DIFFERENCE OF COMPOSITION OPERATORS IN THE POLYDISCS

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Abstract. This paper gives some simple estimates of the essential norm for the difference of composition operators induced by φ and ψ acting on bounded function space in the unit polydiscs U^n, where $\varphi(z)$ and $\psi(z)$be holomorphic self-maps of U^n. As its applications, a characterization of compact difference is given for composition operators acting on the bounded function spaces.

1. Introduction

Let U^n be the unit polydiscs of C^n with boundary ∂U^n. The class of all holomorphic functions on domain U^n will be denoted by $H(U^n)$. Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$ and $\psi(z) = (\psi_1(z), \cdots, \psi_n(z))$ be holomorphic self-maps of U^n. Composition operator is defined by:

$$C_\varphi(f)(z) = f(\varphi(z))$$

for any $f \in H(U^n)$ and $z \in U^n$.

We recall that the essential norm of a continuous linear operator T is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}.$$

Notice that $\|T\|_e = 0$ if and only if T is compact, so that estimates on $\|T\|_e$ lead to conditions for T to be compact.

In the past few years, boundedness and compactness of composition operators between several spaces of holomorphic functions have been studied by many authors. Recently, there have been many papers focused on studying the mapping properties of the difference of two composition operators, i.e., an operator of the form

$$T = C_\varphi - C_\psi.$$

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In [MacOZ], MacCluer et al., characterize the compactness of the difference of composition operator on H^∞ spaces by Poincaré distance. In [Toe] and [GorM], Carl and Gorkin et al., independently extended the results to $H^\infty(B_n)$ spaces, they described compact difference by Carathéodory psedo-distance on the ball, which is the generalization of Poinaré distance. In [Moorhouse], Moorhouse shew that if the pseudo-hyperbloic distance between the image values φ and ψ converges to zero as $z \to \zeta$ for every point ζ at which φ and ψ have finite angular derivative then the difference $C_\varphi - C_\psi$ yields a compact operator. Building on this foundation, this paper gives some simple estimates of the essential norm for the difference of composition operators induced by φ and ψ acting on bounded function space in the unit polydiscs U^n, where $\varphi(z)$ and $\psi(z)$be holomorphic self-maps of U^n. As its applications, a characterization of compact difference is given for composition operators acting on the bounded function spaces by Carathéodory distance on U^n.

2. Notation and background

Let D be the unit disc in C, then the pseudo-hyperbolic distance on D is defined by $\beta(z,w) = |\frac{z-w}{1-\bar{z}w}|$ for $z, w \in D$. By U^n denote the unit polydiscs of C^n, and by H^∞ denote the Banach space of bounded analytic functions on U^n in the sup norm.

Definition 1 The Poincaré distance ρ on D is

$$\rho(z,w) := \tanh^{-1}\beta(z,w)$$

for $z, w \in D$.

Definition 2 The Carathéodory pseudo-distance on a domain $G \subset C^n$ is given by

$$c_G(z,w) := \sup\{\rho(f(z), f(w)) : f \in H(G,D)\}$$

for $z, w \in G$.

If we put $c_G^*(z,w) := \sup\{\beta(f(z), f(w)) : f \in H(G,D)\}$ for $z, w \in G$, it is clear that

$$c_G = \tanh^{-1}(c_G^*) \geq c_G^*.$$

Next we introduce the following pseudo-distance on G

$$d_G(z,w) := \sup\{|f(z) - f(w)| : f \in H(G,D)\}.$$

For the case $G = D$, it is easy to show that

$$d_D(z,w) = \frac{2 - 2\sqrt{1-\beta(z,w)^2}}{\beta(z,w)}.$$

So the Poincaré metirc on D

$$\rho(z,w) := \tanh^{-1}\beta(z,w) = \log\frac{2+d_D(z,w)}{2-d_D(z,w)}.$$

Clearly for $z, w \in G$,

$$\begin{aligned} d_G(z,w) &= \sup\{|g(f(z))-g(f(w))| : g\in H(D,D), f\in H(G,D)\} \\ &= \sup_{f\in H(G,D)} d_D(f(z),f(w)) \end{aligned}$$

Since the map $t \to \log\frac{2+t}{2-t}$ is strictly increasing on $[0,2)$, it follows that

$$\begin{aligned} \log\frac{2+d_G}{2-d_G} &= \sup_{f\in H(G,D)} \log\frac{2+d_D(f(z),f(w))}{2-d_D(f(z),f(w))} \\ &= \sup_{f\in H(G,D)} \rho(f(z),f(w)) \\ &= c_G(z,w) \end{aligned}$$

or equivalently for any domain G and any $z, w \in G$.

$$\begin{aligned} d_G(z,w) &= \frac{2-2\sqrt{1-(\tanh c_G(z,w))^2}}{\tanh c_G(z,w)} \\ &= \frac{2-2\sqrt{1-(c_G^*(z,w))^2}}{c_G^*(z,w)} \end{aligned}$$

It is well known that $c^*_{U^n}(z,w) = \max_{1\le j\le n}\beta(z_j,w_j)$. So we have

$$d_{U^n}(z,w) = \frac{2-2\sqrt{1-(\max_{1\le j\le n}\beta(z_j,w_j))^2}}{\max_{1\le j\le n}\beta(z_j,w_j)}$$

Before proving the main theorem, we give first some symbol. For any $0<\delta<1$, define

$$E^j_\delta := \{z\in U^n|\ \ |\varphi_j(z)|\vee|\psi_j(z)| > 1-\delta\},$$

and we put $E_\delta = \cup_{j=1}^n E^j_\delta$,where $\vee$ means the maximum of two real numbers.

Lemma 1. *Let $\{z_n\}$ be a sequence in D with $|z_n|\to 1$ as $n\to\infty$, then there is a subsequence $\{z_{n_i}\}$,a number $M\ge 1$ and a sequence of functions $f_m \in H^\infty(D)$ such that*

i) $\quad f_m(z_{n_k}) = \delta^k_m$

ii) $\quad \sum_m |f_m(z)| \le M < \infty$ *for any* $z\in U$

(the symbol δ^k_m is equal to 1 if $m=k$ and 0 otherwise)

Proof. By proposition 2 and lemma 12 in [Toe] . □

Lemma 2. *Let Ω be a domain in C^n, $f \in H(\Omega)$. If a compact set K and its neighborhood G satisfy $K \subset G \subset\subset \Omega$ and $\rho = dist(K, \partial G) > 0$, then*

$$\sup_{z\in K} |\frac{\partial f}{\partial z_j}(z)| \leq \frac{\sqrt{n}}{\rho} \sup_{z\in G} |f(z)|.$$

Proof. Since $\rho = dist(K, \partial G) > 0$, for any $a \in K$, the polydisc

$$P_a = \left\{ (z_1, \cdots, z_n) \in C^n : |z_j - a_j| < \frac{\rho}{\sqrt{n}}, j = 1, \cdots, n \right\}$$

is contained in G. Using Cauchy inequality, we have

$$\left| \frac{\partial f}{\partial z_j}(a) \right| \leq \frac{\sqrt{n}}{\rho} \sup_{z\in \partial_0 P_a} |f(z)| \leq \frac{\sqrt{n}}{\rho} \sup_{z\in G} |f(z)|.$$

So the Lemma follows. □

Lemma 3. *For fixed $0 < \delta < 1$, let $F_\delta = \{z \in U^n : \max_{1\leq j\leq n} |z_j| > 1 - \delta\}$. Then*

$$\lim_{r\to 1} \sup_{||f||_\infty = 1} \sup_{z\in F_\delta^c} |f(z) - f(rz)| = 0$$

for any $f \in H^\infty(U^n)$.

Proof.

$$\begin{aligned}
& \sup_{z\in F_\delta^c} |f(z) - f(rz)| \\
= & \sup_{z\in F_\delta^c} |\sum_{j=1}^{n} (f(rz_1, rz_2, \cdots, rz_{j-1}, z_j, \cdots, z_n) \\
- & f(rz_1, rz_2, \cdots, rz_j, z_{j+1}, \cdots, z_n))| \\
\leq & \sup_{z\in F_\delta^c} \sum_{j=1}^{n} \left| \int_r^1 |z_j \frac{\partial f}{\partial z_j}(rz_1, \cdots, rz_{j-1}, tz_j, z_{j+1}, \cdots, z^n) dt \right| \\
\leq & (1-r)n \sup_{z\in F_\delta^c} \left| \frac{\partial f}{\partial z_j}(z) \right|.
\end{aligned}$$

Consider $F_{\delta/2}^c$, then $F_\delta^c \subset F_{\delta/2}^c$ and $dist(F_{\delta/2}^c, \partial U^n) = \frac{\delta}{2}$.

From Lemma 3, we have

$$\sup_{z\in F_\delta^c} \left| \frac{\partial f}{\partial z_j}(z) \right| \leq \frac{2\sqrt{n}}{\delta} \sup_{z\in F_{\delta/2}^c} |f(z)|.$$

then

$$\sup_{z\in F_\delta^c}|f(z)-f(rz)|\le \frac{2(1-r)n\sqrt{n}}{\delta}||f||_\infty.$$

Let $r\to 1$, the conclusion follows. $\square$

3. Main theorem

Theorem. Let $\varphi,\psi : U^n \to U^n$, and $C_\varphi - C_\psi : H^\infty(U^n)\to H^\infty(U^n)$. Then

$$\lim_{\delta\to 0}\sup_{z\in E_\delta}\max_{1\le j\le n}\beta(\varphi_j(z),\psi_j(z))\le ||C_\varphi-C_\psi||_e$$

$$\le \frac{4-4\sqrt{1-\lim\limits_{\delta\to 0}\sup\limits_{z\in E_\delta}\max\limits_{1\le j\le n}\beta(\varphi_j(z),\psi_j(z))^2}}{\lim\limits_{\delta\to 0}\sup\limits_{z\in E_\delta}\max\limits_{1\le j\le n}\beta(\varphi_j(z),\psi_j(z))}.$$

Proof. We consider the upper estimate first. For fixed $0<r<1$, it is easy to check that both $C_{r\varphi}$ and $C_{r\psi}$ are compact operators. Therefore

$$||C_\varphi-C_\psi||_e\le ||C_\varphi-C_\psi-C_{r\varphi}+C_{r\psi}||.$$

Now for any $0<\delta<1$

$$\begin{aligned}
& \|C_\varphi-C_\psi-C_{r\varphi}+C_{r\psi}\|\\
= & \sup_{||f||_\infty=1}||(C_\varphi-C_\psi-C_{r\varphi}+C_{r\psi})f||_\infty\\
= & \sup_{||f||_\infty=1}\sup_{z\in U^n}|f(\varphi(z))-f(\psi(z))-f(r\varphi(z))+f(r\psi(z))|\\
\le & \sup_{||f||_\infty=1}\sup_{z\in E_\delta}|f(\varphi(z))-f(\psi(z))-f(r\varphi(z))+f(r\psi(z))|\\
+ & \sup_{||f||_\infty=1}\sup_{z\in E_\delta^c}|f(\varphi(z))-f(\psi(z))-f(r\varphi(z))+f(r\psi(z))|.
\end{aligned}$$

From Lemma 3, we can choose r sufficiently close to 1 such that the second term of the right hand side is less than any given ε, and denote the first term by I.

Using Schwartz-Pick lemma and the monotony of function $f(x) = \frac{2-2\sqrt{1-x^2}}{x}$, Then

$$\begin{aligned}
I &\leq \sup_{||f||_\infty=1}\sup_{z\in E_\delta}(|f(\varphi(z))-f(\psi(z))|+|-f(r\varphi(z))+f(r\psi(z))|) \\
&= \sup_{z\in E_\delta}\sup_{||f||_\infty=1}(|f(\varphi(z))-f(\psi(z))|+|-f(r\varphi(z))+f(r\psi(z))|) \\
&= \sup_{z\in E_\delta}(d_{U^n}(\varphi(z),\psi(z))+d_{U^n}(r\varphi(z),r\psi(z))) \\
&\leq 2\sup_{z\in E_\delta}\frac{2-2\sqrt{1-\max\limits_{1\leq j\leq n}\beta(\varphi_j(z),\psi_j(z))^2}}{\max\limits_{1\leq j\leq n}\beta(\varphi_j(z),\psi_j(z))} \\
&= \frac{4-4\sqrt{1-\sup\limits_{z\in E_\delta}\max\limits_{1\leq j\leq n}\beta(\varphi_j(z),\psi_j(z))^2}}{\sup\limits_{z\in E_\delta}\max\limits_{1\leq j\leq n}\beta(\varphi_j(z),\psi_j(z))}.
\end{aligned}$$

Let $\delta\to 0$, the upper estimate follows.

Now we turn to the lower estimate.

Define $a_j = \lim\limits_{\delta\to 0}\sup\limits_{z\in E_\delta^j}\beta(\varphi_j(z),\psi_j(z))$. If we set $\delta_m = \frac{1}{m}$, then $\delta_m\to 0$ as $m\to\infty$, and there exists $z_m\in E_{\delta_m}^j$ such that $\lim\limits_{m\to\infty}\beta(\varphi_j(z_m),\psi_j(z_m)) = a_j$.

Without loss of generality, we can assume $|\varphi_j(z_m)|\to 1$. Let $w_m = \varphi_j(z_m)$, by lemma 1, we further assume the subsequence by w_m, there exists a number M_j and a sequence of functions $f_m\in H^\infty(D)$ such that

$$\begin{aligned}
&i) \qquad f_m(w_k) = \delta_m^k \\
&ii) \qquad \sum_m |f_m(w)| \leq M_j < \infty \;\; for\ any\ w\in U
\end{aligned}$$

Now for any $z\in U^n$, we define $\tilde{f_m}(z) := f_m(z^j)$, where z^j is the j^{th} component of z, then $\sum\limits_m |\tilde{f_m}(z)| \leq M_j < \infty$.

Next we claim that $\tilde{f_m}$ converge weakly to 0. In fact, let $\lambda\in H^\infty(U^n)^*$. For any integer N, there exist some unimodular sequence α_m such that

$$\begin{aligned}
\sum_{m=0}^N |\lambda\tilde{f_m}| &= \sum_{m=0}^N \lambda\tilde{f_m}\alpha_m = \lambda(\sum_{m=0}^N \lambda\tilde{f_m}) \\
&\leq ||\lambda||||\sum_{m=0}^N \alpha_m\tilde{f_m}||_\infty \leq ||\lambda||M_j
\end{aligned}$$

Thus $\lambda\tilde{f_m} \to 0$,that is $\tilde{f_m}$ converge weakly to 0.

Putting functions

$$g_m(z) = \frac{\tilde{f_m}(z)}{M_j}\frac{z^j - \psi_j(z_m)}{1 - \overline{\psi_j(z_m)}z^j}$$

then $||g_m||_\infty \leq 1$ and $g_m(z)$ converge weakly to 0. So for any compact operator K, we have $||Kg_m||_\infty \to 0$.

Now we have

$$\begin{aligned}
||C_\varphi - C_\psi - K|| &\geq \limsup_{m\to\infty} ||(C_\varphi - C_\psi - K)g_m||_\infty \\
&\geq \limsup_{m\to\infty}(||(C_\varphi - C_\psi)g_m||_\infty - ||Kg_m||_\infty) \\
&= \limsup_{m\to\infty} \sup_{z\in U^n} |g_m(\varphi(z)) - g_m(\psi(z))| \\
&\geq \frac{1}{M_j} \limsup_{m\to\infty} \sup_{z\in U^n} |\frac{\varphi_j(z) - \psi(z_m)}{1 - \overline{\psi_j(z_m)}\varphi_j(z)}\tilde{f_m}(\varphi(z)) \\
&\quad -\frac{\psi_j(z) - \psi_j(z_m)}{1 - \overline{\psi_j(z_m)}\psi(z)}\tilde{f_m}(\psi(z))| \\
&\geq \frac{1}{M_j} \limsup_{m\to\infty} \frac{|\varphi(z_m) - \psi(z_m)|}{1 - \overline{\psi(z_m)\varphi(z_m)}} = \frac{1}{M_j}a_j.
\end{aligned}$$

For the case$|\psi_j(z)| \to 1$, a similar argument can get the same conclusion except M_j is substituted by a new constant M_j'. If we set $\tilde{M_j} = \max\{M_j, M_j'\}$. Then for $1 \leq j \leq n$, and so we get the following estimate

$$\begin{aligned}
||C_\varphi - C_\psi||_e &\geq \frac{1}{\tilde{M_j}} \max_{1\leq j\leq n} \lim_{\delta\to 0} \sup_{z\in E_\delta^j} \beta(\varphi_j(z), \psi_j(z)) \\
&\geq \tfrac{1}{M} \lim_{\delta\to 0} \sup_{z\in E_\delta} \max_{1\leq j\leq n} \beta(\varphi_j(z), \psi_j(z)).
\end{aligned}$$

where $M = \max_{1\leq j\leq n} \tilde{M_j}$. □

Corollary. $C_\varphi - C_\psi$ is compact if and only if

$$\lim_{\delta\to 0} \sup_{z\in E_\delta} \max_{1\leq j\leq n} \beta(\varphi_j(z), \psi_j(z)) = 0.$$

Proof. By the inequality $\frac{1-\sqrt{1-x^2}}{x} \leq x$ for any $0 < x \leq 1$ and T is compact if and only if $||T||_e = 0$ □

Remark. If for any j, we have $||\varphi_j||_\infty < 1$ and $||\psi_j||_\infty < 1$, then $E_\delta = \emptyset$ when δ is small enough, without loss of generality, we set

$$\lim_{\delta\to 0}\sup_{z\in E_\delta}\max_{1\le j\le n}\beta(\varphi_j(z),\psi_j(z)) = 0.$$

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