Asymptotic Behaviour of Parameter Ideals in Generalized Cohen-Macaulay Modules

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Abstract

The purpose of this paper is to give affirmative answers to two open questions as follows. Let $(R, \mathfrak{m})$ be a generalized Cohen-Macaulay Noetherian local ring. Both questions, the first question was raised by M. Rogers [12] and the second one is due to S. Goto and H. Sakurai [7], ask whether for every parameter ideal $\mathfrak{q}$ contained in a high enough power of the maximal ideal $\mathfrak{m}$ the following statements are true: (1) The index of reducibility $N_R(\mathfrak{q}; R)$ is independent of the choice of $\mathfrak{q}$; and (2) $I^2 = \mathfrak{q}I$, where $I = \mathfrak{q} :_R \mathfrak{m}$.

Key words: index of reducibility, socle, generalized Cohen-Macaulay module, local cohomology module.
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1 Introduction

Let R be a commutative Noetherian local ring with the maximal ideal $\mathfrak{m}$ and residue field $\mathfrak{k} = R/\mathfrak{m}$, and let M be a finitely generated R-module with $\dim M = d$. Recall that a submodule of M is called irreducible if it cannot be written as the intersection of two larger submodules. It is well known that every submodule N of M can be expressed as an irredundant intersection of irreducible submodules, and that the number of irreducible submodules appearing in such an expression depends only on N and not on the expression. Thus for a parameter ideal $\mathfrak{q}$ of M, the number $N_R(\mathfrak{q}; M)$ of irreducible modules that appear in an irredundant irreducible decomposition of $\mathfrak{q}M$ is called the index of reducibility of $\mathfrak{q}$ on M. Let N be an arbitrary R-module. We denote by $\mathrm{Soc}(N)$ the socle of N. Since $\mathrm{Soc}(N) \cong 0 :_N \mathfrak{m} \cong \mathrm{Hom}(\mathfrak{k}, N)$ is a $\mathfrak{k}$-vector space, we set $s(N) = \dim_{\mathfrak{k}} \mathrm{Soc}(N)$ the socle dimension of N. Then we have $N_R(\mathfrak{q}; M) = s(M/\mathfrak{q}M)$.

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In 1957, D. G. Northcott [9, Theorem 3] proved that the index of reducibility of any parameter ideal in a Cohen-Macaulay local ring is dependent only on the ring and not on the choice of the parameter ideal. However, this property of constant index of reducibility of parameter ideals does not characterize Cohen-Macaulay modules. The first example of a non-Cohen-Macaulay Noetherian local ring having constant index of reducibility of parameter ideals was given by S. Endo and M. Narita [5]. In 1984, S. Goto and N. Suzuki [7] considered the supremum $r(M)$ of the index of reducibility of parameter ideals of M and they showed that this number is finite provided M is a generalized Cohen-Macaulay module. Recall that M is said to be a generalized Cohen-Macaulay module, if local cohomology modules $H^i_{\mathfrak{m}}(M)$ of M with respect the maximal ideal $\mathfrak{m}$ is of finite length for $i = 0, 1, \ldots, d-1$. Moreover, they also proved that $r(M) \geqslant \sum\limits_{i=0}^{d} \binom{d}{i} s(H^i_{\mathfrak{m}}(R))$. Later, S. Goto and H. Sakurai in [6, Corollary 3.13] showed that if R is a Buchsbaum ring of positive dimension, then there is a power of the maximal ideal $\mathfrak{m}$ inside which every parameter ideal $\mathfrak{q}$ has the same index of reducibility. J. C. Liu and M. Rogers [8] refer to this by saying R has eventual constant index of reducibility of parameter ideals. Therefore the following question, which was raised first by M. Rogers in [12, Question 1.2] (see also [8, Question 1.3]), is natural: *Does a generalized Cohen-Macaulay rings have eventual constant index of reducibility of parameter ideals?*

Partial answers to this question were proved by Rogers [12, Theorem 1.3] for a generalized Cohen-Macaulay module of dimension $d \leqslant 2$ and by Liu and Rogers [8, Theorem 1.4] for a generalized Cohen-Macaulay module M having $H^i_{\mathfrak{m}}(M) = 0$ for all i with $i \neq 0, t, d$, where t is some integer with $0 < t < d$.

Our first main result in this paper is to provide a completely answer to this question.

Theorem 1.1. *Let M be a generalized Cohen-Macaulay module over a Noetherian local ring $(R, \mathfrak{m})$ with $\dim M = d$. Then there is a positive integer n such that for every parameter ideal $\mathfrak{q}$ of M contained in $\mathfrak{m}^n$ the index of reducibility $N(\mathfrak{q}; M)$ is independent of the choice of $\mathfrak{q}$ and is given by*

$$N(\mathfrak{q}; M) = \sum_{i=0}^{d} \binom{d}{i} s(H^i_{\mathfrak{m}}(M)).$$

In [6], Goto and Sakurai used the study of the index of reducibility of parameter ideals in order to investigate when the equality $I^2 = \mathfrak{q}I$ holds for a parameter ideal $\mathfrak{q}$ of R, where $I = \mathfrak{q} : \mathfrak{m}$. Note that by results of A. Corso, C. Huneke, C. Polini and W. V. Vasconcelos [1, 2, 4] this equality holds for any parameter ideal in a Cohen-Macaulay local ring R which is not regular or dimensional at least 2 and $e(R) > 1$, where $e(R)$ is the multiplicity of R with respect to the maximal ideal $\mathfrak{m}$. Goto and Sakurai generalized this and proved in [6, Theorem 3.11] that if R is a Buchsbaum ring of dimension $\dim R \geq 2$ or $\dim R = 1$ and $e(R) > 1$, then the equality $I^2 = \mathfrak{q}I$ holds true for all parameter ideals $\mathfrak{q}$ contained in a high enough power of the maximal ideal $\mathfrak{m}$. From this point of view, it is natural to ask the following question, which is due to Goto-Sakurai [6, p. 34]: *Let R be a generalized Cohen-Macaulay ring with the multiplicity $e(R) > 1$. Is there a positive integer n such that $I^2 = \mathfrak{q}I$ for every parameter ideal $\mathfrak{q}$ contained in $\mathfrak{m}^n$?*

As a consequence of Theorem 1.1 we obtain the second main result of the paper, which is an affirmative answer to this question.

Theorem 1.2. *Let R be a generalized Cohen-Macaulay ring and assume that* $\dim R \geq 2$ *or* $\dim R = 1$, $e(R) > 1$. *Then there exists a positive integer n such that $I^2 = \mathfrak{q}I$ for every parameter ideal $\mathfrak{q} \subseteq \mathfrak{m}^n$, where $I = \mathfrak{q} : \mathfrak{m}$.*

Our goal for proving Theorem 1.1 is to show by induction on $d = \dim M$ that there is an enough large integer n such that $N(\mathfrak{q}; M) = \sum\limits_{i=0}^{d} \binom{d}{i} s(H^i_{\mathfrak{m}}(M))$ for every parameter ideal $\mathfrak{q} \subseteq \mathfrak{m}^n$. Therefore we give in the Section 2 several lemmata on the asymptotic behaviour of parameter ideals in a generalized Cohen-Macaulay module M in order to prove the following key result in Section 3 (see Theorem 3.3): Let M be a generalized Cohen-Macaulay R-module. Then there exists a enough large integer k such that

$$s(H^i_{\mathfrak{m}}(\frac{M}{(x_1, \ldots, x_{j+1})M}) = s(H^i_{\mathfrak{m}}(\frac{M}{(x_1, \ldots, x_j)M})) + s(H^{i+1}_{\mathfrak{m}}(\frac{M}{(x_1, \ldots, x_j)M})),$$

for every parameter ideal $\mathfrak{q} = (x_1, \ldots, x_d) \subseteq \mathfrak{m}^k$ and for all $0 \leqslant i + j \leqslant d - 1$. The last Section is devoted to prove the main results and their consequences.

2 Some auxiliary lemmata

Throughout this paper we fix the following standard notations: Let R be a Noetherian local commutative ring with maximal ideal $\mathfrak{m}$, $\mathfrak{k} = R/\mathfrak{m}$ the residue field and M a finitely generated R-module with $\dim M = d$. Let $\mathfrak{q} = (x_1, \ldots, x_d)$ be a parameter ideal of module M. We denote by $\mathfrak{q}_i$ the ideal $(x_1, \ldots, x_i)R$ for $i = 1, \ldots, d$ and stipulate that $\mathfrak{q}_0$ is the zero ideal of R.

An R-module M is said to be a *generalized Cohen-Macaulay module* if $H^i_{\mathfrak{m}}(M)$ are of finite length for all $i = 0, 1, \ldots, d - 1$ (see [3]). This condition is equivalent to saying that there exists a parameter ideal $\mathfrak{q} = (x_1, \ldots, x_d)$ of M such that $\mathfrak{q}H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}) = 0$ for all $0 \leq i + j < d$ (see [13]), and such a parameter ideal was called a *standard parameter ideal* of M. It is well-known that if M is a generalized Cohen-Macaulay module, then every parameter ideal of M in a high enough power of the maximal ideal $\mathfrak{m}$ is standard. The following lemma can be easily derived from the basic properties of generalized Cohen-Macaulay modules.

Lemma 2.1. *Let M be a generalized Cohen-Macaulay R-module with* $\dim M = d \geq 1$. *Then there exists a positive integer n_1 such that for all parameter ideals $\mathfrak{q} = (x_1, \ldots, x_d)$ of M contained in $\mathfrak{m}^{n_1}$ we have $\mathfrak{m}^{n_1} H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}) = 0$ for all $0 \leq i + j \leq d - 1$.*

Proof. Since M is a generalized Cohen-Macaulay R-module, there is an integer l such that $\mathfrak{m}^l H^i_{\mathfrak{m}}(M) = 0$ for all $0 \le i \le d-1$. Let $x \in \mathfrak{m}^l$ be a parameter element of M. Since $\ell(0 :_M x) < \infty$, we have isomorphisms $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(\frac{M}{xM})$ for all $i \ge 1$, and so that the sequences

$$0 \longrightarrow H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(\frac{M}{xM}) \longrightarrow H^{i+1}_{\mathfrak{m}}(M) \longrightarrow 0$$

are exact for all $0 \le i \le d-2$. Therefore $\mathfrak{m}^{2l} H^i_{\mathfrak{m}}(\frac{M}{xM}) = 0$ for all $0 \le i \le d-2$. Now, set $n_1 = 2^{d-1}l$. We can use the fact above to prove that for all parameter ideals $\mathfrak{q} = (x_1, \dots, x_d)$ of M contained in $\mathfrak{m}^{n_1}$ and $0 \le i + j \le d-1$, it holds $\mathfrak{m}^{n_1} H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}) = 0$. ☐

In order to prove the next lemma, we need a result of W. V. Vasconcelos on the reduction number of an ideal in local rings. Let J and K be two ideals of R with $J \subseteq K$. The ideal J is called a reduction of K with respect to M if $K^{r+1}M = JK^rM$ for some integer r, and the least of such integers is denoted by $r_J(K, M)$. Then the big reduction number $\mathrm{bigr}(K)$ of K with respect to M was defined by

$$\mathrm{bigr}(K) = \sup\{r_J(K, M)|\ J \text{ is a reduction of } K \text{ with respect to } M\}.$$

It is known that there always exists a reduction ideal for any ideal K provided the residue field $\mathfrak{k}$ of R is infinite. Especially, if K is $\mathfrak{m}$-primary then any minimal reduction ideal of K with respect to M is a parameter ideal of M. Moreover, it was shown by Vasconcelos [14] that $\mathrm{bigr}(K)$ is finite for any ideal K.

Lemma 2.2. *Let M be a generalized Cohen-Macaulay R-module with* $\dim M = d \ge 1$. *Then there exists a positive integer n_2 such that for all parameter ideals $\mathfrak{q} = (x_1, \dots, x_d)$ of M contained in $\mathfrak{m}^{n_2}$ and $0 \le j < d$ we have*

$$\mathfrak{m}^{n_2}\frac{M}{\mathfrak{q}_j M} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}) = 0.$$

Proof. Note first that by the faithfully flat homomorphism $R \to R[X]_{\mathfrak{m}R[X]}$ as a basic change, we can assume without any loss of generality that the residue field $\mathfrak{k}$ of R is infinite. By Lemma 2.1 there is an integer n_1 such that $H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}) = 0 :_{\frac{M}{\mathfrak{q}_j M}} \mathfrak{m}^{n_1}$ for all parameter ideals $\mathfrak{q}$ contained in $\mathfrak{m}^{n_1}$ and $j < d$. Set $K = \mathfrak{m}^{n_1}$ and $n_2 = (\mathrm{bigr}(K) + 1)n_1$. Then for any parameter ideal $\mathfrak{q} = (x_1, \dots, x_d)$ of M contained in $\mathfrak{m}^{n_2}$ and any $0 \le j < d$, there is a parameter ideal $\mathfrak{a} = (a_{j+1}, \dots, a_d)$ of $\frac{M}{\mathfrak{q}_j M}$ contained in K, which is a reduction of K with respect to $\frac{M}{\mathfrak{q}_j M}$, such that

$$\mathfrak{a}K^{r_{\mathfrak{a}}(K, \frac{M}{\mathfrak{q}_j M})}\frac{M}{\mathfrak{q}_j M} = K^{r_{\mathfrak{a}}(K, \frac{M}{\mathfrak{q}_j M})+1}\frac{M}{\mathfrak{q}_j M}.$$

Since $r_{\mathfrak{a}}(K, \frac{M}{\mathfrak{q}_j M}) \le r_{\mathfrak{a}}(K, M) \le \mathrm{bigr}(K) < \infty$, we have

$$\mathfrak{m}^{n_2}\frac{M}{\mathfrak{q}_j M} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}) = \mathfrak{a}K^{\mathrm{bigr}(K)}\frac{M}{\mathfrak{q}_j M} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}) \subseteq \mathfrak{a}\frac{M}{\mathfrak{q}_j M} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}).$$

Therefore it is enough to prove that $\mathfrak{a}\frac{M}{\mathfrak{q}_jM} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_jM}) = 0$. In fact, let $m \in \mathfrak{a}\frac{M}{\mathfrak{q}_jM} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_jM})$. Write $m = a_{j+1}m_{j+1} + \ldots + a_dm_d$, where $m_i \in \frac{M}{\mathfrak{q}_jM}$ for all $i = j+1, \ldots, d$. Since $\frac{M}{\mathfrak{q}_jM}$ is a generalized Cohen-Macaulay module and $\mathfrak{a}$ a standard parameter ideal of $\frac{M}{\mathfrak{q}_jM}$ by Lemma 2.1, we get that

$$m_d \in (a_{j+1}, \ldots, a_{d-1})\frac{M}{\mathfrak{q}_jM} : a_d^2 = (a_{j+1}, \ldots, a_{d-1})\frac{M}{\mathfrak{q}_jM} : a_d.$$

It follows that

$$\mathfrak{a}\frac{M}{\mathfrak{q}_jM} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_jM}) \subseteq (a_{j+1}, \ldots, a_{d-1})\frac{M}{\mathfrak{q}_jM} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_jM}).$$

If $j+1 < d-1$, we can continue the procedure above again so that after $(d-j)$-times we obtain

$$\mathfrak{a}\frac{M}{\mathfrak{q}_jM} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_jM}) \subseteq a_{j+1}\frac{M}{\mathfrak{q}_jM} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_jM}) \subseteq a_{j+1}\frac{M}{\mathfrak{q}_jM} \cap (0 :_{\frac{M}{\mathfrak{q}_jM}} a_{j+1}) = 0$$

as required. □

Lemma 2.3. *Let M be a finitely generated R-module with $\dim M = d \geq 1$. Let k and ℓ be two positive integers. Then there exists an integer $n_3 > \ell$ such that*

$$(\mathfrak{m}^{n_3} + H^0_{\mathfrak{m}}(M)) : \mathfrak{m}^k \subseteq \mathfrak{m}^\ell M + H^0_{\mathfrak{m}}(M).$$

Proof. Let $\overline{M} = \frac{M}{H^0_{\mathfrak{m}}(M)}$. Then there is an $\overline{M}$-regular element a contained in $\mathfrak{m}^k$. By the Artin-Rees Lemma, there exists a positive integer m such that $\mathfrak{m}^{\ell+m}\overline{M} \cap a\overline{M} = \mathfrak{m}^\ell(\mathfrak{m}^m\overline{M} \cap a\overline{M})$. Set $n_3 = \ell + m$. We have

$$a(\mathfrak{m}^{n_3}\overline{M} : \mathfrak{m}^k) \subseteq a(\mathfrak{m}^{n_3}\overline{M} : a) = \mathfrak{m}^{n_3}\overline{M} \cap a\overline{M} = \mathfrak{m}^\ell(\mathfrak{m}^m\overline{M} \cap a\overline{M}),$$

so that $a(\mathfrak{m}^{n_3}\overline{M} : \mathfrak{m}^k) \subseteq a\mathfrak{m}^\ell\overline{M}$. It follows from the regularity of a that $\mathfrak{m}^{n_3}\overline{M} : \mathfrak{m}^k \subseteq \mathfrak{m}^\ell\overline{M}$. Hence $(\mathfrak{m}^{n_3}M + H^0_{\mathfrak{m}}(M)) : \mathfrak{m}^k \subseteq \mathfrak{m}^\ell M + H^0_{\mathfrak{m}}(M)$ as required. □

Lemma 2.4. *Let M be a finitely generated R-module with $\dim M = d \geq 1$. Then there exists a positive integer n_4 such that for all ideals $K \subseteq \mathfrak{m}^{n_4}$ we have*

$$(KM + H^0_{\mathfrak{m}}(M)) : \mathfrak{m} = KM : \mathfrak{m} + H^0_{\mathfrak{m}}(M).$$

Proof. Since $H^0_{\mathfrak{m}}(M)$ have finite length, there exists an integer ℓ such that $\mathfrak{m}^\ell M \cap H^0_{\mathfrak{m}}(M) = 0$. By Lemma 2.3, there is an integer $n_4 > \ell$ such that for all ideals $K \subseteq \mathfrak{m}^{n_4}$ we have

$$(KM + H^0_{\mathfrak{m}}(M)) : \mathfrak{m} \subseteq (\mathfrak{m}^{n_4}M + H^0_{\mathfrak{m}}(M)) : \mathfrak{m} \subseteq \mathfrak{m}^\ell M + H^0_{\mathfrak{m}}(M).$$

Let $b \in (KM + H^0_{\mathfrak{m}}(M)) : \mathfrak{m}$. Write $b = \alpha + \beta$ with $\alpha \in \mathfrak{m}^\ell M$ and $\beta \in H^0_{\mathfrak{m}}(M)$. Then, since $K \subseteq \mathfrak{m}^{n_4} \subseteq \mathfrak{m}^{\ell+1}$,

$$\mathfrak{m}\alpha \subseteq \mathfrak{m}^{\ell+1}M \cap (KM + H^0_{\mathfrak{m}}(M)) = KM + \mathfrak{m}^{\ell+1}M \cap H^0_{\mathfrak{m}}(M) = KM.$$

Thus $\alpha \in KM : \mathfrak{m}$ and so that $(KM + H^0_{\mathfrak{m}}(M)) : \mathfrak{m} = KM : \mathfrak{m} + H^0_{\mathfrak{m}}(M)$. □

Lemma 2.5. *Let M be a generalized Cohen-Macaulay R-module with* $\dim M = d \geq 1$. *Then there exists a positive integer n_5 such that for all parameter ideals $\mathfrak{q} = (x_1, \ldots, x_d)$ of M contained in $\mathfrak{m}^{n_5}$ and $0 \leq j < i \leq d$ we have*

$$[\frac{\mathfrak{q}_i M}{\mathfrak{q}_j M} + H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})] : \mathfrak{m} = \frac{\mathfrak{q}_i M}{\mathfrak{q}_j M} : \mathfrak{m} + H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}).$$

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Proof. Let n_1 and n_2 be two integers as in Lemma 2.1 and Lemma 2.2, respectively. By Lemma 2.3, there always exists an integer $n_5 > n_2$ such that $(\mathfrak{m}^{n_5} M + H^0_{\mathfrak{m}}(M)) : \mathfrak{m}^{n_1+1} \subseteq \mathfrak{m}^{n_2} M + H^0_{\mathfrak{m}}(M)$. Let $\mathfrak{q} = (x_1, \ldots, x_d)$ be a parameter ideal of M contained in $\mathfrak{m}^{n_5}$. For all $0 \leq j < i \leq d$, we have $H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}) = 0 :_{\frac{M}{\mathfrak{q}_j M}} \mathfrak{m}^{n_1}$ by Lemma 2.1, and so that

$$\begin{aligned}(\frac{\mathfrak{q}_i M}{\mathfrak{q}_j M} + H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})) : \mathfrak{m} \subseteq \frac{\mathfrak{m}^{n_5} M}{\mathfrak{q}_j M} &: \mathfrak{m}^{n_1+1} \\ = \frac{\mathfrak{m}^{n_5} M : \mathfrak{m}^{n_1+1}}{\mathfrak{q}_j M} &\subseteq \frac{\mathfrak{m}^{n_2} M}{\mathfrak{q}_j M} + H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}).\end{aligned}$$

Let $b \in (\frac{\mathfrak{q}_i M}{\mathfrak{q}_j M} + H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})) : \mathfrak{m}$. Write $b = \alpha + \beta$ with $\alpha \in \frac{\mathfrak{m}^{n_2} M}{\mathfrak{q}_j M}$ and $\beta \in H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})$. Since $\mathfrak{q}_i \subseteq \mathfrak{m}^{n_5} \subseteq \mathfrak{m}^{n_2+1}$, we get by Lemma 2.2 that

$$\mathfrak{m}\alpha \subseteq \frac{\mathfrak{m}^{n_2+1} M}{\mathfrak{q}_j M} \cap (\frac{\mathfrak{q}_i M}{\mathfrak{q}_j M} + H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})) = \frac{\mathfrak{q}_i M}{\mathfrak{q}_j M} + \frac{\mathfrak{m}^{n_2+1} M}{\mathfrak{q}_j M} \cap H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}) = \frac{\mathfrak{q}_i M}{\mathfrak{q}_j M}.$$

Therefore $\alpha \in \frac{\mathfrak{q}_i M}{\mathfrak{q}_j M} : \mathfrak{m}$, and so that

$$(\frac{\mathfrak{q}_i M}{\mathfrak{q}_j M} + H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})) : \mathfrak{m} = \frac{\mathfrak{q}_i M}{\mathfrak{q}_j M} : \mathfrak{m} + H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})$$

as required. □

3 The socle dimension of local cohomology modules

Let $\mathfrak{q} = (x_1, \ldots, x_d)$ be a parameter ideal of the module M. For each positive integer n, we denote by $\mathfrak{q}(n)$ the ideal $(x_1^n, \ldots, x_d^n)$. Let $K_*(\mathfrak{q}(n))$ be the Koszul complex of R with respect to the ideal $\mathfrak{q}(n)$ and

$$H^*(\mathfrak{q}(n); M) = H^*(\mathrm{Hom}(K_*(\mathfrak{q}(n), M))$$

the Koszul cohomology module of M. Then the family $\{H^i(\mathfrak{q}(n); M)\}_{n \geq 1}$ naturally forms an inductive system of R-modules for every $i \in \mathbb{Z}$, whose inductive limit is just the i-th local cohomology module

$$H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{q}}(M) = \varinjlim_n H^i(\mathfrak{q}(n); M).$$

The following result is due to Goto and Suzuki.

Lemma 3.1 ([7], Lemma 1.7)**.** *Let M be a finitely generated R-module, x an M-regular element and $\mathfrak{q} = (x_1, \ldots, x_r)$ an ideal of R with $x_1 = x$. Then there exists a splitting exact sequence for each $i \in \mathbb{Z}$,*

$$0 \to H^i(\mathfrak{q}; M) \to H^i(\mathfrak{q}; \frac{M}{xM}) \to H^{i+1}(\mathfrak{q}; M) \to 0.$$

The next result is due to Goto and Sakurai.

Lemma 3.2 ([6] Lemma 3.12)**.** *Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $r = \dim R \geq 1$. Let M be a finitely generated R-module. Then there exists a positive integer ℓ such that for all parameter ideals $\mathfrak{q} = (x_1, \ldots, x_d)$ of M contained in $\mathfrak{m}^\ell$ and all $i \in \mathbb{Z}$, the canonical homomorphisms on socles*

$$\mathrm{Soc}(\mathrm{H}^{\mathrm{i}}(\mathfrak{q}, \mathrm{M})) \to \mathrm{Soc}(\mathrm{H}^{\mathrm{i}}_{\mathfrak{m}}(\mathrm{M}))$$

are surjective.

The following theorem is the key to proofs of main results of the paper.

Theorem 3.3. *Let M be a generalized Cohen-Macaulay R-module with* $\dim M = d \geq 1$. *There there exists a positive integer k such that for all parameter ideal $\mathfrak{q}$ of M contained in $\mathfrak{m}^k$ and $d > i + j \geq 0$ we have*

$$s(H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_{j+1}M})) = s(H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})) + s(H^{i+1}_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})),$$

where $s(N) = \dim_{\mathfrak{k}} \mathrm{Soc}(\mathrm{N})$ the socle dimension of the R-module N.

Proof. We set $k = \max\{n_1, n_2, n_5, \ell\} + 1$, where n_1, n_2, n_5 and ℓ are integers as in Lemma 2.1, 2.2 , 2.5, and 3.2, respectively. It will be shown that this integer k is just the required integer of the theorem. Let $\mathfrak{q} = (x_1, \ldots, x_d)$ be a parameter ideal of M contained in $\mathfrak{m}^k$. We denote by M_j the module $\frac{M}{\mathfrak{q}_j M}$ and $\overline{M}_j$ the module $\frac{M_j}{H^0_{\mathfrak{m}}(M_j)}$. It should be noted here that M_j and $\overline{M}_j$ are generalized Cohen-Macaulay modules having $(x_{j+1}, \ldots, x_d)$ as a standard parameter ideal by Lemma 2.1. Then the proof of Theorem 3.3 is divided into two cases.

First case: $i = 0$. Because of the choose of k, the ideal $\mathfrak{q}$ is a standard parameter ideal of M and so that $x_{j+1} H^1_{\mathfrak{m}}(\overline{M}_j) = 0$ for all $0 \leq j < d$. Thus we have

$$H^1_{\mathfrak{m}}(M_j) \cong H^1_{\mathfrak{m}}(\overline{M}_j) \cong H^0_{\mathfrak{m}}(\frac{\overline{M}_j}{x_{j+1}\overline{M}_j}).$$

Therefore, we get by Lemma 2.5 that

$$\begin{aligned} s(H^1_{\mathfrak{m}}(M_j)) = s(H^0_{\mathfrak{m}}(\frac{\overline{M}_j}{x_{j+1}\overline{M}_j})) &= \ell(\frac{(\mathfrak{q}_{j+1}M_j + H^0_{\mathfrak{m}}(M_j)) : \mathfrak{m}}{\mathfrak{q}_{j+1}M_j + H^0_{\mathfrak{m}}(M_j)}) \\ &= \ell(\frac{\mathfrak{q}_{j+1}M_j : \mathfrak{m} + H^0_{\mathfrak{m}}(M_j)}{\mathfrak{q}_{j+1}M_j + H^0_{\mathfrak{m}}(M_j)}) \\ &= \ell(\frac{\mathfrak{q}_{j+1}M_j : \mathfrak{m}}{(\mathfrak{q}_{j+1}M_j : \mathfrak{m}) \cap (\mathfrak{q}_{j+1}M_j + H^0_{\mathfrak{m}}(M_j))}) \\ &= \ell(\frac{\mathfrak{q}_{j+1}M_j : \mathfrak{m}}{\mathfrak{q}_{j+1}M_j + (\mathfrak{q}_{j+1}M_j : \mathfrak{m}) \cap H^0_{\mathfrak{m}}(M_j)}). \end{aligned}$$

Let $a \in (\mathfrak{q}_{j+1}M_j : \mathfrak{m}) \cap H^0_{\mathfrak{m}}(M_j)$. We see by Lemma 2.2 that

$$\mathfrak{m}a \in \mathfrak{q}_{j+1}M_j \cap H^0_{\mathfrak{m}}(M_j) = 0.$$

Therefore $(\mathfrak{q}_{j+1}M_j : \mathfrak{m}) \cap H^0_{\mathfrak{m}}(M_j) = 0 :_{M_j} \mathfrak{m}$, and so that

$$\begin{aligned} s(H^1_{\mathfrak{m}}(M_j)) &= \ell\Big(\frac{\mathfrak{q}_{j+1}M_j : \mathfrak{m}}{\mathfrak{q}_{j+1}M_j + 0 :_{M_j} \mathfrak{m}}\Big) \\ &= \ell\Big(\frac{\mathfrak{q}_{j+1}M_j : \mathfrak{m}}{\mathfrak{q}_{j+1}M_j}\Big) - \ell\Big(\frac{\mathfrak{q}_{j+1}M_j + 0 :_{M_j} \mathfrak{m}}{\mathfrak{q}_{j+1}M_j}\Big) \\ &= \ell\Big(\frac{\mathfrak{q}_{j+1}M_j : \mathfrak{m}}{\mathfrak{q}_{j+1}M_j}\Big) - \ell(0 :_{M_j} \mathfrak{m}) \\ &= s(H^0_{\mathfrak{m}}(M_{j+1})) - s(H^0_{\mathfrak{m}}(M_j)). \end{aligned}$$

Hence, we have $s(H^0_{\mathfrak{m}}(M_{j+1})) = s(H^0_{\mathfrak{m}}(M_j)) + s(H^1_{\mathfrak{m}}(M_j))$ for all $0 \le j < d$.

Second case: $i \ge 1$. We first claim by induction on j that for all $i \ge 1$ and $d > i + j \ge 1$, the canonical homomorphisms on socles

$$\alpha^i_j : \mathrm{Soc}(H^i(\mathfrak{q}, \overline{M}_j)) \to \mathrm{Soc}(H^i_{\mathfrak{m}}(\overline{M}_j))$$

are surjective. For the case $j = 0$, we consider the following commutative diagram

$$\begin{array}{ccc} H^i(\mathfrak{q}; M) & \longrightarrow & H^i(\mathfrak{q}, \overline{M}_0) \\ \big\downarrow f_i & & \big\downarrow g_i \\ H^i_{\mathfrak{m}}(M) & \xrightarrow{\pi_i} & H^i_{\mathfrak{m}}(\overline{M}_0), \end{array}$$

where π_i are isomorphisms for all $i \ge 1$. By Lemma 3.2, the homomorphism f_i induces a surjective homomorphism $\mathrm{Soc}(H^i(\mathfrak{q}, M)) \to \mathrm{Soc}(H^i_{\mathfrak{m}}(M))$ on the socles. Therefore we get by applying the functor $\mathrm{Hom}(\mathfrak{k}, *)$ to the diagram above that

$$\alpha^i_0 : \mathrm{Soc}(H^i(\mathfrak{q}, \overline{M}_0)) \to \mathrm{Soc}(H^i_{\mathfrak{m}}(\overline{M}_0))$$

are surjective for all $i \ge 1$. Now assume that $j \ge 1$. Since $(x_{j+1}, \ldots, x_d)$ is a standard parameter ideal of $\overline{M}_j$ and x_{j+1} an $\overline{M}_j$-regular element, we have for all $d > i + j \ge 1$ the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^i(\mathfrak{q}; \overline{M}_j) & \longrightarrow & H^i(\mathfrak{q}; \frac{\overline{M}_j}{x_{j+1}\overline{M}_j}) & \longrightarrow & H^{i+1}(\mathfrak{q}; \overline{M}_j) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & H^i_{\mathfrak{m}}(\overline{M}_j) & \longrightarrow & H^i_{\mathfrak{m}}(\frac{\overline{M}_j}{x_{j+1}\overline{M}_j}) & \longrightarrow & H^{i+1}_{\mathfrak{m}}(\overline{M}_j) & \longrightarrow & 0 \end{array}$$

with exact rows, where the upper row is split exact by Lemma 3.1. Therefore, by applying the functor $\mathrm{Hom}(\mathfrak{k}, *)$, we obtain for all $d > i + j \ge 1$ the commutative diagram

$$\begin{array}{ccccc}
0 \to \mathrm{Soc}(H^i(\mathfrak{q};\overline{M}_j)) & \longrightarrow & \mathrm{Soc}(H^i(\mathfrak{q};\frac{\overline{M}_j}{x_{j+1}\overline{M}_j})) & \longrightarrow & \mathrm{Soc}(H^{i+1}(\mathfrak{q};\overline{M}_j)) \to 0 \\
\downarrow \alpha_j^i & & \downarrow \beta_{j+1}^i & & \downarrow \alpha_j^{i+1} \\
0 \to \mathrm{Soc}(H^i_{\mathfrak{m}}(\overline{M}_j)) & \longrightarrow & \mathrm{Soc}(H^i_{\mathfrak{m}}(\frac{\overline{M}_j}{x_{j+1}\overline{M}_j})) & \longrightarrow & \mathrm{Soc}(H^{i+1}_{\mathfrak{m}}(\overline{M}_j))
\end{array}$$

with exact rows. By the inductive hypothesis, the homomorphisms α_j^i and α_j^{i+1} are surjective for all $i \geq 1$. Thus β_{j+1}^i are surjective for all $i \geq 1$. Since $\overline{M}_j$ is generalized Cohen-Macaulay, it is easy to check that $H^i_{\mathfrak{m}}(\frac{\overline{M}_j}{x_{j+1}\overline{M}_j}) \cong H^i_{\mathfrak{m}}(\overline{M}_{j+1})$ for all $i \geq 1$. It follows from the commutative diagram

$$\begin{array}{ccc}
\mathrm{Soc}H^i(\mathfrak{q};\frac{\overline{M}_j}{x_{j+1}\overline{M}_j}) & \longrightarrow & \mathrm{Soc}H^i(\mathfrak{q},\overline{M}_{j+1}) \\
\downarrow \beta_{j+1}^i & & \downarrow \alpha_{j+1}^i \\
\mathrm{Soc}H^i_{\mathfrak{m}}(\frac{\overline{M}_j}{x_{j+1}\overline{M}_j}) & \xrightarrow{\cong} & \mathrm{Soc}H^i_{\mathfrak{m}}(\overline{M}_{j+1})
\end{array}$$

that the homomorphism $\alpha_{j+1}^i : \mathrm{Soc}(H^i(\mathfrak{q},\overline{M}_{j+1})) \to \mathrm{Soc}(H^i_{\mathfrak{m}}(\overline{M}_{j+1}))$ are surjective for all $d > i+j \geq 1$, and the claim is proved. Next, from the proof of the claim we obtain exact sequences

$$0 \to \mathrm{Soc}(H^i_{\mathfrak{m}}(\overline{M}_j)) \longrightarrow \mathrm{Soc}(H^i_{\mathfrak{m}}(\frac{\overline{M}_j}{x_{j+1}\overline{M}_j})) \longrightarrow \mathrm{Soc}(H^{i+1}_{\mathfrak{m}}(\overline{M}_j)) \to 0\ ,$$

and so that $s(H^i_{\mathfrak{m}}(\frac{\overline{M}_j}{x_{j+1}\overline{M}_j})) = s(H^i_{\mathfrak{m}}(\overline{M}_j)) + s(H^{i+1}_{\mathfrak{m}}(\overline{M}_j))$ for all $i \geq 1$ and $d > i+j \geq 0$. Therefore, since $H^i_{\mathfrak{m}}(\overline{M}_j) \cong H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})$ for all $i \geq 1$, we have

$$s(H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_{j+1}M})) = s(H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M})) + s(H^{i+1}_{\mathfrak{m}}(\frac{M}{\mathfrak{q}_j M}))$$

for all $i \geq 1$ and $d > i+j \geq 1$, and the proof of Theorem 3.3 is complete. □

4 Proofs of main results

Theorem 1.1 is now an easy consequence of Theorem 3.3.

Proof of Theorem 1.1. By virtue of Theorem 3.3 we can show by induction on d that there exists an integer n such that for every parameter ideal $\mathfrak{q} = (x_1, \ldots, x_d)$ of M contained in $\mathfrak{m}^n$ we have

$$N(\mathfrak{q}; M) = s(H^0_{\mathfrak{m}}(\frac{M}{\mathfrak{q}M})) = \sum_{i=0}^{d} \binom{d}{i} s(H^i_{\mathfrak{m}}(M)).$$

□

Corollary 4.1. *Let M be a generalized Cohen-Macaulay R-module. Then*

$$\sup\{N(\mathfrak{q};M) | \mathfrak{q} \text{ is a standard parameter ideal of } M\} = \sum_{i=0}^{d} \binom{d}{i} s(H^i_{\mathfrak{m}}(M)).$$

Proof. Let $\mathfrak{q} = (x_1, \ldots, x_d)$ be a standard parameter ideal of M. By basic properties of the theory of generalized Cohen-Macaulay modules we can show by induction on t that

$$s(H^i_{\mathfrak{m}}(\frac{M}{(x_1, \ldots, x_t)M})) \leq \sum_{j=0}^{t} \binom{t}{j} s(H^{j+i}_{\mathfrak{m}}(M)).$$

for all $d \geq i + t \geq 0$. Therefore the Corollary follows by the inequality above in the case $t = d$, $i = 0$ and Theorem 1.1. □

In the rest of this paper, we denote

$$S(M) = \sum_{i=0}^{d} \binom{d}{i} s(H^i_{\mathfrak{m}}(M)).$$

Proof of Theorem 1.2. Let $n = \max\{n_1, n_4, k\}$, where n_4 and k are integers in Lemma 2.1, Lemma 2.4 and Theorem 3.3 (for the case $M = R$), respectively. We will prove that $I^2 = \mathfrak{q}I$ for all parameter ideals $\mathfrak{q} = (x_1, \ldots, x_d)$ of R contained in $\mathfrak{m}^n$, where $I = \mathfrak{q} :_R \mathfrak{m}$. Let $\dim R = d$ and $\overline{R} = \frac{R}{H^0_{\mathfrak{m}}(R)}$. Then by Lemma 2.4 we have

$$(\mathfrak{q} + H^0_{\mathfrak{m}}(R)) :_R \mathfrak{m} = \mathfrak{q} :_R \mathfrak{m} + H^0_{\mathfrak{m}}(R),$$

and so that $I\overline{R} = \mathfrak{q}\overline{R} : \mathfrak{m}\overline{R}$.

Case 1: $e(R) = 1$ and $d \geq 2$. Since $\overline{R}$ is unmixed, it is well-known in this case hat $\overline{R}$ is a regular local ring of dimension $d \geq 2$. We have $(I\overline{R})^2 = \mathfrak{q}\overline{R}I\overline{R}$ by Theorem 2.1 in [4]. Therefore $I^2 \subseteq \mathfrak{q}I + H^0_{\mathfrak{m}}(R)$ and so that $I^2 \subseteq \mathfrak{q}I + I^2 \cap H^0_{\mathfrak{m}}(R)$. But, $I^2 \cap H^0_{\mathfrak{m}}(R) \subseteq \mathfrak{q} \cap H^0_{\mathfrak{m}}(R) = 0$ by Lemma 2.2. Thus $I^2 = \mathfrak{q}I$ in this case.

Case 2: $e(R) > 1$. By the choose of n, the parameter ideal $\mathfrak{q}$ is standard Lemma 2.1 and $N(\mathfrak{q}; R) = S(R)$ by Theorem 1.1. Thus, it is enough for us to prove that if $N(\mathfrak{q}; R) = S(R)$ for some standard parameter ideal $\mathfrak{q} = (x_1, \ldots, x_d)$ of R contained in $\mathfrak{m}^n$ then $I^2 = \mathfrak{q}I$. Indeed, we argue by induction d. Let $d = 1$. Then $\overline{R}$ is a non-regular Cohen-Macaulay ring, and the conclusion follows with the same method as used in the proof of case 1. Now assume that $d \geq 2$. Set $R' = \frac{R}{(x_1)}$. By Theorem 3.3, we have $S(R) = S(R')$, and so that $N(\mathfrak{q}R'; R') = S(R')$. Therefore $(IR')^2 = \mathfrak{q}R'IR'$ by the inductive hypothesis. It follows that $I^2 \subseteq (x_2, \ldots, x_d)I + (x_1)$, and so that $I^2 \subseteq (x_2, \ldots, x_d)I + (x_1) \cap I^2$. Let $a \in (x_1) \cap I^2$ and we write $a = x_1 b$ with $b \in R$. Since $e(R) > 1$, by Proposition (2.3) in [6], we have $\mathfrak{m}I^2 = \mathfrak{m}\mathfrak{q}^2$. Therefore $\mathfrak{m}a = x_1\mathfrak{m}b \subseteq (x_1) \cap \mathfrak{q}^2$. Since the parameter ideal $\mathfrak{q}$ is standard, $(x_1) \cap \mathfrak{q}^2 = x_1\mathfrak{q}$ and $H^0_{\mathfrak{m}}(M) = 0 :_R x_1$.

Thus $\mathfrak{m}b \subseteq (x_1\mathfrak{q}) :_R x_1 = \mathfrak{q} + 0 :_R x_1$, and so that $b \in (\mathfrak{q} + 0 :_R x_1) :_R \mathfrak{m} = \mathfrak{q} :_R \mathfrak{m} + 0 :_R x_1$ by Lemma 2.4. Therefore $a \in x_1 I$, and so that $(x_1) \cap I^2 = x_1 I$. Hence $I^2 = \mathfrak{q}I$ as required. □

Corollary 4.2. *Let R be a generalized Cohen-Macaulay local ring with multiplicity $e(R) > 1$. Then for sufficiently large n, we have*

$$\mu(I) = d + S(R)$$

for all parameter ideals $\mathfrak{q}$ contain in $\mathfrak{m}^n$, where $\mu(I)$ is the minimal number of generators of the ideal $I = \mathfrak{q} : \mathfrak{m}$.

Proof. Choose the integer n as in Theorem 1.1 (for the case $M = R$). Then

$$\frac{I}{\mathfrak{q}} \cong \mathrm{Hom}(\mathfrak{k}, \frac{R}{\mathfrak{q}}) \cong \mathfrak{k}^{S(R)}$$

by Theorem 1.1. Since $e(R) > 1$, by Proposition 2.3 in [6], we get that $\mathfrak{m}I = \mathfrak{m}\mathfrak{q}$. Therefore

$$\mu(I) = \ell(\frac{I}{\mathfrak{m}I}) = \ell(\frac{I}{\mathfrak{m}\mathfrak{q}}) = \ell(\frac{I}{\mathfrak{q}}) + \ell(\frac{\mathfrak{q}}{\mathfrak{m}\mathfrak{q}}) = S(R) + d$$

as required. □

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