The solution of the Minkowski problem for open surfaces in Riemannian space.

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Abstract

Author reduces the Minkowski problem to the problem of construction the G-deformations preserving the product of principal curvatures for every point of surface in Riemannian space. G-deformation transfers every normal vector of surface in parallel along the path of the translation for each point of surface. The continuous G-deformations preserving the product of principal curvatures of surface with boundary are considered in this article. The equations of deformations which are obtained in this paper reduce to the nonlinear boundary-value problem. The method of construction continuous G-deformations preserving the product of principal curvatures of surface with boundary and its qualitative analysis are presented in this article.

Introduction

The Minkowski problem (MP) is well known fundamental problem of differential geometry. There have been published a number of articles on this subject since 1903. But all authors have studied this problem in Euclidean or pseudo-Euclidean spaces. They were H. Minkowski, A.V. Pogorelov, A.D. Aleksandrov, W.J. Firey and many others. We know many generalizations of the MP in Euclidean and pseudo-Euclidean spaces.

V.T. Fomenko [21] studied the MP by the methods of deformation theory in Euclidean space.

The MP in Riemannian space differs substantially from the MP in Euclidean space. The MP in Riemannian space is much more complicate than the MP in Euclidean space.

Author of this article created for the first time the method of finding solutions of the MP in Riemannian space using deformation theory.

Author have studied AG-deformations in Euclidean spaces in [1-16]. It was very hard to obtain the equation system of AG-deformations in Riemannian space and much harder to solve it. V.T. Fomenko in [24] studied infinitesimal ARG-deformations in Riemannian space. The methods developed in [1-16] are very useful for finding the solution of the Minkowski problem.

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§1. Basic definitions. Statement of the main result.

Let R^3 be the three-dimensional Riemannian space with metric tensor $\tilde{a}_{\alpha\beta}$, F be the two-dimensional simply connected oriented surface in R^3 with the boundary ∂F.

Let $F \in C^{m,\nu}, \nu \in (0;1), m \geq 4$, $\partial F \in C^{m+1,\nu}$. Let F has all strictly positive principal curvatures k_1 and k_2. Let F be oriented so that mean curvature H is strictly positive. Denote $K = k_1 k_2$.

Let F be given by immersion of the domain $D \subset E^2$ into R^3 by the equation: $y^\sigma = f^\sigma(x), x \in D$, $f : D \to R^3$. Denote by $d\sigma(x) = \sqrt{g}dx^1 \wedge dx^2$ the area element of the surface F. We identify the points of immersion of surface F with the corresponding coordinate sets in R^3. Without loss of generality we assume that D is unit disk. Let x^1, x^2 be the Cartesian coordinates.

Symbol $_{,i}$ denotes covariant derivative in metric of surface F. Symbol ∂_i denotes partial derivative by variable x^i. We will assume $\dot{f} \equiv \frac{df}{dt}$. We define $\Delta(f) \equiv f(t) - f(0)$.

Indices denoted by Greek alphabet letters define tensor coordinates in Riemannian space R^3. We use the following rule: a formula is valid for all admissible values of indices if there are no instructions for which values of indices it is valid. We use the Einstein rule. We assume that integer m_1 satisfies the condition $0 \leq m_1 \leq m - 2$.

We consider continuous deformation of the surface F: $\{F_t\}$ defined by the equations

$$y_t^\sigma = y^\sigma + z^\sigma(t), z^\sigma(0) \equiv 0, t \in [0; t_0], t_0 > 0. \tag{1.1}$$

Definition 1 . *Deformation $\{F_t\}$ is called the continuous deformation preserving the product of principal curvatures (or $M-$deformation [21]) if the following condition holds: $\Delta(K) = 0$ and $z^\sigma(t)$ is continuous by t.*

The deformation $\{F_t\}$ generates the following set of paths in R^3

$$u^{\alpha_0}(\tau) = (y^{\alpha_0} + z^{\alpha_0}(\tau)), \tag{1.2}$$

where $z^{\alpha_0}(0) \equiv 0, \tau \in [0; t], t \in [0; t_0], t_0 > 0$.

Definition 2 . *The deformation $\{F_t\}$ is called the $G-$deformation if every normal vector of surface transfers in parallel along the path of the translation for each point of surface.*

Let, along the ∂F, be given vector field tangent to F. We denote it by the following formula:

$$v^\alpha = l^i y^\alpha_{,i}. \tag{1.3}$$

We consider the boundary-value condition:

$$\tilde{a}_{\alpha\beta} z^\alpha v^\beta = \tilde{\gamma}(s,t), s \in \partial D. \tag{1.4}$$

Let v^α and $\tilde{\gamma}$ be of class $C^{m-2,\nu}$.

We denote:

$$\tilde{\lambda}_k = \tilde{a}_{\alpha\beta} y^\alpha_{,k} v^\beta, k = 1, 2. \tag{1.5}$$

$$\lambda_k = \frac{\tilde{\lambda}_k}{(\tilde{\lambda}_1)^2 + (\tilde{\lambda}_2)^2}, k = 1, 2. \tag{1.6}$$

$$\lambda(s) = \lambda_1(s) + i\lambda_2(s), s \in \partial D. \tag{1.7}$$

Let n be the index of the given boundary-value condition

$$n = \frac{1}{2\pi}\Delta_{\partial D} \arg \lambda(s). \tag{1.8}$$

Theorem 1 . *Let $F \in C^{m,\nu}, \nu \in (0;1), m \geq 4$, $\partial F \in C^{m+1,\nu}$. Let $\tilde{a}_{\alpha\beta} \in C^{m,\nu}$, $\exists M_0 = const > 0$ such that $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$. Let $v^\beta, \tilde{\gamma} \in C^{m-2,\nu}(\partial D)$, $\tilde{\gamma}$ is continuously differentiable by t. Let, at the point $(x^1_{(0)}, x^2_{(0)})$ of the domain D, the following condition holds: $\forall t : z^\sigma(t) \equiv 0$.*

Then the following statements hold:

1) if $n > 0$ then there exist $t_0 > 0$ and $\varepsilon(t_0) > 0$ such that for any admissible $\tilde{\gamma}$ satisfying the condition: $|\dot{\tilde{\gamma}}|_{m-2,\nu} \leq \varepsilon$ for all $t \in [0, t_0)$ there exists $(2n-1)-$parametric $MG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

2) if $n < 0$ then there exist $t_0 > 0$ and $\varepsilon(t_0) > 0$ such that for any admissible $\tilde{\gamma}$ satisfying the condition: $|\dot{\tilde{\gamma}}|_{m-2,\nu} \leq \varepsilon(t_0)$ for all $t \in [0, t_0)$ there exists nor more than one $MG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

3) if $n = 0$ then there exist $t_0 > 0$ and $\varepsilon(t_0) > 0$ such that for any admissible $\tilde{\gamma}$ satisfying the condition: $|\dot{\tilde{\gamma}}|_{m-2,\nu} \leq \varepsilon$ for all $t \in [0, t_0)$ there exists one $MG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

We denote:
$\tilde{a}_{\alpha\beta}(t) \equiv \tilde{a}_{\alpha\beta}(y^\sigma + z^\sigma(t))$, $\tilde{a}_{\alpha\beta}(0) \equiv \tilde{a}_{\alpha\beta}$.
$\Gamma^\gamma_{\alpha\beta}(t) \equiv \Gamma^\gamma_{\alpha\beta}(y^\sigma + z^\sigma(t))$, $\Gamma^\gamma_{\alpha\beta}(0) \equiv \Gamma^\gamma_{\alpha\beta}$.
$b_{ij}(t) \equiv b_{ij}(y^\sigma + z^\sigma(t))$, $b_{ij}(0) \equiv b_{ij}$. $b(t) \equiv b(y^\sigma + z^\sigma(t))$, $b(0) \equiv b$.
$g_{ij}(t) \equiv g_{ij}(y^\sigma + z^\sigma(t))$, $g_{ij}(0) \equiv g_{ij}$. $g(t) \equiv g(y^\sigma + z^\sigma(t))$, $g(0) \equiv g$.
$a^j(t) \equiv a^j, c(t) \equiv c$, $z^\sigma(t) \equiv z^\sigma$.

§2. Deduction of $G-$deformation formulas for surfaces in Riemannian space.

The deformation $\{F_t\}$ of surface F is defined by (1.1).

We denote:

$$z^\sigma(t) = a^j(t)y^\sigma{}_{,j} + c(t)n^\sigma, \tag{2.1}$$

where $a^j(0) \equiv 0, c(0) \equiv 0$. Therefore the deformation of surface F is defined by functions a^j and c.

We denote:

$$\nabla^*_i z^\alpha = z^\alpha_{,i} + \Gamma^\alpha_{\gamma\sigma}(0) z^\sigma y^\gamma{}_{,i}\,. \tag{2.2}$$

Then we have (see [27]):

$$\nabla_j^* z^\alpha = (a^i b_{ij} + c_{,j})n^\alpha + (a^i{}_{,j} - cb_{jm}g^{mi})y^\alpha{}_{,i} \tag{2.3}$$

The condition of $G-$deformation is equivalent to the following equality:

$$\tilde{a}_{\alpha_0\beta} A_i^{*\alpha_0}(t) n^\beta = 0, \tag{2.4}$$

where $A_i^{*\alpha_0}(t)$ is the result of parallel translation of tensor $(y_{,i}^{\alpha_0} + z_{,i}^{\alpha_0}(t))$ from the point $(y^{\alpha_0} + z^{\alpha_0}(t))$ to the point (y^{α_0}) along the path of the translation for each point of surface by deformation, i.e. along the following curve:

$$u^{\alpha_0}(\tau) = (y^{\alpha_0} + z^{\alpha_0}(\tau)), \tag{2.5}$$

where $z^{\alpha_0}(0) \equiv 0, \tau \in [0;t]$.

Then parallel translation of tensor $y_{,i}^{\alpha_0} + z_{,i}^{\alpha_0}(t)$ reduces to the following Cauchy problem: find $A_i^{\alpha_0}(t,\tau)$ satisfying the equations:

$$\frac{dA_i^{\alpha_0}(t,\tau)}{d\tau} + \Gamma_{\beta\gamma}^{\alpha_0}(\tau)\dot{z}^\beta(\tau) A_i^\gamma(t,\tau) = 0, \alpha_0 = 1,2,3, \tau \in [0;t], \tag{2.6}$$

with initial boundary value conditions:

$$A_i^{\alpha_0}(t,t) = y_{,i}^{\alpha_0} + z_{,i}^{\alpha_0}(t). \tag{2.7}$$

We denote:

$$A_{(1)\gamma_0}^{\alpha_0}(t,\tau) = \int_\tau^t \Gamma_{\beta_0\gamma_0}^{\alpha_0}(\tau_0)\dot{z}^{\beta_0}(\tau_0) d\tau_0. \tag{2.8}$$

For $k \geq 2$ we denote:

$$A_{(k)\gamma_{k-1}}^{\alpha_0}(t,\tau) = \int_\tau^t \int_{\tau_0}^t \ldots \int_{\tau_{k-2}}^t (\prod_{j=0}^{k-1} \Gamma_{\beta_j\gamma_j}^{\alpha_j}(\tau_j)\dot{z}^{\beta_j}(\tau_j)) d\tau_0 d\tau_1 \ldots d\tau_{k-1}. \tag{2.9}$$

and for $j \geq 1, k \geq 2$ we assume

$$\alpha_j \equiv \gamma_{j-1}. \tag{2.10}$$

Lemma 2.1. *The following inequalities hold:*

$$\|A_{(1)\gamma_0}^{\alpha_0}(t,\tau)\|_{m_1,\nu} \leq t\|\Gamma\|_{m_1,\nu}^{(t)} \|\dot{z}\|_{m_1,\nu}^{(t)}, \tau \in [0;t]. \tag{2.11}$$

$$\|A_{(k)\gamma_0}^{\alpha_0}(t,\tau)\|_{m_1,\nu} \leq t^k (\|\Gamma\|_{m_1,\nu}^{(t)})^k (\|\dot{z}\|_{m_1,\nu}^{(t)})^k, \tau \in [0;t]. \tag{2.12}$$

The proof follows from aspects of $A_{(k)\gamma_0}^{\alpha_0}(t,\tau)$.

Lemma 2.2. *Let the following conditions hold:*

1) metric tensor in R^3 satisfies the conditions: $\exists M_0 = const > 0$ such that $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$.

2) $\exists t_0 > 0$ *such that* $c(t), c_{,i}(t), a^k(t), \partial_i a^k(t)$ *are continuous by* $t, \forall t \in [0, t_0]$, $c(0) \equiv 0, c_{,i}(0) \equiv 0, a^k(0) \equiv 0, \partial_i a^k(0) \equiv 0$.

3) $\exists t_0 > 0$ *such that* $z^\alpha(t) \in C^{m-2,\nu}$, $z^\alpha_{,i}(t) \in C^{m-3,\nu}$, $\forall t \in [0, t_0]$.

Then there exists $t_* > 0$ *such that* $\forall t \in [0, t_*)$ *exists the unique result* $A^{*\alpha_0}_i(t)$ *of translation in parallel of the tensor* $(y^{\alpha_0}_{,i} + z^{\alpha_0}_{,i}(t))$ *from the point* $(y^{\alpha_0} + z^{\alpha_0}(t))$ *to the point* (y^{α_0}) *along the path of the translation for each point of surface by deformation.* $A^{*\alpha_0}_i(t)$ *has the following representation*

$$A^{*\alpha_0}_i(t) = y^{\alpha_0}_{,i} + z^{\alpha_0}_{,i}(t) + \sum_{k=1}^{\infty}(y^{\gamma_{k-1}}_{,i} + z^{\gamma_{k-1}}_{,i}(t))A^{\alpha_0}_{(k)\gamma_{k-1}}(t, 0). \tag{2.13}$$

$A^{*\alpha_0}_i(t)$ *is of class* $C^{m-3,\nu}$ *and continuous by* t.

Proof. Finding the result of translation in parallel of the tensor along the given curve brings to the Cauchy problem of differential equation system. Using the methods represented in [19, p. 56] we reduce the differential equation system to the integral equation system which is resolved by the method of successive approximations.

The null approximation is:

$$A^{*\alpha_0}_{(0)i}(t) = y^{\alpha_0}_{,i} + z^{\alpha_0}_{,i}(t). \tag{2.14}$$

The $p-$th $(p > 0)$ approximation of Cauchy problem is:

$$A^{*\alpha_0}_{(p)i}(t) = y^{\alpha_0}_{,i} + z^{\alpha_0}_{,i}(t) + \sum_{k=1}^{p}(y^{\gamma_{k-1}}_{,i} + z^{\gamma_{k-1}}_{,i}(t))A^{\alpha_0}_{(k)\gamma_{k-1}}(t, 0). \tag{2.15}$$

Taking into account that $C^{m,\nu}$ is complete normed space, lemma 2.1. and using reasonings that are similar to the ones from [19, p. 56] for solution of this Cauchy problem we get the proof of lemma 2.2.

Lemma 2.3. *Let the conditions of lemma 2.2 hold. Then there exists* $t_* > 0$ *such that* $\forall t \in [0, t_*)$ *the following holds:*

$$\dot{A}^{*\alpha_0}_i(t) = \dot{z}^{\alpha_0}_{,i}(t) + \sum_{k=1}^{\infty}\dot{z}^{\gamma_{k-1}}_{,i}(t)A^{\alpha_0}_{(k)\gamma_{k-1}}(t, 0) + \sum_{k=1}^{\infty}(y^{\gamma_{k-1}}_{,i} + z^{\gamma_{k-1}}_{,i}(t))\dot{A}^{\alpha_0}_{(k)\gamma_{k-1}}(t, 0). \tag{2.16}$$

$\dot{A}^{*\alpha_0}_i(t)$ *is of class* $C^{m-3,\nu}$ *and continuous by* t.

The proof follows from the rules of termwise differentiation of functional series, lemmas 2.1., 2.2. and the properties of space $C^{m-3,\nu}$.

Let obtain the equations of $G-$deformation and transform them to the appropriate for our method form.

We denote $\sigma \equiv \gamma_{k-1}$. Then we have:

$$A^{*\alpha_0}_i(t) = y^{\alpha_0}_{,i} + z^{\alpha_0}_{,i}(t) + (y^{\sigma}_{,i} + z^{\sigma}_{,i}(t))\sum_{k=1}^{\infty}A^{\alpha_0}_{(k)\sigma}(t, 0). \tag{2.17}$$

We denote:

$$\nabla^*_i z^{\alpha_0}(t) = z^{\alpha_0}_{,i}(t) + \Gamma^{\alpha_0}_{\beta_0\sigma}(0)y^{\sigma}_{,i}z^{\beta_0}(t). \tag{2.18}$$

We can write (2.17) in the following form:

$$A_i^{*\alpha_0}(t) = y_{,i}^{\alpha_0} + z_{,i}^{\alpha_0}(t) + \Gamma_{\beta_0\sigma}^{\alpha_0}(0)y_{,i}^{\sigma}z^{\beta_0}(t)+$$

$$(y_{,i}^{\sigma} + z_{,i}^{\sigma}(t))A_{(1)\sigma}^{\alpha_0}(t,0) - \Gamma_{\beta_0\sigma}^{\alpha_0}(0)y_{,i}^{\sigma}z^{\beta_0}(t) + (y_{,i}^{\sigma} + z_{,i}^{\sigma}(t))\sum_{k=2}^{\infty} A_{(k)\sigma}^{\alpha_0}(t,0). \tag{2.19}$$

Inserting (2.18) into (2.19), we get:

$$A_i^{*\alpha_0}(t) = y_{,i}^{\alpha_0} + \nabla_i^* z^{\alpha_0}(t) + z_{,i}^{\sigma}(t)A_{(1)\sigma}^{\alpha_0}(t,0)+$$

$$y_{,i}^{\sigma}(A_{(1)\sigma}^{\alpha_0}(t,0) - \Gamma_{\beta_0\sigma}^{\alpha_0}(0)z^{\beta_0}(t)) + (y_{,i}^{\sigma} + z_{,i}^{\sigma}(t))\sum_{k=2}^{\infty} A_{(k)\sigma}^{\alpha_0}(t,0). \tag{2.20}$$

Therefore we can write (2.20) as:

$$A_i^{*\alpha_0}(t) = y_{,i}^{\alpha_0} + \nabla_i^* z^{\alpha_0}(t) + z_{,i}^{\sigma}(t)\sum_{k=1}^{\infty} A_{(k)\sigma}^{\alpha_0}(t,0)+$$

$$y_{,i}^{\sigma}(A_{(1)\sigma}^{\alpha_0}(t,0) - \Gamma_{\beta_0\sigma}^{\alpha_0}(0)z^{\beta_0}(t)) + y_{,i}^{\sigma}\sum_{k=2}^{\infty} A_{(k)\sigma}^{\alpha_0}(t,0). \tag{2.21}$$

We denote:

$$S_{(1)\sigma}^{\alpha_0}(t,0) = \sum_{k=1}^{\infty} A_{(k)\sigma}^{\alpha_0}(t,0). \tag{2.22}$$

$$S_{(2)\sigma}^{\alpha_0}(t,0) = \sum_{k=2}^{\infty} A_{(k)\sigma}^{\alpha_0}(t,0). \tag{2.23}$$

Using (2.22), (2.23) from (2.21), we obtain:

$$A_i^{*\alpha_0}(t) = y_{,i}^{\alpha_0} + \nabla_i^* z^{\alpha_0}(t) + z_{,i}^{\sigma}(t)S_{(1)\sigma}^{\alpha_0}(t,0)+$$

$$y_{,i}^{\sigma}(A_{(1)\sigma}^{\alpha_0}(t,0) - \Gamma_{\beta_0\sigma}^{\alpha_0}(0)z^{\beta_0}(t)) + y_{,i}^{\sigma}S_{(2)\sigma}^{\alpha_0}(t,0). \tag{2.24}$$

We denote:

$$S_{(3)i}^{\alpha_0}(t,0) = y_{,i}^{\sigma}(A_{(1)\sigma}^{\alpha_0}(t,0) - \Gamma_{\beta_0\sigma}^{\alpha_0}(0)z^{\beta_0}(t)) + y_{,i}^{\sigma}S_{(2)\sigma}^{\alpha_0}(t,0). \tag{2.25}$$

From (2.24) and (2.25) we get:

$$A_i^{*\alpha_0}(t) = y_{,i}^{\alpha_0} + \nabla_i^* z^{\alpha_0}(t) + z_{,i}^{\sigma}(t)S_{(1)\sigma}^{\alpha_0}(t,0) + S_{(3)i}^{\alpha_0}(t,0). \tag{2.26}$$

From (2.1), we get:

$$z^{\sigma}{}_{,i}(t) = a^j{}_{,i}\, y^{\sigma}{}_{,j} + c_{,i}\, n^{\sigma} + a^j y_{,j,i}^{\sigma} + c n_{,i}^{\sigma}. \tag{2.27}$$

Inserting (2.27) into (2.26), we have:

$$A_i^{*\alpha_0}(t) = y_{,i}^{\alpha_0} + \nabla_i^* z^{\alpha_0}(t) + S_{(3)i}^{\alpha_0}(t,0) + (a^j{}_{,i}\, y^{\sigma}{}_{,j} + c_{,i}\, n^{\sigma})S_{(1)\sigma}^{\alpha_0}(t,0)+$$

$$(a^j y^\sigma_{,j,i} + cn^\sigma_{,i}) S^{\alpha_0}_{(1)\sigma}(t,0). \tag{2.28}$$

We denote:

$$S^{\alpha_0}_{(4)i}(t,0) = S^{\alpha_0}_{(3)i}(t,0) + (a^j y^\sigma_{,j,i} + cn^\sigma_{,i}) S^{\alpha_0}_{(1)\sigma}(t,0). \tag{2.29}$$

Insertion (2.29) into (2.28), we obtain:

$$A^{*\alpha_0}_i(t) = y^{\alpha_0}_{,i} + \nabla^*_i z^{\alpha_0}(t) + S^{\alpha_0}_{(4)i}(t,0) + (a^j{}_{,i}\, y^\sigma{}_{,j} + c_{,i}\, n^\sigma) S^{\alpha_0}_{(1)\sigma}(t,0). \tag{2.30}$$

Consider the following formula:

$$a^j{}_{,i} = \partial_i(a^j) + \Gamma^j_{pi} a^p. \tag{2.31}$$

Then, form (2.30) and (2.31), we have:

$$A^{*\alpha_0}_i(t) = y^{\alpha_0}_{,i} + \nabla^*_i z^{\alpha_0}(t) + (\partial_i(a^j) y^\sigma{}_{,j} + c_{,i}\, n^\sigma) S^{\alpha_0}_{(1)\sigma}(t,0) +$$
$$S^{\alpha_0}_{(4)i}(t,0) + \Gamma^j_{pi} a^p y^\sigma{}_{,j}\, S^{\alpha_0}_{(1)\sigma}(t,0). \tag{2.32}$$

We denote:

$$S^{\alpha_0}_{(5)i}(t,0) = S^{\alpha_0}_{(4)i}(t,0) + \Gamma^j_{pi} a^p y^\sigma{}_{,j}\, S^{\alpha_0}_{(1)\sigma}(t,0). \tag{2.33}$$

$$T^{\alpha_0}_0(t,0) = n^\sigma S^{\alpha_0}_{(1)\sigma}(t,0). \tag{2.34}$$

$$T^{\alpha_0}_j(t,0) = y^\sigma{}_{,j}\, S^{\alpha_0}_{(1)\sigma}(t,0), j = 1,2. \tag{2.35}$$

Using (2.33), (2.34) and (2.35), we can write (2.32) in the following form:

$$A^{*\alpha_0}_i(t) = y^{\alpha_0}_{,i} + \nabla^*_i z^{\alpha_0}(t) + c_{,i}\, T^{\alpha_0}_0(t,0) + \partial_i(a^j) T^{\alpha_0}_j(t,0) + S^{\alpha_0}_{(5)i}(t,0). \tag{2.36}$$

For $G-$deformation the following condition holds:

$$\tilde{a}_{\alpha_0\beta_0} A^{*\alpha_0}_i(t) n^{\beta_0} = 0. \tag{2.37}$$

Insertion (2.36) into (2.37), we have:

$$\tilde{a}_{\alpha_0\beta_0} A^{*\alpha_0}_i(t) n^{\beta_0} = a^l b_{li} + c_{,i} + c_{,i}\, \tilde{a}_{\alpha_0\beta_0} T^{\alpha_0}_0(t,0) n^{\beta_0} +$$
$$\partial_i(a^j) \tilde{a}_{\alpha_0\beta_0} T^{\alpha_0}_j(t,0) n^{\beta_0} + \tilde{a}_{\alpha_0\beta_0} S^{\alpha_0}_{(5)i}(t,0) n^{\beta_0}. \tag{2.38}$$

We denote:

$$N_j(t,0) = \tilde{a}_{\alpha_0\beta_0} T^{\alpha_0}_j(t,0) n^{\beta_0}, j = 0,1,2,$$
$$Q_i(t,0) = \tilde{a}_{\alpha_0\beta_0} S^{\alpha_0}_{(5)i}(t,0) n^{\beta_0}. \tag{2.39}$$

Then the equations of $G-$deformation are:

$$a^l b_{li} + (1 + N_0(t,0)) c_{,i} + \partial_i a^j N_j(t,0) + Q_i(t,0) = 0, i = 1,2. \tag{2.40}$$

§3. The estimations of norms.

We denote:
$\|S_{(p)}\|^{(t)}_{m_1,\nu} = \max_{\alpha_0,\sigma} \|S^{\alpha_0}_{(p)\sigma}(t,0)\|_{m_1,\nu}, p = 1,2.$
$\|S_{(l)}\|^{(t)}_{m_1,\nu} = \max_{\alpha_0,i} \|S^{\alpha_0}_{(l)i}(t,0)\|_{m_1,\nu}, l = 3,4,5.$
$\|T\|^{(t)}_{m_1,\nu} = \max_{\alpha_0,j=0,1,2} \|T^{\alpha_0}_j(t,0)\|_{m_1,\nu}.$
$\|N\|^{(t)}_{m_1,\nu} = \max_{j=0,1,2} \|N_j(t,0)\|_{m_1,\nu}.$
$\|Q\|^{(t)}_{m_1,\nu} = \max_i \|Q_i(t,0)\|_{m_1,\nu}.$

Lemma 3.1. *The following estimations hold:*
1) $\|S_{(1)}\|^{(t)}_{m_1,\nu} \le \sum_{k=1}^{\infty}(t\|\Gamma\|^{(t)}_{m_1,\nu}\|\dot z\|^{(t)}_{m_1,\nu})^k.$
2) $\|S_{(2)}\|^{(t)}_{m_1,\nu} \le \sum_{k=2}^{\infty}(t\|\Gamma\|^{(t)}_{m_1,\nu}\|\dot z\|^{(t)}_{m_1,\nu})^k.$
3) $\|S_{(3)}\|^{(t)}_{m_1,\nu} \le Kt\|z\|^{(t)}_{m_1,\nu}\|\dot z\|^{(t)}_{m_1,\nu} + K_2\sum_{k=2}^{\infty}(t\|\Gamma\|^{(t)}_{m_1,\nu}\|\dot z\|^{(t)}_{m_1,\nu})^k.$
4) $\|S_{(4)}\|^{(t)}_{m_1,\nu} \le \|S_{(3)}\|^{(t)}_{m_1,\nu} + M_9\|z\|^{(t)}_{m_1,\nu}\|S_{(1)}\|^{(t)}_{m_1,\nu}.$
5) $\|S_{(5)}\|^{(t)}_{m_1,\nu} \le \|S_{(4)}\|^{(t)}_{m_1,\nu} + M_{10}\|S_{(1)}\|^{(t)}_{m_1,\nu}.$
6) $\|T\|^{(t)}_{m_1,\nu} \le M_{11}\|S_{(1)}\|^{(t)}_{m_1,\nu}.$
7) $\|N\|^{(t)}_{m_1,\nu} \le M_{14}\|S_{(1)}\|^{(t)}_{m_1,\nu}.$
8) $\|Q\|^{(t)}_{m_1,\nu} \le M_{15}\|S_{(5)}\|^{(t)}_{m_1,\nu}.$

The proof of lemma follows from the forms of estimated functions and properties of norms in the space $C^{m_1,\nu}$.

Lemma 3.2. *Let the following conditions hold:*
1) metric tensor in R^3 *satisfies the conditions:* $\exists M_0 = const > 0$ *such that* $\|\tilde a_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial\tilde a_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2\tilde a_{\alpha\beta}\|_{m,\nu} < M_0$.
2) $\exists t_0 > 0$ *such that* $c(t), c_{,i}(t), a^k(t), \partial_i a^k(t)$ *are continuous by* $t, \forall t \in [0,t_0]$, $c(0) \equiv 0, c_{,i}(0) \equiv 0, a^k(0) \equiv 0, \partial_i a^k(0) \equiv 0$.
3) $\exists t_0 > 0$ *such that* $z^\alpha(t) \in C^{m-2,\nu}$, $z^\alpha_{,i}(t) \in C^{m-3,\nu}$, $\forall t \in [0,t_0]$.
Then $\forall\varepsilon > 0 \exists t_0 > 0$ *such that*
1) $\|S_{(i)}\|^{(t)}_{m-2,\nu} \le \varepsilon, \forall t \in [0,t_0], i = \overline{1,5}.$
2) $\|T\|^{(t)}_{m-2,\nu} \le \varepsilon, \forall t \in [0,t_0].$
3) $\|N\|^{(t)}_{m-2,\nu} \le \varepsilon, \forall t \in [0,t_0].$
4) $\|Q\|^{(t)}_{m-2,\nu} \le \varepsilon, \forall t \in [0,t_0].$

The proof of lemma follows from the forms of considered functions and properties of the space $C^{m,\nu}$ and previous lemmas.

§4. Transformation of the $G-$deformations equations.

We introduce conjugate isothermal coordinate system where $b_{ii} = V, i = 1,2, b_{12} = b_{21} = 0$. Then we have the equation system from (2.40):

$$c_{,1}(1 + N_0) + Va^1 + N_k\partial_1 a^k + Q_1 = 0$$

$$c_{,2}(1 + N_0) + Va^2 + N_k\partial_2 a^k + Q_2 = 0 \qquad (4.1)$$

We differentiate the first equation by x^2, the second one by x^1, and subtract from the first equation the second one. Then we obtain:

$$V\partial_2 a^1 - V\partial_1 a^2 + c_{,1}\partial_2 N_0 - c_{,2}\partial_1 N_0 + \partial_1 a^k \partial_2 N_k - \partial_2 a^k \partial_1 N_k +$$

$$\partial_2 V a^1 - \partial_1 V a^2 + \partial_2 Q_1 - \partial_1 Q_2 = 0. \qquad (4.2)$$

We denote

$$\Psi_1 = -(c_{,1}\partial_2 N_0 - c_{,2}\partial_1 N_0 + \partial_1 a^k \partial_2 N_k - \partial_2 a^k \partial_1 N_k + \partial_2 Q_1 - \partial_1 Q_2)/V. \qquad (4.3)$$

Then, from (4.2) and (4.3), we have the following equation:

$$\partial_2 a^1 - \partial_1 a^2 + p_k a^k = \Psi_1, \qquad (4.4)$$

where $p_1 = \partial_2(\ln V), p_2 = -\partial_1(\ln V)$. Note that p_k do not depend on t.

Differentiating the equation (4.4) by t we obtain the following equation:

$$\partial_2 \dot{a}^1 - \partial_1 \dot{a}^2 + p_k \dot{a}^k = \dot{\Psi}_1. \qquad (4.5)$$

§5. Solution of the equation system (4.1): finding function $\dot{c}$ on functions $\dot{a}^i$.

We will solve the equation system (4.1) assuming that functions a^1 and a^2 are given. Note that N_k, Q_i depend only on $c, \dot{c}, a^i, \dot{a}^i$. Function V does not depend on c, a^i. We will use the following formulas.

$$c(x^1, x^2, t) = \int_0^t \dot{c}(x^1, x^2, \tau)d\tau, (c(x^1, x^2, 0) = 0). \qquad (5.1)$$

$$a^i(x^1, x^2, t) = \int_0^t \dot{a}^i(x^1, x^2, \tau)d\tau, (a^i(x^1, x^2, 0) = 0). \qquad (5.2)$$

For functions $a^i_{,j}$ we will use the following formula:

$$a^i_{,k}(x^1, x^2, t) = \int_0^t \dot{a}^i_{,k}(x^1, x^2, \tau)d\tau, (a^i_{,k}(x^1, x^2, 0) = 0). \qquad (5.3)$$

Formulas (5.1), (5.2) and (5.3) establish the connections between functions c, a^i and $\dot{c}, \dot{a}^i$. It means that if the functions $\dot{c}, \dot{a}^i$ are found then the functions c, a^i are found also.

Therefore we pass on to the new equation system (5.4) where there we will consider functions $\dot{c}, \dot{a}^i, \dot{c}_{,i}, \dot{a}^i_{,j}$. We differentiate the equation system (4.1) by t and get (5.4). Note that $N_k, Q_i, \dot{N}_k, \dot{Q}_i$, depend only on $c, \dot{c}, a^i, \dot{a}^i$ and therefore depend only on $\dot{c}, \dot{a}^i$. We can show this by differentiating N_k, Q_i by t.

Then we obtain equation system for $\dot{c}$.

$$\dot{c}_{,1} = -\frac{d}{dt}\left(\frac{Va^1 + N_k\partial_1 a^k + Q_1}{(1+N_0)}\right)$$

$$\dot{c}_{,2} = -\frac{d}{dt}\left(\frac{Va^2 + N_k\partial_2 a^k + Q_2}{(1+N_0)}\right) \tag{5.4}$$

We can present equation system (5.4) as following:

$$\dot{c}_{,i} = -V\dot{a}^i - \left(\frac{-V\dot{a}^i N_0 + N_k\partial_i\dot{a}^k + \dot{N}_k\partial_i a^k + \dot{Q}_i}{(1+N_0)}\right) + \frac{\dot{N}_0(Va^i + N_k\partial_i a^k + Q_i)}{(1+N_0)^2} \tag{5.5}$$

Then we transform (5.5) into integral equation relative to function $\dot{c}$. Let l^* be arbitrary admissible curve in D starting at the point $(x^1_{(0)}, x^2_{(0)})$ and given by the equations $x^1 = x^1(s), x^2 = x^2(s)$. Then we have the following equation.

$$\dot{c}(x^1,x^2,t) =$$

$$\int\limits_{(x^1_{(0)},x^2_{(0)})}^{(x^1,x^2)} \left(-\frac{-V\dot{a}^1 N_0 + N_k\partial_1\dot{a}^k + \dot{N}_k\partial_1 a^k + \dot{Q}_1}{(1+N_0)} + \frac{\dot{N}_0(Va^1 + N_k\partial_1 a^k + Q_1)}{(1+N_0)^2}\right)d\tilde{x}^1 +$$

$$\left(-\frac{-V\dot{a}^2 N_0 + N_k\partial_2\dot{a}^k + \dot{N}_k\partial_2 a^k + \dot{Q}_2}{(1+N_0)} + \frac{\dot{N}_0(Va^2 + N_k\partial_2 a^k + Q_2)}{(1+N_0)^2}\right)d\tilde{x}^2 +$$

$$\int\limits_{(x^1_{(0)},x^2_{(0)})}^{(x^1,x^2)} \left(-V\dot{a}^1\right)d\tilde{x}^1 + \left(-V\dot{a}^2\right)d\tilde{x}^2 \tag{5.6}$$

Then the equation (5.6) along l^* takes the form:

$$\dot{c}(x^1,x^2,t) =$$

$$\int\limits_0^s \Bigg(\left(-\frac{-V\dot{a}^1 N_0 + N_k\partial_1\dot{a}^k + \dot{N}_k\partial_1 a^k + \dot{Q}_1}{(1+N_0)} + \frac{\dot{N}_0(Va^1 + N_k\partial_1 a^k + Q_1)}{(1+N_0)^2}\right)x^{1\prime}(s1) +$$

$$\left(-\frac{-V\dot{a}^2 N_0 + N_k\partial_2\dot{a}^k + \dot{N}_k\partial_2 a^k + \dot{Q}_2}{(1+N_0)} + \frac{\dot{N}_0(Va^2 + N_k\partial_2 a^k + Q_2)}{(1+N_0)^2}\right)x^{2\prime}(s1)\Bigg)ds1 +$$

$$\int\limits_0^s \left(-V\dot{a}^1(s1)x^{1\prime}(s1) - V\dot{a}^2(s_1)x^{2\prime}(s1)\right)ds1. \tag{5.7}$$

The equation (5.7) is nonlinear integral equation. We will show that (5.7) has unique solution of class of continuous functions for any continuous functions $\dot{a}^i$ and $\partial_p\dot{a}^i$.

The equation (5.7) takes the form $\dot{c} = L_a(\dot{c}) + \gamma_t$, where operator L_a has explicit form.

$$\gamma_t(s) = \int\limits_0^s \Big(-V\dot{a}^1(s1)x^{1\prime}(s1) - V\dot{a}^2(s_1)x^{2\prime}(s1)\Big)ds1. \tag{5.8}$$

Therefore every pair of functions $\dot{a}^i \in C^{m-2,\nu}$ corresponds to the unique function $\dot{c} \in C^{m-2,\nu}$ and therefore to the unique function $c \in C^{m-2,\nu}$:
$c(x^1, x^2, t) = \int\limits_0^t \dot{c}(x^1, x^2, \tau)d\tau$, $(c(x^1, x^2, 0) = 0)$.

Then the equation along l^* takes the form:

$$\dot{c}(x^1, x^2, t) = \int\limits_0^s \Big(K_a(s_1, \dot{c}(s_1))\Big)ds_1 + \gamma_t(s), \tag{5.9}$$

where

$$K_a(s_1, \dot{c}(s_1)) =$$
$$\Bigg(\Bigg(-\frac{-V\dot{a}^1N_0 + N_k\partial_1\dot{a}^k + \dot{N}_k\partial_1 a^k + \dot{Q}_1}{(1+N_0)} + \frac{\dot{N}_0(Va^1 + N_k\partial_1 a^k + Q_1)}{(1+N_0)^2}\Bigg)x^{1\prime}(s1) +$$
$$\Bigg(-\frac{-V\dot{a}^2N_0 + N_k\partial_2\dot{a}^k + \dot{N}_k\partial_2 a^k + \dot{Q}_2}{(1+N_0)} + \frac{\dot{N}_0(Va^2 + N_k\partial_2 a^k + Q_2)}{(1+N_0)^2}\Bigg)x^{2\prime}(s1)\Bigg). \tag{5.10}$$

We denote:

$$L_a(\dot{c}) = \int\limits_0^s \Big(K_a(s_1, \dot{c}(s_1))\Big)ds_1. \tag{5.11}$$

We will investigate the decidability problem of the equation in the space $C^{m-2,\nu}(\bar{D})$:

$$\dot{c} = L_a(\dot{c}) + \gamma_t. \tag{5.12}$$

We will solve equation (5.12) by the method of successive approximations.

Lemma 5.1. *Let the following conditions hold:*

1) metric tensor in R^3 *satisfies the conditions:* $\exists M_0 = const > 0$ *such that* $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$.

2) $\exists t_0 > 0$ *such that* $a^k(t), \partial_i a^k(t), \dot{a}^k(t), \partial_i\dot{a}^k(t)$ *are continuous by* $t, \forall t \in [0, t_0]$, $a^k(0) \equiv 0, \partial_i a^k(0) \equiv 0$.

3) $\exists t_0 > 0$ *such that* $a^i(t) \in C^{m-2,\nu}, \partial_k a^i(t) \in C^{m-3,\nu}$, $\forall t \in [0, t_0]$.

Then $\exists t_* > 0$ *such that the equation* $\dot{c} = L_a(\dot{c}) + \gamma_t$ $\forall t \in [0, t_*]$. *has unique solution of class* $C^{m-2,\nu}$ *continuous by* t.

Proof.

We construct the sequence of functions $\{\dot{c}^{(k)}\}$: we find function $\dot{c}^{(1)}$ from the equation

$$\dot{c}^{(1)}(x^1, x^2, t) = \int\limits_0^s \Big(-V\dot{a}^1(s1)x^{1\prime}(s1) - V\dot{a}^2(s_1)x^{2\prime}(s1)\Big)ds1, \tag{5.13}$$

we find function $\dot{c}^{(k)}, k > 1$ from the equation

$$\dot{c}^{(k)} = L_a(\dot{c}^{(k-1)}) + \gamma_t. \tag{5.14}$$

The sequence of functions $\{\dot{c}^{(k)}\}$ is determined uniquely and functions $\dot{c}^{(k)}$ are of class $C^{m-2,\nu}(\bar{D})$.

We will show that the sequence of functions $\{\dot{c}^{(k)}\}$ is bounded in the space $C^{m-2,\nu}(\bar{D})$.

For any $\varepsilon > 0$ there exists $t_0 > 0$ such that for all $t \in [0, t_0)$ the following inequality holds: $\|\dot{c}^{(k)}\|_{m-2,\nu} < \varepsilon$. This inequality is proved by the method of mathematical induction.

Therefore the sequence $\{\dot{c}^{(k)}\}$ is bounded in the space $C^{m-2,\nu}(\bar{D})$.

We will show that the sequence $\{\dot{c}^{(k)}\}$ is convergent in the space $C^{m-2,\nu}(\bar{D})$. Consider the equations:

$$\dot{c}^{(k)} = L_a\dot{c}^{(k)} + \gamma_t, \tag{5.15}$$

$$\dot{c}^{(k+1)} = L_a\dot{c}^{(k+1)} + \gamma_t. \tag{5.16}$$

Subtracting from the second equation the first one we obtain the equation:

$$\dot{c}^{(k+1)} - \dot{c}^{(k)} = L_a(\dot{c}^{(k+1)}) - L_a(\dot{c}^{(k)}). \tag{5.17}$$

Using the explicit form of L_a we have the estimate:

$$\|\dot{c}^{(k+1)} - \dot{c}^{(k)}\|_{m-2,\nu} \le K_3(t)\|\dot{c}^{(k)} - \dot{c}^{(k-1)}\|_{m-2,\nu}, \tag{5.18}$$

where we can choose t_0 such that the following condition holds $K_3(t) < 1$ for all $t \in [0, t_0)$. Then the sequence $\{\dot{c}^{(k)}\}$ is Cauchy sequence in the space $C^{m-2,\nu}(\bar{D})$ and therefore is convergent since the space $C^{m-2,\nu}(\bar{D})$ is complete.

We will show that obtained solution is continuous by t. We have:

$$\dot{c}(t_1) - \dot{c}(t_2) = L_a(\dot{c}(t_1)) - L_a(\dot{c}(t_2)) + \gamma_{t_1} - \gamma_{t_2}. \tag{5.19}$$

Then there is the estimate:

$$\|\dot{c}(t_1) - \dot{c}(t_2)\|_{m-2,\nu} \le \delta_1(t_1, t_2) + \delta_2(t_1, t_2)\|\dot{c}(t_1) - \dot{c}(t_2)\|_{m-2,\nu}, \tag{5.20}$$

where function δ_1 converges to zero if $|t_1 - t_2|$ converges to zero. Function $\delta_2(t_1, t_2)$ is such that for any $N > 0$ we can choose such $t_0 > 0$ that for any t_1 and $t_2 \in [0, t_0)$ the following inequality holds $|\delta_2(t_1, t_2)| < N$. Therefore we obtain the continuity of solution.

We will show that the equation $\dot{c} = L_a\dot{c} + \gamma_t$ has unique solution of class $C^{m-2,\nu}(\bar{D})$ for all sufficiently small $t \ge 0$. Let there exist two different solutions $\dot{c}_{(1)}, \dot{c}_{(2)}$ of class $C^{m-2,\nu}(\bar{D})$.

Consider the equations:

$$\dot{c}_{(1)} = L_a\dot{c}_{(1)} + \gamma_t, \tag{5.21}$$

$$\dot{c}_{(2)} = L_a\dot{c}_{(2)} + \gamma_t. \tag{5.22}$$

Subtracting from the second equation the first one we obtain the equation:

$$\dot{c}_{(2)} - \dot{c}_{(1)} = L_a(\dot{c}_{(2)}) - L_a(\dot{c}_{(1)}). \tag{5.23}$$

Using the explicit form of L_a we have the estimate:

$$\|\dot c_{(2)} - \dot c_{(1)}\|_{m-2,\nu} \leq K_{17}(t)\|\dot c_{(2)} - \dot c_{(1)}\|_{m-2,\nu}. \qquad (5.24)$$

We can choose t_0 such that the following condition holds $K_{17}(t) < 1$ for all $t \in [0, t_0)$. Therefore we have contradiction. Therefore $\dot c_{(1)} \equiv \dot c_{(2)}$ for all sufficiently small $t \geq 0$.

Lemma 5.1. is proved.

Since curve l^* is arbitrary admissible in D therefore the equation (5.4) is solvable uniquely for any continuous functions $\dot a^i$ and $\partial_p \dot a^i$.

Corollary. *Let the conditions of lemma 5.1. hold.*
Then the function $\dot c$ takes the form:

$$\dot c(x^1, x^2, t) = \int\limits_{(x^1_{(0)}, x^2_{(0)})}^{(x^1, x^2)} \left(-V\dot a^1\right) d\tilde x^1 + \left(-V\dot a^2\right) d\tilde x^2 + P(\dot a^1, \dot a^2), \qquad (5.25)$$

and for P the following inequality holds:

$$\|P(\dot a^1_{(1)}, \dot a^2_{(1)}) - P(\dot a^1_{(2)}, \dot a^2_{(2)})\|_{m-2,\nu} \leq K_8(t)(\|\dot a^1_{(1)} - \dot a^1_{(2)}\|_{m-2,\nu} + \|\dot a^2_{(1)} - \dot a^2_{(2)}\|_{m-2,\nu}),$$

where for any $\varepsilon > 0$ there exists $t_0 > 0$ such that for all $t \in [0, t_0)$ the following inequality holds: $K_8(t) < \varepsilon$.

The proof follows from construction of function $\dot c$.

§6. Deduction the formulas of deformations preserving the product of principal curvatures.

§6.1. Deduction the formula of $\Delta(g)$.

Consider the following

$$\Delta(g) = g_t - g, \qquad (6.1.1)$$

where g_t is determinant of the first fundamental form matrix of hypersurface F_t.

We will calculate $\Delta(g_{ij})$. Deformation $\{F_t\}$ of surface F is defined by the formula (1.1). We will use (2.1), (2.2), (2.3), where

$$a^j(0) \equiv 0, c(0) \equiv 0. \qquad (6.1.2)$$

Notice that deformation of surface F determines by the functions a^j and c.

Let $\tilde a_{\alpha\beta}(t)$ be metric tensor of Riemannian space at the point $(y^\sigma + z^\sigma(t))$, $\tilde a_{\alpha\beta}(t) \equiv \tilde a_{\alpha\beta}(y^\sigma + z^\sigma(t))$, $\tilde a_{\alpha\beta}(0) \equiv \tilde a_{\alpha\beta}(y^\sigma)$. $\tilde a_{\alpha\beta} \equiv \tilde a_{\alpha\beta}(0)$. The designations $\Gamma^\gamma_{\beta\sigma}(0)$ and $\Gamma_{\alpha\sigma,\beta}(0)$ mean that the Christoffel symbols are calculated at the point (y^σ).

$$\Delta(g_{ij}) = \tilde a_{\alpha\beta}(t)(y^\alpha{}_{,i} + z^\alpha{}_{,i})(y^\beta{}_{,j} + z^\beta{}_{,j}) - \tilde a_{\alpha\beta}(0) y^\alpha{}_{,i} y^\beta{}_{,j}\,. \qquad (6.1.3)$$

Then we obtain:

$$\Delta(g_{ij}) = (\tilde{a}_{\alpha\beta}(0) + \frac{\partial \tilde{a}_{\alpha\beta}}{\partial y^\sigma}(0)z^\sigma)(y^\alpha,_i + z^\alpha,_i)(y^\beta,_j + z^\beta,_j) - \tilde{a}_{\alpha\beta}(0)y^\alpha,_i y^\beta,_j +$$

$$(\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0) - \frac{\partial \tilde{a}_{\alpha\beta}}{\partial y^\sigma}(0)z^\sigma)(y^\alpha,_i + z^\alpha,_i)(y^\beta,_j + z^\beta,_j). \tag{6.1.4}$$

Therefore we have:

$$\Delta(g_{ij}) = \tilde{a}_{\alpha\beta}(0)(y^\alpha,_i + z^\alpha,_i)(y^\beta,_j + z^\beta,_j) - \tilde{a}_{\alpha\beta}(0)y^\alpha,_i y^\beta,_j +$$

$$(\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0))(y^\alpha,_i + z^\alpha,_i)(y^\beta,_j + z^\beta,_j) + \frac{\partial \tilde{a}_{\alpha\beta}}{\partial y^\sigma}(0)z^\sigma y^\alpha,_i y^\beta,_j -$$

$$\frac{\partial \tilde{a}_{\alpha\beta}}{\partial y^\sigma}(0)z^\sigma y^\alpha,_i y^\beta,_j. \tag{6.1.5}$$

Hence we have:

$$\Delta(g_{ij}) = \tilde{a}_{\alpha\beta}(0)(y^\alpha,_i z^\beta,_j + y^\beta,_j z^\alpha,_i) + \frac{\partial \tilde{a}_{\alpha\beta}}{\partial y^\sigma}(0)z^\sigma y^\alpha,_i y^\beta,_j + \tilde{a}_{\alpha\beta}(0)z^\alpha,_i z^\beta,_j +$$

$$(\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0))(y^\alpha,_i + z^\alpha,_i)(y^\beta,_j + z^\beta,_j) - \frac{\partial \tilde{a}_{\alpha\beta}}{\partial y^\sigma}(0)z^\sigma y^\alpha,_i y^\beta,_j. \tag{6.1.6}$$

Consider the formula:

$$\frac{\partial \tilde{a}_{\alpha\beta}}{\partial y^\sigma}(0) = \Gamma_{\alpha\sigma,\beta}(0) + \Gamma_{\beta\sigma,\alpha}(0) = \tilde{a}_{\gamma\beta}\Gamma^\gamma_{\alpha\sigma}(0) + \tilde{a}_{\gamma\alpha}\Gamma^\gamma_{\beta\sigma}(0). \tag{6.1.7}$$

where $\Gamma_{\alpha\sigma,\beta}(0), \Gamma^\gamma_{\beta\sigma}(0)$ are calculated at the point (y^σ).

Then we obtain:

$$\frac{\partial \tilde{a}_{\alpha\beta}}{\partial y^\sigma}(0)z^\sigma y^\alpha,_i y^\beta,_j = \tilde{a}_{\gamma\beta}\Gamma^\gamma_{\alpha\sigma}(0)z^\sigma y^\alpha,_i y^\beta,_j + \tilde{a}_{\gamma\alpha}\Gamma^\gamma_{\beta\sigma}(0)z^\sigma y^\alpha,_i y^\beta,_j. \tag{6.1.8}$$

We change the positions of indices α and γ in the first term in the right part of the equation (6.1.8) and we also change the positions of indices β and γ in the second term.

Therefore we have:

$$\frac{\partial \tilde{a}_{\alpha\beta}}{\partial y^\sigma}(0)z^\sigma y^\alpha,_i y^\beta,_j = \tilde{a}_{\alpha\beta}\Gamma^\alpha_{\gamma\sigma}(0)z^\sigma y^\gamma,_i y^\beta,_j + \tilde{a}_{\alpha\beta}\Gamma^\beta_{\gamma\sigma}(0)z^\sigma y^\alpha,_i y^\gamma,_j. \tag{6.1.9}$$

Considering the following formula (2.2) we have:

$$\Delta(g_{ij}) = \tilde{a}_{\alpha\beta}(0)y^\alpha,_i \nabla^*_j z^\beta + \tilde{a}_{\alpha\beta}(0)y^\beta,_j \nabla^*_i z^\alpha + (\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0) - \frac{\partial \tilde{a}_{\alpha\beta}}{\partial y^\sigma}(0)z^\sigma)y^\alpha,_i y^\beta,_j +$$

$$(\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0))(y^\alpha,_i z^\beta,_j + y^\beta,_j z^\alpha,_i) + \tilde{a}_{\alpha\beta}(t)z^\alpha,_i z^\beta,_j. \tag{6.1.10}$$

Then we obtain:

$$g^{ij}\Delta(g_{ij}) = 2g^{ij}\tilde{a}_{\alpha\beta}(0)y^{\alpha},_{i}\nabla^{*}_{j}z^{\beta} + (\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0) - \frac{\partial\tilde{a}_{\alpha\beta}}{\partial y^{\sigma}}(0)z^{\sigma})g^{ij}y^{\alpha},_{i}y^{\beta},_{j} + 2(\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0))g^{ij}y^{\alpha},_{i}z^{\beta},_{j} + \tilde{a}_{\alpha\beta}(t)g^{ij}z^{\alpha},_{i}z^{\beta},_{j}\,. \quad (6.1.11)$$

Denote:

$$W_1 = (\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0) - \frac{\partial\tilde{a}_{\alpha\beta}}{\partial y^{\sigma}}(0)z^{\sigma})g^{ij}y^{\alpha},_{i}y^{\beta},_{j} + 2(\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0))g^{ij}y^{\alpha},_{i}z^{\beta},_{j} + \tilde{a}_{\alpha\beta}(t)g^{ij}z^{\alpha},_{i}z^{\beta},_{j}\,. \quad (6.1.12)$$

Using the properties of determinant we have:

$$\Delta(g) = gg^{ij}\Delta(g_{ij}) + W_2. \quad (6.1.13)$$

where

$$W_2 = \Delta(g_{11})\Delta(g_{22}) - (\Delta(g_{12}))^2. \quad (6.1.14)$$

Then the equation (6.1.2) takes the form:

$$\Delta(g) = 2gg^{ij}\tilde{a}_{\alpha\beta}y^{\alpha},_{i}\nabla^{*}_{j}z^{\beta} + gW_1 + W_2. \quad (6.1.15)$$

Using the equation (6.1.3) we write the equation (6.1.15) as:

$$\frac{\Delta(g)}{2g} = a^{l},_{l} - cb_{lm}g^{ml} + \frac{W_1}{2} + \frac{W_2}{2g}. \quad (6.1.16)$$

Using the formula $\partial_i(\ln\sqrt{g}) = \Gamma^{j}_{ij}$, where Γ^{k}_{ij} are the Christoffel symbols for hypersurface F in the metric g_{ij} and formula of mean curvature $2H = g^{im}b_{im}$ we write the equation (6.1.16) as

$$\frac{\Delta(g)}{2\sqrt{g}} = \partial_l(\sqrt{g}a^{l}) - 2Hc\sqrt{g} + \frac{\sqrt{g}W_1}{2} + \frac{W_2}{2\sqrt{g}}. \quad (6.1.17)$$

The equation (6.1.17) is required equation for functions a^i and c, determining continuous $A-$deformation of hypersurface F.

Equation (6.1.17) takes the form:

$$\frac{\Delta(g)}{2g} = \partial_1 a^1 + \partial_2 a^2 + a^1\partial_1(\ln\sqrt{g}) + a^2\partial_2(\ln\sqrt{g}) - \Psi_2, \quad (6.1.18)$$

where

$$\Psi_2 = 2Hc - \frac{W_1}{2} - \frac{W_2}{2g}. \quad (6.1.19)$$

Then we obtain:

$$\Delta(g) = 2g(\partial_1 a^1 + \partial_2 a^2 + q_k a^k - \Psi_2), \quad (6.1.20)$$

where

$$q_1 = \partial_1(\ln\sqrt{g}), q_2 = \partial_2(\ln\sqrt{g}). \quad (6.1.21)$$

Note that q_k do not depend on t.

Equation (6.1.20) determines $\Delta(g)$ for deformations of surface F in R^3.

§6.2. Deduction the formulas of deformations preserving the product of principal curvatures.

Deformation $\{F_t\}$ of surface F is determined by (2.1). We will deduct the formulas of changing of the second fundamental form determinant.

The condition of preservation the product of principal curvatures takes the following form:

$$\Delta(g) = \frac{g}{b}\Delta(b). \tag{6.2.1}$$

$$\Delta(b) = bb^{ij}\Delta(b_{ij}) + W_2^{(b)}. \tag{6.2.2}$$

We have the formula:

$$\Delta(K) = \frac{1}{b(t)}(\Delta(g) - \frac{g}{b}\Delta(b)), b(t) = b + \Delta(b). \tag{6.2.3}$$

We introduce conjugate isothermal coordinate system where

$$b_{ii} = V, i = 1,2, b_{12} = b_{21} = 0, b^{ii} = \frac{1}{V}, i = 1,2, b^{12} = b^{21} = 0. \tag{6.2.4}$$

Then we have:

$$\Delta(b) = V(\Delta(b_{11}) + \Delta(b_{22})) + W_2^{(b)}, \tag{6.2.5}$$

where

$$W_2^{(b)} = \Delta(b_{11})\Delta(b_{22}) - (\Delta(b_{12}))^2. \tag{6.2.6}$$

Therefore the condition of preservation the product of principal curvatures takes the following form:

$$\Delta(g) = \frac{g}{V}(\Delta(b_{11}) + \Delta(b_{22})) + \frac{g}{V^2}W_2^{(b)}. \tag{6.2.7}$$

We have the following formula:

$$b_{ij}(0) = -\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}\nabla^*_j n^{\beta}(0). \tag{6.2.8}$$

$$b_{ij}(t) = -\tilde{a}_{\alpha\beta}(t)(y^{\alpha}_{,i} + z^{\alpha}_{,i})\nabla^*_j \tilde{n}^{\beta}(t), \tag{6.2.9}$$

where $\tilde{n}^{\beta}(t)$ is unit normal vector at the point $(y^{\alpha} + z^{\alpha})$.

Then we obtain:

$$b_{ij}(t) = -\tilde{a}_{\alpha\beta}(t)(y^{\alpha}_{,i} + z^{\alpha}_{,i})(\tilde{n}^{\beta}_{,j}(t) + \Gamma^{\beta}_{\mu\sigma}(t)(y^{\mu}_{,j} + z^{\mu}_{,j})\tilde{n}^{\sigma}(t)). \tag{6.2.10}$$

Let $n^{\beta}(t)$ be result of parallel transfer of unite normal vector $n^{\beta}(0)$ to the point $(y^{\alpha} + z^{\alpha})$ along the path of translation by deformation. Therefore we have the following formula for all sufficiently small t:

$$n^{\alpha}(t) = n^{\alpha}(0) + n^{\sigma}(0)\sum_{k=1}^{\infty} A^{\alpha}_{(k)\sigma}(0,t). \tag{6.2.11}$$

Use the following formula:

$$\tilde{n}^{\beta}(t) = \frac{n^{\beta}(t)}{\sqrt{\tilde{a}_{\alpha_0\beta_0}(t)n^{\alpha_0}(t)n^{\beta_0}(t)}}. \tag{6.2.12}$$

Denote:

$$\|n(t)\| = \sqrt{\tilde{a}_{\alpha_0\beta_0}(t)n^{\alpha_0}(t)n^{\beta_0}(t)}. \tag{6.2.13}$$

Then we have:

$$b_{ij}(t) = -\tilde{a}_{\alpha\beta}(t)z^{\alpha}_{,i}\nabla^{*}_{j}\tilde{n}^{\beta}(t) - \tilde{a}_{\alpha\beta}(t)y^{\alpha}_{,i}\nabla^{*}_{j}\tilde{n}^{\beta}(t). \tag{6.2.14}$$

Using the formulas (2.27) and (2.31) we obtain:

$$-\tilde{a}_{\alpha\beta}(t)z^{\alpha}_{,i}\nabla^{*}_{j}\tilde{n}^{\beta}(t) = -\tilde{a}_{\alpha\beta}(t)\partial_i(a^k)y^{\alpha}{}_{,k}\,\nabla^{*}_{j}\tilde{n}^{\beta}(t) + M^1_{ij}, \tag{6.2.15}$$

where

$$M^1_{ij} = -\tilde{a}_{\alpha\beta}(t)(\Gamma^k_{pi}a^p y^{\alpha}{}_{,k} + c_{,i}\, n^{\alpha} + a^k y^{\alpha}_{,k,i} + c n^{\alpha}_{,i})\nabla^{*}_{j}\tilde{n}^{\beta}(t). \tag{6.2.16}$$

Then we have:

$$-\tilde{a}_{\alpha\beta}(t)z^{\alpha}_{,i}\nabla^{*}_{j}\tilde{n}^{\beta}(t) = -\tilde{a}_{\alpha\beta}(0)\partial_i(a^k)y^{\alpha}{}_{,k}\,\nabla^{*}_{j}n^{\beta}(0) -$$

$$\tilde{a}_{\alpha\beta}(t)\partial_i(a^k)y^{\alpha}{}_{,k}\,\nabla^{*}_{j}\tilde{n}^{\beta}(t) + \tilde{a}_{\alpha\beta}(0)\partial_i(a^k)y^{\alpha}{}_{,k}\,\nabla^{*}_{j}n^{\beta}(0) + M^1_{ij}, \tag{6.2.17}$$

Define:

$$M^2_{ij} = -\tilde{a}_{\alpha\beta}(t)\partial_i(a^k)y^{\alpha}{}_{,k}\,\nabla^{*}_{j}\tilde{n}^{\beta}(t) + \tilde{a}_{\alpha\beta}(0)\partial_i(a^k)y^{\alpha}{}_{,k}\,\nabla^{*}_{j}n^{\beta}(0) + M^1_{ij}. \tag{6.2.18}$$

Consequently we get:

$$-\tilde{a}_{\alpha\beta}(t)z^{\alpha}_{,i}\nabla^{*}_{j}\tilde{n}^{\beta}(t) = -\tilde{a}_{\alpha\beta}(0)\partial_i(a^k)y^{\alpha}{}_{,k}\,\nabla^{*}_{j}n^{\beta}(0) + M^2_{ij}. \tag{6.2.19}$$

We use the following equation:

$$\nabla^{*}_{j}n^{\beta}(0) = -b_{jk}g^{kl}y^{\beta}_{,l}. \tag{6.2.20}$$

Then we obtain:

$$-\tilde{a}_{\alpha\beta}(t)z^{\alpha}_{,i}\nabla^{*}_{j}\tilde{n}^{\beta}(t) = \partial_i(a^k)b_{jk} + M^2_{ij}. \tag{6.2.21}$$

Using the fact $b_{12} = 0$ we have:

$$-\tilde{a}_{\alpha\beta}(t)z^{\alpha}_{,1}\nabla^{*}_{1}\tilde{n}^{\beta}(t) = V\partial_1(a^1) + M^2_{11}, \tag{6.2.22}$$

$$-\tilde{a}_{\alpha\beta}(t)z^{\alpha}_{,2}\nabla^{*}_{2}\tilde{n}^{\beta}(t) = V\partial_2(a^2) + M^2_{22}. \tag{6.2.23}$$

We have the expression:

$$\nabla^{*}_{j}\tilde{n}^{\beta}(t) = \tilde{n}^{\beta}_{,j}(t) + \Gamma^{\beta}_{\mu\sigma}(t)(y^{\mu}_{,j} + z^{\mu}_{,j})\tilde{n}^{\sigma}(t) = \left(\frac{n^{\beta}(t)}{\|n(t)\|}\right)_{,j} + \Gamma^{\beta}_{\mu\sigma}(t)(y^{\mu}_{,j} + z^{\mu}_{,j})\left(\frac{n^{\sigma}(t)}{\|n(t)\|}\right). \tag{6.2.24}$$

Then we obtain:

$$\nabla^*_j \tilde{n}^\beta(t) = \frac{\nabla^*_j n^\beta(t)}{\|n(t)\|} + n^\beta(t)\left(\frac{1}{\|n(t)\|}\right)_{,j}. \tag{6.2.25}$$

Consider the equation:

$$-\tilde{a}_{\alpha\beta}(t)y^\alpha_{,i}\nabla^*_j\tilde{n}^\beta(t) = -\tilde{a}_{\alpha\beta}(t)y^\alpha_{,i}\nabla^*_j n^\beta(t)\left(\frac{1}{\|n(t)\|}\right) - \tilde{a}_{\alpha\beta}(t)y^\alpha_{,i}n^\beta(t)\left(\frac{1}{\|n(t)\|}\right)_{,j}. \tag{6.2.26}$$

Consider the formula:

$$\nabla^*_j n^\beta(t) = n^\beta_{,j}(t) + \Gamma^\beta_{\mu\sigma}(t)(y^\mu_{,j} + z^\mu_{,j})n^\sigma(t). \tag{6.2.27}$$

Then we have:

$$\begin{gathered}\nabla^*_j n^\beta(t) = n^\beta_{,j}(t) + \Gamma^\beta_{\mu\sigma}(t)y^\mu_{,j}n^\sigma(t) + \Gamma^\beta_{\mu\sigma}(t)z^\mu_{,j}n^\sigma(t) = \\ = n^\beta_{,j}(t) + \Gamma^\beta_{\mu\sigma}(0)y^\mu_{,j}n^\sigma(t) + (\Gamma^\beta_{\mu\sigma}(t) - \Gamma^\beta_{\mu\sigma}(0))y^\mu_{,j}n^\sigma(t) + \\ \Gamma^\beta_{\mu\sigma}(0)z^\mu_{,j}n^\sigma(t) + (\Gamma^\beta_{\mu\sigma}(t) - \Gamma^\beta_{\mu\sigma}(0))z^\mu_{,j}n^\sigma(t).\end{gathered} \tag{6.2.28}$$

We will use the following formula:

$$n^\beta(t) = n^\beta(0) + n^\sigma(0)\sum_{k=1}^{\infty} A^\beta_{(k)\sigma}(0,t). \tag{6.2.29}$$

Denote:

$$A^\beta_1(t) = n^\sigma(0)\sum_{k=1}^{\infty} A^\beta_{(k)\sigma}(0,t), \tag{6.2.30}$$

$$A^\beta_2(t) = n^\sigma(0)\sum_{k=2}^{\infty} A^\beta_{(k)\sigma}(0,t). \tag{6.2.31}$$

We use the equation:

$$n^\beta(t) = n^\beta(0) + n^\sigma(0)A^\beta_{(1)\sigma}(0,t) + A^\beta_2(t). \tag{6.2.32}$$

We have:

$$n^\beta_{,j}(t) = n^\beta_{,j}(0) + n^\sigma_{,j}(0)A^\beta_{(1)\sigma}(0,t) + n^\sigma(0)A^\beta_{(1)\sigma,j}(0,t) + A^\beta_{2,j}(t). \tag{6.2.33}$$

Hence:

$$\begin{gathered}\nabla^*_j n^\beta(t) = n^\beta_{,j}(0) + n^\sigma_{,j}(0)A^\beta_{(1)\sigma}(0,t) + n^\sigma(0)A^\beta_{(1)\sigma,j}(0,t) + A^\beta_{2,j}(t) + \\ \Gamma^\beta_{\mu\tau}(0)y^\mu_{,j}n^\tau(0) + \Gamma^\beta_{\mu\tau}(0)y^\mu_{,j}n^\sigma(0)A^\tau_{(1)\sigma}(0,t) + \Gamma^\beta_{\mu\tau}(0)y^\mu_{,j}A^\tau_2(t) + \\ (\Gamma^\beta_{\mu\tau}(t) - \Gamma^\beta_{\mu\tau}(0))y^\mu_{,j}n^\tau(0) + (\Gamma^\beta_{\mu\tau}(t) - \Gamma^\beta_{\mu\tau}(0))y^\mu_{,j}A^\tau_1(t) +\end{gathered}$$

$$\Gamma^{\beta}_{\mu\tau}(0)z^{\mu}_{,j}n^{\tau}(0)+\Gamma^{\beta}_{\mu\tau}(0)z^{\mu}_{,j}A^{\tau}_{1}(t)+(\Gamma^{\beta}_{\mu\tau}(t)-\Gamma^{\beta}_{\mu\tau}(0))z^{\mu}_{,j}n^{\tau}(t). \tag{6.2.34}$$

Denote:

$$T^{\beta}_{j}=n^{\sigma}_{,j}(0)A^{\beta}_{(1)\sigma}(0,t)+A^{\beta}_{2,j}(t)+\Gamma^{\beta}_{\mu\tau}(0)y^{\mu}_{,j}n^{\sigma}(0)A^{\tau}_{(1)\sigma}(0,t)+\Gamma^{\beta}_{\mu\tau}(0)y^{\mu}_{,j}A^{\tau}_{2}(t)+$$
$$(\Gamma^{\beta}_{\mu\tau}(t)-\Gamma^{\beta}_{\mu\tau}(0))y^{\mu}_{,j}n^{\tau}(0)+(\Gamma^{\beta}_{\mu\tau}(t)-\Gamma^{\beta}_{\mu\tau}(0))y^{\mu}_{,j}A^{\tau}_{1}(t)+$$
$$\Gamma^{\beta}_{\mu\tau}(0)z^{\mu}_{,j}A^{\tau}_{1}(t)+(\Gamma^{\beta}_{\mu\tau}(t)-\Gamma^{\beta}_{\mu\tau}(0))z^{\mu}_{,j}n^{\tau}(t). \tag{6.2.35}$$

Then we get:

$$\nabla^{*}_{j}n^{\beta}(t)=n^{\beta}_{,j}(0)+\Gamma^{\beta}_{\mu\tau}(0)y^{\mu}_{,j}n^{\tau}(0)+n^{\sigma}(0)A^{\beta}_{(1)\sigma,j}(0,t)+\Gamma^{\beta}_{\mu\tau}(0)z^{\mu}_{,j}n^{\tau}(0)+T^{\beta}_{j}=$$
$$\nabla^{*}_{j}n^{\beta}(0)+n^{\sigma}(0)A^{\beta}_{(1)\sigma,j}(0,t)+\Gamma^{\beta}_{\mu\tau}(0)z^{\mu}_{,j}n^{\tau}(0)+T^{\beta}_{j}. \tag{6.2.36}$$

Consider the expression:

$$-\tilde{a}_{\alpha\beta}(t)y^{\alpha}_{,i}\nabla^{*}_{j}n^{\beta}(t)=-\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}\nabla^{*}_{j}n^{\beta}(t)-(\tilde{a}_{\alpha\beta}(t)-\tilde{a}_{\alpha\beta}(0))y^{\alpha}_{,i}\nabla^{*}_{j}n^{\beta}(t)=$$
$$\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}b_{jk}g^{kl}y^{\beta}_{,l}-\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}(n^{\sigma}(0)A^{\beta}_{(1)\sigma,j}(0,t)+\Gamma^{\beta}_{\mu\tau}(0)z^{\mu}_{,j}n^{\tau}(0))-$$
$$\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}T^{\beta}_{j}-(\tilde{a}_{\alpha\beta}(t)-\tilde{a}_{\alpha\beta}(0))y^{\alpha}_{,i}\nabla^{*}_{j}n^{\beta}(t). \tag{6.2.37}$$

Then we obtain:

$$-\tilde{a}_{\alpha\beta}(t)y^{\alpha}_{,i}\nabla^{*}_{j}n^{\beta}(t)=b_{ji}-\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}(n^{\sigma}(0)A^{\beta}_{(1)\sigma,j}(0,t)+\Gamma^{\beta}_{\mu\tau}(0)z^{\mu}_{,j}n^{\tau}(0))-$$
$$\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}T^{\beta}_{j}-(\tilde{a}_{\alpha\beta}(t)-\tilde{a}_{\alpha\beta}(0))y^{\alpha}_{,i}\nabla^{*}_{j}n^{\beta}(t). \tag{6.2.38}$$

Therefore:

$$-\tilde{a}_{\alpha\beta}(t)y^{\alpha}_{,i}\nabla^{*}_{j}n^{\beta}(t)\left(\frac{1}{\|n(t)\|}\right)=$$
$$b_{ji}\left(\frac{1}{\|n(t)\|}\right)-\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}(n^{\sigma}(0)A^{\beta}_{(1)\sigma,j}(0,t)+\Gamma^{\beta}_{\mu\tau}(0)z^{\mu}_{,j}n^{\tau}(0))\left(\frac{1}{\|n(t)\|}\right)-$$
$$\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}T^{\beta}_{j}\left(\frac{1}{\|n(t)\|}\right)-(\tilde{a}_{\alpha\beta}(t)-\tilde{a}_{\alpha\beta}(0))y^{\alpha}_{,i}\nabla^{*}_{j}n^{\beta}(t)\left(\frac{1}{\|n(t)\|}\right). \tag{6.2.39}$$

We change the form of last expression:

$$-\tilde{a}_{\alpha\beta}(t)y^{\alpha}_{,i}\nabla^{*}_{j}n^{\beta}(t)\left(\frac{1}{\|n(t)\|}\right)=$$
$$b_{ji}+\frac{b_{ji}(1-\|n(t)\|)}{\|n(t)\|}-\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}(n^{\sigma}(0)A^{\beta}_{(1)\sigma,j}(0,t)+\Gamma^{\beta}_{\mu\tau}(0)z^{\mu}_{,j}n^{\tau}(0))\left(\frac{1}{\|n(t)\|}\right)-$$
$$\tilde{a}_{\alpha\beta}(0)y^{\alpha}_{,i}T^{\beta}_{j}\left(\frac{1}{\|n(t)\|}\right)-(\tilde{a}_{\alpha\beta}(t)-\tilde{a}_{\alpha\beta}(0))y^{\alpha}_{,i}\nabla^{*}_{j}n^{\beta}(t)\left(\frac{1}{\|n(t)\|}\right). \tag{6.2.40}$$

Define:

$$M^3_{ij} = \frac{b_{ji}(1 - \|n(t)\|)}{\|n(t)\|} - \tilde{a}_{\alpha\beta}(0) y^\alpha_{,i}(n^\sigma(0) A^\beta_{(1)\sigma,j}(0,t) + \Gamma^\beta_{\mu\tau}(0) z^\mu_{,j} n^\tau(0)) \left(\frac{1}{\|n(t)\|}\right) -$$

$$\tilde{a}_{\alpha\beta}(0) y^\alpha_{,i} T^\beta_j \left(\frac{1}{\|n(t)\|}\right) - (\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0)) y^\alpha_{,i} \nabla^*_j n^\beta(t) \left(\frac{1}{\|n(t)\|}\right). \qquad (6.2.41)$$

Then we have:

$$-\tilde{a}_{\alpha\beta}(t) y^\alpha_{,i} \nabla^*_j n^\beta(t) \left(\frac{1}{\|n(t)\|}\right) = b_{ij} + M^3_{ij}. \qquad (6.2.42)$$

Consequently:

$$b_{ij}(t) = -\tilde{a}_{\alpha\beta}(t) z^\alpha_{,i} \nabla^*_j \tilde{n}^\beta(t) - \tilde{a}_{\alpha\beta}(t) y^\alpha_{,i} \nabla^*_j \tilde{n}^\beta(t) =$$

$$\partial_i(a^k) b_{jk} + M^2_{ij} + b_{ij} + M^3_{ij} - \tilde{a}_{\alpha\beta}(t) y^\alpha_{,i} n^\beta(t) \left(\frac{1}{\|n(t)\|}\right)_{,j}. \qquad (6.2.43)$$

Denote:

$$M^4_{ij} = M^2_{ij} + M^3_{ij} - \tilde{a}_{\alpha\beta}(t) y^\alpha_{,i} n^\beta(t) \left(\frac{1}{\|n(t)\|}\right)_{,j}. \qquad (6.2.44)$$

Hence:

$$b_{ij}(t) = \partial_i(a^k) b_{jk} + b_{ij} + M^4_{ij}. \qquad (6.2.45)$$

Therefore:

$$\Delta(b_{ij}) = \partial_i(a^k) b_{jk} + M^4_{ij}. \qquad (6.2.46)$$

Then we have:

$$\Delta(b_{11}) = V \partial_1(a^1) + M^4_{11}, \qquad (6.2.47)$$

$$\Delta(b_{22}) = V \partial_2(a^2) + M^4_{22}. \qquad (6.2.48)$$

Hence the condition of preservation the product of principal curvatures takes the following form:

$$\Delta(g) = g(\partial_1(a^1) + \partial_2(a^2)) + \frac{g}{V}(M^4_{11} + M^4_{22}) + \frac{g}{V^2} W^{(b)}_2. \qquad (6.2.49)$$

Using the formula (6.1.20) we obtain the equation of preservation the product of principal curvatures:

$$\partial_1 a^1 + \partial_2 a^2 + 2q_k a^k - 2\Psi_2 = \frac{1}{V}(M^4_{11} + M^4_{22}) + \frac{1}{V^2} W^{(b)}_2. \qquad (6.2.50)$$

Then we have:

$$\partial_1 a^1 + \partial_2 a^2 + 2q_k a^k = 2\Psi_2 + \frac{1}{V}(M^4_{11} + M^4_{22}) + \frac{1}{V^2} W^{(b)}_2. \quad (6.2.51)$$

We differentiate the equation (6.2.51) by t. Then we have:

$$\partial_1 \dot{a}^1 + \partial_2 \dot{a}^2 + 2q_k \dot{a}^k = 2\dot{\Psi}_2 + \frac{1}{V}(\dot{M}^4_{11} + \dot{M}^4_{22}) + \frac{1}{V^2} \dot{W}^{(b)}_2. \quad (6.2.52)$$

The equation takes the following form:

$$\partial_1 \dot{a}^1 + \partial_2 \dot{a}^2 + q^{(b)}_k \dot{a}^k = \dot{\Psi}^{(b)}_2, \quad (6.2.53)$$

where $\dot{\Psi}^{(b)}_2 = q^{(b)}_0 \dot{c} - P_0(\dot{a}^1, \dot{a}^2, \partial_i \dot{a}^j)$. P_0 has explicit form. Notice that $q^{(b)}_k \in C^{m-3,\nu}$, $q^{(b)}_0 \in C^{m-3,\nu}$ and do not depend on t.

Lemma 6.2.1. *Let the following conditions hold:*

1) metric tensor in R^3 satisfies the conditions: $\exists M_0 = const > 0$ such that $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial \tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2 \tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$.

2) $\exists t_0 > 0$ such that $a^k(t), \partial_i a^k(t), \dot{a}^k(t), \partial_i \dot{a}^k(t)$ are continuous by $t, \forall t \in [0, t_0]$, $a^k(0) \equiv 0, \partial_i a^k(0) \equiv 0$.

3) $\exists t_0 > 0$ such that $a^i(t) \in C^{m-2,\nu}, \partial_k a^i(t) \in C^{m-3,\nu}$, $\forall t \in [0, t_0]$.

Then $\exists t_* > 0$ such that for all $t \in [0, t_*)$ $P_0 \in C^{m-3,\nu}$ and the following inequality holds:

$$\|P_0(\dot{a}^1_{(1)}, \dot{a}^2_{(1)}) - P_0(\dot{a}^1_{(2)}, \dot{a}^2_{(2)})\|_{m-2,\nu} \leq K_9(t)(\|\dot{a}^1_{(1)} - \dot{a}^1_{(2)}\|_{m-1,\nu} + \|\dot{a}^2_{(1)} - \dot{a}^2_{(2)}\|_{m-1,\nu}),$$

where for any $\varepsilon > 0$ there exists $t_0 > 0$ such that for all $t \in [0, t_0)$ the following inequality holds: $K_9(t) < \varepsilon$.

The proof follows from construction of function P_0 and lemmas of §7 and §8.

The equation (6.2.53) determines deformations of surface F preserving the product of principal curvatures with condition of $G-$deformation.

6.2.1. The formulas of $\Delta(K)$ and $\dot{\Delta}(K)$.

Consider the following formula:

$$\Delta(K) = \frac{1}{b(t)}(\Delta(g) - \frac{g}{b}\Delta(b)) =$$

$$\frac{g}{b(t)}(\partial_1 a^1 + \partial_2 a^2 + 2q_k a^k - (2\Psi_2 + \frac{1}{V}(M^4_{11} + M^4_{22}) + \frac{1}{V^2} W^{(b)}_2)), \quad (6.2.54)$$

$$b(t) = b + \Delta(b). \quad (6.2.55)$$

We have:

$$K(t) = K + \Delta(K). \quad (6.2.56)$$

Therefore:

$$\dot{b}(t) = \dot{\Delta}(b), \dot{K}(t) = \dot{\Delta}(K). \tag{6.2.57}$$

Then we obtain:

$$\dot{\Delta}(K) = -\frac{g\dot{\Delta}b(t)}{(b(t))^2}(\partial_1 a^1 + \partial_2 a^2 + 2q_k a^k - (2\Psi_2 + \frac{1}{V}(M_{11}^4 + M_{22}^4) + \frac{1}{V^2}W_2^{(b)}))+$$

$$\frac{g}{b(t)}(\partial_1 \dot{a}^1 + \partial_2 \dot{a}^2 + 2q_k \dot{a}^k - (2\dot{\Psi}_2 + \frac{1}{V}(\dot{M}_{11}^4 + \dot{M}_{22}^4) + \frac{1}{V^2}\dot{W}_2^{(b)})). \tag{6.2.58}$$

We finally obtain the following formula:

$$\dot{\Delta}(K) = -\frac{g\dot{\Delta}b(t)}{(b(t))^2}(\partial_1 a^1 + \partial_2 a^2 + 2q_k a^k - (2\Psi_2 + \frac{1}{V}(M_{11}^4 + M_{22}^4) + \frac{1}{V^2}W_2^{(b)}))+$$

$$\frac{g}{b(t)}(\partial_1 \dot{a}^1 + \partial_2 \dot{a}^2 + q_k^{(b)} \dot{a}^k - \dot{\Psi}_2^{(b)}). \tag{6.2.59}$$

§7. Auxiliary estimations of norms.

Denote:
$\|\partial z\|_{m_1,\nu}^{(t)} = \max_{\alpha,i} \|z_{,i}^{\alpha}\|_{m_1,\nu}^{(t)} = \max_{\alpha,i} \max_{\tau\in[0;t]} \|z_{,i}^{\alpha}(\tau)\|_{m_1,\nu}$.

Lemma 7.1. it The following estimations hold:

1) $\|z\|_{m_1,\nu}^{(t)} \leq M_5(\|a\|_{m_1,\nu}^{(t)} + \|c\|_{m_1,\nu}^{(t)})$,

2) $\|a\|_{m_1,\nu}^{(t)} \leq M_6\|z\|_{m_1,\nu}^{(t)}$,

3) $\|c\|_{m_1,\nu}^{(t)} \leq M_7\|z\|_{m_1,\nu}^{(t)}$,

4) $\|\partial z\|_{m_1,\nu}^{(t)} \leq M_8\|z\|_{m_1+1,\nu}^{(t)}$,

where constants M_5, M_6, M_7, M_8 are determined by surface F and do not depend on t.

Proof of lemma follows from properties of norm in the space $C^{m_1,\nu}$.

Lemma 7.2. *The following estimations hold:*

1) $\|W_1\|_{m_1,\nu}^{(t)} \leq M_2((\|z\|_{m_1,\nu}^{(t)})^2 + \|z\|_{m_1,\nu}^{(t)}\|\partial z\|_{m_1,\nu}^{(t)} + (\|\partial z\|_{m_1,\nu}^{(t)})^2)$,

2)
$\|\Delta(g_{ij})\|_{m_1,\nu}^{(t)} \leq M_3(\|z\|_{m_1,\nu}^{(t)} + \|\partial z\|_{m_1,\nu}^{(t)} + (\|z\|_{m_1,\nu}^{(t)})^2 + \|z\|_{m_1,\nu}^{(t)}\|\partial z\|_{m_1,\nu}^{(t)} + (\|\partial z\|_{m_1,\nu}^{(t)})^2)$.

3) $\|W_2\|_{m_1,\nu}^{(t)} \leq M_4(\max_{i,j} \|\Delta(g_{ij})\|_{m_1,\nu}^{(t)})^2$,

where constants M_2, M_3, M_4 are determined by surface F and do not depend on t.

Proof of lemma follows from properties of norms in the space $C^{m_1,\nu}$.

Lemma 7.3. *Let the following conditions hold:*

1) metric tensor of R^3 satisfies the conditions: $\exists M_0 = const > 0$, such that $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$.

2) $\exists t_0 > 0$, such that $c(t), c_{,i}(t), a^k(t), \partial_i a^k(t)$ are continuous by $t, \forall t \in [0, t_0]$, $c(0) \equiv 0, c_{,i}(0) \equiv 0, a^k(0) \equiv 0, \partial_i a^k(0) \equiv 0$.

3) $\exists t_0 > 0$, such that $z^{\alpha}(t) \in C^{m-2,\nu}, z_{,i}^{\alpha}(t) \in C^{m-3,\nu}$, $\forall t \in [0, t_0]$.

Then $\forall \varepsilon > 0 \exists t_0 > 0$ such that

1) $\|W_1\|_{m-3,\nu}^{(t)} \leq \varepsilon, \forall t \in [0, t_0]$.

2) $\|W_2\|^{(t)}_{m-3,\nu} \le \varepsilon, \forall t \in [0, t_0]$.

3) $\|\Psi_2\|^{(t)}_{m-3,\nu} \le \varepsilon, \forall t \in [0, t_0]$.

Proof of lemma follows from the form of functions W_1, W_2, Ψ_2, properties of space $C^{m,\nu}$ and previous lemmas.

§8. Properties of functions $\dot W_1, \dot W_2, \dot\Psi_2$.

§8.1. Formula of function $\dot W_1$.

We have:

$$\dot W_1 = (\dot{\tilde a}_{\alpha\beta}(t) - \frac{\partial \tilde a_{\alpha\beta}}{\partial y^\sigma}(0)\dot z^\sigma) g^{ij} y^\alpha{}_{,i} y^\beta{}_{,j} +$$

$$2\dot{\tilde a}_{\alpha\beta}(t) g^{ij} y^\alpha{}_{,i} z^\beta{}_{,j} + 2(\tilde a_{\alpha\beta}(t) - \tilde a_{\alpha\beta}(0)) g^{ij} y^\alpha{}_{,i} \dot z^\beta{}_{,j} + \dot{\tilde a}_{\alpha\beta}(t) g^{ij} z^\alpha{}_{,i} z^\beta{}_{,j} +$$

$$\tilde a_{\alpha\beta}(t) g^{ij} \dot z^\alpha{}_{,i} z^\beta{}_{,j} + \tilde a_{\alpha\beta}(t) g^{ij} z^\alpha{}_{,i} \dot z^\beta{}_{,j}\,. \tag{8.1}$$

Consider the formula:

$$\dot{\tilde a}_{\alpha\beta}(t) = \frac{\partial \tilde a_{\alpha\beta}(t)}{\partial y^\sigma} \dot z^\sigma. \tag{8.2}$$

Using (8.2) we obtain:

$$\dot W_1 = (\frac{\partial \tilde a_{\alpha\beta}(t)}{\partial y^\sigma}\dot z^\sigma - \frac{\partial \tilde a_{\alpha\beta}}{\partial y^\sigma}(0)\dot z^\sigma) g^{ij} y^\alpha{}_{,i} y^\beta{}_{,j} +$$

$$2\frac{\partial \tilde a_{\alpha\beta}(t)}{\partial y^\sigma}\dot z^\sigma g^{ij} y^\alpha{}_{,i} z^\beta{}_{,j} + 2(\tilde a_{\alpha\beta}(t) - \tilde a_{\alpha\beta}(0)) g^{ij} y^\alpha{}_{,i} \dot z^\beta{}_{,j} + \frac{\partial \tilde a_{\alpha\beta}(t)}{\partial y^\sigma}\dot z^\sigma g^{ij} z^\alpha{}_{,i} z^\beta{}_{,j} +$$

$$\tilde a_{\alpha\beta}(t) g^{ij} \dot z^\alpha{}_{,i} z^\beta{}_{,j} + \tilde a_{\alpha\beta}(t) g^{ij} z^\alpha{}_{,i} \dot z^\beta{}_{,j}\,. \tag{8.3}$$

§8.2. Formula of function $\dot W_2$.

$$\dot W_2 = \dot\Delta(g_{11})\Delta(g_{22}) + \Delta(g_{11})\dot\Delta(g_{22}) - 2\Delta(g_{12})\dot\Delta(g_{12}). \tag{8.4}$$

$$\dot\Delta(g_{ij}) = \tilde a_{\alpha\beta}(0) y^\alpha{}_{,i} \nabla^*_j \dot z^\beta + \tilde a_{\alpha\beta}(0) y^\beta{}_{,j} \nabla^*_i \dot z^\alpha + (\dot{\tilde a}_{\alpha\beta}(t) - \frac{\partial \tilde a_{\alpha\beta}}{\partial y^\sigma}(0)\dot z^\sigma) y^\alpha{}_{,i} y^\beta{}_{,j} +$$

$$\dot{\tilde a}_{\alpha\beta}(t)(y^\alpha{}_{,i} z^\beta{}_{,j} + y^\beta{}_{,j} z^\alpha{}_{,i}) + (\tilde a_{\alpha\beta}(t) - \tilde a_{\alpha\beta}(0))(y^\alpha{}_{,i} \dot z^\beta{}_{,j} + y^\beta{}_{,j} \dot z^\alpha{}_{,i}) +$$

$$\dot{\tilde a}_{\alpha\beta}(t) z^\alpha{}_{,i} z^\beta{}_{,j} + \tilde a_{\alpha\beta}(t) \dot z^\alpha{}_{,i} z^\beta{}_{,j} + \tilde a_{\alpha\beta}(t) z^\alpha{}_{,i} \dot z^\beta{}_{,j}\,. \tag{8.5}$$

Then we have:

$$\dot\Delta(g_{ij}) = \tilde a_{\alpha\beta}(0) y^\alpha{}_{,i} \nabla^*_j \dot z^\beta + \tilde a_{\alpha\beta}(0) y^\beta{}_{,j} \nabla^*_i \dot z^\alpha + (\frac{\partial \tilde a_{\alpha\beta}(t)}{\partial y^\sigma}\dot z^\sigma - \frac{\partial \tilde a_{\alpha\beta}}{\partial y^\sigma}(0)\dot z^\sigma) y^\alpha{}_{,i} y^\beta{}_{,j} +$$

$$\frac{\partial \tilde{a}_{\alpha\beta}(t)}{\partial y^\sigma} \dot{z}^\sigma (y^\alpha{}_{,i} z^\beta{}_{,j} + y^\beta{}_{,j} z^\alpha{}_{,i}) + (\tilde{a}_{\alpha\beta}(t) - \tilde{a}_{\alpha\beta}(0))(y^\alpha{}_{,i} \dot{z}^\beta{}_{,j} + y^\beta{}_{,j} \dot{z}^\alpha{}_{,i}) +$$

$$\frac{\partial \tilde{a}_{\alpha\beta}(t)}{\partial y^\sigma} \dot{z}^\sigma z^\alpha{}_{,i} z^\beta{}_{,j} + \tilde{a}_{\alpha\beta}(t) \dot{z}^\alpha{}_{,i} z^\beta{}_{,j} + \tilde{a}_{\alpha\beta}(t) z^\alpha{}_{,i} \dot{z}^\beta{}_{,j} \,. \tag{8.6}$$

§8.3. The inequalities for norms of functions $\dot{W}_1, \dot{W}_2, \dot{\Psi}_2$.

Denote:
$\|\nabla^* z\|^{(t)}_{m_1,\nu} = \max_{\alpha,i} \|\nabla^*_i z^\alpha\|^{(t)}_{m_1,\nu} = \max_{\alpha,i} \max_{\tau\in[0;t]} \|\nabla^*_i z^\alpha(\tau)\|_{m_1,\nu}$.
$\|\nabla^* \dot{z}\|^{(t)}_{m_1,\nu} = \max_{\alpha,i} \|\nabla^*_i \dot{z}^\alpha\|^{(t)}_{m_1,\nu} = \max_{\alpha,i} \max_{\tau\in[0;t]} \|\nabla^*_i \dot{z}^\alpha(\tau)\|_{m_1,\nu}$.

Lemma 8.3.1. *The following estimations hold:*

1) $\|\dot{W}_1\|^{(t)}_{m_1,\nu} \le M_{20}(\|\dot{z}\|^{(t)}_{m_1,\nu}\|z\|^{(t)}_{m_1,\nu} + \|\dot{z}\|^{(t)}_{m_1,\nu}\|\partial z\|^{(t)}_{m_1,\nu} + \|z\|^{(t)}_{m_1,\nu}\|\partial \dot{z}\|^{(t)}_{m_1,\nu} + \|\dot{z}\|^{(t)}_{m_1,\nu}(\|\partial z\|^{(t)}_{m_1,\nu})^2 + \|\partial \dot{z}\|^{(t)}_{m_1,\nu}\|\partial z\|^{(t)}_{m_1,\nu})$.

2) $\|\nabla^* z\|^{(t)}_{m_1,\nu} \le M_{21}(\|\partial z\|^{(t)}_{m_1,\nu} + \|z\|^{(t)}_{m_1,\nu})$.

3) $\|\nabla^* \dot{z}\|^{(t)}_{m_1,\nu} \le M_{22}(\|\partial \dot{z}\|^{(t)}_{m_1,\nu} + \|\dot{z}\|^{(t)}_{m_1,\nu})$.

4) $\|\dot{\Delta}(g_{ij})\|^{(t)}_{m_1,\nu} \le M_{23}(\|\nabla^* \dot{z}\|^{(t)}_{m_1,\nu} + \|\dot{z}\|^{(t)}_{m_1,\nu}\|z\|^{(t)}_{m_1,\nu} + \|\dot{z}\|^{(t)}_{m_1,\nu}\|\partial z\|^{(t)}_{m_1,\nu} + \|z\|^{(t)}_{m_1,\nu}\|\partial \dot{z}\|^{(t)}_{m_1,\nu} + \|\dot{z}\|^{(t)}_{m_1,\nu}(\|\partial z\|^{(t)}_{m_1,\nu})^2 + \|\partial \dot{z}\|^{(t)}_{m_1,\nu}\|\partial z\|^{(t)}_{m_1,\nu})$.

5) $\|\dot{W}_2\|^{(t)}_{m_1,\nu} \le M_{24}(\max_{i,j} \|\dot{\Delta}(g_{ij})\|^{(t)}_{m_1,\nu})(\max_{i,j} \|\Delta(g_{ij})\|^{(t)}_{m_1,\nu})$.

Proof of lemma follows from the forms of functions $W_1, W_2, \dot{W}_1, \dot{W}_2$, properties of space $C^{m_1,\nu}$ and previous lemmas.

Lemma 8.3.2. *Let the conditions of lemma 7.3. hold:*

1) metric tensor of R^3 *satisfies the conditions:* $\exists M_0 = const > 0$, *such that* $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial \tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2 \tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$.

2) $\exists t_0 > 0$, *such that* $c(t), c_{,i}(t), a^k(t), \partial_i a^k(t)$ *are continuous by* $t, \forall t \in [0, t_0]$, $c(0) \equiv 0, c_{,i}(0) \equiv 0, a^k(0) \equiv 0, \partial_i a^k(0) \equiv 0$.

3) $\exists t_0 > 0$, *such that* $z^\alpha(t) \in C^{m-2,\nu}$, $z^\alpha_{,i}(t) \in C^{m-3,\nu}$, $\forall t \in [0, t_0]$.

Then $\forall \varepsilon > 0 \exists t_0 > 0$, *such that*

1) $\|\dot{W}_1\|^{(t)}_{m-3,\nu} \le \varepsilon, \forall t \in [0, t_0]$.

2) $\|\dot{W}_2\|^{(t)}_{m-3,\nu} \le \varepsilon, \forall t \in [0, t_0]$.

3) $\|\dot{\Psi}_2\|^{(t)}_{m-3,\nu} \le \varepsilon, \forall t \in [0, t_0]$.

Proof of lemma follows from the form of functions $W_1, W_2, \dot{W}_1, \dot{W}_2$, properties of space $C^{m_1,\nu}$ and previous lemmas.

§9. Decidability of boundary-value problem A.

We have the following equation system of elliptic type:

$$\partial_2 \dot{a}^1 - \partial_1 \dot{a}^2 + p_k \dot{a}^k = \dot{\Psi}_1,$$

$$\partial_1 \dot{a}^1 + \partial_2 \dot{a}^2 + q^{(b)}_k \dot{a}^k = \dot{\Psi}^{(b)}_2, \tag{9.1}$$

where we use (6.2.53). $\dot{\Psi}_2^{(b)} = q_0^{(b)}\dot{c} - P_0$. Note that $q_k^{(b)}$ do not depend on t.

Without loss of generality we denote x^1 as x^2 and x^2 as x^1.

We write (9.1) as:

$$\partial_1 \dot{a}^1 - \partial_2 \dot{a}^2 + p_k \dot{a}^k = \dot{\Psi}_1,$$
$$\partial_2 \dot{a}^1 + \partial_1 \dot{a}^2 + q_k^{(b)} \dot{a}^k = \dot{\Psi}_2^{(b)}, \tag{9.2}$$

Define: $w = a^1 + ia^2, z = x^1 + ix^2$.

Therefore we have boundary-value problem for generalized analytic functions.

$$\partial_{\bar{z}}\dot{w} + A\dot{w} + B\bar{\dot{w}} = \dot{\Psi}_0, \qquad Re\{\overline{\lambda}\dot{w}\} = \dot{\varphi} \quad on \quad \partial D, \tag{9.3}$$

where

$$\partial_{\bar{z}}\dot{w} = \frac{1}{2}(\dot{w}_x + i\dot{w}_y), A = \frac{1}{4}(p_1 + q_2^{(b)} + iq_1^{(b)} - ip_2),$$
$$B = \frac{1}{4}(p_1 - q_2^{(b)} + iq_1^{(b)} + ip_2), \dot{\Psi}_0 = \frac{1}{2}(\dot{\Psi}_1 + i\dot{\Psi}_2^{(b)}). \tag{9.4}$$

We change the form of obtained boundary-value problem (9.4). Consider the following:

$$\dot{\Psi}_2^{(b)} = q_0^{(b)}\dot{c} - P_0 =$$
$$q_0^{(b)}\Big(\int\limits_{(x^1_{(0)},x^2_{(0)})}^{(x^1,x^2)} \Big(-V\dot{a}^1\Big)d\tilde{x}^1 + \Big(-V\dot{a}^2\Big)d\tilde{x}^2\Big) + q_0^{(b)}P(\dot{a}^1, \dot{a}^2) - P_0. \tag{9.5}$$

Denote:

$$\dot{\Psi}_3 = q_0^{(b)}P(\dot{a}^1, \dot{a}^2) - P_0. \tag{9.6}$$

We define: $\dot{\Psi} = \frac{1}{2}(\dot{\Psi}_1 + i\dot{\Psi}_3)$.

By (9.5), (9.6), the boundary-value problem (9.4) takes the form:

$$\partial_{\bar{z}}\dot{w} + A\dot{w} + B\bar{\dot{w}} + i\frac{q_0^{(b)}}{2}\int\limits_{(x^1_{(0)},x^2_{(0)})}^{(x^1,x^2)} \Big(V\dot{a}^1\Big)d\tilde{x}^1 + \Big(V\dot{a}^2\Big)d\tilde{x}^2 = \dot{\Psi}, \tag{9.7}$$

$Re\{\overline{\lambda}\dot{w}\} = \dot{\varphi}$ on ∂D.

Therefore we have:

$$\partial_{\bar{z}}\dot{w} + A\dot{w} + B\bar{\dot{w}} +$$
$$\int\limits_{(x^1_{(0)},x^2_{(0)})}^{(x^1,x^2)} \Big(i\frac{q_0^{(b)}(x^1,x^2)}{2}V(\tilde{x}^1,\tilde{x}^2)\dot{a}^1\Big)d\tilde{x}^1 + \Big(i\frac{q_0^{(b)}(x^1,x^2)}{2}V(\tilde{x}^1,\tilde{x}^2)\dot{a}^2\Big)d\tilde{x}^2 = \dot{\Psi}, \tag{9.8}$$

$Re\{\overline{\lambda}\dot{w}\} = \dot{\varphi}$ on ∂D.

Using the formulas: $\dot{a}^1 = \frac{1}{2}(\dot{w} + \bar{\dot{w}})$, $\dot{a}^2 = \frac{i}{2}(\bar{\dot{w}} - \dot{w})$,
and denoting: $E_0 = i\frac{q_0^{(b)}(x^1,x^2)}{2}V(\tilde{x}^1, \tilde{x}^2)$ we obtain the following form of desired boundary-value problem:

$$\partial_{\bar{z}}\dot{w} + A\dot{w} + B\bar{\dot{w}} + \int\limits_{(x^1_{(0)},x^2_{(0)})}^{(x^1,x^2)} \left(\frac{E_0}{2}(\dot{w} + \bar{\dot{w}})\right)d\tilde{x}^1 + \left(\frac{iE_0}{2}(\bar{\dot{w}} - \dot{w})\right)d\tilde{x}^2 = \dot{\Psi}, \tag{9.9}$$

$Re\{\overline{\lambda}\dot{w}\} = \dot{\varphi}$ on ∂D.

We denote:

$$E(\dot{w}) = \int\limits_{(x^1_{(0)},x^2_{(0)})}^{(x^1,x^2)} \left(\frac{E_0}{2}(\dot{w} + \bar{\dot{w}})\right)d\tilde{x}^1 + \left(\frac{iE_0}{2}(\bar{\dot{w}} - \dot{w})\right)d\tilde{x}^2. \tag{9.10}$$

Then we finally have the form of desired boundary-value problem:

$$\partial_{\bar{z}}\dot{w} + A\dot{w} + B\bar{\dot{w}} + E(\dot{w}) = \dot{\Psi}, \qquad Re\{\overline{\lambda}\dot{w}\} = \dot{\varphi} \quad on \quad \partial D. \tag{9.11}$$

Let, along the ∂F, be given vector field tangent to F. We denote it by the following formula:

$$v^\alpha = l^i y^\alpha_{,i}. \tag{9.12}$$

We consider the boundary-value condition:

$$\tilde{a}_{\alpha\beta} z^\alpha v^\beta = \tilde{\gamma}(s,t), s \in \partial D. \tag{9.13}$$

Define: $\tilde{\lambda}_k = \tilde{a}_{\alpha\beta} y^\alpha_{,k} v^\beta, k = 1,2$.
Then boundary condition takes the form: $Re\{(\dot{a^1} + i\dot{a^2})(\tilde{\lambda}_1 - i\tilde{\lambda}_2)\} = \dot{\tilde{\gamma}}$ on ∂F.

Denote: $\lambda_k = \frac{\tilde{\lambda}_k}{(\tilde{\lambda}_1)^2 + (\tilde{\lambda}_2)^2}, k = 1,2$. $\dot{\varphi} = \frac{\dot{\tilde{\gamma}}}{(\tilde{\lambda}_1)^2 + (\tilde{\lambda}_2)^2}$.

Then boundary-value condition takes the form: $Re\{\overline{\lambda}\dot{w}\} = \dot{\varphi}$ on ∂F, where $|\lambda| = 1$.

We analyze the decidability of the following boundary-value problem (A):

$$\partial_{\bar{z}}\dot{w} + A\dot{w} + B\bar{\dot{w}} + E(\dot{w}) = \dot{\Psi}, \qquad Re\{\overline{\lambda}\dot{w}\} = \dot{\varphi} \quad on \quad \partial D, \tag{9.14}$$

$\lambda = \lambda_1 + i\lambda_2$, $|\lambda| \equiv 1$, $\lambda, \dot{\varphi} \in C^{m-2,\nu}(\partial D)$.

We use the fact that $\dot{\Psi} = \dot{\Psi}(\dot{w}, z, t)$, $E(\dot{w}) = E(\dot{w}, z, t)$, $\dot{w} = \dot{w}(t)$,
$\dot{\varphi} = \dot{\varphi}(s,t), s \in \partial D$, $\lambda = \lambda(s), s \in \partial D$.

Let n be index of obtained boundary-value problem

$$n = \frac{1}{2\pi}\Delta_{\partial D} \arg \lambda(s). \tag{9.15}$$

Theorem 9.1. *Let t be fixed.*
Let $A(z), B(z), \dot{\Psi}(z) \in C^{m-3,\nu}(\bar{D})$, $\lambda(s), \dot{\varphi} \in C^{m-2,\nu}(\partial D)$, $|\lambda(s)| \equiv 1$.

Let $\dot\Psi(0,z)=0$, $\|\dot\Psi(\dot w_1,z)-\dot\Psi(\dot w_2,z)\|_{m-2,\nu}\le\mu(\rho)\|\dot w_1-\dot w_2\|_{m-1,\nu}$, *for* $\|\dot w_1\|_{m-2,\nu}\le\rho$, $\|\dot w_2\|_{m-2,\nu}\le\rho$, $\lim_{\rho\to 0}\mu(\rho)=0$.

Then, assuming that t *is fixed, the following holds:*

1) if $n\ge 0$ *then there exist* ρ *and* $\varepsilon(\rho)>0$ *such that for* $\|\dot\varphi\|_{m-2,\nu}\le\varepsilon$ *the boundary-value problem has* $(2n+1)-$*parametric solution of class* $C^{m-2,\nu}(\bar D)$ *for any admissible* $\dot\varphi$.

2) if $n<0$ *then there exist* $\rho>0$ *and* $\varepsilon(\rho)>0$ *such that for* $\|\dot\varphi\|_{m-2,\nu}\le\varepsilon(\rho)$ *the boundary-value problem has nor more than one solution of class* $C^{m-2,\nu}(\bar D)$ *for any admissible* $\dot\varphi$*. For* $\dot\varphi\equiv 0$ *boundary-value problem with condition:* $\|\dot w\|_{m-2,\nu}\le\rho$ *has only zero solution.*

Proof. Consider the following boundary-value problem (A_0):

$$\partial_{\bar z}\dot w+A\dot w+B\bar{\dot w}=\dot\Psi,\qquad Re\{\overline{\lambda}\dot w\}=\dot\varphi\quad on\quad \partial D, \tag{9.16}$$

$|\lambda|\equiv 1$, $\dot\varphi\in C^{m-2,\nu}(\partial D)$, $\dot\Psi\in C^{m-3,\nu}(\bar D)$.

Consider the operator:

$$I(\dot\Psi,z)=-\frac{1}{\pi}\int\int_D(\Omega_1(z,\zeta)\dot\Psi(\zeta)+\Omega_2(z,\zeta)\overline{\dot\Psi(\zeta)})d\xi d\eta,\zeta=\xi+i\eta, \tag{9.17}$$

where Ω_1,Ω_2 are principal kernels of the equation $\partial_{\bar z}\dot w+A(z)\dot w+B(z)\bar{\dot w}=0$.

It is well known [17,18] that operator $I(\dot\Psi,z)$ takes the form:

$$I(\dot\Psi,z)=T(\dot\Psi)-\frac{1}{\pi}\int\int_D(K_1(z,\zeta)\dot\Psi(\zeta)+\Omega_2(z,\zeta)\overline{\dot\Psi(\zeta)})d\xi d\eta,\zeta=\xi+i\eta, \tag{9.18}$$

$$T(\dot\Psi)=-\frac{1}{\pi}\int\int_D\frac{\dot\Psi(\zeta)}{\zeta-z}d\xi d\eta, \tag{9.19}$$

where operator $T(\dot\Psi)$ is completely continuous [17,18].

Consider the operator:

$$A(\dot\Psi,z)=I(\dot\Psi,z)+\int_{\partial D}Re\{\overline{\lambda(s)}I(\dot\Psi,s)\}M_0(z,s)ds, \tag{9.20}$$

where $M_0(z,s)$ is kernel of boundary-value problem (do not depend on $\dot\varphi$)

$$\partial_{\bar z}\dot w+A(z)\dot w+B(z)\bar{\dot w}=0, Re\{\overline{\lambda(s)}w(s)\}=\dot\varphi, s\in\partial D. \tag{9.21}$$

Consider the operator

$$A_2(\dot w)=A_1(\dot w)=A(\dot\Psi(\dot w,z)). \tag{9.22}$$

According to the results from [20], theorem 9.1. is valid for problem (A_0). For the case $n\ge 0$ problem (A_0) is solved as:

$$\dot w=A_2(\dot w)+\int_{\partial D}\dot\varphi(s)M_0(z,s)ds+\sum_{i=1}^{2n+1}c_i\dot w_i. \tag{9.23}$$

Therefore for the case $n \geq 0$ problem (A) is solved as:

$$\dot{w} = A_2(\dot{w}) + \int\limits_{\partial D} \dot{\varphi}(s) M_0(z,s) ds + \sum_{i=1}^{2n+1} c_i \dot{w}_i + A_2(E(\dot{w})). \qquad (9.24)$$

Then for this equation we use theory of Fredholm operator of index zero and theory of Volterra operator equation. Therefore we can solve (9.24) by the method of successive approximations.

For the case $n < 0$ we solve the problem (A_0) as equation system consisting of $-2n$ equations:

$$\dot{w} = A_2(\dot{w}) + \int\limits_{\partial D} \dot{\varphi}(s) M_0(z,s) ds, \qquad (9.25)$$

$$\int\limits_{\partial D} (\dot{\varphi}(s) + Re\{\overline{\lambda(s)} I(\dot{\Psi}, s)\}) \dot{w}'_j(s) \lambda(s) ds = 0, \quad j = \overline{1, -2n-1},$$

where $\dot{w}'_j$ are complete system of solutions of the following problem:

$$\partial_{\bar{z}} \dot{w}' - A(z) \dot{w}' - \bar{B}(z) \overline{\dot{w}'} = 0, \quad Re\{\lambda(z) \frac{dz(s)}{ds} \dot{w}'(z)\} = 0 \quad on \quad \partial D$$

Then for the case $n < 0$ we solve the problem (A) as equation system consisting of $-2n$ equations:

$$\dot{w} = A_2(\dot{w}) + \int\limits_{\partial D} \dot{\varphi}(s) M_0(z,s) ds + A_2(E(\dot{w})), \qquad (9.26)$$

$$\int\limits_{\partial D} (\dot{\varphi}(s) + Re\{\overline{\lambda(s)} I(\dot{\Psi}, s)\}) \dot{w}'_j(s) \lambda(s) ds = 0, \quad j = \overline{1, -2n-1},$$

where $\dot{w}'_j$ are complete system of solutions of the following problem:

$$\partial_{\bar{z}} \dot{w}' - A(z) \dot{w}' - \bar{B}(z) \overline{\dot{w}'} = 0, \quad Re\{\lambda(z) \frac{dz(s)}{ds} \dot{w}'(z)\} = 0 \quad on \quad \partial D$$

Then for this equation system we use theory of Fredholm operator of index zero and theory of Volterra operator equation.

By modifying standard method from [20], using the method of successive approximations and principle of contractive mapping, we obtain the proof theorem 9.1. for boundary-value problem (A).

Theorem 9.2. *Let* $F \in C^{m,\nu}, \nu \in (0;1), m \geq 4, \partial F \in C^{m+1,\nu}$.

Then the following holds:

1) if $n \geq 0$ *then there exists* $t_0 > 0$ *and exists* $\varepsilon(t_0) > 0$ *such that for* $\|\dot{\varphi}\|_{m-2,\nu} \leq \varepsilon$ *boundary-value problem (A) for all* $t \in [0, t_0)$ *has* $(2n+1)-$*parametric solution of class* $C^{m-2,\nu}(\bar{D})$ *continuous by* $t \in [0, t_0)$ *for any admissible* $\dot{\varphi}$.

2) if $n < 0$ *then exists* $t_0 > 0$ *and exists* $\varepsilon(t_0) > 0$ *such that for* $\|\dot{\varphi}\|_{m-2,\nu} \leq \varepsilon(t_0)$ *boundary-value problem (A) for all* $t \in [0, t_0)$ *has nor more than one solution of class* $C^{m,\nu}(\bar{D})$

continuous by $t \in [0, t_0)$ *for any admissible* $\dot{\varphi}$. *For* $\dot{\varphi} \equiv 0$ *the boundary-value problem has only zero solution.*

Proof follows from theorem 9.1., form of function $\dot{\Psi}$ and the fact that for all sufficiently small t the conditions of theorem 9.1 hold.

§10. Proof of theorem 1.

Proof of theorem 1 follows from theorem 9.2., formulas of $MG-$deformation and formulas of finding function $\dot{c}$ on functions $\dot{a}^j$. Using the condition of theorem 1: at the point $(x^1_{(0)}, x^2_{(0)})$ of the domain D, the following condition holds: $\forall t : a^i(t) \equiv 0, c(t) \equiv 0$. Therefore in case 1) $n > 0$ boundary-value problem (A) has $(2n-1)-$parametric solution. Using similar reasonings we prove theorem 1 for the cases 2) and 3).

The theorem 1 is proved.

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