Non-Abelian Global Strings at Chiral Phase Transition

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Abstract

We construct non-Abelian global string solutions in the $U(N)_{\mathrm{L}} \times U(N)_{\mathrm{R}}$ linear sigma model. These strings are the most fundamental objects which are expected to form during the chiral phase transitions, because the Abelian η' string is marginally decomposed into N of them. We point out Nambu-Goldstone modes of $\mathbf{C}P^{N-1}$ for breaking of $U(N)_{\mathrm{V}}$ arise around a non-Abelian vortex.

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1 Introduction

Formation of topological defects is inevitable when a symmetry is spontaneously broken at a phase transition. Especially strings [1] are often formed in various area of physics. They appear in the standard model [2], (SUSY) GUTs [3] and other models in particle physics, and superconductors, superfluids, nematic liquid crystals, cold atom and so on in condensed matter physics. In cosmology they are proposed as cosmic strings [1]. Depending on whether a broken symmetry is global, local (gauged) or both, strings are called local, global, or semi-local. Local and global $U(1)$ (namely Abelian) strings were extensively studied in cosmology because these cosmic strings were proposed to play a role in structure formation [1]. Although this possibility was rejected once, recently there have been a revival of cosmic strings from superstring theory, brane inflation and possible observation of these strings [4]. More recently non-Abelian local strings have been found in superstring theory [5] and in supersymmetric QCD [6]. These strings are BPS *i.e.* at the critical coupling, and so the most generic solutions and their moduli space have been obtained [7] in the moduli matrix formalism [8]. These strings have been proposed as non-Abelian cosmic strings; the reconnection (intercommutation) rate has been shown to be unity [9], and gravity coupling [10] and thermal effects [11] have been investigated. Furthermore non-Abelian *semi-local* strings have also been extensively studied [8, 12].

Another kind of non-Abelian strings not yet studied much is a non-Abelian *global* string. Non-Abelian global strings are not only theoretically interesting as a candidate for cosmic strings, but they might have been formed during the chiral phase transition of QCD where the chiral symmetry $U(N)_{\rm L} \times U(N)_{\rm R}$ is spontaneously broken to its diagonal subgroup $U(N)_{\rm V}$. Strings at this transition were first discussed by Brandenberger et al [13]–[17] where two types of global strings were proposed. One, the pion string, is made of the pions associated to broken $SU(N)_{\rm A}$ symmetry and is topologically unstable[1]. The other, an *Abelian* string called the η' string, is made of the η' meson associated to broken $U(1)_{\rm A}$ symmetry and is topologically stable at least at high temperature where the anomaly term disappears [19]. At low temperature the anomaly term arises, the η' meson acquires mass [20], and the string is attached by domain walls [21] as in the case of axion strings [1]. The existence of *non-Abelian* global strings was first pointed out by Balachandran and Digal in a seminal paper [22][2]. Non-Abelian strings are the most elementary objects at the chiral phase transition since the η' string is a composite state made of N of them. Although the authors in [22] constructed solutions with domain walls

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[1] Similar unstable string exists in the Skyrme model [18].

[2] Non-Abelian *global* strings with $SU(N)_{\rm L}$ gauged appear in high density QCD [23]. Similar global strings appear in the B-phase of ^{3}He superfluids, in which chiral symmetry is replaced by $U(1) \times SO(3)^2$ broken down to $SO(3)_{\rm V}$ [24], and in spinor Bose-Einstein condensate in cold atom.

in the presence of the anomaly, they did not construct pure string solutions in the absence of the anomaly.

In this Letter we construct non-Abelian string solutions with the axial symmetry in the $U(N)_\mathrm{L} \times U(N)_\mathrm{R}$ linear sigma model in the absence of the anomaly. Fine structure of these profile functions will be important to discuss the detailed dynamics of strings at the chiral phase transition. We find that Nambu-Goldstone bosons of $\mathbf{C}P^{N-1}$ appear due to the breaking of $U(N)_\mathrm{V}$ around the fundamental non-Abelian strings, as in the case of local strings [6]. The interaction of these strings with general orientations $\mathbf{C}P^{N-1}$ will be reported in [26].

2 The Linear Sigma Model

The Lagrangian of the linear sigma model is given by

$$\mathcal{L} = \mathrm{tr}\,(\partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi) - \lambda_1 [\mathrm{tr}\,(\Phi^\dagger \Phi)]^2 - \lambda_2\,\mathrm{tr}\,[(\Phi^\dagger \Phi)^2] \tag{1}$$

where Φ is an N by N matrix of complex scalar fields and λ_1 and λ_2 are the coupling constants [25]. The chiral symmetry $U(N)_\mathrm{L} \times U(N)_\mathrm{R}$ acts on Φ as

$$\Phi \to U_\mathrm{L} \Phi U_\mathrm{R}^\dagger, \quad (U_\mathrm{L}, U_\mathrm{R}) \in U(N)_\mathrm{L} \times U(N)_\mathrm{R}. \tag{2}$$

The static energy density is $\mathcal{E} = \mathrm{tr}(\partial_i \Phi^\dagger \partial_i \Phi) + V$ with the potential

$$V(\Phi^\dagger, \Phi) = m^2 \mathrm{tr}(\Phi^\dagger \Phi) + \lambda_1 [\mathrm{tr}(\Phi^\dagger \Phi)]^2 + \lambda_2 \mathrm{tr}(\Phi^\dagger \Phi)^2. \tag{3}$$

The field equations of Φ for static solutions read

$$\delta \mathcal{E}/\delta \Phi^\dagger = [-\nabla^2 + m^2 + 2\lambda_1 \mathrm{tr}(\Phi^\dagger \Phi) + 2\lambda_2 \Phi \Phi^\dagger]\Phi = 0. \tag{4}$$

We consider the parameter region

$$m^2 < 0, \quad \lambda_2 > 0, \quad N\lambda_1 + \lambda_2 > 0, \tag{5}$$

in which the vacuum $\langle \Phi \rangle$ is taken to be

$$\langle \Phi \rangle = \mathrm{diag}\,(v, \cdots, v), \quad v \equiv \sqrt{-m^2/2(N\lambda_1 + \lambda_2)} \tag{6}$$

by using the symmetry (2). The chiral symmetry (2) is spontaneously broken down to a diagonal $U(N)_\mathrm{V}$;

$$\langle \Phi \rangle \to U \langle \Phi \rangle U^\dagger, \quad U \in U(N)_\mathrm{V}. \tag{7}$$

This breaking results in the appearance of N^2 Nambu-Goldstone bosons associated with the coset space $[U(N)_\mathrm{L} \times U(N)_\mathrm{R}]/U(N)_\mathrm{V} \simeq U(N)_\mathrm{A}$, which are interpreted as pions and η' meson. (The latter is massive in the presence of the anomaly of $U(1)_\mathrm{A}$ [20].)

3 String Solutions

Since the fundamental homotopy group of the chiral symmetry breaking is nontrivial, $\pi_1(G/H) = \pi_1(U(N)_A) \simeq \mathbf{Z}$, there exist vortex-string solutions. Let us consider an axially symmetric ansatz with multiple winding numbers $n_i \in \mathbf{Z}$ $(i = 1, \cdots, N)$

$$\Phi = \mathrm{diag}\,(f_1(r)e^{in_1\theta}, f_2(r)e^{in_2\theta}, \cdots, f_N(r)e^{in_N\theta}) \tag{8}$$

where r and θ are radial and angular coordinates of two codimensions of vortices. We call the solution of this form an $(n_1, n_2, \cdots, n_N)$-vortex. The total winding number of π_1 is $\sum_i n_i$. With this ansatz, the tension, or the static energy per unit length, of strings is written as

$$E = 2\pi \sum_{i=1}^{N} \int \left[f_i'^2 + \frac{1}{r^2}(n_i^2 f_i^2) + m^2 f_i^2 + \lambda_1 (\sum_{j=1}^{N} f_j^2) f_i^2 + \lambda_2 f_i^4 \right] r\, dr\,. \tag{9}$$

The field equation (4) for f_i reduces to

$$0 = -f_i'' - \frac{1}{r} f_i' + \frac{n_i^2}{r^2} f_i + m^2 f_i + 2\lambda_1 (\sum_{j \neq i}^{N} f_j^2) f_i + 2(\lambda_1 + \lambda_2) f_i^3\,. \tag{10}$$

One can see from (10) that the asymptotic forms are

$$f_i = a_{n_i} r^{n_i} + O(r^{n_i+2}), \quad f_i = v + \frac{\alpha_i}{r^2} + O(r^{-4}) \tag{11}$$

for small and large r, respectively, where a_{n_i} is a shooting parameter determined by the boundary condition at infinity, and

$$\alpha_i = -\frac{n_i^2 v}{2pv^2 + m^2} - \frac{4\lambda_1(N-1)v^2}{2pv^2 + m^2} \sum_{j \neq i}^{N} \alpha_j\,, \tag{12}$$

with $p \equiv (N+2)\lambda_1 + 3\lambda_2$. Using these asymptotic forms, one can compute the leading order of the static energy:

$$E = \mathrm{const.} + 2\pi \sum_{i=1}^{N} n_i^2\, v \log\Lambda\,, \tag{13}$$

where Λ is an infrared cutoff parameter. It is logarithmically divergent as generally expected in the theory of global vortices. This formula implies that there exists repulsive

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force between vortices in the same component which is well-known in the case of Abelian strings [1], while no force is exerted between vortices in different components, which is a new feature of non-Abelian strings.

Abelian solutions are obtained by the $(1, 1, \cdots, 1)$ ansatz with the tension

$$E = \text{const.} + 2\pi N v \log\Lambda \,. \tag{14}$$

The profile is the same with that of the Abelian η' string [21]. The pions do not contribute to this string.

Genuine non-Abelian strings as the minimal solution are obtained by the $(1, 0, \cdots, 0)$ ansatz:

$$\Phi_0(r, \theta) = \text{diag}\left(f(r)e^{i\theta}, g(r), \cdots, g(r)\right). \tag{15}$$

The field equation in (10) becomes

$$0 = -f'' - \frac{1}{r}f' + \frac{1}{r^2}f + m^2 f + 2\lambda_1(N-1)g^2 f + 2(\lambda_1 + \lambda_2)f^3, \tag{16}$$

$$0 = -g'' - \frac{1}{r}g' + m^2 g + 2\lambda_1 f^2 g + 2[(N-1)\lambda_1 + \lambda_2]g^3 \,. $$

The tension

$$E = \text{const.} + 2\pi v \log\Lambda \tag{17}$$

is $1/N$ of (14) for the Abelian η' string, so non-Abelian strings are sometimes called $1/N$ (fractional) strings. It implies that an η' string can be marginally decomposed into N non-Abelian strings without the cost of energy. The fractional property of $1/N$ strings comes from the fact that the string (15) winds around a circle in $U(N)_\text{A}$ generated by a linear combination of $U(1)_\text{A}$ and $T \sim \text{diag}(1 - N, 1, \cdots, 1)$ of $SU(N)_\text{A}$ [22]. Therefore $1/N$ strings are composed of both η' and pions.

One can see from Eqs. (16) that when $\lambda_1 = 0$ the profile g is flat, $g = v$, and the profile f is identical to that of the Abelian string. In this case, a non-Abelian solution is simply obtained by embedding an Abelian string to the corner of the N by N matrix Φ in (6).

Let us discuss zero modes in the presence of the string (15). In addition to two translational zero modes, some internal zero modes appear because the solution (15) breaks vacuum symmetry $U(N)_\text{V}$ given in (7) down to its subgroup $U(N-1) \times U(1)$. The Nambu-Goldstone modes for this breaking parametrize internal space of the coset space

$$\mathbf{C}P^{N-1} = \frac{U(N)_\text{V}}{U(N-1)_\text{V} \times U(1)_\text{V}}, \tag{18}$$

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which is known as the complex projective space. This was pointed out [6] in the case of local non-Abelian strings. More explicitly, the most general minimal solution is given by $U\Phi_0(r,\theta)U^\dagger$ with Φ_0 in Eq. (15) and U the elements of $\mathbf{C}P^{N-1}$. For instance $\Phi_0 = \mathrm{diag}(fe^{i\theta}, g)$ for $N = 2$ is transformed by $U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$ (with $|\alpha|^2 + |\beta|^2 = 1$) to $\begin{pmatrix} |\alpha|^2 fe^{i\theta} + |\beta|^2 g & \alpha\beta(-fe^{i\theta} + g) \\ \alpha^*\beta^*(-fe^{i\theta} + g) & |\beta|^2 fe^{i\theta} + |\alpha|^2 g \end{pmatrix}$. This does not depend on the overall phase of α, β, so the space is $\mathbf{C}P^1 \simeq S^2$. These modes are non-normalizable (in an infinite region) whereas the corresponding modes are normalizable in the local case [6]. However in a finite volume they are normalizable and are relevant to dynamics.

4 Numerical computation

In this section we show the numerical results for the fundamental string $(1, 0, 0)$ and a composite string $(1, 1, 0)$ for $N = 3$. For that purpose, let us introduce dimensionless quantities,

$$\hat{x} = \hat{m}x, \quad \hat{\Phi} = \frac{\sqrt{\lambda_2}}{\hat{m}}\Phi; \quad \hat{m} \equiv \sqrt{-m^2}. \tag{19}$$

The Lagrangian (1) is rescaled as

$$\mathcal{L} = \mathrm{tr}\,(\partial_\mu \hat{\Phi}^\dagger \partial^\mu \hat{\Phi} + \hat{\Phi}^\dagger \hat{\Phi}) - \kappa\,[\mathrm{tr}\,(\hat{\Phi}^\dagger \hat{\Phi})]^2 - \mathrm{tr}\,(\hat{\Phi}^\dagger \hat{\Phi})^2 \tag{20}$$

where from Eq. (5) we have $\kappa \equiv \lambda_1/\lambda_2 > -1/N$.

A. The $(1, 0, 0)$-vortex. The ansatz is given by

$$\hat{\Phi}(r, \theta) = (\hat{f}(\hat{r})e^{i\theta}, \hat{g}(\hat{r}), \hat{g}(\hat{r})), \tag{21}$$

with $\hat{r} = \hat{m}r$ and $\hat{f}$ and $\hat{g}$ rescaled as $\hat{\Phi}$. The field equations (16) become

$$0 = -\hat{f}'' - \frac{1}{\hat{r}}\hat{f}' + \frac{1}{\hat{r}^2}\hat{f} - \hat{f} + 4\kappa\hat{g}^2\hat{f} + 2\,(1 + \kappa)\,\hat{f}^3,$$

$$0 = -\hat{g}'' - \frac{1}{\hat{r}}\hat{g}' - \hat{g} + 2\kappa\hat{f}^2\hat{g} + 2(1 + 2\kappa)\hat{g}^3 \tag{22}$$

with prime denoting differentiation with respect to $\hat{r}$. The asymptotic forms for small $\hat{r}$ are derived as

$$\hat{f} = a_1\hat{r} + a_3\hat{r}^3 + O(\hat{r}^4), \quad \hat{g} = b_0 + b_2\hat{r}^2 + O(\hat{r}^3) \tag{23}$$
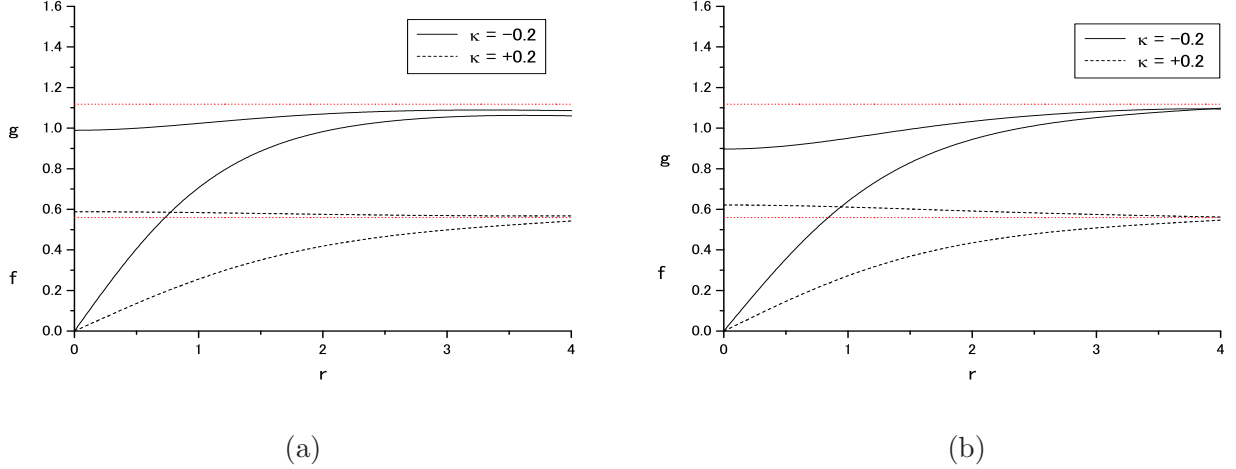
Figure 1: The profile functions f and g as functions of r (with all hats omitted) in the cases of (a) $\kappa = -0.2, 0.2$ for the $(1,0,0)$-vortex, and (b) $\kappa = -0.2, 0.2$ for the $(1,1,0)$-vortex. The horizontal broken lines denote v to which f and g converge.

with a_1 and b_0 are shooting parameters and

$$a_3 = \frac{1}{8}(4\kappa b_0^2 - 1)a_1, \quad b_2 = \frac{1}{4}\left[2(1 + 2\kappa)b_0^2 - 1\right]b_0. \tag{24}$$

Fig. 1-(a) shows the profile functions for the $(1,0,0)$-vortex. The g changes its behavior drastically depending on the sign of κ: it is concave, constant ($g = v$) or convex, for $\kappa < 0$, $\kappa = 0$ or $\kappa > 0$, respectively. For the realistic case of the chiral symmetry breaking in the absence of anomaly κ is negative [25].

B. The $(1,1,0)$-vortex. The ansatz is given by

$$\hat{\Phi}(\hat{r},\theta) = (\hat{f}(\hat{r})e^{i\theta}, \hat{f}(\hat{r})e^{i\theta}, \hat{g}(\hat{r})) \tag{25}$$

The corresponding field equations are

$$0 = -\hat{f}'' - \frac{1}{\hat{r}}\hat{f}' + \frac{1}{\hat{r}^2}\hat{f} - \hat{f} + 2\kappa\hat{g}^2\hat{f} + 2(1 + 2\kappa)\hat{f}^3,$$

$$0 = -\hat{g}'' - \frac{1}{\hat{r}}\hat{g}' - \hat{g} + 4\kappa\hat{f}^2\hat{g} + 2(1 + \kappa)\hat{g}^3. \tag{26}$$

The asymptotic forms for small $\hat{r}$ are given by (23) with

$$a_3 = \frac{1}{8}(2\kappa b_0^2 - 1)a_1, \quad b_2 = \frac{1}{4}\left[2(1 + \kappa)b_0^2 - 1\right]b_0. \tag{27}$$

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The profiles behave similarly to the $(1, 0, 0)$-vortex profiles as shown in Fig. 1-(b).

The both figures show that the transverse size of the strings is of order $\hat{x} \sim 1$ $(x \sim \hat{m}^{-1})$ as expected. From the profile $\hat{g}$, we confirm that unlike the Abelian case the chiral symmetry is recovered only partially in the core of strings as speculated in [22]; $U(1)_\mathrm{A}$ for the $(1, 0, 0)$ vortex and $U(2)_\mathrm{A}$ for the $(1, 1, 0)$ vortex.

5   Conclusion and Discussion

Taking examples of $(1, 0, 0)$ and $(1, 1, 0)$ vortices in the $N = 3$ model, we have constructed axially symmetric non-Abelian string solutions in the linear sigma model. The numerical result shows that the profile function g without a winding is concave, flat or convex with respect to the radius, depending on the sign of the coupling constant κ. The profile function f with non-zero winding number vanishes at the center of the string as expected. We have also shown that additional Nambu-Goldstone bosons of $\mathbf{C}P^{N-1}$ appear around the fundamental non-Abelian strings, as observed in the local strings [6].

While Abelian global strings emit the $U(1)$ Nambu-Goldstone bosons [1], non-Abelian strings will emit both the η' meson and the pions when they oscillate. We expect that this radiation gives a signal which can be detected by a heavy-ion collider or in the early Universe. Interactions between two strings will be described by a direct calculation [26] or by exchange of the pions. Interaction between a string and baryons as Skyrmions is also an interesting problem.

The obvious extension in the context of the chiral symmetry breaking is the inclusion of the anomaly term [21, 22] and the pion masses. In application to cosmology, a calculation of the reconnection (intercommutation) rate will be important. These strings may play some roles in the early Universe such as structure formation and the primordial magnetic field (see [14] for the Abelian case). The ring of vortex can be a candidate of the dark matter (see [16] for the pion string). The thermal effect and gravity coupling also remain as future problems.

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