infty$ we have two different geometrical cases:

1) when $r_1 = constant$ and $r_2^2 = qr_1^2 \to \infty$, the outer cylinder grows unlimited so that we have just the one cylinder problem. Complex velocity (39) by replacement $r_2^2 = qr_1^2$, taking the limit $q \to \infty$ and using (89),(91) gives finite result

$$\bar{V}(z) = \frac{i\kappa}{z - z_0} - \frac{i\kappa}{z}\frac{r_1^2/\bar{z}_0}{z - r_1^2/\bar{z}_0} \tag{101}$$

which is exactly the result of the circle theorem (2). For angular velocity we have using (81) and the limits (89),(91),

$$\omega = \kappa \frac{r_1^2}{|z_0|^2(|z_0|^2 - r_1^2)}. \tag{102}$$

For the complex potential we have

$$F(z) = i\kappa \ln\left[(z - z_0)\frac{E_q\left(\frac{z}{(1-q)z_0}\right)E_q\left(\frac{z_0}{(1-q)z}\right)}{E_q\left(\frac{z\bar{z}_0}{(1-q)r_1^2}\right)E_q\left(\frac{q\, r_1^2}{(1-q)z\bar{z}_0}\right)}\right]. \tag{103}$$

Applying the limits (99),(100) we obtain two images as required by the circle theorem

$$F(z) = i\kappa\left[\ln(z - z_0) - \ln\left(z - \frac{r_1^2}{\bar{z}_0}\right) + \ln z\right]. \tag{104}$$

2) when $r_2 = constant$ and $r_1^2 = r_2^2/q \to 0$, the inner cylinder decreases until the thin string and then disappears. In this limiting case in complex velocity (39) we replace $r_1^2 = r_2^2/q$, and use formulas (89),(91) so that we get expression (3). For angular velocity of one vortex from (81) by limits (89),(91) we have

$$\omega = -\kappa \frac{1}{r_2^2 - |z_0|^2}. \tag{105}$$

Applying above limits (99),(100) to the complex potential

$$F(z) = i\kappa \ln\left[(z - z_0)\frac{E_q\left(\frac{z}{(1-q)z_0}\right)E_q\left(\frac{z_0}{(1-q)z}\right)}{E_q\left(\frac{q\, z\bar{z}_0}{(1-q)r_2^2}\right)E_q\left(\frac{r_2^2}{(1-q)z\bar{z}_0}\right)}\right]. \tag{106}$$

we obtain just one image as expected from (3)

$$F(z) = i\kappa\left[\ln(z - z_0) - \ln\left(z - \frac{r_2^2}{\bar{z}_0}\right) + \ln\left(-\frac{r_2^2}{\bar{z}_0}\right)\right]. \tag{107}$$

8 Conclusions

We have shown that the infinite set of images for a vortex in annular domain between two coaxial cylinders is given completely in terms of q-logarithmic function for complex velocity and in terms of the Jackson q-exponential function for the complex potential. Recent results on Pade approximation for the q-elementary functions [11] could be efficient approximation in the vortex image problem, restricting number of images. For example in paper [13] in order to compute the q-logarithm, with reference on P. Sebah, the next formula proposed

$$ln_q(1+z) = z\sum_{n=1}^{N}\frac{1}{q^n+z} + r_N(z) \tag{108}$$

where

$$r_N(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^n}{q^{Nn(q^n-1)}} \tag{109}$$

with N being any positive integer. Then application of this formula to our problem gives N-vortex images approximation which could be useful in applications.

By conformal mapping of the annular region to the rectangular one and composing q-exponents, the stream function has been represented in terms of the Jacobi elliptic function and it coincides exactly with result of Johnson, Mc Donald. Since annular region can be conformally mapped to exterior of two cylinders in the plane, our solution provides also solution of the last problem in terms of q-elementary functions. Results of calculations are in preparation [10].

Finally we like to note that elementary relation between the hydrodynamical problem and q-calculus considered in this paper could be applied to several physical situations with the same geometry, as the electrostatic problem or the problem of anyons [8]. The image picture allow us to construct Green function in the domain in terms of q-elementary functions and apply it to other problems like the vortex dynamics.

9 Acknowledgements

This work was partially supported by TUBITAK under the Grant No. 106T447 and by Izmir Institute of Technology.

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