

# The Set of Prime Numbers\*

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## Abstract

In this work we show that the prime distribution is deterministic. Indeed the set of prime numbers  $P$  can be expressed in terms of two subsets of  $\mathbb{N}$  using three specific selection rules, acting on two sets of prime candidates. The prime candidates are obtained in terms of the first perfect number. The asymptotic behaviour is also considered.

We obtain for the first time an explicit relation for generating the full set  $P$  of prime numbers smaller than  $n$  or equal to  $n^1$ .

## 1 Introduction

The sequence of prime numbers is of fundamental importance in many fields in general, and in Mathematics in particular. During the last two centuries many mathematicians have attempted to solve this problem using different methods (see for example [1]-[9]).

The arithmetic of prime numbers has a crucial role in the present Cryptography and Information Security. Indeed, many cryptosystems, such as RSA, XTR, ECC (Elliptic Curve Cryptosystems), are based on our historical ignorance about the inner nature of primality.

In addition, many papers are devoted to primes in Physics; a sort of mathematical blueprint seems to guide the evolution of structures at different length scales [10], [11].

While in [12] we built an explicating approach based on dynamical processes and genetic algorithm, here we analyze the analytic properties of the proposed sets of prime candidates. Then we consider the selection rules to obtain two pure sets of primes, containing all prime numbers with the exception of the first two.

A number of efficient algorithms have been known for a long time (for details see [13], [14], [15], [16], [17], [18]). The algorithms of Rabin, and Solovay and Strassen are randomized. In addition the algorithm of Adleman et al. requires (slightly) super-polynomial time, and the algorithm of Miller is in P only under an unproved number-theoretic hypothesis [14]. A relevant contribution was given by Agrawal, Kayal and Saxena in 2004 [13]. Indeed, they proved that the problem is in P. We analyze the asymptotic behaviour too and show that we can write a deterministic second degree polynomial algorithm if we want numerically verify our results to obtain primes.

In writing this work I was encouraged by a historical sentence by Fields Medalist Prof. Bombieri, that is, more or less the following: 'When things become too complex, sometime it makes sense to stop and ask: is my question right?' Moreover, another encouragement comes from the well known sentence in the *Fine Hall* of Princeton: "Raffiniert ist der Herr Gott, aber boshaft ist Er nicht". Then, the main idea used as starting point of this work are:

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<sup>1</sup>At the present the patent procedure is under definition. Consequently, each a Computer Science procedure/algorithm, software solution or firmware/hardware solution for generating prime candidates and selection rules according to the results presented below for Public, Military/Defence and Business aims, must be defined in agreement with the author of the present paper. The solutions which are obtained from the relations shown in what follows can be used for Public, Military/Defence and Business aims exclusively in agreement with the author of the present paper.

- the use of a simple language to represent the objects of interest (i.e. primes);
- Nature manifests itself through beautiful symmetries and harmonies based on primes in such a way that we can understand some interesting results thanks to primality.

The paper is organized as follows. In Sect.2 we give an useful partition of  $\mathbb{N}$  in terms of the first perfect number, then we write the two explicit maximum sets of prime candidates and using some specific selection rules we reduce the previous sets to the two explicit maximum sets of prime numbers; Sect.3 is devoted to study some asymptotic properties, while we consider the algorithms and their computational complexity to verify the results numerically in Sect.4. The conclusions are drawn in Sect.5.

## 2 Prime Candidates and Prime Sets

### 2.1 The Sets of Prime Candidates

It is well known a perfect number is defined as an integer which is the sum of its proper positive divisors, that is, the sum of the positive divisors not including the number. Equivalently, a perfect number is a number that is half the sum of all of its positive divisors, or  $\sigma(n) = 2n$ . It is very interesting to stress that the first perfect number is 6, because 1, 2 and 3 are its proper positive divisors and  $1 + 2 + 3 = 6$ . For the reasons which were expressed in [12] let us write the set of natural number  $\mathbb{N}$  in terms of the first perfect number. Indeed, we consider the following sets of positive integers

$$ONE = \{o_k = 6k - 5 : k \in \mathbb{N}\}, \tag{2.1}$$

$$TWO = \{t_k = 6k - 4 : k \in \mathbb{N}\}, \tag{2.2}$$

$$THREE = \{r_k = 6k - 3 : k \in \mathbb{N}\}, \tag{2.3}$$

$$FOUR = \{f_k = 6k - 2 : k \in \mathbb{N}\}, \tag{2.4}$$

$$A = \{\alpha_k = 6k - 1 : k \in \mathbb{N}\}, \tag{2.5}$$

$$SIX = \{s_k = 6k : k \in \mathbb{N}\}, \tag{2.6}$$

$$B = \{\beta_k = 6k + 1 : k \in \mathbb{N}\}, \tag{2.7}$$

.....

Then, we can write the set of natural numbers  $\mathbb{N}$  as follows

$$\mathbb{N} = \{1\} \cup TWO \cup THREE \cup FOUR \cup A \cup SIX \cup B \tag{2.8}$$

Indeed, *ONE* has the same elements of *B* with the exception of the first element, that is 1. Moreover, also the sets which follows *B* have got the same elements of the previous sets (*ONE - B*), since their elements are just obtained shifting the elements into the previous sets. The following table shows the first values of the previous sets.

$6k - 4$	$6k - 3$	$6k - 2$	<i>A</i>	$6k$	<i>B</i>
2	3	4	5	6	7
8	9	10	11	12	13
14	15	16	17	18	19
20	21	22	23	24	25
26	27	28	29	30	31
32	33	34	35	36	37
...	...	...	...	...	...

**Lemma 1.** *The set TWO is made of composite numbers with the exception of the first element, that is 2.*

**Proof.** The set *TWO* contains elements of the form  $6k - 4$  for construction. For each a  $k \in \mathbb{N}$ ,  $6k$  is an even number and the same thing happens for  $6k - 4$ , since the difference between two even numbers is an even number too. Then only the first element of *TWO* can be prime and this is the case due to the fact that the number 2 is the first prime number.

□

**Lemma 2.** *The set THREE is made of composite numbers with the exception of the first element, that is 3.*

**Proof.** The set *THREE* contains elements of the form  $6k - 3$  for construction. For each a  $k \in \mathbb{N}$ ,  $6k$  is an even number and  $6k - 3$  is an odd multiple of 3; that is each an element is odd and  $6k - 3 = 0 \pmod{3}$ . Then only the first element of *THREE* can be prime and this is the case due to the fact that the number 3 is the second prime number.

□

**Lemma 3.** *The sets FOUR and SIX are made of composite numbers with no exception.*

**Proof.** The proof is trivial due to the fact that the elements of the set *SIX* are multiple of the first perfect number, that is an even number (that is  $6k = 0 \pmod{6}$ ), while the elements of *FOUR* have the form  $6k - 2$  and the difference between two even numbers is an even number too.

□

**Theorem 1 (The Prime Candidates).** The set of the prime candidates  $\tilde{P}$  can be written as

$$\begin{aligned} \tilde{P} &= (\mathbb{N} \cap \{2\}) \cup (\mathbb{N} \cap \{3\}) \cup (\mathbb{N} \setminus (TWO \cup THREE \cup FOUR \cup SIX)) = \\ &= \{2\} \cup \{3\} \cup A \cup B \end{aligned} \tag{2.9}$$

**Proof.** The proof follows from the Lemma 1-3 *trivially*. Indeed note that for construction only the sets *A* and *B* among the others can be made of primes, since *FOUR* and *SIX* are made of composite numbers, while *TWO* and *THREE* contain one prime only, that is the first element of each a set.

□

Unfortunately, the sets *A* and *B* also contain composite numbers such as the positive integers which are multiple of 5, and so on. Consequently, we introduce the selection rules for obtaining the full set of prime.

## 2.2 Selection Rules and the Full Set of Prime Numbers

If we look at the sequence of prime candidates  $x \geq 5$  coming from *A* and *B* and compare them with the sequence of primes we see that, with some exceptions, *A* and *B* alternate each others. For example, if we consider  $k = 1, \dots, 10$  we obtain

	$6k - 1$	$6k + 1$
$k = 1$	5	7
$k = 2$	11	13
$k = 3$	17	19
$k = 4$	23	<b>25</b>
$k = 5$	29	31
$k = 6$	<b>35</b>	37
$k = 7$	41	43
$k = 8$	47	<b>49</b>
$k = 9$	53	<b>55</b>
$k = 10$	59	61

where the numbers in bold face are composite. The total list of prime numbers is in the following table. The second column of this table contain the symbol "−" if the corresponding prime candidate comes from the set *A*, and "+" if it comes from the set *B*.

5 -  
7 +  
11 -  
13 +  
17 -  
19 +  
23 -  
29 -  
31 +  
37 +  
41 -  
43 +  
47 -  
53 -  
59 -  
61 +

It seems clear that the class transition, between two consecutive primes, appears to be random. Indeed, in a first approximation it can be modeled through a Random Process. But a deep analysis show that the generation process is deterministic and can be realized by a recursive combination of two sub-processes. In [12] the author showed that we must use a process that produces a jump between the classes  $A$  and  $B$  (we can call it *zig-zag*) and a second process which switches off a number into a class for a fixed  $k$  if it is composite (we can call it *intermittence*). The intermittence can be made using the selection rules presented below.

Let us introduce the following two subsets  $A^{(-)} \subset A$  and  $B^{(-)} \subset B$ :

$$A^{(-)} = \{ \alpha_{k_{ij}} \in A : \alpha_{k_{ij}} = 36ij - 6i + 6j - 1, \forall k, i, j \in \mathbb{N} \}, \quad (2.10)$$

$$B^{(-)} = \{ \beta_{k_{ij}} \in B : \beta_{k_{ij}} = 36ij - 6i - 6j + 1 \text{ and } \beta_{k_{ij}} = 36ij + 6i + 6j + 1 \forall k, i, j \in \mathbb{N} \}. \quad (2.11)$$

Then we can prove the following Lemmas.

**Lemma 4.** *The positive integer numbers  $\alpha_{k_{ij}} \in A^{(-)}$  are composite numbers .*

**Proof.** Multiplying  $\alpha_k \in A$  to  $\beta_k \in B$  we obtain

$$\begin{aligned} \alpha_{k_{ij}} &= \alpha_k \beta_k = (6i + 1)(6j - 1) = \\ &= 36ij - 6i + 6j - 1 = 6S_{ij}^{(1)} - 1, \end{aligned}$$

where

$$S_{ij}^{(1)} = (6ij - i + j) \forall i, j \in \mathbb{N}. \quad (2.12)$$

Choosing  $(6ij + i - j) = k$  we see that  $(6i - 1)(6j + 1) = 6k - 1 \in A$ . Consequently, the product  $\alpha_k \beta_k$  with  $\alpha_k \in A$  and  $\beta_k \in B$  gives  $\alpha_{k_{ij}} \in A^{(-)}$ . This means that the numbers  $\alpha_{k_{ij}} \in A$  are composite numbers, since they can be written as product of two numbers.  $\square$

**Remarks 1.** The prime numbers  $\alpha_k \in A$  must be different from  $\alpha_{k_{ij}}$ .

**Lemma 5.** *The positive integer numbers  $\beta_{k_{ij}} \in B^{(-)}$  are composite numbers.*

**Proof.** Multiplying two elements  $\beta_k, \beta_{k^*} \in B$ , we obtain

$$\begin{aligned} \beta_{k_{ij}} &= \beta_k \beta_{k^*} = (6i + 1)(6j + 1) = \\ &= 36ij + 6i + 6j + 1 = 6S_{ij}^{(2)} + 1, \end{aligned}$$

where

$$S_{ij}^{(2)} = (6ij + i + j) \quad \forall i, j \in \mathbb{N}. \quad (2.13)$$

Choosing  $(6ij + i + j) = k$  we see that  $(6i + 1)(6j + 1) = 6k + 1 \in B$ . Consequently, the product  $\beta_k \beta_{k^*}$  with  $\beta_k, \beta_{k^*} \in B$  gives  $\beta_{k_{ij}} \in B^{(-)}$ . This means that the numbers  $\beta_{k_{ij}} \in B$  are composite numbers, since they can be written as product of two numbers.

Similarly, multiplying two elements  $\alpha_k, \alpha_{k^*} \in A$ , we obtain

$$\begin{aligned} \beta_{k_{ij}} &= \alpha_k \alpha_{k^*} = (6i - 1)(6j - 1) = \\ &= 36ij - 6i - 6j + 1 = 6S_{ij}^{(3)} + 1, \end{aligned}$$

where

$$S_{ij}^{(3)} = (6ij - i - j) \quad \forall i, j \in \mathbb{N}. \quad (2.14)$$

Consequently, the product  $\alpha_k \alpha_{k^*}$  with  $\alpha_k, \alpha_{k^*} \in A$  gives  $\beta_{k_{ij}} \in B$ . This means that the numbers  $\beta_{k_{ij}} \in B^{(-)}$  are composite numbers too, since they can be written as product of two numbers.

□

**Remark 2.** The primes  $\beta_k \in B$  must be different from  $\beta_{k_{ij}}$ .

**Remark 3.** The Lemmas 4-5 show that the numbers  $\alpha_k$  and  $\beta_k$ , which are expressed as  $\alpha_{k_{ij}}$  and  $\beta_{k_{ij}}$  are composite numbers. Now let us prove that they are the unique composite numbers in  $A$  and  $B$ .

**Lemma 6.** The composite numbers  $\alpha_{k_{ij}} \in A$  can only be written in terms of the product  $\alpha_k \beta_{k^*}$ .

**Proof.** From the Lemma 5, it follows that

$$\alpha_k \alpha_{k^*} = 0 \pmod{6k + 1},$$

that is  $\alpha_{kk^*} = \alpha_k \alpha_{k^*} \in B \quad \forall \alpha_k, \alpha_{k^*} \in A$ . Moreover,

$$t_k t_{k^*} = f_k f_{k^*} = s_k s_{k^*} = t_k f_{k^*} = t_k s_{k^*} = f_k s_{k^*} = 0 \pmod{2}.$$

that is the product of two elements of the sets *TWO*, *FOUR* and *SIX* is an even number.

In addition

$$r_k r_{k^*} = 0 \pmod{3} \text{ and it is an odd number,}$$

that is  $r_k r_{k^*} \in \textit{THREE}$ . Furthermore,

$$r_k t_{k^*} = r_k f_{k^*} = r_k s_{k^*} = 0 \pmod{2},$$

that is the product of an element of the set *THREE* to an element of the sets *TWO* (or *FOUR* or *SIX*) is an even number. We also obtain that

$$\alpha_k t_{k^*} = \alpha_k f_{k^*} = \alpha_k s_{k^*} = 0 \pmod{2},$$

that is the product of an  $\alpha_k \in A$  to an element of the set *TWO* (or *FOUR* or *SIX*) is an even number, and also

$$\alpha_k r_{k^*} = 0 \pmod{3} \text{ and it is an odd number,}$$

that is  $\alpha_k r_{k^*} \in \textit{THREE}$ . Moreover,

$$\beta_k \beta_{k^*} = 0 \pmod{6k + 1},$$

that is  $\beta_k \beta_{k*} = \beta_k \beta_{k*} \in B \quad \forall \beta_k, \beta_{k*} \in B$ , and

$$\beta_k t_{k*} = \beta_k f_{k*} = \beta_k s_{k*} = 0 \pmod{2},$$

and

$$\beta_k r_{k*} = 0 \pmod{3} \text{ and it is an odd number.}$$

Consequently, we obtain that a composite element  $\alpha_k \in A$  can be written only in terms of the product  $\alpha_k \beta_{k*}$ .

□

**Lemma 7.** *The composite numbers  $\beta_k \in B$  can only be written in terms of the products  $\alpha_k \alpha_{k*}$ ,  $\beta_k \beta_{k*}$ .*

**Proof.** The proof is obtained using the results of the Lemmas 4-6 and going head in the same way like in Lemma 6.

□

**Theorem 2 (about the Selection Rules).** *The natural numbers  $\alpha_k \in A$ ,  $\beta_k \in B$  are composite if and only if  $\alpha_k \in A^{(-)}$ ,  $\beta_k \in B^{(-)}$ .*

**Proof.** The proof of the theorem follows from the previous four Lemmas trivially.

□

From (2.8), and the theorem 2 we obtain the following theorem.

**Theorem 3. (The Full Set of Primes).** *The full set of primes has got the following minimum explicit representation*

$$P = \{2\} \cup \{3\} \cup A' \cup B', \quad (2.15)$$

where

$$A' = \{\alpha_k \in \mathbb{N} : \alpha_k = 6k - 1 \text{ and } k \neq 6ij - i + j, \forall k, i, j \in \mathbb{N}\}, \quad (2.16)$$

$$B' = \{\beta_k \in \mathbb{N} : \beta_k = 6k + 1 \text{ and } k \neq 6ij + i + j \text{ or } k \neq 6ij - i - j, \forall k, i, j \in \mathbb{N}\}. \quad (2.17)$$

**Remark 4.** We can also note that

$$A' = A \setminus A^{(-)} \text{ and } B' = B \setminus B^{(-)}. \quad (2.18)$$

then

$$P = \{2\} \cup \{3\} \cup (A \setminus A^{(-)}) \cup (B \setminus B^{(-)}). \quad (2.19)$$

### 3 Some Asymptotic Consequences

Let us consider some asymptotic behaviours.

**Theorem 4 (Equipartition of the Cuts).** If we call  $\#_A(s)$  the first  $s$  elements of  $A$  and  $\#_B(s)$  the first  $s$  elements of  $B$ , then  $\#_{A^{(-)}}(s) = s^2$  and  $\#_{B^{(-)}}(s) = s^2 + s$ . Moreover, for  $s \rightarrow \infty$  we have  $\#_{A^{(-)}}(s) \approx \#_{B^{(-)}}(s)$ .

**Proof.** Considering that each an element of  $A^{(-)}$  is given multiplying an element of  $A$  to an element of  $B$  we obtain  $\#_{A^{(-)}}(s) = s^2$  trivially. Considering that each an element of  $B^{(-)}$  is given multiplying two elements of  $A$  or two elements of  $B$ , then

$$\#_{B^{(-)}}(s) = \binom{s+2-1}{2} + \binom{s+2-1}{2} = s^2 + s.$$

Moreover,

$$\lim_{k \rightarrow \infty} \frac{\#_{B^{(-)}}(s)}{\#_{A^{(-)}}(s)} = \lim_{k \rightarrow \infty} \frac{s^2 + s}{s^2} = 1. \quad (3.1)$$

□

**Theorem 5 (Equipower of Primes into the Sets of Primes).** If we call  $\#_{A'}(k)$  the number of the first  $k$  primes into  $A'$ , and  $\#_{B'}(k)$  the number of the first  $k$  primes into  $B'$ , then

$$\lim_{k \rightarrow \infty} \frac{\#_{B'}(k)}{\#_{A'}(k)} = l, \text{ with } l < \infty$$

**Proof.** For a fixed  $k$  we obtain  $k$  elements into  $A$ , that is  $\#_A(k) = k$ ; while to evaluate the number of selectors we must estimate the maximum indexes  $i$  and  $j$ . Indeed from (2.12) we obtain

$$i_{\max}^{(S1)} = \left\lfloor \frac{k-1}{5} \right\rfloor, \quad (3.2)$$

corresponding to  $j = 0$  and

$$j_{\max}^{(S1)} = \left\lfloor \frac{k+1}{7} \right\rfloor, \quad (3.3)$$

corresponding to  $i = 0$ . Consequently the number of selectors smaller then  $k$  or equal to  $k$ , with respect to (2.12), is

$$\mathfrak{X}^{(S1)}(k) = i_{\max}^{(S1)} j_{\max}^{(S1)} = \left\lfloor \frac{k-1}{5} \right\rfloor \left\lfloor \frac{k+1}{7} \right\rfloor. \quad (3.4)$$

Then the number of primes into  $A$  for a fixed  $k$  will be

$$\begin{aligned} \#_{A'}(k) &= \#_A(k) - \mathfrak{X}^{(S1)}(k) = \\ &= k - \left\lfloor \frac{k-1}{5} \right\rfloor \left\lfloor \frac{k+1}{7} \right\rfloor \end{aligned} \quad (3.5)$$

For the same  $k$  we obtain  $k$  elements into  $B$  too, that is  $\#_B(k) = k$ ; while to evaluate the number of selectors we must estimate the maximum indexes  $i$  and  $j$  coming from (2.13) and (2.14). Indeed for the selectors of  $B$ , we stress that when  $i$  and  $j$  run from 1 to their max value we obtain a symmetric matrix of selectors. Then arbitrarily for one of the two indexes we must choose the maximum with  $\lceil \cdot \rceil$ , instead of  $\lfloor \cdot \rfloor$ . In other words from (2.13), we have

$$i_{\max}^{(S2)} = \left\lceil \frac{k-1}{7} \right\rceil, \quad (3.6)$$

and

$$j_{\max}^{(S2)} = \left\lceil \frac{k-1}{7} \right\rceil. \quad (3.7)$$

Consequently the number of selectors smaller then  $k$  or equal to  $k$ , with respect to (2.13), is

$$\mathfrak{X}^{(S2)}(k) = i_{\max}^{(S2)} j_{\max}^{(S2)} = \left\lceil \frac{k-1}{7} \right\rceil \left\lceil \frac{k-1}{7} \right\rceil. \quad (3.8)$$

Moreover from (2.14), we obtain

$$i_{\max}^{(S3)} = \left\lfloor \frac{k+1}{5} \right\rfloor, \quad (3.9)$$

and

$$j_{\max}^{(S3)} = \left\lfloor \frac{k+1}{5} \right\rfloor.$$

Then the number of selectors smaller then  $k$  or equal to  $k$ , with respect to (2.14), is

$$\mathfrak{X}^{(S3)}(k) = i_{\max}^{(S3)} j_{\max}^{(S3)} = \left\lfloor \frac{k+1}{5} \right\rfloor \left\lfloor \frac{k+1}{5} \right\rfloor. \quad (3.10)$$

Consequently the number of primes into  $B$  for a fixed  $k$  will be

$$\begin{aligned}\#_{B'}(k) &= \#_B(k) - \mathfrak{X}^{(S2)}(k) - \mathfrak{X}^{(S3)}(k) = \\ &= k - \left\lfloor \frac{k-1}{7} \right\rfloor \left\lceil \frac{k-1}{7} \right\rceil - \left\lfloor \frac{k+1}{5} \right\rfloor \left\lceil \frac{k+1}{5} \right\rceil.\end{aligned}\quad (3.11)$$

Without losing generality for our purpose, let us approximate

$$\left\lfloor \frac{k \pm 1}{\delta} \right\rfloor = \left\lceil \frac{k \pm 1}{\delta} \right\rceil \approx \left( \frac{k \pm 1}{\delta} \right) \text{ with } \delta = 5, 7; \quad (3.12)$$

$$\begin{aligned}\lim_{k \rightarrow \infty} \frac{\#_{B'}(k)}{\#_{A'}(k)} &= \lim_{k \rightarrow \infty} \frac{k - \lfloor \frac{k-1}{7} \rfloor \lceil \frac{k-1}{7} \rceil - \lfloor \frac{k+1}{5} \rfloor \lceil \frac{k+1}{5} \rceil}{k - \lfloor \frac{k-1}{5} \rfloor \lceil \frac{k+1}{7} \rceil} \approx \\ &\approx \lim_{k \rightarrow \infty} \frac{\mu k^2 + \lambda k + \omega}{\rho k^2 + \vartheta k + \sigma} = \mu/\rho = l,\end{aligned}\quad (3.13)$$

with  $\mu, \lambda, \omega, \rho, \vartheta, \sigma$ -constants; in particular in the present case  $\mu = 74$  and  $\rho = 1$ ; this means  $\#_{A'}(k) \approx \frac{1}{\mu} \#_{B'}(k)$  for  $k \rightarrow \infty$ .  
□

**Remark 5.** *Of course the approximation (3.12) is not acceptable for evaluating  $\pi(n)$ . Indeed in this case we could solve the inverse problem: the quantities  $\pi(n)$  and  $\#(n)$  could be considered as known quantities for estimating  $\mathfrak{X}(n)$  (for details see [12]).*

## 4 Algorithms and their Computational Complexity

If we use the definition of prime numbers we have immediately a way of determining if a number  $n$  is prime. Indeed, try dividing  $n$  by every number  $m \leq \sqrt{n}$  then if any  $m$  divides  $n$  then the last one is composite, otherwise it is prime. As it is well known this test is inefficient, since it takes  $\Omega(\sqrt{n})$  steps to determine if  $n$  is prime. As anticipated and shown in [13] an unconditional deterministic polynomial-time algorithm, for determining whether an input is prime or composite, can be obtained. In what follows we show that our not-optimized algorithm has a computational complexity  $C(n) \in O(n^2)$  if we want verify whether  $n$  is prime or composite, while a second algorithm to verify the same things according to the sets  $A'$  and  $B'$  has a computational complexity  $C(n) \in O(1)$ .

It is trivial to prove that following the transformation  $r = i - 1$ ,  $s = j - 1$ , the previous three selection rules (2.12), (2.13), (2.14) can be written respectively

$$S_{rs}^{(1)} = 6 + 7s + (5 + 6s)r \quad \forall r, s \in \mathbb{N}_0, \quad (4.1)$$

$$S_{rs}^{(2)} = 8 + 7s + (7 + 6s)r \quad \forall r, s \in \mathbb{N}_0, \quad (4.2)$$

$$S_{rs}^{(3)} = 4 + 5s + (5 + 6s)r \quad \forall r, s \in \mathbb{N}_0. \quad (4.3)$$

The not-optimized algorithm to verify whether  $n$  is prime can be written as follows.



<p>Input: integer <math>n &gt; 3</math></p> <p>Step 1. If <math>(n + 1 = 0 \pmod{6})</math> <math>k = (n + 1)/6</math>, <math>r_{\max} = \lfloor \frac{k-6}{5} \rfloor</math>, <math>s_{\max} = \lfloor \frac{k-6}{7} \rfloor</math></p> <p style="padding-left: 20px;">For <math>s = 0</math> to <math>s_{\max}</math> do</p> <p style="padding-left: 40px;">For <math>r = 0</math> to <math>r_{\max}</math> do</p> <p style="padding-left: 60px;"><math>k(r, s) = 6 + 7s + (5 + 6s)r</math></p> <p style="padding-left: 60px;">If <math>(k(r, s) = k)</math> output COMPOSITE</p> <p style="padding-left: 60px;">Else output PRIME</p> <p>Step 2. If <math>(n - 1 = 0 \pmod{6})</math> <math>k = (n - 1)/6</math>, <math>l_{\max} = \lfloor \frac{k-4}{5} \rfloor</math>, <math>m_{\max} = l_{\max} + 1</math></p> <p style="padding-left: 20px;">For <math>m = 0</math> to <math>m_{\max}</math> do</p> <p style="padding-left: 40px;">For <math>l = m</math> to <math>l_{\max}</math> do</p> <p style="padding-left: 60px;"><math>k(l, m) = 4 + 5m + (5 + 6m)l</math></p> <p style="padding-left: 60px;">If <math>(k(l, m) = k)</math> output COMPOSITE</p> <p style="padding-left: 40px;">Else <math>c_{\max} = \lfloor \frac{k-8}{7} \rfloor</math>, <math>d_{\max} = d_{\max} + 1</math></p> <p style="padding-left: 20px;">For <math>d = 0</math> to <math>d_{\max}</math> do</p> <p style="padding-left: 40px;">For <math>c = d</math> to <math>c_{\max}</math> do</p> <p style="padding-left: 60px;"><math>k(c, d) = 8 + 7d + (7 + 6d)c</math></p> <p style="padding-left: 60px;">If <math>(k(c, d) = k)</math> output COMPOSITE</p> <p style="padding-left: 60px;">Else output PRIME</p> <p>Step 3 Else output COMPOSITE</p>
---

We are taking into account that the matrices of the selectors  $k(l, m)$  and  $k(c, d)$  are symmetric.

**Theorem 6.** *The algorithm above returns PRIME if and only if  $n$  is prime.*

**Proof.** If  $n \notin A$  or  $n \notin B$  Step 3, and so the algorithm, returns COMPOSITE. Moreover if  $n \in A$  and  $k(r, s) = k$  then Step 1 returns COMPOSITE. For each a  $n \in B$  and  $k(l, m) = k$  (or  $k(c, d) = k$ ) Step 2 returns COMPOSITE. Consequently, the algorithm returns PRIME if only  $n \in A'$  or  $n \in B'$ , that is if only  $n$  is prime.

Vice versa if  $n$  is prime then  $n \in A'$  or  $n \in B'$ .

If  $n \in A'$  then  $n \in A$  since  $A' \subset A$ . If  $n \in A$  the algorithm returns COMPOSITE for each a  $n$  such that  $k(r, s) = k$ , but if  $n$  is prime, then  $n \notin A^{(-)}$ . Then the output of Step 1 must be PRIME.

Similarly if  $n \in B'$  then  $n \in B$  since  $B' \subset B$ . If  $n \in B$  the algorithm returns COMPOSITE for each a  $n$  such that  $k(l, m) = k$  or  $k(c, d) = k$ , but if  $n$  is prime, then  $n \notin B^{(-)}$ . Then the output of Step 2 must be PRIME.

□

By looking at the previous algorithm we can evaluate  $O(n)$  for recognizing if  $n$  is prime or composite.

This algorithm is not optimized, since our aim is just to show that the computational complexity is a second degree deterministic polynomial.

**Theorem 7.** *The computational complexity for recognizing if  $n$  is prime or composite is*

$$C(n) \in O(n^2) \tag{4.4}$$

**Proof.** When  $n \in A$  is prime, the number of operations can be evaluated as follows. The Step 1 costs two basic operations (that is a sum and a division), to verify  $n \in A$ . In addition we have six basic operations to estimate the indexes maximum values and  $k$ ,  $6 \times r_{\max} \times s_{\max}$  operations to evaluate  $k(r, s)$  and other  $r_{\max} \times s_{\max}$  operations to compare  $k(r, s)$  with  $k$ . Consequently,  $O_{\min}(r_{\max}, s_{\max}) = 8 + 7 \times r_{\max} \times s_{\max}$ . Considering that  $r_{\max} = \lfloor \frac{k-6}{5} \rfloor = \lfloor \frac{n-35}{5} \rfloor$  and  $s_{\max} = \lfloor \frac{k-6}{7} \rfloor = \lfloor \frac{n-35}{42} \rfloor$  we obtain

$$C(n) \in O(n^2) \tag{4.5}$$

Similarly when  $n \in B$  is prime, the Step 2 requires two basic operations to verify  $n \in B$ . In addition we have five basic operations to estimate the indexes maximum values and  $k$ ,  $7 \times r_{\max} \times (r_{\max} + 1)$  operations

to evaluate  $k(r, s)$  and to compare  $k(r, s)$  with  $k$ . Consequently,  $O_{\max}(r, s) = 7 + 7 \times r_{\max} \times (r_{\max} + 1)$ . Considering that  $r_{\max} = \lfloor \frac{k-4}{5} \rfloor = \lfloor \frac{n-25}{30} \rfloor$  or  $r_{\max} = \lfloor \frac{k-8}{7} \rfloor = \lfloor \frac{n-49}{42} \rfloor$ , it follows

$$C(n) \in O(n^2) \quad (4.6)$$

□

**Remark 6.** *If  $n$  is composite, then  $C(n) \in \Omega(1)$ .*

**Remark 9.** *Theorem 7 is useful just to verify that  $C(n) \in O(n^2)$ .*

To evaluate if  $n$  is prime or composite we use the following algorithm based on the knowledge of  $A'$  and  $B'$ .

```

Input: integer  $n > 3$ 
Step 1. If  $(n + 1 = 0 \pmod{6})$   $k = (n + 1)/6$ 
        read  $k(r, s)$ 
        If  $(k(r, s) = k)$  output COMPOSITE
        Else output PRIME
Step 2. If  $(n - 1 = 0 \pmod{6})$   $k = (n - 1)/6$ 
        read  $k(l, m)$ 
        If  $(k(l, m) = k)$  output COMPOSITE
        Else read  $k(c, d)$ 
        If  $(k(c, d) = k)$  output COMPOSITE
        Else output PRIME
Step 3. Else output COMPOSITE

```

**Theorem 8 (Primality Test based on Pre-computed Selection Rules).** *The computational complexity for recognizing if  $n$  is prime or composite is*

$$C(n) \in O(1) \quad (4.7)$$

**Proof.** The proof is trivial using the rule of the sum for the computational complexity and assuming that the arithmetic operations, the function print and the function read have a computational complexity  $C(n) \in O(1)$ .

□

**Remark 10.** *Clearly we have to take into account the computational complexity to generate the pre-computed selectors  $k$ . Trivially considering that we have two nested for loops, it is in  $O(n^2)$ . We must stress this evaluation is off-line. In other words, it is not in the procedure of testing, but it is into the procedure of generation.*

## 5 Conclusion

In this work we defined two maximum explicit sets of prime candidates. Then we showed that they can be reduced to two maximum explicit sets of primes. In conclusion the sets obtained thanks to  $6k \pm 1$   $k \in \mathbb{N}$ , with their selection rules, that is

where

$$A' = \{\alpha_k \in \mathbb{N} : \alpha_k = 6k - 1 \text{ and } k \neq 6ij - i + j, \forall k, i, j \in \mathbb{N}\}, \quad (5.1)$$

$$B' = \{\beta_k \in \mathbb{N} : \beta_k = 6k + 1 \text{ and } k \neq 6ij + i + j \text{ or } k \neq 6ij - i - j, \forall k, i, j \in \mathbb{N}\}. \quad (5.2)$$

together with  $\{2\}$  and  $\{3\}$  give us the full set of prime numbers (in its reduced form), that is

$$P = \{2\} \cup \{3\} \cup A' \cup B'.$$

We also show the selection rules have the same weight acting on the two sets of candidates and the sets of primes have the same power.

Moreover basing on the results shown above, the algorithm is a second degree deterministic polynomial procedure.

Thanks to the discovery of the sets  $A'$  and  $B'$ , we have obtained, for the first time an explicit expression of the full set  $P$  of prime numbers smaller than  $n$  or equal to  $n$ , which are generated with a specific rule and without the use of some test.

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