EXTENDED CESÁRO OPERATORS BETWEEN GENERALIZED BESOV SPACES AND BLOCH TYPE SPACES IN THE UNIT BALL

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Abstract. Let g be a holomorphic map of B, where B is the unit ball of C^n. Let $0 < p < +\infty, -n-1 < q < +\infty$, $q > -1$ and $\alpha > 0$. This paper gives some necessary and sufficient conditions for the Extended Cesáro Operators induced by g to be bounded or compact between generalized Besov space $B(p,q)$ and α-Bloch space $\mathcal{B}^\alpha$.

1. Introduction

Let $f(z)$ be a holomorphic function on the unit disc D with $Taylor$ expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$, the classical Cesáro operator acting on f is

$$\mathcal{C}[f](z) = \sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^{j} a_k \right) z^j.$$

In the recent years, boundedness and compactness of extended Cesáro operator between several spaces of holomorphic functions have been studied by many mathematicians. It is known that the operator $\mathcal{C}$ is bounded on the usual Hardy spaces $H^p(D)$ for $0 < p < \infty$. Basic results facts on Hardy spaces can be found in [Durn]. For $1 \le p < \infty$, Siskais [Sis1] studied the spectrum of $\mathcal{C}$, as a by-product he obtained that $\mathcal{C}$ is bounded on $H^p(D)$. For $p = 1$, the boundedness of $\mathcal{C}$ was given also by Sisakis [Sis3] by a particularly elegant method, independent of spectrum theory, a different proof of the result can be found in [GalM]. After that, for $0 < p < 1$, Miao [Mia] proved $\mathcal{C}$ is also bounded. For $p = \infty$, the boundedness of $\mathcal{C}$ was given by Danikas and Siskais in [DanS]. It has been also shown that the operator $\mathcal{C}$ ia also bounded on the Bergman space (in [Sis4]) as well as on the weighted Bergman spaces (in [AS] and [BC]). But the operator $\mathcal{C}$ is not always bounded, in [ShiR], Shi and Ren gave a sufficient and necessary condition for the operator $\mathcal{C}$ to be bounded on mixed norm spaces in the unit disc.

The generalized Cesáro operators $\mathcal{C}^\gamma$ acting on f in the unit disc were first introduced in [St] and have been subsequently studied in [And] and [Xia]. The adjoint operator operator of $\mathcal{C}^\gamma$ was considered in [And], [Gal],[Sis1], [St] and [Xia]. Note that when $\gamma = 0$, $\mathcal{C}^0 = \mathcal{C}$. Stempak proved that $\mathcal{C}^\gamma$ is bounded on $H^p(D)$ for $0 < p \le 2$. For $0 < p \le 1$, his method is similar to that of Miao; for $p = 2$, it is

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based on the boundedness of an appropriate sequence transformation, and an interpolation then yields the result for $1 < p < 2$. After that, Andersen [And] and Xiao [Xia] prove the boundedness of $\mathcal{C}^\gamma$, on $H^p(D)$ for $p > 2$ using different methods.

More recently, there have been many papers focused on studying the same problems for n-dimensional case, for the unit polydisc, we refer the reader to see [CS], where they prove the boundedness of the generalized Cesáro operator on Hardy space $H^p(D^n)$ and the generalized Bergman space.

Let dv be the *Lebesegue* measure on the unit ball B of C^n normalized so that $v(B) = 1$. $H(B)$ is the class of all holomorphic functions on B.

A little calculation shows $\mathcal{C}[f](z) = \frac{1}{z}\int_0^z f(t)(\log\frac{1}{1-t})'dt$. From this point of view, if $g \in H(B)$, it is natural to consider the extended Cesáro operator T_g on $H(B)$ defined by

$$T_g(f)(z) = \int_0^1 f(tz)g(tz)\frac{dt}{t},$$

where $f \in H(B), z \in B$.

It is easy to show that T_g take $H(B)$ into itself. In general, there is no easy way to determine when a extended Cesáro operator is bounded or compact.

Motivated by [ShiR], Hu gave some sufficient and necessary conditions for the extended $\mathcal{C}$ to be bounded and compact on mixed norm spaces, Bloch space as well as Dirichlet space in the unit ball (see [Hu1],[Hu2] and [Zha]).

For $a \in B$, let $g(z,a) = \log|\varphi_a(z)|^{-1}$ be the *Green's* function on B with logarithmic singularity at a, where φ_a is the *Möbius* transformation of B with $\varphi_a(0) = a, \varphi_a(a) = 0, \varphi_a = \varphi_a^{-1}$.

Let $0 < p, s < +\infty, -n-1 < q < +\infty$ and $q + s > -1$. We say $f \in F(p,q,s)$ provided that $f \in H(B)$ and

$$\|f\|_{F(p,q,s)} = |f(0)| + \{\sup_{a\in B}\int_B |\nabla f(z)|^p(1-|z|^2)^q g^s(z,a)dv(z)\}^{\frac{1}{p}} < +\infty,$$

where

$$\nabla f(z) = (\frac{\partial f(z)}{\partial z_1}, \cdots, \frac{\partial f(z)}{\partial z_n}),$$

$F(p,q,s)$ is defined first by [Zhao], we also refer the reader to see [ZhoCh].

Let $0 < p < +\infty$, $-n-1 < q < +\infty$ and $q > -1$. We say $f \in B(p,q)$ provided that $f \in H(B)$ and

$$\|f\|_{(p,q)} = \{\int_B |\nabla f(z)|^p(1-|z|^2)^q dv(z)\}^{\frac{1}{p}} < +\infty,$$

where

$$\nabla f(z) = (\frac{\partial f(z)}{\partial z_1}, \cdots, \frac{\partial f(z)}{\partial z_n}).$$

It is obvious that $B(p,q) = F(p,q,0)$ if we take $s = 0$. In fact, $B(p,q)$ is also classical Besov space if we take special parameters of p, q. It is not hard to show that is a *Banach* space under the norm $\|f\|_{B(p,q)} = |f(0)| + \|f\|_{(p,q)}$, we refer the reader to see Zhu's book [Zhu1]. From Exercises 2.2 in [Zhu1] we know that a holomorphic function $f \in B(p,q)$ if and only if $\int_B |Rf(z)|^p(1-|z|^2)^q < +\infty$, where $Rf(z) =< \nabla f(z), \bar{z} >= \sum_{j=1}^n z_j \frac{\partial f(z)}{\partial z_j}$.

For $\alpha \geq 0$, f is said to be in the *Bloch* space $\mathcal{B}^\alpha$ provided that $f \in H(B)$ and

$$\|f\|_\alpha = \sup_{z\in B}(1-|z|^2)^\alpha|\nabla f(z)| < +\infty.$$

As we all know, $\mathcal{B}^\alpha$ is a *Banach* space when $\alpha \geq 1$ under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$. The spaces $\mathcal{B}^1$ and $\mathcal{B}^\alpha (0 < \alpha < 1)$ are just the *Bloch* space and the *Lipschitz* spaces $L_{1-\alpha}$ respectively. From [YaOuy] we know that a holomorphic function $f \in \mathcal{B}^\alpha$ if and only if $\sup_{z\in B}(1-|z|^2)^\alpha |Rf(z)| < +\infty$.

Furthermore, by the Norm Equivalent Theorem we have

$$\|f\|_{\mathcal{B}^\alpha} \approx |f(0)| + \sup_{z\in B}(1-|z|^2)^\alpha |Rf(z)|,$$

where $M \approx N$ means the two quantities M and N are comparable, that is there exist two positive constants C_1 and C_2 such that $C_1 M \leq N \leq C_2 M$.

For $p > 0, z \in B$, denote the function

$$G_p(z) = \begin{cases} 1, & 0 < p < 1; \\ \log \frac{2}{1-|z|^2}, & p = 1; \\ \left(\frac{1}{1-|z|^2}\right)^{\alpha-1}, & p > 1. \end{cases}$$

In this paper, we discussed the extended Cesáro operator between the generalized Besov space $B(p,q)$ and Bloch type space $\mathcal{B}^\alpha$ on the unit ball, and gave some sufficient and necessary conditions for the operator to be bounded and compact. The main results of the paper are the following:

Theorem 1. $0 < p < +\infty$, $-n-1 < q < +\infty$, $q > -1, \alpha \geq 0$, $g \in H(B)$, T_g is bounded from $B(p,q)$ to $\mathcal{B}^\alpha$ if and only if

$$\sup_{z\in B}(1-|z|^2)^\alpha G_{\frac{n+1+q}{p}}(z)|Rg(z)| < \infty.$$

Theorem 2. For $0 < p < +\infty, -n-1 < q < +\infty$, $q > -1, \alpha \geq 0$, $g \in H(B)$, T_g is compact from $B(p,q)$ to $\mathcal{B}^\alpha$ if and only if

(1) If $0 < \frac{n+1+q}{p} < 1$, $g \in \mathcal{B}^\alpha$;

(2) If $\frac{n+1+q}{p} \leq 1$, $\lim_{|z|\to 1^-}(1-|z|^2)^\alpha G_{\frac{n+1+q}{p}}(z)|Rg(z)| = 0$.

2. Some Lemmas

In the following, we will use the symbol c or C to denote a finite positive number which does not depend on variable z and may depend on some norms and parameters p, q, n, α, x, f etc, not necessarily the same at each occurrence.

In order to prove the main result, we will give some Lemmas first.

Lemma 1. *If* $0 < p < +\infty$, $-n-1 < q < +\infty$, $q > -1$, *then* $B(p,q) \subset \mathcal{B}^{\frac{n+1+q}{p}}$ *and* $\exists\, c > 0$ *s.t. for* $\forall f \in B(p,q)$,

$$\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \leq c\|f\|_{B(p,q)}.$$

Proof. Suppose $f \in B(p,q)$. Fixed $0 < r_0 < 1$, since $(Rf)\circ\varphi_a \in H(B)$, so $|(Rf)\circ\varphi_a|^p$ is subharmonic in B. That is

$$\begin{aligned} &|Rf(a)|^p = |(Rf)\circ\varphi_a(0)|^p \\ &\leq \frac{1}{r_0^{2n}}\int_{r_0 B}|(Rf)\circ\varphi_a(\omega)|^p dv(\omega) \\ &= \frac{1}{r_0^{2n}}\int_{\varphi_a(r_0 B)}|(Rf(z))|^p \frac{(1-|a|^2)^{n+1}}{|1-<z,a>|^{(2n+2)}}dv(z). \end{aligned}$$

From (5) in [ZhuOuy], we have

$$\frac{1-r_0}{1+r_0}(1-|a|^2) \le (1-|z|^2) \le \frac{1+r_0}{1-r_0}(1-|z|^2)$$

as $z \in \varphi_a(r_0 B)$. Thus

$$\frac{(1-|a|^2)^{n+1}}{|1-<z,a>|^{2n+2}(1-|z|^2)^q} \le \frac{4^{n+1}}{(1-|a|^2)^{n+1+q}}(\frac{1+r_0}{1-r_0})^{|q|}.$$

Therefore, we get

$$\begin{array}{l} |Rf(a)|^p \le \frac{1}{r_0^{2n}}\int_{\varphi_a(r_0B)}|Rf(z)|^p\frac{(1-|a|^2)^{n+1}}{|1-<z,a>|^{2n+2}}dv(z) \\ = \frac{1}{r_0^{2n}}\int_{\varphi_a(r_0B)}|Rf(z)|^p(1-|z|^2)^q\frac{(1-|a|^2)^{n+1}}{|1-<z,a>|^{2n+2}(1-|z|^2)^q}dv(z) \\ \le \frac{4^{n+1}r_0^{-2n}}{(1-|a|^2)^{n+1+q}}(\frac{1+r_0}{1-r_0})^{|q|}\|f\|^p_{B(p,q)}. \end{array}$$

This shows that $f \in \mathcal{B}^{\frac{n+1+q}{p}}$ and $\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \le c\|f\|_{B(p,q)}$.

Lemma 2. *Let $p > 0$, then there is a constant $c > 0$, for $\forall f \in \mathcal{B}^p$ and $\forall z \in B$, the estimate*

$$|f(z)| \le cG_p(z)\|f\|_{\mathcal{B}^p},$$

holds, where $G_p(z)$ is the function defined in Introduction.

Proof. This Lemma can be easily obtained by some integral estimates. For the convenience of the reader, we will still give the proof here.

For $\forall f \in \beta^p(B_n)$, since $||f||_{\beta^p} = |f(0)| + \sup_{z\in B_n}(1-|z|^2)^p|\nabla f(z)|$, we have

$$|f(0)| \le ||f||_{\beta^p}, \quad \text{and} \quad |\nabla f(z)| \le \frac{||f||_{\beta^p}}{(1-|z|^2)^p}.$$

but

$$f(z) = f(0) + \int_0^1 <z, \overline{\nabla f(tz)}> dt.$$

therefore

$$|f(z)| \le |f(0)| + \int_0^1 |z|\,|\nabla f(tz)|dt$$

$$\le \quad ||f||_{\beta^p} + ||f||_{\beta^p}\int_0^1 \frac{1}{(1-|tz|^2)^p}dt \le ||f||_{\beta^p}\Big(1+\int_0^{|z|}\frac{dt}{(1-t^2)^p}\Big).$$

when $p = 1$, $\int_0^{|z|}\frac{dt}{1-t^2} = \frac{1}{2}\ln\frac{1+|z|}{1-|z|} \le \frac{1}{2}\ln\frac{4}{1-|z|^2}$, therefore

$$|f(z)| \le \Big(1+\frac{1}{2}\ln\frac{4}{1-|z|^2}\Big)||f||_{\beta^p}.$$

If $p \ne 1$, then

$$\int_0^{|z|}\frac{dt}{(1-t^2)^p} = \int_0^{|z|}\frac{dt}{(1-t)^p(1+t)^p} \le \int_0^{|z|}\frac{dt}{(1-t)^p} = \frac{1-(1-|z|)^{1-p}}{1-p},$$

therefore when $0 < p < 1$, notice that $\int_0^{|z|}\frac{dt}{(1-t^2)^p} \le \frac{1}{1-p}$ we get

$$|f(z)| \le \Big(1+\frac{1}{1-p}\Big)||f||_{\beta^p}.$$

and when $p > 1$

$$\int_0^{|z|} \frac{dt}{(1-t^2)^p} \leq \frac{1-(1-|z|)^{1-p}}{1-p}$$
$$= \frac{1-(1-|z|)^{p-1}}{(p-1)(1-|z|)^{p-1}} \leq \frac{2^{p-1}}{(p-1)(1-|z|^2)^{p-1}}$$

so

$$|f(z)| \leq \Big(1 + \frac{2^{p-1}}{(p-1)(1-|z|^2)^{p-1}}\Big)||f||_{\mathcal{B}^p}.$$

Lemma 3. *Let $0 < p < 1$, $\{f_j\}$ is any bounded sequence in $\mathcal{B}^p$ and $f_j(z) \to 0$ on any compact subset of B. Then*

$$\lim_{j\to\infty} \sup_{z\in B} |f_j(z)| = 0.$$

Proof. This lemma has been given by [Zha].

Lemma 4. *There is a constant $c > 0$ such that for $\forall\ t > -1$ and $z \in B$,*

$$\int_B |\log \frac{1}{1-<z,w>}|^2 \frac{(1-|w|^2)^t}{(1-<z,w>)^{n+1+t}} dv(w) \leq C\big(\log \frac{1}{1-|z|^2}\big)^2.$$

Proof. This Lemma can be proved by Stirling formula and some complex integral estimates. For the convenience of the reader, we will still give the proof here.

Denote the right term as I_t and let $2\lambda = t+n+1$. By Taylor expansion

$$|\log \frac{1}{1-<z,w>}|^2 = \sum_{u,v=1}^{+\infty} \frac{<z,w>^u<w,z>^v}{uv}$$

and

$$\frac{1}{|1-<z,w>|^{2\lambda}} = \sum_{k,l=0}^{+\infty} \frac{\Gamma(\lambda+k)\Gamma(\lambda+l)}{k!l!\Gamma(\lambda)^2} <z,w>^k<w,z>^l,$$

therefore

$$\begin{aligned} I_t &= \int_B \sum_{u,v=1}^{+\infty} \sum_{k,l=0}^{+\infty} \tfrac{\Gamma(\lambda+k)\Gamma(\lambda+l)}{uvk!l!\Gamma(\lambda)^2} <z,w>^{k+u}<w,z>^{l+v} (1-|w|^2)^t dv(w) \\ &= \sum_{u=1}^{+\infty} \sum_{k=0}^{+\infty} \sum_{l=0}^{u+k-1} \tfrac{\Gamma(\lambda+k)\Gamma(\lambda+l)}{u(u+k-l)k!l!\Gamma(\lambda)^2} \int_B |<z,w>|^{2(u+k)}(1-|w|^2)^t dv(w) \end{aligned}$$

without lost of generality, let $z = |z|e_1$, then

$$\begin{aligned} \int_B & |<z,w>|^{2(u+k)}(1-|w|^2)^t dv(w) \\ &= \int_B (|z|w_1)^{2(u+k)}(1-|w|^2)^t dv(w) \\ &= 2n\int_0^1 \int_{\partial B} \rho^{2n-1}|z|^{2(u+k)}|\rho\xi_1|^{2(u+k)}(1-\rho^2)^t d\rho d\delta_n(\xi) \\ &= 2n|z|^{2(u+k)} \int_0^1 \rho^{2(u+k+n-1)+1}(1-\rho^2)^t d\rho \int_{\partial B} |\xi_1|^{2(u+k)} d\delta(\xi) \\ &= n|z|^{2(u+k)} \tfrac{\Gamma(u+k+n)\Gamma(t+1)}{\Gamma(u+k+n+t+1)} \tfrac{(n-1)!(u+k)!}{(u+k+n-1)!} \\ &= \tfrac{\Gamma(t+1)\Gamma(u+k+1)n!}{\Gamma(2\lambda+u+k)} |z|^{2(u+k)}, \end{aligned}$$

so

$$\begin{aligned} I_t &= \sum_{u=1}^{+\infty}\sum_{k=0}^{+\infty}\sum_{l=0}^{u+k-1}\tfrac{\Gamma(\lambda+k)\Gamma(\lambda+l)}{u(u+k-l)k!l!\Gamma(\lambda)^2}\tfrac{\Gamma(t+1)\Gamma(u+k+1)n!}{\Gamma(2\lambda+u+k)}|z|^{2(u+k)} \\ &= \sum_{u=1}^{+\infty}\sum_{k=0}^{+\infty}\tfrac{n!\Gamma(t+1)\Gamma(\lambda+k)\Gamma(u+k+1)}{uk!\Gamma(\lambda)^2\Gamma(2\lambda+u+k)}\sum_{l=0}^{u+k-1}\tfrac{\Gamma(\lambda+l)}{(u+k-l)l!}|z|^{2(u+k)} \\ &= \sum_{u=1}^{+\infty}\sum_{k=1}^{+\infty}\tfrac{n!\Gamma(t+1)\Gamma(\lambda+k)\Gamma(u+k+1)}{uk!\Gamma(\lambda)^2\Gamma(2\lambda+u+k)}\sum_{l=0}^{u+k-1}\tfrac{\Gamma(\lambda+l)}{(u+k-l)l!}|z|^{2(u+k)} \\ &\quad+\sum_{u=1}^{+\infty}\tfrac{n!\Gamma(t+1)\Gamma(u+1)}{u\Gamma(\lambda)\Gamma(2\lambda+u)}\sum_{l=0}^{u-1}\tfrac{\Gamma(\lambda+l)}{(u-l)l!}|z|^{2u} \\ &= I_1+I_2, \end{aligned}$$

by Stirling formula, there is an absolute constant C_1 s.t.

$$\frac{\Gamma(\lambda+l)}{l!}\leq C_1 l^{\lambda-1}, \frac{\Gamma(u+k+1)}{\Gamma(2\lambda+u+k)}\leq C_1(u+k)^{1-2\lambda},$$

$$\frac{\Gamma(u+k+1)}{\Gamma(2\lambda+u)}\leq C_1 u^{1-2\lambda}, \frac{\Gamma(\lambda+k)}{k!}\leq C_1 k^{\lambda-1}$$

for all $l,u,k\geq 1$, then

$$I_1\leq C_1^3\sum_{u=1}^{+\infty}\sum_{k=1}^{+\infty}\frac{n!\Gamma(t+1)k^{\lambda-1}(u+k)^{1-2\lambda}}{u\Gamma(\lambda)^2}\sum_{l=1}^{u+k-1}\frac{l^{\lambda-1}}{(u+k-l)}|z|^{2(u+k)}$$

and

$$I_2\leq C_1^2\sum_{u=1}^{+\infty}\frac{n!\Gamma(t+1)u^{1-2\lambda}}{u\Gamma(\lambda)}\sum_{l=1}^{u-1}\frac{l^{\lambda-1}}{(u-l)}|z|^{2u}.$$

Notice that

$$\sum_{l=1}^{M-1}\frac{l^{(\lambda-1)}}{M-l}\approx M^{\lambda-2}\log M$$

for any $M\geq 2$, then there is constant C, s.t.

$$\begin{aligned} I_1\leq\ & C\sum_{u=1}^{+\infty}\sum_{k=1}^{+\infty}\tfrac{n!\Gamma(t+1)k^{\lambda-1}(u+k)^{1-2\lambda}}{\Gamma(\lambda)^2 u}(u+k)^{\lambda-2}\log(u+k)|z|^{2(u+k)} \\ &= C\sum_{u=1}^{+\infty}\sum_{k=1}^{+\infty}\tfrac{n!\Gamma(t+1)}{\Gamma(\lambda)^2}\tfrac{k^\lambda}{(u+k)^\lambda}\tfrac{\log(u+k)}{u+k}\tfrac{1}{uk}|z|^{2(u+k)} \\ &\leq C\sum_{u=1}^{+\infty}\sum_{k=1}^{+\infty}\tfrac{1}{uk}|z|^{2(u+k)}=C\big(\log\tfrac{1}{1-|z|^2}\big)^2 \end{aligned}$$

and

$$\begin{aligned} I_2\leq\ & C\sum_{u=1}^{+\infty}\tfrac{n!\Gamma(t+1)u^{1-2\lambda}}{\Gamma(\lambda)u}u^{\lambda-2}\log u|z|^{2u} \\ &= C\sum_{u=1}^{+\infty}\tfrac{n!\Gamma(t+1)}{\Gamma(\lambda)}\tfrac{1}{u^{\lambda+1}}\tfrac{\log u}{u}|z|^{2u}, \end{aligned}$$

then it is clearly that I_2 can be control by $\big(\log\frac{1}{1-|z|^2}\big)^2$. This ends the proof of the lemma.

Lemma 5. *Let g be a holomorphic self-map of B, K is an arbitrary compact subset of B. Then $T_g: B(p,q)\to\mathcal{B}^\alpha$ is compact if and only if for any uniformly bounded sequence $\{f_j\}(j\in N)$ in $B(p,q)$ which converges to zero uniformly for z on K when $j\to\infty$, $\|T_g f_j\|_{\mathcal{B}^\alpha}\to 0$ holds.*

Proof. Assume that T_g is compact and suppose $\{f_j\}$ is a sequence in $B(p,q)$ with $\sup_{j\in N}\|f_j\|_{B(p,q)} < \infty$ and $f_j \to 0$ uniformly on compact subsets of B. By the compactness of T_g we have that $\{T_g f_j\}$ has a subsequence $\{T_g f_{j_m}\}$ which converges in β^α, say, to h. By Lemma 2 we have that for any compact set $K \subset B$, there is a positive constant C_K independent of f such that

$$|T_g f_j(z) - h(z)| \leq C_K \|T_g f_j - h\|_{\beta^\alpha}$$

for all $z \in K$. This implies that $T_g f_j(z) - h(z) \to 0$ uniformly on compact sets of B. Since K is a compact subset of B, by the hypothesis and the definition of T_g, $T_g f_j(z)$ converges to zero uniformly on K. It follows from the arbitrary of K that the limit function h is equal to 0. Since it's true for arbitrary subsequence of $\{f_j\}$, we see that $T_g f_j \to 0$ in β^α.

Conversely, $\{f_j\} \in K_r = B_{B(p,q)}(0,r)$, where $B_{B(p,q)}(0,r)$ is a ball in $B(p,q)$, then by Lemma 2, $\{f_j\}$ is uniformly bounded in arbitrary compact subset M of B. By $Montel's$ Lemma, $\{f_j\}$ is a normal family , therefore there is a subsequence $\{f_{j_m}\}$ which converges uniformly to $f \in H(B)$ on compact subsets of B. It follows that $\nabla f_{j_m} \to \nabla f$ uniformly on compact subsets of B.

Denote $B_k = B(0, 1-\frac{1}{k}) \subset C^n$, then

$$\begin{aligned}&\textstyle\int_B |\nabla f|^p (1-|z|^2)^q dv(z)\\ &= \lim_{k\to+\infty} \textstyle\int_{B_k} \lim_{m\to+\infty} |\nabla f_{j_m}|^p (1-|z|^2)^q dv(z)\\ &\leq \lim_{k\to+\infty} \lim_{m\to+\infty} \textstyle\int_{B_k} |\nabla f_{j_m}|^p (1-|z|^2)^q dv(z).\end{aligned}$$

But $\{f_{j_m}\} \subset B_{B(p,q)}(0,r)$, then

$$\int_{B_k} |\nabla f_{j_m}|^p (1-|z|^2)^q dv(z) < r^p,$$

therefore

$$\int_B |\nabla f|^p (1-|z|^2)^q dv(z) \leq r^p.$$

So $\|f\|_{B(p,q)} \leq r$, and $f \in B(p,q)$. Hence the sequence $\{f_{j_m} - f\}$ is such that $\|f_{j_m} - f\| \leq 2r < \infty$ and converges to 0 on compact subsets of B, by the hypothesis of this lemma, we have that

$$T_g f_{j_m} \to T_g f$$

in $\mathcal{B}^\alpha$. Thus the set $T_g(K_r)$ is relatively compact, finishing the proof.

Lemma 6. *Let $g \in H(B)$, then*

$$R[T_g f](z) = f(z) Rg(z)$$

for any $f \in H(B)$ and $z \in B$.

Proof. Suppose the holomorphic function fRg has the $Taylor$ expansion

$$(fRg)(z) = \sum_{|\alpha|\geq 1} a_\alpha z^\alpha.$$

Then we have

$$R(T_g f)(z) = R\int_0^1 f(tz) R(tz) \frac{dt}{t} = R\int_0^1 \sum_{|\alpha|\geq 1} a_\alpha (tz)^\alpha \frac{dt}{t}$$

$$= R[\sum_{|\alpha|\geq 1} \frac{a_\alpha z^\alpha}{|\alpha|}] = \sum_{|\alpha|\geq 1} a_\alpha z^\alpha = (fRg)(z).$$

3. The Proof Of Theorem 1

Suppose $\sup_{z\in B}(1-|z|^2)^\alpha G_{\frac{n+1+q}{p}}(z)|Rg(z)|<\infty$. $\forall f\in H(B)$ then by Lemma 1, Lemma 2 and Lemma 6, we have

$$\begin{aligned}&(1-|z|^2)^\alpha|R[T_gf](z)|\\&=(1-|z|^2)^\alpha|f(z)||Rg(z)|\\&\le c(1-|z|^2)^\alpha G_{\frac{n+1+q}{p}}(z)|Rg(z)|\\&\le c\|f\|_{B(p,q)}(1-|z|^2)^\alpha G_{\frac{n+1+q}{p}}(z)|Rg(z)|\\&\le c\|f\|_{B(p,q)}.\end{aligned}$$

Therefore, T_g is bounded .

On the other hand, suppose T_g is bounded, with

$$\|T_gf\|_{\mathcal{B}^\alpha}\le c\|f\|_{B(p,q)}.$$

(1) If $0<\frac{n+1+q}{p}<1$, it's very easy to show that the function $f(z)=1$ are in $B(p,q)$, therefore T_gf must be in $\mathcal{B}^\alpha$,
namely

$$\begin{aligned}&\sup_{z\in B}(1-|z|^2)^\alpha|RT_gf(z)|\\&=\sup_{z\in B}(1-|z|^2)^\alpha|Rg(z)|<\infty.\end{aligned}$$

(2) If $\frac{n+1+q}{p}>1$, we need to prove that $\sup_{z\in B}(1-|z|^2)^\alpha(\frac{1}{1-|z|^2})^{\frac{n+1+q}{p}-1}|Rg(z)|<\infty$.

For $w\in B$, take the test function

$$f_w(z)=\frac{1-|w|^2}{(1-<z,w>)^{\frac{n+1+q}{p}}}.$$

It is easy to see that

$$\int_B(1-|z|^2)^q|\nabla f_w(z)|^pdv(z)\le c(1-|w|^2)^p\int_B\frac{(1-|z|^2)^q}{|1-<z,w>|^{n+1+q+p}}dv(z)\le c.$$

The last inequality follws from [Zhu1], so $f_w\in B(p,q)$ for any $w\in B$. With the boundedness of T_g, we get

$$\begin{aligned}&(1-|z|^2)^\alpha(\frac{1}{1-|z|^2})^{\frac{n+1+q}{p}-1}|Rg(z)|\\&=(1-|z|^2)^\alpha|f_z(z)||Rg(z)|\\&=(1-|z|^2)^\alpha|R(T_gf_z)(z)|\\&\le\|T_gf_z\|_{\mathcal{B}^\alpha}\le c\|T_g\|<\infty.\end{aligned}$$

(3) If $\frac{n+1+q}{p}=1$, namely $p=n+1+q$, we need to prove

$$\sup_{z\in B}(1-|z|^2)^\alpha\log\frac{2}{1-|z|^2}|Rg(z)|<\infty.$$

For $w\in B$, take the test function

$$f_w(z)=(\log\frac{1}{1-|w|^2})^{-\frac{2}{p}}(\log\frac{1}{1-<z,w>})^{1+\frac{2}{p}}.$$

It is easy to show that $f_w \in B(p,q)$ from Lemma 4. The same discussion as the case (2) gives the needed result, and we omit it here. So, the proof of Theorem 1 is completed.

4. The Proof Of Theorem 2

$\{f_j\}$ is an uniformly bounded sequence in $B(p,q)$ which converges to zero uniformly on any compact subset of B when $j \to \infty$.

(1) If T_g is compact, we have got that $g \in \mathcal{B}^\alpha$.

On the other hand, from Lemma 1, we know that $\|f_j\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \le c\|f_j\|_{B(p,q)}$, thus $\{f_j\}$ is unformly bounded in $\mathcal{B}^{\frac{n+1+q}{p}}$. Then by the hypothesis and Lemma 3, we get that

$$\lim_{j\to\infty}\sup_{z\in B}|f_j(z)| = 0.$$

Therefore

$$\|T_g f_j\|_{\mathcal{B}^\alpha} \le c\sup_{z\in B}(1-|z|^2)^\alpha |f_j(z)Rg(z)| \le c\|g\|_{\mathcal{B}^\alpha}\sup_{z\in B}|f_j(z)|.$$

Then when $j \to \infty$, $\|T_g f_j\|_{\mathcal{B}^\alpha} \to 0$. So T_g is compact from Lemma 5.

(2) If $\lim_{|z|\to 1^-}(1-|z|^2)^\alpha G_{\frac{n+1+q}{p}}(z)|Rg(z)| = 0$, then $\forall \varepsilon > 0$, $\exists r \in (0,1)$, such that

$$(1-|z|^2)^\alpha G_{\frac{n+1+q}{p}}(z)|Rg(z)| < \varepsilon, r < |z| < 1.$$

Then

$$\begin{aligned}
&\|T_g f_j\|_{\mathcal{B}^\alpha} \le c\sup_{|z|\le r}(1-|z|^2)^\alpha |f_j(z)Rg(z)| + c\sup_{r<|z|<1}(1-|z|^2)^\alpha |f_j(z)Rg(z)| \\
&\le c\sup_{|z|\le r}(1-|z|^2)^\alpha |Rg(z)||f_j(z)| + c\sup_{r<|z|<1}(1-|z|^2)^\alpha G_{\frac{n+1+q}{p}}(z)|Rg(z)|\|f_j\|_{B(p,q)} \\
&\le c\sup_{|z|\le r}(1-|z|^2)^\alpha |Rg(z)||f_j(z)| + c\varepsilon\|f_j\|_{B(p,q)} \\
&\le c\varepsilon,
\end{aligned}$$

if j is sufficiently large. This means $\|T_g f_j\|_{\mathcal{B}^\alpha} \to 0$ as j tends to ∞.

On the other hand, if $\frac{n+1+q}{p} = 1$, it is sufficient to prove

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |Rg(z)|\log\frac{1}{1-|z|^2} = 0.$$

Suppose that $\lim_{|z|\to 1}(1-|z|^2)^\alpha |Rg(z)|\log\frac{1}{1-|z|^2} \ne 0$, then there exists $\varepsilon_0 > 0$, $\{z^j\} \in B$, such that

$$(1-|z^j|^2)^\alpha |Rg(z^j)|\log\frac{1}{1-|z^j|^2} \ge \varepsilon_0.$$

Let

$$f_j(z) = (\log\frac{1}{1-|z^j|^2})^{-\frac{2}{p}}(\log\frac{1}{1-<z,z^j>})^{1+\frac{2}{p}}.$$

We have shown that $f_j \in B(p,q)$ with $\|f_j\|_{B(p,q)} \leq c$, and it is obvious that $f_j \to 0$ uniformly on any compact subset of B as $j \to \infty$. While

$$\begin{aligned}
&\|T_g f_j\|_{\mathcal{B}^\alpha} \\
&\geq (1-|z^j|^2)^\alpha |f_j(z^j)||Rg(z^j)| \\
&= \{(1-|z^j|^2)^\alpha |Rg(z^j)| \log \frac{1}{1-|z^j|^2}\}|f_j(z^j)|(\log \frac{1}{1-|z^j|^2})^{-1} \\
&\geq \varepsilon_0 |f_j(z^j)|(\log \frac{1}{1-|z^j|^2})^{-1} \\
&= \varepsilon_0,
\end{aligned}$$

then $\|T_g f_j\|_{\mathcal{B}^\alpha}$ doesn't tend to 0 when $j \to \infty$. It's a contraction. So

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |Rg(z)| \log \frac{1}{1-|z|^2} = 0.$$

Meanwhile, as $\lim_{|z|\to 1} \log \frac{1}{1-|z|^2} = \infty$, it is easy to see that $\lim_{|z|\to 1}(1-|z|^2)^\alpha |Rg(z)| = 0$.

Therefore, we have

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |Rg(z)| \log \frac{2}{1-|z|^2} = 0.$$

If $\frac{n+1+q}{p} > 1$, just let

$$f_j(z) = \frac{1-|z^j|^2}{(1-<z,z^j>)^{\frac{n+1+q}{p}}},$$

and use the same method as the situation of $\frac{n+1+q}{p} = 1$, we can also prove that the theorem holds. So, the proof of Theorem 2 is completed.

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