

# A covariant entropy conjecture on cosmological dynamical horizon

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**ABSTRACT:** We here propose a covariant entropy conjecture on cosmological dynamical horizon. After the formulation of our conjecture, we test its validity in adiabatically expanding universes with open, flat and closed spatial geometry, where our conjecture can also be viewed as a cosmological version of the generalized second law of thermodynamics in some sense.

**KEYWORDS:** Space-Time Symmetries, Black Holes, Classical Theories of Gravity, Spacetime Singularities.

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## Contents

1. Introduction and motivation	1
2. Covariant entropy conjecture related to cosmological dynamical horizon	2
3. Covariant entropy conjecture tested by adiabatically expanding universes	3
4. Conclusions and discussions	7

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## 1. Introduction and motivation

Recently Bousso has conjectured that in a spacetime satisfying Einstein's equation with the dominant energy condition holding for matter, the entropy flux  $S$  through any null hypersurface generated by geodesics with non-positive expansion emanating orthogonally from some two-dimensional spacelike surface of area  $A$  must satisfy[1, 2, 3, 4]

$$S \leq \frac{A}{4}. \tag{1.1}$$

Not only does this conjecture improve an earlier suggestion of Fischler and Susskind on cosmic holography[5], but also reduces to the spacelike entropy bound whenever the latter is expected to hold. Furthermore, this conjecture can be interpreted as a statement of the so called holographic principle, which is believed to be manifest in an underlying quantum theory of gravity[6, 7].

Later a generalized covariant entropy bound was suggested by Flanagan, Marolf, and Wald[8]. Namely, if one allows the geodesics generating the null hypersurface to terminate at another two-dimensional spacelike surface of area  $A'$  before coming to a caustic, boundary or singularity of spacetime, one can replace the above conjecture Eq.(1.1) by

$$S \leq \frac{A - A'}{4}. \tag{1.2}$$

Obviously, this more general bound implies both the original Bousso entropy bound and generalized second law of thermodynamics for any process of black hole formation.

It is noteworthy that in highly dynamical spacetimes, since the dynamical horizon, foliated by the apparent horizons, generally divides the normal region from the trapped or antitrapped one, it plays a subtle role in constructing the null hypersurface mentioned above in such spacetimes as growing black holes and expanding universes[4]. In addition, as shown by Ashtekar and Krishnan, there are series of intriguing properties related to the dynamical horizon itself, such as the area balance law[9, 10, 11]. All of these motivate us to propose a covariant entropy bound conjecture related to the dynamical horizon in a direct way. In particular, as a first step, we here shall suggest such a entropy bound on the dynamical horizon in the cosmological context, which will be formulated in the next section. In Section 3, its validity is demonstrated in adiabatically expanding universes. Conclusions and discussions are presented in the end.

## 2. Covariant entropy conjecture related to cosmological dynamical horizon

Start from the FRW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad (2.1)$$

which describes homogeneous and isotropic universes, including, to a good degree of approximation, the portion we have seen of our own universe. In terms of the conformal time  $\eta$  and the comoving coordinate, i.e.,

$$d\eta = \frac{dt}{a(t)}, d\chi = \frac{dr}{\sqrt{1-kr^2}}, \quad (2.2)$$

the FRW metric takes the form

$$ds^2 = a^2(\eta)[-d\eta^2 + d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (2.3)$$

Here  $k = -1, 0, 1$  and  $f(\chi) = \sinh \chi, \chi, \sin \chi$  correspond to open, flat, and closed universes, respectively.

Next, to identify cosmological dynamical horizon, let us firstly compute the initial expansion of the future directed null congruences orthogonal to an arbitrary sphere characterized by some value of  $(\eta, \chi)$ . Accordingly one finds[4]

$$\theta_{\pm} = \frac{\dot{a}}{a} \pm \frac{f'}{f}, \quad (2.4)$$

where the dot(prime) denotes differentiation with respect to  $\eta(\chi)$ , and the sign  $+(-)$  represents the null congruence is directed at larger(smaller) values of  $\chi$ . Note that the first term in Eq.(2.4) is positive when the universe expands and negative if it contracts. In addition, the second term is given by  $\coth \chi, \frac{1}{\chi}, \cot \chi$  for open, flat, and closed universes, respectively. Especially, this term diverges when  $\chi \rightarrow 0$ , and it also diverges when  $\chi \rightarrow \pi$  for a closed universe.

Now cosmological dynamical horizon is defined geometrically as a three-dimensional hypersurface foliated by those spheres at which at least there exists one orthogonal null congruence with vanishing expansion. Thus cosmological dynamical horizon  $\chi(\eta)$  can be identified by solving the equation

$$\frac{\dot{a}}{a} = \pm \frac{f'}{f}. \quad (2.5)$$

There is one solution for open and flat universes while for a closed universe, there are generally two solutions, which are symmetrically related to each other by  $\chi_2(\eta) = \pi - \chi_1(\eta)$ . Then a cosmological version of our conjecture can be proposed as follows: *Let  $A(\eta)$  be the area of cosmological dynamical horizon at the conformal time  $\eta$ , then the entropy flux  $S$  through cosmological dynamical horizon between the conformal times  $\eta$  and  $\eta'$  must satisfy  $S \leq \frac{|A(\eta) - A(\eta')|}{4}$  if the dominant energy condition holds for matter.*

Note that the description of our above conjecture is well defined and obviously covariant. In the subsequent section, we shall test its validity in adiabatically expanding universes.

### 3. Covariant entropy conjecture tested by adiabatically expanding universes

The matter content of FRW universes is most generally described by a perfect fluid, with the energy momentum tensor

$$T_{ab} = a^2(\eta) \{ \rho(\eta) (d\eta)_a (d\eta)_b + p(\eta) [ (d\chi)_a (d\chi)_b + f^2(\chi) ( (d\theta)_a (d\theta)_b + \sin^2 \theta (d\phi)_a (d\phi)_b ) ] \}. \quad (3.1)$$

Later, we shall assume that the pressure  $p$  and energy density  $\rho$  is related by a fixed equation of state

$$p = wp, \quad (3.2)$$

where the constant  $w$  is controlled within the range  $-1 \leq w \leq 1$  by the dominant energy condition. Furthermore, we shall restrict ourselves in the case that FRW universes are in the expanding phase. Thus, Einstein equation can be solved. As a result, the

cosmological scale factor is given by

$$a = f^q\left(\frac{\eta}{q}\right), \quad (3.3)$$

where

$$q = \frac{2}{1+3w} \quad (3.4)$$

is confined within the range  $q \geq \frac{1}{2}$  or  $q \leq -1$  due to the dominant energy condition requirement. The corresponding cosmological dynamical horizon is located at

$$\chi = \frac{\eta}{q} \quad (3.5)$$

in all cases. An additional mirror horizon lies at  $\chi = \pi - \frac{\eta}{q}$  in the closed case<sup>1</sup>. Note that  $\eta$  is positive for  $q \geq \frac{1}{2}$  and negative in the case of  $q \leq -1$ . Moreover,  $\frac{\eta}{q} < \frac{\pi}{2}$  is required in the closed case.

To proceed, we further assume that the evolution of matter in FRW universes is adiabatical. Therefore the entropy current of matter can be written as

$$s^a = \frac{s}{a^3} \left(\frac{\partial}{\partial t}\right)^a, \quad (3.6)$$

which implies the conservation of the entropy current, i.e.,  $\nabla_a s^a = 0$ . Note that  $s$  is actually the ordinary comoving entropy density, constant in space and time.

We shall now check whether our conjecture is satisfied. However, as shown in Figure 1, there is an obvious difference between  $q \geq \frac{1}{2}$  and  $q \leq -1$ . Thus let  $\eta' \geq \eta$ , then by the conservation of the entropy and Gauss theorem, our conjecture can be reformulated as

$$\frac{A(\eta')}{4} - S(\eta') \geq \frac{A(\eta)}{4} - S(\eta) \quad (3.7)$$

for  $q \geq \frac{1}{2}$ , and

$$\frac{A(\eta')}{4} + S(\eta') \geq \frac{A(\eta)}{4} + S(\eta) \quad (3.8)$$

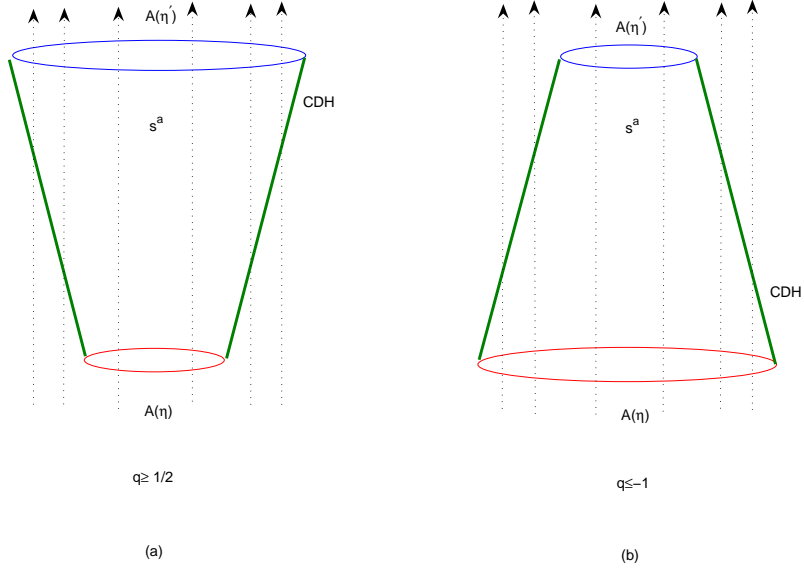
for  $q \leq -1$ <sup>2</sup>. Here we have invoked the fact that the area

$$A(\eta) = 4\pi f^{2(q+1)} \left(\frac{\eta}{q}\right) \quad (3.9)$$

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<sup>1</sup>Due to this space mirror symmetry, for a closed universe, we only need to focus on cosmological dynamical horizon near the north pole  $\chi = 0$  in the following discussions.

<sup>2</sup>If we endow cosmological dynamical horizon with a geometrical entropy of  $\frac{A}{4}$  and take into account the direction of the entropy current illustrated in Figure 1, it seems reasonable to assume that the total entropy of universe is the geometrical entropy of cosmological dynamical horizon plus that of matter in the antitrapped region for  $q \geq \frac{1}{2}$  while for  $q \leq -1$  the total entropy is assumed to be a sum of the geometrical entropy and that for matter in the normal region. Then it is obvious that our conjecture can be regarded as the generalized second law of thermodynamics for expanding universes.



**Figure 1:** (a) For  $q \geq \frac{1}{2}$ , the entropy current flows across cosmological dynamical horizon from the normal region to the antitrapped one. (b) For  $q \leq -1$ , the entropy current flows across cosmological dynamical horizon from the antitrapped region to the normal one.

is always an increasing function of  $\eta$ . In addition,  $S(\eta)$  denotes the entropy flux through the normal region at the conformal time  $\eta$ , given by

$$S(\eta) = 4\pi s \int_0^{\frac{\eta}{q}} d\chi f^2(\chi). \quad (3.10)$$

Furthermore one finds our conjecture is equivalent to require that

$$H(\eta) \equiv \frac{1}{4}A(\eta) - S(\eta) \quad (3.11)$$

be an increasing function of  $\eta$  for  $q \geq \frac{1}{2}$ . Substituting Eq.(3.9) and (3.10) into Eq.(3.11), It follows

$$(q+1)f^{2q-1}\left(\frac{\eta}{q}\right) - 2s \geq 0. \quad (3.12)$$

It is obvious that the LHS of inequality is an increasing function of  $\eta$  in all cases considered here. Whence we know that once the inequality holds at some initial moment,

it will continue to be valid for all later moments. Note that at Planck epoch,  $\frac{\eta}{q}$  is of order one in all the cases. So the first term in the LHS of the inequality is of the same order as  $q + 1$  which is larger than  $\frac{3}{2}$ . On the other hand, since the scale factor is of the Planck size as well, the comoving entropy density  $s$  becomes the entropy per Planck volume which can not exceed one quarter, as generally believed to be a generic feature for any underlying theory of quantum gravity. Therefore in case of  $q \geq \frac{1}{2}$  our conjecture holds for open, flat and closed expanding universes at Planck epoch and remains to be valid henceforth.

Now let us turn to the case of  $q \leq -1$ , where our conjecture means that the function

$$Z(\eta) \equiv \frac{1}{4}A(\eta) + S(\eta) \tag{3.13}$$

is required as an increasing function of  $\eta$ , which follows

$$(q + 1)f^{2q-1}\left(\frac{\eta}{q}\right) + 2s \leq 0. \tag{3.14}$$

Similar to the case of  $q \geq \frac{1}{2}$ , the inequality will hold for all the rest of the expansion epoch of the universe once it is valid at an early time, say, Planck time.

At such an early time, the conformal time has decreased to  $\frac{\eta}{q} \sim O(1)$ , thus the inequality holds if  $s$  is bounded by  $-\left(\frac{q+1}{2}\right)$  which is non-negative. In particular, if  $q = -1$ , which describes a de-Sitter universe, the inequality holds trivially due to  $s = 0$  for a cosmological constant.

Generally the bound for  $s$  is  $q$ -dependent if we expect that our conjecture holds at Planck time<sup>3</sup>. If  $s$  turns out to be too large for our conjecture to hold at Planck time, the validity of our conjecture will not be ruined, since it will be valid at some later time and continue to hold henceforth. The only difference is that the initial moment our conjecture becomes valid is postponed due to a larger  $s$ . In other words, any reasonable value of  $s$  will yield a reasonable starting time for our conjecture to hold, which is expected to be not too much later than Planck time.

In all, our conjecture holds universally for open, flat and closed universes which are adiabatically expanding with  $q \geq \frac{1}{2}$  and  $q \leq -1$ , back to a very early time near Planck epoch (the exact moment depends on  $q$ ). That is to say, once it is valid at some early time, it continues to be.

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<sup>3</sup>Actually this is also the case for  $q \geq \frac{1}{2}$ , although there is a reasonable universal bound  $\frac{3}{4}$  for  $s$  there. It should be borne in mind that the  $q$ -dependent bound for  $s$  does not surprise us because  $q$  is a parameter associated with the matter and it is definitely related to the entropy density of matter. In fact, in [4] the discussion is simplified by ignoring factors containing  $q$ , and a more careful consideration will also lead to a  $q$ -dependent bound for  $s$  in test of Bousso entropy bound.

## 4. Conclusions and discussions

We have proposed a covariant entropy conjecture on cosmological dynamical horizon. Its validity has also been demonstrated in the case of adiabatically expanding universes. In other words, if our conjecture holds at some time in early universes, where classical general relativity may be replaced by a quantum theory of gravity, then it will hold for all the later times. The validity for our conjecture at a time as early as near Planck time is also verified under the assumption that  $s$  is reasonably bounded. On the other hand, although we restrict our discussion in expanding universes, it is obvious that our conjecture also applies to collapsing universes due to the time reversal symmetry.

All of these further encourage us to extend our conjecture to more general contexts, especially to black hole dynamical horizons, whose test is more difficult and challenging than that in cosmological contexts, which will be reported elsewhere[12]. In addition, it is highly desirable to provide some reasonable local conditions sufficient for proof of our proposal and this work is also on progress[13].

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