

Parabolic equations with the second order Cauchy conditions on the boundary

Nikolai Dokuchaev

Department of Mathematics, Trent University, Ontario, Canada

September 18, 2021

Abstract

The paper studies some ill-posed boundary value problems on semi-plane for parabolic equations with homogenous Cauchy condition at initial time and with the second order Cauchy condition on the boundary of the semi-plane. A class of inputs that allows some regularity is suggested and described explicitly in frequency domain. This class is everywhere dense in the space of square integrable functions.

Key words: ill-posed problems, parabolic equations, second order Cauchy condition, regularity, solution in frequency domain, Hardy spaces, smoothing kernel.

AMS 2000 classification : 35K20, 35Q99, 32A35, 47A52

Parabolic equations such as heat equations have fundamental significance for natural sciences, and various boundary value problems for them were widely studied including well-posed problems as well as the so-called ill-posed problems that are often significant for applications. The present paper introduces and investigates a special boundary value problem on semi-plane for parabolic equations with homogenous Cauchy condition at initial time and with second order Cauchy condition on the boundary of the semi-plane. The problem is ill-posed. A set of solvability, or a class of inputs that allows some regularity in a form of prior energy type estimates is suggested and described explicitly in frequency domain. This class is everywhere dense in the class of L_2 -integrable functions. This result looks counterintuitive, since these boundary conditions are unusual; solvability of this boundary value problem for a wider class of inputs is inconsistent with basic theory.

1 The problem setting

Let us consider the following boundary value problem

$$\begin{aligned}
 a \frac{\partial u}{\partial t}(x, t) &= \frac{\partial^2 u}{\partial x^2}(x, t) + b \frac{\partial u}{\partial x}(x, t) + cu(x, t) + f(x, t), \\
 u(x, 0) &\equiv 0, \\
 u(0, t) &\equiv g_0(t), \quad \frac{\partial u}{\partial x}(0, t) \equiv g_1(t).
 \end{aligned} \tag{1}$$

Here $x > 0$, $t > 0$, and $a > 0, b, c \in \mathbf{R}$ are constants, $g_k \in L_2(0, +\infty)$, $k = 1, 2$, and f is a measurable function such that $\int_0^y dx \int_0^\infty |f(x, t)|^2 dt < +\infty$ for all $y > 0$.

This problem is ill-posed (see Tikhonov and Arsenin (1977)).

Let $\mu \triangleq b^2/4 - c$. We assume that $\mu > 0$. Note that this assumption does not reduce generality for the cases when we are interested in solution on a finite time interval, since we can rewrite the parabolic equation as the one with c replaced by $c - M$ for any $M > 0$ and $g_k(t)$ replaced by $e^{-Mt}g_k(t)$; the solution u_M of the new equation related to the solution u of the old one as $u_M(x, t) = e^{-Mt}u(x, t)$.

Definitions and special functions

Let $\mathbf{R}^+ \triangleq [0, +\infty)$, $\mathbf{C}^+ \triangleq \{z \in \mathbf{C} : \operatorname{Re} z > 0\}$. For $v \in L_2(\mathbf{R})$, we denote by $\mathcal{F}v$ and $\mathcal{L}v$ the Fourier and the Laplace transforms respectively

$$V(i\omega) = (\mathcal{F}v)(i\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\omega t} v(t) dt, \quad \omega \in \mathbf{R}, \tag{2}$$

$$V(p) = (\mathcal{L}v)(p) \triangleq \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-pt} v(t) dt, \quad p \in \mathbf{C}^+. \tag{3}$$

Let H^r be the Hardy space of holomorphic on \mathbf{C}^+ functions $h(p)$ with finite norm $\|h\|_{H^r} = \sup_{k>0} \|h(k + i\omega)\|_{L_r(\mathbf{R})}$, $r \in [1, +\infty]$ (see, e.g., Duren (1970)).

For $y > 0$, let $\mathcal{W}(y)$ be the Banach space of the functions $u : (0, y) \times \mathbf{R}^+ \rightarrow \mathbf{R}$ with the finite norm

$$\begin{aligned}
 \|u\|_{\mathcal{W}(y)} \triangleq \sup_{x \in (0, y)} & \left(\|u(x, \cdot)\|_{L_2(\mathbf{R}^+)} + \left\| \frac{\partial u}{\partial x}(x, \cdot) \right\|_{L_2(\mathbf{R}^+)} + \left\| \frac{\partial^2 u}{\partial x^2}(x, \cdot) \right\|_{L_2(\mathbf{R}^+)} \right. \\
 & \left. + \left\| \frac{\partial u}{\partial t}(x, \cdot) \right\|_{L_2(\mathbf{R}^+)} \right).
 \end{aligned}$$

The class $\mathcal{W}(y)$ is such that all the equations presented in problem (1) are well defined for any $u \in \mathcal{W}(y)$ and in the domain $(0, y) \times \mathbf{R}^+$. For instance, If $v \in \mathcal{W}(y)$, then, for any $t_* > 0$, we have that $v|_{[0, y] \times [0, t_*]} \in C([0, t_*], L_2(0, y))$ as a function of $t \in [0, t_*]$. Hence the initial condition at time $t = 0$ is well defined as an equality in $L_2([0, y])$. Further, we have that $v|_{[0, y] \times \mathbf{R}^+} \in C([0, y], L_2(\mathbf{R}^+))$ and $\frac{\partial v}{\partial x}|_{[0, y] \times \mathbf{R}^+} \in C([0, y], L_2(\mathbf{R}^+))$ as functions of $x \in [0, y]$. Hence the functions $v(0, t)$, $\frac{dv}{dx}(x, t)|_{x=0}$ are well defined as elements of $L_2(\mathbf{R}^+)$, and the boundary value conditions at $x = 0$ are well defined as equalities in $L_2(\mathbf{R}^+)$.

Special smoothing kernel

Let us introduce the set of the following special function:

$$K(p) = K_{\alpha, \beta, q}(p) \triangleq e^{-\alpha(p+\beta)^q}, \quad p \in \mathbf{C}^+. \quad (4)$$

Here $\alpha > 0$, $\beta > 0$ are reals, and $q \in (\frac{1}{2}, 1)$ is a rational number. We mean the branch of $(p + \beta)^q$ such that its argument is $q \text{Arg}(p + \beta)$, where $\text{Arg} z \in (-\pi, \pi]$ denotes the principal value of the argument of $z \in \mathbf{C}$.

The functions $K_{\alpha, \beta, q}(p)$ are holomorphic in \mathbf{C}^+ , and

$$\ln |K(p)| = -\text{Re}(\alpha(p + \beta)^q) = -\alpha|p + \beta|^q \cos[q \text{Arg}(p + \beta)].$$

In addition, there exists $M = M(\beta, q) > 0$ such that $\cos[q \text{Arg}(p + \beta)] > M$ for all $p \in \mathbf{C}^+$. It follows that

$$|K(p)| \leq e^{-\alpha M |p + \beta|^q} < 1, \quad p \in \mathbf{C}^+. \quad (5)$$

Hence $K \in H^r$ for all $r \in [1, +\infty]$.

Proposition 1 *Let $\beta > 0$ and a rational number $q \in (\frac{1}{2}, 1)$ be given. Let $v \in L_2(\mathbf{R}^+)$, $V = \mathcal{L}v \in H^2$. For $\alpha > 0$, set $V_\alpha \triangleq K_{\alpha, \beta, q}V$, $v_\alpha \triangleq \mathcal{F}^{-1}V_\alpha(i\omega)|_{\omega \in \mathbf{R}}$. Then $V_\alpha \in H^2$ and $v_\alpha \rightarrow v$ in $L_2(\mathbf{R}^+)$ as $\alpha \rightarrow 0$, $\alpha > 0$.*

Proof. Clearly, $V_\alpha(i\omega) \rightarrow V(i\omega)$ as $\alpha \rightarrow 0$ for a.e. $\omega \in \mathbf{R}$. By (4), $V_\alpha \in H^2$. In addition, $|K_{\alpha, \beta, q}(i\omega)| \leq 1$. Hence $|V_\alpha(i\omega) - V(i\omega)| \leq 2|V(i\omega)|$. We have that $\|V(i\omega)\|_{L_2(\mathbf{R})} = \|v\|_{L_2(\mathbf{R}^+)} < +\infty$. By Lebesgue Dominance Theorem, it follows that

$$\|V_\alpha(i\omega) - V(i\omega)\|_{L_2(\mathbf{R})} \rightarrow 0 \quad \text{as } \alpha \rightarrow 0.$$

Hence $v_\alpha \rightarrow v$ in $L_2(\mathbf{R}^+)$ as $\alpha \rightarrow 0$. Then the proof follows. \square

The inverse Fourier transform $k(t) = \mathcal{F}^{-1}K_{\alpha,\beta,q}(i\omega)|_{\omega \in \mathbf{R}}$ can be viewed as a smoothing kernel; $k(t) = 0$ for $t < 0$. It can be seen that k has derivatives of any order.

Denote by \mathcal{C} the set of functions $v : \mathbf{R}^+ \rightarrow \mathbf{R}$ such that there exist $\alpha > 0$, $\beta > 0$, and a rational number $q \in (\frac{1}{2}, 1)$, such that $\widehat{V} \in H^2$, where $\widehat{V}(p) = K_{\alpha,\beta,q}(p)^{-1}V(p)$, $V = \mathcal{L}v$.

The set \mathcal{C} includes outputs of the convolution integral operators with the kernels $k(t)$. By Proposition 1, it follows that the set \mathcal{C} is everywhere dense in $L_2(\mathbf{R}^+)$.

2 The main result

Set $F(x, \cdot) \triangleq \mathcal{L}f(x, \cdot)$, where $x > 0$ is given, and $G_k \triangleq \mathcal{L}g_k$, $k = 0, 1$.

Theorem 1 *Let the functions f and g_k are such that there exists $y > 0$, $\alpha > 0$, $\beta > 0$, a rational number $q \in (\frac{1}{2}, 1)$, such that $\widehat{G}_k \in H^2$, $\widehat{F}(x, \cdot) \in H^2$ for a.e. $x > 0$ and $\int_0^y \|\widehat{F}(s, \cdot)\|_{H^2} ds < +\infty$, where*

$$\widehat{F}(x, p) \triangleq \frac{F(x, p)}{K(p)}, \quad \widehat{G}_k(p) \triangleq \frac{G_k(p)}{K(p)}, \quad (6)$$

and where the function $K = K_{\alpha,\beta,q}$ is defined by (4) (in particular, this means that $g_k \in \mathcal{C}$ and $f(x, \cdot) \in \mathcal{C}$ for a.e. $x \in [0, y]$). Then there exists an unique solution $u(x, t)$ of problem (1) in the domain $(0, y) \times \mathbf{R}^+$ in the class $\mathcal{W}(y)$. Moreover, there exists a constant $C(y) = C(a, b, c, \alpha, \beta, q, y)$ such that

$$\|u\|_{\mathcal{W}(y)} \leq C(y) \left(\|\widehat{G}_1\|_{H^2} + \|\widehat{G}_2\|_{H^2} + \int_0^y \|\widehat{F}(s, \cdot)\|_{H^2} ds \right).$$

Remark 1 *Theorem 1 requires that functions f and g_k are smooth in t ; in particular, they belong to C^∞ in t . However, it is not required that $f(x, t)$ is smooth in x .*

Proof of Theorem 1. Instead of (1), consider the following problems for $p \in \mathbf{C}^+$:

$$\begin{aligned} apU(x, p) &= \frac{\partial^2 U}{\partial x^2}(x, p) + b\frac{\partial U}{\partial x}(x, p) + cU(x, p) + F(x, p), \quad x > 0, \\ U(0, p) &\equiv G_0(p), \quad \frac{\partial U}{\partial x}(0, p) \equiv G_1(p). \end{aligned} \quad (7)$$

Let $\lambda_k = \lambda_k(p)$ be the roots of the equation $\lambda^2 + b\lambda + (c - ap) = 0$. Clearly, $\lambda_{1,2} \triangleq -b/2 \pm \sqrt{ap + \mu}$. Recall that $\mu > 0$. It follows that the functions $(\lambda_1(p) - \lambda_2(p))^{-1}$ and $\lambda_k(p)(\lambda_1(p) - \lambda_2(p))^{-1}$, $k = 1, 2$, belong to H^∞ .

For $x \in (0, y]$, the solution of (7) is

$$U(x, p) = \frac{1}{\lambda_1 - \lambda_2} \left((G_1(p) - \lambda_2 G_0(p)) e^{\lambda_1 x} - (G_1(p) - \lambda_1 G_0(p)) e^{\lambda_2 x} - \int_0^x e^{\lambda_1(x-s)} F(s, p) ds + \int_0^x e^{\lambda_2(x-s)} F(s, p) ds \right). \quad (8)$$

This can be derived, for instance, using Laplace transform method applied to linear ordinary differential equation (7), and having in mind that

$$\frac{1}{\lambda^2 + b\lambda + c - ap} = \frac{1}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = \frac{1}{\lambda_1 - \lambda_2} \left(\frac{1}{\lambda - \lambda_1} - \frac{1}{\lambda - \lambda_2} \right),$$

$$\frac{\lambda}{\lambda^2 + b\lambda + c - ap} = \frac{\lambda}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = \frac{1}{\lambda_1 - \lambda_2} \left(\frac{\lambda_1}{\lambda - \lambda_1} - \frac{\lambda_2}{\lambda - \lambda_2} \right).$$

Let $x \in (0, y)$, $s \in [0, x]$. The functions $e^{(x-s)\lambda_k(p)}$, $k = 1, 2$, are holomorphic in \mathbf{C}^+ .

We have

$$\ln |e^{(x-s)\lambda_k(p)}| = \operatorname{Re}((x-s)\lambda_k(p)) = (x-s) \left(-\frac{b}{2} \pm |ap + \mu|^{1/2} \cos \frac{\operatorname{Arg}(ap + \mu)}{2} \right),$$

where $k = 1, 2$, $p \in \mathbf{C}^+$. It follows that

$$|K(p)e^{(x-s)\lambda_k(p)}| \leq e^{(x-s)[-b/2 + |ap + \mu|^{1/2}] - \alpha M |p + \beta|^q},$$

$k = 1, 2$, $p \in \mathbf{C}^+$. Similarly,

$$|K(p)e^{\lambda_k x}| \leq e^{x[-b/2 + |ap + \mu|^{1/2}] - \alpha M |p + \beta|^q}.$$

Since $q > 1/2$, it follows that $K(p)e^{\lambda_k x} \in H^r$, $K(p)e^{(x-s)\lambda_k(p)} \in H^r$, $pK(p)e^{\lambda_k x} \in H^r$, and $pK(p)\Psi_k(p) \in H^r$, for $r = 2$ and $r = +\infty$. Moreover, we have

$$\sup_{s \in [0, x]} \|p^m e^{\lambda_k(p)s} G_k(p)\|_{H^2} \leq C_1(x) \|\tilde{G}_k\|_{H^2},$$

$$\sup_{s \in [0, x]} \|p^m e^{\lambda_k(p)s} K(p)\|_{H^\infty} \leq C_2(x),$$

where $m = 0, 1$. Hence

$$\begin{aligned} & \sup_{x \in [0, y]} \left\| p^m \int_0^x e^{(x-s)\lambda_k} F(s, p) ds \right\|_{H^2} \leq \sup_{x \in [0, y]} \int_0^x \left\| e^{(x-s)\lambda_k} p^m F(s, p) \right\|_{H^2} ds \\ & \leq \sup_{x \in [0, y]} \int_0^x \|p^m e^{\lambda_k(x-s)} K(s)\|_{H^\infty} \|\tilde{F}(s, p)\|_{H^2} ds \leq C_2(y) \int_0^y \|\hat{F}(s, p)\|_{H^2} ds, \end{aligned}$$

where $m = 0, 1$. Here $C_1(x), C_2(x)$ are constants that depend on $a, b, c, \alpha, \beta, q, x$. It follows that $p^m e^{\lambda_k x} G_m(p) \in H^2$ and $p^m \int_0^x e^{(x-s)\lambda_k} F(p, s) ds \in H^2$ for any $x > 0$, $m = 0, 1, k = 1, 2$.

Recall that $\lambda_k = \lambda_k(p)$. Let

$$N \triangleq \left\| \frac{1}{\lambda_1 - \lambda_2} \right\|_{H^\infty} + \sum_{k=1,2} \left\| \frac{\lambda_k}{\lambda_1 - \lambda_2} \right\|_{H^\infty}.$$

It follows from the above estimates that

$$\|p^m U(x, p)\|_{H^2} \leq N \left(C_1(y) \sum_{k=1,2} \|\widehat{G}_k\|_{H^2} + C_2(y) \int_0^x \|\widehat{F}(s, p)\|_{H^2} ds \right), \quad m = 0, 1. \quad (9)$$

It follows that the corresponding inverse Fourier transforms $u(x, \cdot) = \mathcal{F}^{-1}U(x, i\omega)|_{\omega \in \mathbf{R}}, \frac{\partial u}{\partial t}(x, \cdot) = \mathcal{F}^{-1}(pU(x, i\omega))|_{\omega \in \mathbf{R}}$ are well defined and are vanishing for $t < 0$. In addition, we have that $\overline{U(x, i\omega)} = U(x, -i\omega)$ (for instance, $\overline{K(i\omega)} = K(-i\omega)$, $\overline{e^{(x-s)\lambda_k(i\omega)}} = e^{(x-s)\lambda_k(-i\omega)}$, etc). It follows that the inverse of Fourier transform $u(x, \cdot) = \mathcal{F}^{-1}U(x, \cdot)$ is real.

Further, we have that

$$\begin{aligned} \frac{\partial U}{\partial x}(x, p) &= \frac{1}{\lambda_1 - \lambda_2} \left((G_1(p) - \lambda_2 G_0(p)) \lambda_1 e^{\lambda_1 x} - (G_1(p) - \lambda_1 G_0(p)) \lambda_2 e^{\lambda_2 x} \right. \\ &\quad \left. - \lambda_1 \int_0^x e^{\lambda_1(x-s)} F(s, p) ds + \lambda_2 \int_0^x e^{\lambda_2(x-s)} F(s, p) ds \right). \end{aligned} \quad (10)$$

Since $\lambda_1(p)\lambda_2(p) = c - ap$, we obtain again that

$$\left\| \frac{\partial U}{\partial x}(x, p) \right\|_{H^2} \leq C_3(y) \left(\sum_{k=1,2} \|\widehat{G}_k\|_{H^2} + \int_0^x \|\widehat{F}(s, p)\|_{H^2} ds \right). \quad (11)$$

By (7), $\partial^2 U / \partial x^2$ can be expressed as a linear combination of $F, G_k, U, pU, \partial U / \partial x$. By (9)-(11),

$$\left\| \frac{\partial^2 U}{\partial x^2}(x, p) \right\|_{H^2} \leq C_4(y) \left(\left\| \frac{\partial U}{\partial x}(x, p) \right\|_{H^2} + \sum_{m=0,1} \|p^m U(x, p)\|_{H^2} + \|F(x, p)\|_{H^2} \right).$$

We have that $|K(p)| < 1$ on C^+ and $\|F(s, p)\|_{H^2} \leq \|\widehat{F}(s, p)\|_{H^2}$. It follows that

$$\left\| \frac{\partial^2 U}{\partial x^2}(x, p) \right\|_{H^2} \leq C_5(y) \left(\sum_{k=1,2} \|\widehat{G}_k\|_{H^2} + \int_0^x \|\widehat{F}(s, p)\|_{H^2} ds \right). \quad (12)$$

Here $C_k(y)$ are constants that depend on $a, b, c, \alpha, \beta, q, y$. By (9)-(12), estimate (6) holds.

Therefore, $u(x, \cdot) = \mathcal{F}^{-1}U(x, i\omega)|_{\omega \in \mathbf{R}}$ is the solution of (1) in $\mathcal{W}(y)$. The uniqueness is ensured by the linearity of the problem, by estimate (6), and by the fact that $\mathcal{L}u(x, \cdot)$, $\mathcal{L}(\partial^k u(x, \cdot)/\partial x^k)$, and $\mathcal{L}(\partial u(x, \cdot)/\partial t)$ are well defined on \mathbf{C}^+ for any $u \in \mathcal{W}(y)$. This completes the proof of Theorem 1. \square

Remark 2 *It can be seen from the proof that it is crucial that $u(x, 0) \equiv 0$. Non-zero initial conditions can not be included.*

References

- Duren, P. *Theory of H^p -Spaces*. 1970. Academic Press, New York.
- Tikhonov, A. N. and Arsenin, V. Y. *Solutions of Ill-posed Problems*. 1977. W. H. Winston, Washington, D. C.