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EXTENDED CESÁRO OPERATORS ON ZYGMUND SPACES IN THE UNIT BALL

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Abstract. Let g be a holomorphic function of the unit ball B in the n-dimensional space, and denote by T_g and I_g the induced extended Cesáro operator and another integral operator. The boundedness and compactness of T_g and I_g acting on the Zygmund spaces in the unit ball are discussed and necessary and sufficient conditions are given in this paper.

1. Introduction

Let $f(z)$ be a holomorphic function on the unit disc D with $Taylor$ expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$, the classical Cesáro operator acting on f is

$$\mathcal{C}[f](z) = \sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^{j} a_k \right) z^j.$$

In the past few years, many authors focused on the boundedness and compactness of extended Cesáro operator between several spaces of holomorphic functions . It is well known that the operator $\mathcal{C}$ is bounded on the usual Hardy spaces $H^p(D)$ for $0 < p < \infty$ and Bergman space, we recommend the interested readers refer to [10, 12, 8, 2, 13]. But the operator $\mathcal{C}$ is not always bounded, in [15], Shi and Ren gave a sufficient and necessary condition for the operator $\mathcal{C}$ to be bounded on mixed norm spaces in the unit disc. Recently, Siskakis and Zhao in [14] obtained sufficient and necessary conditions for Volterra type operator, which is a generalization of $\mathcal{C}$, to be bounded or compact between $BMOA$ spaces in the unit disc. It is a natural question to ask what are the conditions for higher dimensional case.

Let dv be the *Lebesgue* measure on the unit ball B of C^n normalized so that $v(B) = 1$,and $dv_\beta = c_\beta(1 - |z|^2)^\beta dv$,where c_β is a normalizing constant so that dv_β is a probability measure. The class of all holomorphic functions on B is defined by $H(B)$. For $f \in H(B)$ we write

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

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A little calculation shows $\mathcal{C}[f](z) = \frac{1}{z}\int_0^z f(t)(\log\frac{1}{1-t})'dt$. From this point of view, if $g \in H(B)$, it is natural to consider the extended Cesáro operator (also called Volterra-type operator or Riemann-Stieltijes type operator) T_g on $H(B)$ defined by

$$T_g(f)(z) = \int_0^1 f(tz)Rg(tz)\frac{dt}{t}.$$

It is easy to show that T_g take $H(B)$ into itself. In general, there is no easy way to determine when an extended Cesáro operator is bounded or compact.

Motivated by [15], Hu and Zhang [6, 7, 17] gave some sufficient and necessary conditions for the extended $\mathcal{C}$ to be bounded and compact on mixed norm spaces, Bloch space as well as Dirichlet space in the unit ball.

Another natural integral operator is defined as follows:

$$I_g(f)(z) = \int_0^1 Rf(tz)g(tz)\frac{dt}{t}.$$

The importance of them comes from the fact that

$$T_g(f) + I_g(f) = M_g f - f(0)g(0) \tag{1}$$

where the multiplication operator is defined by

$$M_g(f)(z) = g(z)f(z), f \in H(B), z \in B.$$

Now we introduce some spaces first. Let H^∞ denote the space of all bounded holomorphic functions on the unit ball, equipped with the norm $||f||_\infty = \sup_{z\in B}|f(z)|$.

The Bloch space $\mathcal{B}$ is defined as the space of holomorphic functions such that

$$||f||_{\mathcal{B}} = \sup\{(1-|z|^2)|Rf(z)| : z \in B\} < \infty.$$

It is easy to check that if $f \in \mathcal{B}$ then

$$|f(z)| \le C \log\frac{2}{1-|z|^2}\|f\|_{\mathcal{B}}. \tag{2}$$

We define weighted Bloch space $\mathcal{B}_{log}$ as the space of holomorphic functions $f \in H(B)$ such that

$$||f||_{\mathcal{B}_{log}} = \sup\{(1-|z|^2)|Rf(z)|\log\frac{2}{1-|z|^2} : z \in B\} < \infty.$$

The Zygmund space $\mathcal{Z}$ [18] in the unit ball consists of those functions whose first order partial derivatives are in the Bloch space.

It is well known that (Theorem 7.11 in [18]) $f \in \mathcal{Z}$ if and only if $Rf \in \mathcal{B}$, and $\mathcal{Z}$ is a Banach space with the norm

$$||f|| = |f(0)| + ||Rf||_{\mathcal{B}}. \tag{3}$$

The purpose of this paper is to discuss the boundedness and compactness of extended Cesáro operator T_g and another integral operator I_g on the Zygmund space in the unit ball.

2. Some Lemmas

In the following, we will use the symbol C to denote a finite positive number which does not depend on variable z and f.

In order to prove the main results, we will give some Lemmas first.

Lemma 1. *Assume $f \in \mathcal{Z}$, then we have*

$$||f||_\infty \leq C||f||$$

Proof. Since $f \in \mathcal{Z}$ implies that $Rf \in \mathcal{B}$, it follows from (2) that

$$|Rf(z)| \leq C \log \frac{2}{1-|z|^2} \|Rf\|_{\mathcal{B}} \leq C \log \frac{2}{1-|z|^2} ||f||. \tag{4}$$

Furthermore by $\lim\limits_{|z|\to 1} (1-|z|^2) \log \frac{2}{1-|z|^2} = 0$ we have

$$(1-|z|^2)|Rf(z)| \leq C(1-|z|^2) \log \frac{2}{1-|z|^2} ||f|| < \infty, \tag{5}$$

so $f \in \mathcal{B}$. It follows from Theorem 2.2 in [18] that

$$Rf(z) = \int_B \frac{Rf(z) dv_\beta(w)}{(1- < z, w >)^{n+1+\beta}}$$

where β is a sufficiently large positive constant. Since $Rf(0) = 0$,

$$f(z) - f(0) \;\; = \;\; \int_0^1 \frac{Rf(tz)}{t} dt = \int_B Rf(w) L(z, w) dv_\beta(w)$$

where the kernel

$$L(z, w) = \int_0^1 \left(\frac{1}{(1-t < z, w >)^{n+1+\beta}} - 1\right) \frac{dt}{t}$$

satisfies

$$|L(z, w)| \leq \frac{C}{|1- < z, w > |^{n+\beta}}$$

for all z and w in B. Note that $t^{1/2} \log \frac{2}{t} \leq \frac{2}{e} \cdot (1 - \log 2)$ for all $t \in (0, 1]$, then

$$\begin{aligned} |f(z) - f(0)| &= C \int_B \frac{(1-|w|^2)|Rf(w)| dv_{\beta-1}(w)}{|1- < z, w > |^{n+\beta}} \\ &\leq C \int_B \frac{(1-|w|^2) \log \frac{2}{1-|w|^2} ||f|| dv_{\beta-1}(w)}{|1- < z, w > |^{n+\beta}} \\ &\leq C \int_B \frac{(1-|w|^2)^{1-1/2} ||f|| dv_{\beta-1}(w)}{|1- < z, w > |^{n+\beta}} \\ &\leq C||f||. \end{aligned}$$

The last inequality holds since $\int_B \frac{(1-|w|^2)^t dv(w)}{|1-<z,w>|^{n+1+t+c}}$ is bounded for $c < 0$. This completes the proof of Lemma 1.

By Lemma 1, Montel theorem and the definition of compact operator, the following lemma follows.

Lemma 2. *Assume that $g \in H(B)$. Then T_g (or I_g) : $\mathcal{Z} \to \mathcal{Z}$ is compact if and only if T_g (or I_g) is bounded and for any bounded sequence $(f_k)_{k\in N}$ in $\mathcal{Z}$ which converges to zero uniformly on $\overline{B}$ as $k \to \infty$, $||T_g f_k|| \to 0$(or $||I_g f_k|| \to 0$) as $k \to \infty$.*

Lemma 3. *If $(f_k)_{k\in N}$ is a bounded sequence in $\mathcal{Z}$ which converges to zero uniformly on compact subsets of B as $k \to \infty$, then $\lim\limits_{k\to\infty} \sup\limits_{z\in B} |f_k(z)| = 0$.*

proof. Assume $||f_k|| \le M$. For any given $\epsilon > 0$, there exists $0 < \eta < 1$ such that $\frac{\sqrt{1-\eta}}{\eta} < \epsilon$. Note that $t^{1/2} \log \frac{2}{t} \le \frac{2}{e} \cdot (1 - \log 2)$ for all $t \in (0, 1]$, then when $\eta < |z| < 1$, it follows from (4) that

$$\begin{aligned} |f_k(z) - f_k(\frac{\eta}{|z|}z)| &= \left| \int_{\frac{\eta}{|z|}}^1 Rf_k(tz)\frac{dt}{t} \right| \le C \int_{\frac{\eta}{|z|}}^1 \log \frac{2}{1-|tz|^2} ||f_k|| \frac{dt}{t} \\ &\le C\frac{|z|}{\eta} \int_{\frac{\eta}{|z|}}^1 \frac{||f_k||dt}{(1-|tz|^2)^{1/2}} \le C\frac{M}{\eta} \int_{\frac{\eta}{|z|}}^1 \frac{|z|dt}{(1-t|z|)^{1/2}} \\ &\le 2CM\frac{(1-\eta)^{1/2}}{\eta} < C\epsilon. \end{aligned}$$

So we get $\sup\limits_{\eta<|z|<1} |f_k(z)| \le C\epsilon + \sup\limits_{|w|=\eta} |f_k(w)|$. Thus, we have

$$\lim_{k\to\infty} \sup_{z\in B} |f_k(z)| \le \lim_{k\to\infty} (\sup_{|z|\le\eta} |f_k(z)| + \sup_{\eta<|z|<1} |f_k(z)|) \le C\epsilon.$$

Now we finish the proof of this lemma.

Lemma 4. *Let $g \in H(B)$, then*

$$R[T_g f](z) = f(z)Rg(z)$$

for any $f \in H(B)$ and $z \in B$.

Proof. Suppose the holomorphic function fRg has the $Taylor$ expansion

$$(fRg)(z) = \sum_{|\alpha|\ge1} a_\alpha z^\alpha.$$

Then we have

$$R(T_g f)(z) = R\int_0^1 f(tz)R(tz)\frac{dt}{t} = R\int_0^1 \sum_{|\alpha|\ge1} a_\alpha (tz)^\alpha \frac{dt}{t}$$

$$= R[\sum_{|\alpha|\ge1} \frac{a_\alpha z^\alpha}{|\alpha|}] = \sum_{|\alpha|\ge1} a_\alpha z^\alpha = (fRg)(z).$$

3. Main Theorems

Theorem 1. Suppose $g \in H(B)$, then the following conditions are all equivalent:

(a) T_g is bounded on $\mathcal{Z}$;
(b) T_g is compact on $\mathcal{Z}$;

(c) $g \in \mathcal{Z}$.

Proof. $b \Longrightarrow a$ is obvious. For $a \Longrightarrow c$ we just take the test function given by $f(z) \equiv 1$.

We are going to prove $c \Longrightarrow b$. Now assume that $g \in \mathcal{Z}$ and that $(f_k)_{k\in N}$ is a sequence in $\mathcal{Z}$ such that $\sup_{k\in N} ||f_k|| \leq M$ and that $f_k \to 0$ uniformly on on $\overline{B}$ as $k \to \infty$. Now note that $T_g g_k(0) = 0$ and for every $\epsilon > 0$, there is a $\delta \in (0,1)$, such that

$$(1-|z|^2)(\ln \frac{2}{1-|z|^2})^2 < \epsilon$$

whenever $\delta < |z| < 1$. Let $K = \{z \in B : |z| \leq \delta\}$, it follows from Lemma 4 and (4) that

$$\begin{aligned}
||T_g f_k|| &= \sup_{z\in B}(1-|z|^2)\,|R(R(T_g f_k))| \\
&= \sup_{z\in B}(1-|z|^2)|Rf_k \cdot Rg + f_k \cdot R(Rg)| \\
&\leq \sup_{z\in B}(1-|z|^2)(|Rf_k \cdot Rg| + |f_k \cdot R(Rg)|) \\
&\leq \sup_{z\in K}(1-|z|^2)|Rf_k \cdot Rg| + \sup_{z\in B-K}(1-|z|^2)(|Rf_k \cdot Rg| \\
&\quad + \sup_{z\in B}(1-|z|^2)|f_k \cdot R(Rg)| \\
&\leq C||g|| \sup_{z\in K}(1-|z|^2)|Rf_k(z)| \log \frac{2}{1-|z|^2} \\
&\quad + C||f_k|| \cdot ||g|| \sup_{z\in B-K}(1-|z|^2)(\log \frac{2}{1-|z|^2})^2 + ||g|| \cdot \sup_{z\in B}|f_k(z)|.
\end{aligned}$$

With the uniform convergence of f_k to 0 and the Cauchy estimate, the conclusion follows by letting $k \to \infty$.

Theorem 2. Suppose $g \in H(B)$, $I_g : \mathcal{Z} \to \mathcal{Z}$. Then I_g is bounded if and only if $g \in H^\infty \cap \mathcal{B}_{log}$.

Proof. First we assume that $g \in H^\infty \cap \mathcal{B}_{log}$. Notice that $I_g f(0) = 0$ and $R(I_g f) = fRg$, it follows from (4) that

$$\begin{aligned}
(1-|z|^2)|RR(I_g f)(z)| &= (1-|z|^2)|R(Rf(z) \cdot g(z))| \\
&= (1-|z|^2)|R(Rf)(z) \cdot g(z) + Rf(z) \cdot Rg(z)| \\
&\leq \|Rf(z)\|_{\mathcal{B}}\|g\|_\infty + |Rf(z)|(1-|z^2|)|Rg(z)| \\
&\leq C||f|| \cdot ||g||_\infty + C||f||(1-|z|^2)|Rg(z)| \log \frac{2}{1-|z|^2} \\
&\leq C||f|| \cdot ||g||_\infty + C||f|| \cdot ||g||_{\mathcal{B}_{log}}.
\end{aligned}$$

The boundedness of I_g follows.

Conversely, assume that I_g is bounded, then there is a positive constant C such that

$$||I_g f|| \leq C||f|| \tag{6}$$

for every $f \in \mathcal{Z}$. Setting

$$h_a(z) = (\log \frac{2}{1-|a|^2})^{-1}(< z, a > -1)[(1 + \log \frac{2}{1- < z, a >})^2 + 1]$$

for $a \in B$ such that $|a| \geq \sqrt{1 - 2/e}$, then

$$Rh_a(z) = < z, a > (\log \frac{2}{1- < z, a >})^2 (\log \frac{2}{1-|a|^2})^{-1}$$

and

$$RRh_a(z) = \{< z, a > (\log \frac{2}{1- < z, a >})^2 + \frac{2 < z, a >^2}{1- < z, a >} \log \frac{2}{1- < z, a >}\}(\log \frac{2}{1-|a|^2})^{-1}$$

It is easy to check that $M = \sup_{\sqrt{1-2/e} \leq |a| < 1} ||h_a|| < \infty$. Therefore, we have that

$$\begin{aligned}
\infty &> \|I_g\| \|h_a\| \geq ||I_g h_a|| \\
&\geq \sup_{z \in B} (1-|z|^2)|RRh_a(z) \cdot g(z) + Rh_a(z) \cdot Rg(z)| \\
&\geq (1-|a|^2)|\frac{2|a|^4}{1-|a|^2} g(a) + |a|^2 \log \frac{2}{1-|a|^2} g(a) + |a|^2 Rg(a) \log \frac{2}{1-|a|^2}| \\
&\geq -\{2|a|^4 + |a|^2 \frac{2}{e}(1 - \log 2)\}|g(a)| + |a|^2(1-|a|^2)|Rg(a)| \log \frac{2}{1-|a|^2} \\
&\geq -(2 + \frac{2}{e}(1 - \log 2))|a|^2 + |a|^2(1-|a|^2)|Rg(a)| \log \frac{2}{1-|a|^2}. \qquad (7)
\end{aligned}$$

Next let

$$f_a(z) = h_a(z) - \int_0^1 < z, a > \log \frac{2}{1 - t < z, a >} dt$$

then

$$Rf_a(z) = < z, a > \{(\log \frac{2}{1- < z, a >})^2 (\log \frac{2}{1-|a|^2})^{-1} - \log \frac{2}{1- < z, a >}\}$$

$$RRf_a(z) = RRh_a(z) - < z, a > \log \frac{2}{1- < z, a >} - \frac{< z, a >^2}{1- < z, a >}$$

and consequently $N = \sup_{\sqrt{1-2/e} \leq |a| < 1} ||f_a|| < \infty$. Note that $Rf_a(a) = 0$ and $RRf_a(a) = \frac{|a|^4}{1-|a|^2}$, we have

$$\begin{aligned}
\infty &> ||I_g|| \cdot ||f_a|| \geq ||I_g f_a|| \\
&\geq \sup_{z \in B} (1-|z|^2)|RRf_a(z) \cdot g(z) + Rf_a(z) \cdot Rg(z)| \\
&\geq (1-|a|^2)|RRf_a(a) g(a) + Rf_a(a) Rg(a)| = |a|^4 |g(a)|. \qquad (8)
\end{aligned}$$

From the maximum modulus theorem, we get $g \in H^\infty$. So it follows from (7) and (8) that

$$\sup_{\sqrt{1-2/e} \leq |a| < 1} (1-|a|^2)|Rg(a)| \log \frac{2}{1-|a|^2} < \infty. \qquad (9)$$

On the other hand, we have

$$\begin{aligned}
&\sup_{|a|\le\sqrt{1-2/e}}(1-|a|^2)|Rg(a)|\log\frac{2}{1-|a|^2}\\
&\le \frac{2}{e}\cdot(1-\log 2)\max_{|a|=\sqrt{1-2/e}}|Rg(a)|\\
&\le \sup_{\sqrt{1-2/e}\le|a|<1}(1-|a|^2)|Rg(a)|\log\frac{2}{1-|a|^2}<+\infty. \qquad (10)
\end{aligned}$$

Combining (9) and (10), we finish the proof of Theorem 2.

Corollary The multiplication operator $M_g:\mathcal{Z}\to\mathcal{Z}$ is bounded if and only if $g\in\mathcal{Z}$.

Proof. If M_g is bounded on $\mathcal{Z}$, then setting the test function $f\equiv 1$, we have $M_g f=g\in\mathcal{Z}$.

Conversely, if $g\in\mathcal{Z}$, from Lemma 1 and (5), it is easy to see that $g\in H^\infty\cap\mathcal{B}_{log}$, so by Theorems 1 and 2, both T_g and I_g are bounded, it follows from (1) that M_g is also bounded.

Theorem 3. Suppose $g\in H(B)$, $I_g:\mathcal{Z}\to\mathcal{Z}$. Then I_g is compact if and only if $g=0$.

Proof. The sufficiency is obvious. We just need to prove the necessity. Suppose that I_g is compact, for any given sequence $(z_k)_{k\in N}$ in B such that $|z_k|\to 1$ as $k\to\infty$, if we can show $g(z_k)\to 0$ as $k\to\infty$, then by the maximum modulus theorem we have $g\equiv 0$. In fact, setting

$$f_k(z)=h_{z_k}(z)-(\log\frac{2}{1-|z_k|})^{-2}\int_0^1 <z,z_k>(\log\frac{2}{1-t<z,z_k>})^3dt.$$

Using the same way as in Theorem 2, we can show $\sup_{k\in N}||f_k||\le C$ and f_k converges to 0 uniformly on compact subsets of B. Since I_g is compact, we have $||I_gf_k||\to 0$ as $k\to\infty$. Note that $Rf_k(z_k)=0$ and $RRf_k(z_k)=-\frac{|z_k|^4}{1-|z_k|^2}$, it follows that

$$\begin{aligned}
|z_k|^4|g(z_k)| &\le \sup_{z\in B}(1-|z|^2)|RRf_k(z)\cdot g(z)+Rf_k(z)\cdot Rg(z)|\\
&\le \sup_{z\in B}(1-|z|^2)|RR(I_gf_k)(z)|\le ||I_gf_k||\to 0
\end{aligned}$$

as $k\to\infty$. This ends the proof of Theorem 3.

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