A mean ergodic theorem for actions of amenable quantum groups

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Abstract
We prove a weak form of the mean ergodic theorem for actions of amenable locally compact quantum groups in the von Neumann algebra setting.

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1 Introduction

The following mean ergodic theorem is well-known: Let G be a locally compact group with right Haar measure μ, and assume that it contains a Følner net (Λ_λ), i.e. a net of Borel sets in G such that $0 < \mu(\Lambda_\lambda) < \infty$ and $\lim_\lambda \mu(\Lambda_\lambda \Delta (\Lambda_\lambda g)) / \mu(\Lambda_\lambda) = 0$ for all $g \in G$. Furthermore, let U_g be a contraction on a Hilbert space H such that $U_g U_h = U_{gh}$ for all $g, h \in G$, and $G \ni g \mapsto \langle U_g x, y \rangle$ is Borel measurable for all $x, y \in H$. Take P to be the projection of H onto $V := \{x \in \mathfrak{H} : U_g x = x \text{ for all } g \in G\}$. Then

$$\lim_\lambda \frac{1}{\mu(\Lambda_\lambda)} \int_{\Lambda_\lambda} U_g x d\mu(g) = Px \tag{1.1}$$

for all $x \in H$. A standard proof for the case $G = \mathbb{Z}$ can be found for example in [4] and [9], but it can be extended to the more general case without much effort.

In this paper we prove a version of this theorem for the action of an amenable locally compact quantum group on a von Neumann algebra. We use the von Neumann algebra setting for quantum groups, as developed by Kusterman and Vaes [8] building on earlier work on Kac algebras (see for example [3]).

In this setting a *locally compact quantum group* is defined to be a von Neumann algebra M with a unital normal $*$-homomorphism $\Delta : M \to M \otimes M$ (where $M \otimes N$ denotes the von Neumann algebraic tensor product of two von Neumann algebras), such that $(\Delta \otimes \iota_M) \circ \Delta = (\iota_M \otimes \Delta) \circ \Delta$ (where ι_M denotes the identity map on M), and on which there exists normal semi-finite faithful (n.s.f.) weights φ and ψ such that $\varphi((\theta \otimes \iota_M) \circ \Delta(a)) = \varphi(a)\theta(1)$ for all $a \in \mathcal{M}_\varphi^+$ and $\psi((\iota_M \otimes \theta) \circ \Delta(a)) = \psi(a)\theta(1)$ for all $a \in \mathcal{M}_\psi^+$, for all $\theta \in M_*^+$, where M_*^+ is the positive normal linear functionals on M, and $\mathcal{M}_\varphi^+ = \{a \in M^+ : \varphi(a) < \infty\}$. This quantum group is denoted as (M, Δ). We refer the reader to the papers [5, 6, 7] for background and motivation for this definition. If furthermore there exists a net (φ_λ) of normal states on M such that $\|\theta * \varphi_\lambda - \varphi_\lambda\|$ converges to 0 for all $\theta \in M_*$ with $\theta(1) = 1$, then we call (M, Δ) *amenable*; see for example [2]. Here $\mu * \nu := (\mu \otimes \nu) \circ \Delta$ for any $\mu, \nu \in M_*^+$.

An *action* of (M, Δ) on another von Neumann algebra A is defined to be a normal injective unital $*$-homomorphism $\alpha : A \to M \otimes A$ such that $(\iota_M \otimes \alpha) \circ \alpha = (\Delta \otimes \iota_A) \circ \alpha$; see [12].

Given such an action, we will assume the presence of a normal state ω on A which is *invariant* under the action, by which we mean that $(\theta \otimes \omega) \circ \alpha = \omega$ for all normal states θ on M. In Section 2 we show how to set up the analogue of the integral in (1.1) for a quantum group action, and in Section 3 we state and prove a mean ergodic theorem for such actions, however only in a weak form analogous to

$$\lim_\lambda \left\langle x, \frac{1}{\mu(\Lambda_\lambda)} \int_{\Lambda_\lambda} U_g y d\mu(g) \right\rangle = \langle x, Py \rangle$$

for all $x, y \in H$. Our approach is to set the problem up in a suitable Hilbert space framework, closely related to that of (1.1), and then to follow the basic structure of (1.1)'s proof.

We will not need the full force of the theory of locally compact quantum groups as developed in [7, 8], and therefore it will be convenient to formulate our results in an abstract setting incorporating only the concepts from locally compact quantum groups that we need, modelled on the definitions discussed above. We will focus on this abstract setting, rather than on concrete examples.

2 A suitable integration theory

In this section we develop the tools and notation that we need in order to formulate and prove the mean ergodic theorem in the next section. Throughout this section and the next we will use the following notation: R will be an arbitrary von Neumann algebra, its unit denoted by 1_R, and its normal states by $(R_*^+)_1$. By ω we will mean an arbitrary normal state on a von Neumann algebra A. We will denote the GNS construction of (A,ω) by (H,γ), by which we mean that H is a Hilbert space and $\gamma : A \to H$ a linear mapping such that $\langle \gamma(a), \gamma(b) \rangle = \omega(a^* b)$ and with $\gamma(A)$ dense in H.

We remind the reader that we will use the notation $R \otimes A$ to indicate the von Neumann algebraic tensor product, often written as $R \overline{\otimes} A$ in the literature. The algebraic tensor product will be written as $R \odot A$. We will constantly use tensor products of mappings on von Neumann algebras, and a useful reference regarding this topic is [11]. For example, if θ is a normal state on R while ι_A is the identity map $A \to A$, then we can define $\theta \otimes \iota_A : R \otimes A \to A$ as the tensor product of conditional expectations, in which case $\theta \otimes \iota_A$ itself is a conditional expectation, which is also normal, i.e. σ-weakly continuous; see [11, Section 9].

We are going to view R as a noncommutative measurable space, and roughly speaking we will be integrating A valued "functions" over R.

Note that the integral in (1.1) is an integral of a bounded function $f : G \to H$ which can be defined via the Riesz representation theorem by $\left\langle \int_\Lambda f d\mu, x \right\rangle = \int_\Lambda \langle f(g), x \rangle \, d\mu(g)$. We now mimic this construction for A valued "functions" on R, in other words for elements of $R \otimes A$.

Proposition 2.1. *Let μ be a normal positive linear functional on R. Then there is a unique function*

$$\tilde{\mu} : R \otimes A \to H$$

such that

$$\langle \gamma(d), \tilde{\mu}(T) \rangle = \mu \otimes \omega \left([1_R \otimes d]^* T \right) \tag{2.1}$$

for all $T \in R \otimes A$ and $d \in A$. Furthermore, $\tilde{\mu}$ is linear, $\|\tilde{\mu}\| \leq \|\mu\|$ and

$$\langle \gamma(d), \tilde{\mu}(T) \rangle = \omega \left(d^* \left(\mu \otimes \iota_A \right) (T) \right)$$

for all $T \in R \otimes A$ and $d \in A$.

Proof. For any $T \in R \otimes A$, define the linear functional

$$f_T : \gamma(A) \to \mathbb{C} : \gamma(d) \mapsto \overline{\mu \otimes \omega \left([1_R \otimes d]^* T \right)}$$

which is indeed well defined, since if $\gamma(d) = 0$ we have $\mu \otimes \omega \left([1_R \otimes d]^* T \right) = 0$ as follows: First consider any $T = \sum_{j=1}^n r_j \otimes a_j \in R \odot A$, then

$$\left| \mu \otimes \omega \left([1_R \otimes d]^* T \right) \right| \leq \sum_{j=1}^n \left| \mu \left(r_j \right) \right| \left| \omega \left(d^* a_j \right) \right|$$

but $|\omega(d^*a_j)| \leq \sqrt{\omega(d^*d)}\sqrt{\omega(a_j^*a_j)} = 0$, since $\omega(d^*d) = \|\gamma(d)\|^2$, therefore $\mu \otimes \omega([1_R \otimes d]^* T) = 0$. For a general $T \in R \otimes A$ there is a net $T_\lambda \in R \odot A$ converging σ-weakly to T, according to von Neumann's density theorem (see for example [1, Section 2.4.2]). Hence $[1_R \otimes d]^* T_\lambda$ converges σ-weakly to $[1_R \otimes d]^* T$, but $\mu \otimes \omega$ is σ-weakly continuous (i.e. normal), so $\mu \otimes \omega([1_R \otimes d]^* T) = 0$.

Clearly f_T is linear, and $\|f_T\| \leq \|\mu\| \|T\|$ since

$$\begin{aligned} |f_T(\gamma(d))| &\leq \sqrt{\mu \otimes \omega([1_R \otimes d]^*[1_R \otimes d])}\sqrt{\mu \otimes \omega(T^*T)} \\ &\leq \sqrt{\mu(1_R)}\sqrt{\omega(d^*d)}\sqrt{\|\mu \otimes \omega\| \|T^*T\|} \\ &= \sqrt{\|\mu\|} \|\gamma(d)\| \sqrt{\|\mu\|} \|T\| \end{aligned}$$

Therefore f_T can be linearly extended uniquely to H without changing its norm. By the Riesz represention theorem and since $\gamma(A)$ is dense in H, there is a unique element $\tilde{\mu}(T)$ in H such that $f_T(\gamma(d)) = \langle \tilde{\mu}(T), \gamma(d) \rangle$ for all $d \in A$. Furthermore $\|\tilde{\mu}(T)\| = \|f_T\|$. Hence we obtain a unique function $\tilde{\mu} : R \otimes A \to H$ such that (2.1) holds. Clearly $\tilde{\mu}$ is linear and $\|\tilde{\mu}(T)\| \leq \|\mu\| \|T\|$.

Lastly, for $r \in R$ and $a \in A$ we have

$$\mu \otimes \omega([1_R \otimes d]^*(r \otimes a)) = \omega(d^*(\mu \otimes \iota_A)(r \otimes a))$$

hence $\mu \otimes \omega([1_R \otimes d]^* T) = \omega(d^*(\mu \otimes \iota_A) T)$ for all $T \in R \odot A$ by linearity. But again by σ-denseness, and by σ-weak continuity, this extends to all $T \in R \otimes A$. □

We now take this a step further by finding an analogue of the linear operator $H \to H : x \mapsto \int_\Lambda U_g x d\mu(g)$ that appears in (1.1).

Proposition 2.2. *Consider the situation in Proposition 2.1 and furthermore assume that we have a $*$-homomorphism $\alpha : A \to R \otimes A$ which leaves ω invariant in the following sense:*

$$(\mu \otimes \omega) \circ \alpha = \mu(1_R)\,\omega \tag{2.2}$$

for the given μ. Then there exists a unique linear operator $\tilde{\mu}^\alpha : H \to H$ such that

$$\tilde{\mu}^\alpha(\gamma(a)) = \tilde{\mu} \circ \alpha(a)$$

for all $a \in A$. Furthermore, $\|\tilde{\mu}^\alpha\| \leq \|\mu\|$, and if α is unital, then $\|\tilde{\mu}^\alpha\| = \|\mu\|$.

Proof. The operator $\tilde{\mu}^{\alpha}$ is well defined on $\gamma(A)$ since $\tilde{\mu} \circ \alpha(a) = 0$ when $\gamma(a) = 0$ as we now show: For any $d \in A$ we have from Proposition 2.1 that

$$\begin{aligned} |\langle \gamma(d), \tilde{\mu} \circ \alpha(a) \rangle|^2 &= \left| \mu \otimes \omega \left([1_R \otimes d]^2 \alpha(a) \right) \right|^2 \\ &\leq \mu \otimes \omega \left([1_R \otimes d]^* [1_R \otimes d] \right) \mu \otimes \omega \left(\alpha(a^* a) \right) \\ &= 0 \end{aligned}$$

by (2.2) and since $\omega(a^*a) = \|\gamma(a)\|^2 = 0$. But $\gamma(A)$ is dense in H, so $\tilde{\mu} \circ \alpha(a) = 0$. Clearly $\tilde{\mu}^{\alpha}$ is linear, and as in the above calculation we have for any $a, d \in A$ that

$$|\langle \gamma(d), \tilde{\mu}^{\alpha}(\gamma(a)) \rangle| \leq \mu(1_R) \|\gamma(d)\| \|\gamma(a)\|$$

so $\|\tilde{\mu}^{\alpha}\| \leq \mu(1_R) = \|\mu\|$. Hence $\tilde{\mu}^{\alpha}$ has a unique bounded linear extension to H, with the same norm. If α is unital, then by (2.1) we have

$$\begin{aligned} \langle \gamma(d), \tilde{\mu}^{\alpha}(\gamma(1_A)) \rangle &= \langle \gamma(d), \tilde{\mu}(1_R \otimes 1_A) \rangle \\ &= \mu(1_R) \omega(d^* 1_A) \\ &= \langle \gamma(d), \mu(1_R) \gamma(1_A) \rangle \end{aligned}$$

so $\tilde{\mu}^{\alpha}(\gamma(1_A)) = \mu(1_R) \gamma(1_A)$ from which $\|\tilde{\mu}^{\alpha}\| = \|\mu\|$ follows. □

Lastly we will need the following important property in the proof of the mean ergodic theorem. Note that by a *normal* $*$-homomorphism from one von Neumann algebra to another, we mean a $*$-homomorphism that is σ-weakly continuous.

Proposition 2.3. *Consider the situation in Propositions 2.1 and 2.2. Furthermore, let ν be another normal positive linear functional on R satisfying $(\nu \otimes \omega) \circ \alpha = \nu(1_R) \omega$. Also assume that α is normal, and that $\Delta : R \to R \otimes R$ is a normal $*$-homomorphism such that*

$$(\iota_R \otimes \alpha) \circ \alpha = (\Delta \otimes \iota_A) \circ \alpha$$

Write

$$\mu * \nu := (\mu \otimes \nu) \circ \Delta$$

then it follows that

$$\widetilde{\mu * \nu}^{\alpha} = \tilde{\nu}^{\alpha} \tilde{\mu}^{\alpha}$$

Proof. For any $r \in R$ and $a \in A$ we have for all $d \in A$ that

$$\begin{aligned} \omega\left(d^* [\mu \otimes (\nu \otimes \iota_A)] \circ (\iota_R \otimes \alpha)(r \otimes a)\right) &= \omega\left(d^* \mu(r) (\nu \otimes \iota_A) \circ \alpha(a)\right) \\ &= \langle \gamma(d), \tilde{\nu} \circ \alpha(\mu(r) a) \rangle \\ &= \langle \gamma(d), \tilde{\nu}^{\alpha}(\tilde{\mu}(r \otimes a)) \rangle \end{aligned}$$

by Propositions 2.1 and 2.2, hence by linearity

$$\omega\left(d^*\left[\mu\otimes(\nu\otimes\iota_A)\right]\circ(\iota_R\otimes\alpha)(T)\right)=\left\langle(\tilde{\nu}^\alpha)^*\gamma(d),\tilde{\mu}(T)\right\rangle \tag{2.3}$$

for all $T\in R\odot A$. The left hand side of (2.3) is a σ-weakly continuous linear functional of $T\in R\otimes A$, since $\iota_R\otimes\alpha$ is the tensor product of two σ-weakly continuous $*$-homomorphisms, and $(\mu/\|\mu\|)\otimes((\nu/\|\nu\|)\otimes\iota_A)$ that of two σ-weakly continuous conditional expectations (the case $\mu=0$ or $\nu=0$ being trivial). The right hand side of (2.3) is also a σ-weakly continuous linear functional of $T\in R\otimes A$. To see this, consider any net $T_\lambda\in R\otimes A$ converging σ-weakly to T. For any $c\in A$ one has

$$\langle\gamma(c),\tilde{\mu}(T_\lambda)\rangle=\mu\otimes\omega\left([1_R\otimes c]^*T_\lambda\right)\to\mu\otimes\omega\left([1_R\otimes c]^*T\right)=\langle\gamma(c),\tilde{\mu}(T)\rangle$$

in the λ limit, since $\mu\otimes\omega$ is σ-weakly continuous. However, the σ-weak topology is a weak* topology, hence by the resonance theorem (uniform boundedness) the net (T_λ) is bounded in the norm of $R\otimes A$. Since $\gamma(A)$ is dense in H, it therefore follows that $\langle x,\tilde{\mu}(T_\lambda)\rangle\to\langle x,\tilde{\mu}(T)\rangle$ for all $x\in H$, so indeed (2.3)'s right hand side is σ-weakly continuous in T. But $R\odot A$ is σ-weakly dense in $R\otimes A$, therefore (2.3) holds for all $T\in R\otimes A$, in particular for $T=\alpha(a)$, so

$$\begin{aligned}\langle\gamma(d),\tilde{\nu}^\alpha\tilde{\mu}^\alpha(\gamma(a))\rangle&=\omega\left(d^*\left[\mu\otimes(\nu\otimes\iota_A)\right]\circ(\iota_R\otimes\alpha)\circ\alpha(a)\right)\\&=\omega\left(d^*\left[(\mu\otimes\nu)\otimes\iota_A\right]\circ(\Delta\otimes\iota_A)\circ\alpha(a)\right)\\&=\omega\left(d^*\left\{\left[(\mu\otimes\nu)\circ\Delta\right]\otimes\iota_A\right\}\circ\alpha(a)\right)\\&=\left\langle\gamma(d),\widetilde{\mu*\nu}^\alpha(\gamma(a))\right\rangle\end{aligned}$$

for any $a\in A$, by Propositions 2.1 and 2.2, and since Δ is normal (which ensures that $[(\mu\otimes\nu)\otimes\iota_A]\circ(\Delta\otimes\iota_A)=[(\mu\otimes\nu)\circ\Delta]\otimes\iota_A$ on $R\otimes A$). Since $\gamma(A)$ is dense in H, we obtain $\tilde{\nu}^\alpha\tilde{\mu}^\alpha=\widetilde{\mu*\nu}^\alpha$. □

3 The mean ergodic theorem

Continuing with Section 2's notation, we can now formulate and prove a mean ergodic theorem:

Theorem 3.1. *Consider two normal $*$-homomorphisms $\Delta:R\to R\otimes R$ and $\alpha:A\to R\otimes A$ such that $(\iota_R\otimes\alpha)\circ\alpha=(\Delta\otimes\iota_A)\circ\alpha$ and $(\theta\otimes\omega)\circ\alpha=\omega$ for all $\theta\in(R_*^+)_1$. Assume the existence of a net (φ_λ) in $(R_*^+)_1$ such that $\|\theta*\varphi_\lambda-\varphi_\lambda\|\to0$ for all $\theta\in(R_*^+)_1$. Let P the projection of H on $V:=\left\{x\in H:\tilde{\theta}^\alpha x=x \text{ for all } \theta\in(R_*^+)_1\right\}$. Then*

$$\lim_\lambda\langle x,\tilde{\varphi}_\lambda^\alpha y\rangle=\langle x,Py\rangle$$

for all $x, y \in H$.

Proof. Set

$$N = \overline{\operatorname{span}\left\{x - \tilde{\theta}^{\alpha}x : x \in H, \theta \in \left(R_*^+\right)_1\right\}}$$

and note that $\left\|\tilde{\theta}^{\alpha}\right\| \leq \|\theta\| = 1$, i.e. $\tilde{\theta}^{\alpha}$ is a contraction, then by a standard argument $N = V^{\perp}$ (see for example [4, Section 1.1]). Keep in mind that $(R \otimes A)_* = R_* \otimes_* A_*$ where by $\otimes_*$ we mean the tensor product of Banach spaces with the completion taken in the dual norm of the spatial C*-norm on $R \odot A$ (see for example [10, Section 1.22]); this will be useful in the following calculation. Note that this dual norm is a cross norm. For any $a, d \in A$ and $\theta \in (R_*^+)_1$ it follows from Proposition 2.3 that

$$\begin{aligned}
&\left|\left\langle \gamma(d), \tilde{\varphi}_{\lambda}^{\alpha}\left(\gamma(a) - \tilde{\theta}^{\alpha}\gamma(a)\right)\right\rangle\right| \\
&= \left|\left\langle \gamma(d), \tilde{\varphi}_{\lambda}^{\alpha}\gamma(a) - \widetilde{\theta * \varphi_{\lambda}}^{\alpha}\gamma(a)\right\rangle\right| \\
&= \left|\varphi_{\lambda} \otimes \omega\left([1_R \otimes d]^* \alpha(a)\right) - (\theta * \varphi_{\lambda}) \otimes \omega\left([1_R \otimes d]^* \alpha(a)\right)\right| \\
&= \left|(\varphi_{\lambda} - \theta * \varphi_{\lambda}) \otimes \omega\left([1_R \otimes d]^* \alpha(a)\right)\right| \\
&\leq \|\varphi_{\lambda} - \theta * \varphi_{\lambda}\| \, \|\omega\| \, \|[1_R \otimes d]^* \alpha(a)\| \\
&\to 0
\end{aligned}$$

Furthermore $\left\|\tilde{\varphi}_{\lambda}^{\alpha} - \tilde{\varphi}_{\lambda}^{\alpha}\tilde{\theta}^{\alpha}\right\| \leq 2$ by Proposition 2.2, so $\left(\tilde{\varphi}_{\lambda}^{\alpha} - \tilde{\varphi}_{\lambda}^{\alpha}\tilde{\theta}^{\alpha}\right)$ is a bounded net, while $\gamma(A)$ is dense in H, hence

$$\left\langle x, \tilde{\varphi}_{\lambda}^{\alpha}\left(y - \tilde{\theta}^{\alpha}y\right)\right\rangle \to 0$$

for all $x, y \in H$ and $\theta \in (R_*^+)_1$. Since $\|\tilde{\varphi}_{\lambda}^{\alpha}\| \leq 1$, we conclude from the definition of N that

$$\langle x, \tilde{\varphi}_{\lambda}^{\alpha}y\rangle \to 0$$

for all $x \in H$ and all $y \in N$. So for any $x, y \in H$ we obtain

$$\begin{aligned}
\langle x, \tilde{\varphi}_{\lambda}^{\alpha}y\rangle &= \langle x, \tilde{\varphi}_{\lambda}^{\alpha}Py\rangle + \langle x, \tilde{\varphi}_{\lambda}^{\alpha}(1-P)y\rangle \\
&= \langle x, Py\rangle + \langle x, \tilde{\varphi}_{\lambda}^{\alpha}(1-P)y\rangle \\
&\to \langle x, Py\rangle
\end{aligned}$$

by the definition of P and since $(1-P)y \in V^{\perp} = N$. □

In particular this result holds in the situation presented in Section 1, where $R = M$ is an amenable locally compact quantum group. This is our main and final result, and we now conclude with a few brief remarks to give some indication of the relation with classical ergodic theory.

Note that if α is unital in Theorem 3.1, then one has $P\Omega = \Omega$, where $\Omega := \gamma(1_A)$ is the (non-zero) cyclic vector of (A, ω)'s GNS construction, since

$$\left\langle \gamma(d), \tilde{\theta}^{\alpha}\Omega \right\rangle = \left\langle \gamma(d), \tilde{\theta}(1_R \otimes 1_A) \right\rangle = \langle \gamma(d), \Omega \rangle$$

for all $\theta \in (R_*^+)_1$. This is essentially the same situation as in classical ergodic theory. Extending the classical case, it seems reasonable to say that the *dynamical system* (A, ω, α) is *ergodic* when $\dim PH = 1$, i.e. $PH = \mathbb{C}\Omega$. Using Theorem 3.1, this is easily seen to be equivalent to

$$\lim_{\lambda} \varphi_\lambda \otimes \omega\left([1_R \otimes a]\alpha(b)\right) = \omega(a)\omega(b)$$

again paralleling the situation in classical ergodic theory.

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