KOPPELMAN FORMULAS ON GRASSMANNIANS

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ABSTRACT. We construct Koppelman formulas on Grassmannians for forms with values in any holomorphic line bundle as well as in the tautological vector bundle and its dual. As a consequence we obtain some vanishing theorems of the Bott-Borel-Weil type. We also relate the projection part of our formulas to the Bergman kernels associated to the line bundles.

1. INTRODUCTION

The Cauchy integral formula in one complex variable is of vast importance in many respects. It provides a way of representing a holomorphic function as a superposition of simple rational functions, and gives an explicit solution to the equation $\bar{\partial}u = f$. Furthermore, it is an important tool in function theory. For our purposes it is convenient to note that Cauchy's formula is equivalent to the current equation $\bar{\partial}u = [z]$, where $u = (2\pi i)^{-1}d\zeta/(\zeta - z)$ is the Cauchy form, and $[z]$ is the Dirac measure at z considered as a $(1,1)$-current. This point of view is well adapted for generating weighted Cauchy formulas. For instance, by computing $\bar{\partial}\big(((1 - |\zeta|^2)/(1 - z\bar{\zeta}))^\alpha u\big)$ in the current sense, one obtains (for suitable α) the weighted representation formula

$$f(z) = \frac{\alpha}{\pi} \int_{\{|\zeta|<1\}} f(\zeta) \frac{(1 - |\zeta|^2)^{\alpha-1}}{(1 - z\bar{\zeta})^{\alpha+1}} d\lambda(\zeta),$$

for holomorphic functions on the unit disc with certain limited growth at the boundary. The integral kernel is the reproducing kernel for a weighted Bergman space; and this shows that there is a connection between Cauchy kernels and Bergman kernels. Both these kernels are also intimately linked with the symmetry of the disc. Recall that the group

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in M_{22}(\mathbb{C})\big| \, |a|^2 - |b|^2 = 1 \right\}$$

acts holomorphically and transitively on the unit disc by $z \mapsto (az+b)/(\bar{b}z + \bar{a})$. The stabilizer of the origin is the subgroup

$$K := \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\} \cong S^1,$$

and hence the disc can be viewed as the homogeneous space $SU(1,1)/S^1$. The kernels are then invariant under certain actions on functions which are induced from the natural action on the closed disc. From the point of view of representation theory, the Bergman kernels are interesting since the corresponding weighted Bergman spaces form a family of unitary representation spaces for $SU(1,1)$, and moreover, these kernels can be described entirely in terms of the Lie-theoretic structure of the group. This discussion indicates two possible directions of generalizations; namely to domains in $\mathbb{C}^n$, and to complex homogeneous spaces. In the latter case, the class of bounded symmetric domains have been studied extensively from the Lie-theoretic point of view. Hua, [10], computed the Cauchy kernels and Bergman kernels for the classical domains using the explicit description of their symmetry groups. Later, more abstract group theoretic machinery has been used to describe both Bergman kernels (cf. [15]) and the generalized Cauchy-Szeg kernels, [11]. For compact Hermitian symmetric spaces, Bergman kernels for line bundles can be described explicitly in terms of the polynomial models for the spaces of global holomorphic sections, [21].

Complex analysts have mainly been concerned with domains in $\mathbb{C}^n$. The Bochner-Martinelli kernel represents holomorphic functions in any domain but has the drawback of not being holomorphic, a property which is highly useful in applications. The Cauchy-Fantappi-Leray kernel is holomorphic in domains where we can find a holomorphic support function, for example strictly pseudoconvex domains. More flexibility is afforded by using weighted formulas, which was first done in [6], and such formulas have been widely used in applications such as interpolation, division, obtaining estimates for solutions to the $\bar{\partial}$-equation, etc. See, e.g., [1] and [3] and the references therein. Some work has also been done on generalizing integral formulas to complex manifolds, see, e.g., [9], [5], [4]. Of these, the paper [4] by Berndtsson will be of particular importance for us; see below.

More recently, in [1] was introduced a general method for generating weighted formulas for domains in $\mathbb{C}^n$, both for holomorphic functions and (p,q)-forms. For future reference, we will describe this method in the former case in some detail. First, recall that the Cauchy kernel, u, in one variable satisfies $\bar{\partial}u = [z]$, but less obviously, we also have $\delta_{\zeta-z}u = 1$, where $\delta_{\zeta-z}$ denotes contraction with the vector field $2\pi i(\zeta - z)\partial/\partial\zeta$. These equations can be combined into the single equation

$$(1) \qquad\qquad \nabla_{\zeta-z}u = 1 - [z],$$

where $\nabla_{\zeta-z}$ is the operator

$$\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar{\partial}.$$

To generalize this to $\mathbb{C}^n$, we define $\delta_{\zeta-z}$ as contraction with

$$(2) \qquad\qquad 2\pi i \sum(\zeta_j - z_j)\frac{\partial}{\partial\zeta_j},$$

and if we construe equation (1) as being in $\mathbb{C}^n$, the right hand side of (1) now contains one form of bidegree $(0,0)$ and one of bidegree (n,n), so we must in

fact have $u = u_{1,0} + u_{2,1} + \ldots + u_{n,n-1}$, where $u_{k,k-1}$ has bidegree $(k, k-1)$. We can then write the $\nabla_{\zeta-z}$-equation (1) as the system of equations

$$\delta_{\zeta-z}u_{1,0} = 1, \qquad \delta_{\zeta-z}u_{1,2} - \bar{\partial}u_{1,0} = 0, \qquad \ldots, \qquad \bar{\partial}u_{n,n-1} = [z].$$

In that case, $u_{n,n-1}$ will satisfy $\bar{\partial}u_{n,n-1} = [z]$ and will give a kernel for a representation formula. One advantage of this approach, as opposed to just solving $\bar{\partial}u_{n,n-1} = [z]$, is that it easily allows for weighted integral formulas. We define $g = g_{0,0} + \cdots + g_{n,n}$ to be a weight if $\nabla g = 0$ and $g_{0,0}(z,z) = 1$. It is easy to see that $\nabla(u \wedge g) = g - [\Delta]$, and this yields a representation formula

$$\phi(z) = \int_{\partial D} \phi(\zeta)(u \wedge g)_n + \int_D \phi g_n$$

if $\phi \in \mathcal{O}(\overline{D})$ and $z \in D$. Note that if g_1 and g_2 are weights, then $g_1 \wedge g_2$ is also a weight.

In the case of compact manifolds one is naturally led to consider holomorphic line bundles and representation formulas for holomorphic sections as well as smooth bundle-valued forms. In this setting the integral kernels must be operator valued, and the integrals become superpositions of contributions from all fibres. Our method for achieving this has two crucial components; the above mentioned ∇-formalism, and Berndtsson's method from [4]. Indeed, Berndtsson gave a method for obtaining integral formulas for (p,q)-forms on n-dimensional manifolds X which admit a vector bundle of rank n over $X \times X$ such that the diagonal has a defining section η; and to get formulas for forms with values in bundles the ∇-method is well suited. In fact, by generalizing it to manifolds one realizes that it allows for operator valued weights. We then need something to substitute for the vector field (2), and this is where Berndtsson's assumption comes in: we will use the section η to contract with, and define $\nabla_\eta := \delta_\eta - \bar{\partial}$. It is of independent interest to note that ∇_η in fact is a superconnection in the sense of Quillen, [14]. In the recent article [8] by the first author, this general theory for integral formulas on manifolds has been developed to a large degree, and explicit formulas have been constructed on $\mathbb{CP}^n$ yielding explicit proofs of vanishing theorems for its line bundles. Such proofs could be of interest also for representation theoretic purposes. Indeed, in view of the by now firmly established goal, initiated by the Bott-Borel-Weil theorem and further fortified by the conjecture of Langlands, [12], and Schmid's proof of it, [16], of wanting to realize representations of Lie groups in Dolbeault cohomology (or, rather L^2-cohomology in the non-compact case), (cf. also [19] and [20]), it is our hope that explicit integral formulas could give further insight into the underlying group theory.

In this paper, we extend the method in [8] to the vector bundle setting and we apply the technique to complex Grassmannians, $Gr(k,N)$. We find a suitable vector bundle, with a section η as above, and natural weights for the line bundles and for the tautological k-plane bundle. We thus get Koppelman formulas for (p,q)-forms with values in any holomorphic line bundle as well as in the tautological bundle and its dual. The construction is

uniform in the sense that it uses the explicit description of the Picard group of holomorphic line bundles and reduces the problem to that of finding a weight for the generator. The generator in turn, is the determinant of the tautological bundle; by certain algebraic properties of weights, it thus suffices to construct a weight for the tautological bundle. As an application, we give explicit proofs of certain vanishing theorems of Bott-Borel-Weil type [1] for the cohomology groups associated with these line bundles. We also relate the projection part of our Koppelman formulas to Bergman kernels; thus giving a geometric interpretation of the latter ones.

This paper is organized as follows: In Section 2 we recapture the general method for finding weighted Koppelman formulas on manifolds from [8]. The only difference is that we allow for forms with values in vector bundles and state a slightly more general Koppelman formula. The proofs have been omitted since they are straightforward generalizations of the proofs in [8]. Section 3 describes some general operations on weights. In Section 4 we construct the ingredients necessary to generate weighted formulas on Grassmannians according to the general framework. In Section 5 we review the representation theoretic description of the Picard group and we prove a certain invariance property for the weights, which will be useful for the applications. We also prove that the bundle E restricted to the diagonal is equivalent to the holomorphic cotangent bundle over $Gr(k, N)$. In the last section, Section 6, we discuss some applications; we obtain vanishing theorems for the line bundles over Grassmann, and we give a geometric interpretation of the Bergman kernels associated to the line bundles.

2. A general method for finding weighted Koppelman formulas on manifolds

Let X be a complex manifold of dimension n. We want to find Koppelman formulas for differential forms on X with values in a given vector bundle $H \to X$. The method described in this section is taken from [8], except for the generalization which yields formulas for a general vector bundle H instead of for a line bundle.

We begin by noting that Stokes' theorem holds also for sections of vector bundles, which is easily proved. Let M be any complex manifold, and $G \to M$ a holomorphic Hermitian vector bundle over M. Let D_{G^*} and D_G be the Chern connections for G^* and G respectively. If u is a differential form taking values in G^* and ϕ is a test form with values in G, we have

$$(3) \qquad \int_M D_{G^*} u \wedge \phi = (-1)^{\deg u + 1} \int_M u \wedge D_G \phi,$$

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[1] These are not given in the form including the ρ-shift which is common in representation theory.

where $\wedge$ denotes taking the natural pairing between the factors in G^* and G, and taking the wedge product between the factors which are differential forms. If u is instead a current, we can take (3) as a definition. In the same way, we also have

$$(4) \qquad \int_M \bar{\partial}u \wedge \phi = (-1)^{\deg u + 1} \int_M u \wedge \bar{\partial}\phi.$$

Let Δ be the diagonal in $X_z \times X_\zeta$. Let H_z denote $\pi_z^*(H)$, where π_z is the projection from $X_z \times X_\zeta$ to X_z, and analogously for H_ζ. Let $g_{0,0}$ be a section of $H_z \otimes H_\zeta^* = \mathrm{Hom}(H_\zeta, H_z)$ such that $g_{0,0}(z, z) = \mathrm{Id}$ for all z. If $[\Delta]$ denotes the current of integration over the diagonal and $\omega(\zeta, z)$ is a differential form with values in $H_z^* \otimes H_\zeta$, then we let

$$[\Delta]_{g_{0,0}}(\omega) := [\Delta].((g_{0,0} \otimes \mathrm{Id})\omega),$$

where Id acts on the differential forms in ω, and we take the natural pairing $(H_z^* \otimes H_\zeta) \times (H_z \otimes H_\zeta^*) \to \mathbb{C}$. Note that this does not depend on which $g_{0,0}$ we choose, since the values on the diagonal are the only ones that matter. The reason for the subscript on $g_{0,0}$ will become apparent later on.

Proposition 1 (Koppelman's formula). *Assume that $D \subset X_\zeta$, $\phi \in \mathcal{E}_{p,q}(\bar{D}, H_\zeta)$, and that the current $K(z, \zeta)$ and the smooth form $P(z, \zeta)$ take values in $H_z \otimes H_\zeta^* = Hom(H_\zeta, H_z)$ and solve the equation*

$$(5) \qquad \bar{\partial}K = [\Delta]_{g_{0,0}} - P.$$

We then have

$$(6) \qquad \phi(z) = \int_{\partial D} K \wedge \phi + \int_D K \wedge \bar{\partial}\phi + \bar{\partial}_z \int_D K \wedge \phi + \int_D P \wedge \phi,$$

where the integrals are taken over the ζ variable.

The proof of this uses (4) but is otherwise just like the usual proof of the Koppelman formula. Note that if ϕ in (6) is a $\bar{\partial}$-closed form and the first and fourth terms of the right hand side of Koppelman's formula vanish, we get a solution to the $\bar{\partial}$-problem for ϕ.

Our purpose now is to find K and P that satisfy (5) in a special type of manifold. To begin with, we will let H be the trivial line bundle. Assume that we can find a holomorphic vector bundle $E \to X_z \times X_\zeta$ of rank n, such that there exists a holomorphic section η of E that defines the diagonal Δ. In other words, η must vanish to the first order on Δ and be non-zero elsewhere. Let $\{e_i\}$ be a local frame for E, and $\{e_i^*\}$ the dual local frame for E^*. Contraction with η is an operation on E^* which we denote by δ_η; if $\eta = \sum \eta_i e_i$ then

$$\delta_\eta \left(\sum \sigma_i e_i^* \right) = \sum \eta_i \sigma_i.$$

We define the operator

$$\nabla_\eta = \delta_\eta - \bar{\partial}.$$

Choose a Hermitian metric h for E, let D_E be the Chern connection on E, and D_{E^*} the induced connection on E^*. Consider the bundle

$$G_E = \Lambda[T^*(X \times X) \oplus E \oplus E^*] \to X \times X$$

and $\Gamma(X \times X, G_E)$, the space of C^∞ sections of G_E (note the change of notation compared to [8]). If A lies in $\Gamma(X \times X, T^*(X \times X) \otimes E \otimes E^*))$, then we define $\tilde{A}$ as the corresponding element in $\Gamma(X \times X, G_E)$, arranged with the differential form first, then the section of E and finally the section of E^*. For example, if $A = dz_1 \otimes e_1 \otimes e_1^*$, then $\tilde{A} = dz_1 \wedge e_1 \wedge e_1^*$.

To define a derivation D on $\Gamma(X \times X, G_E)$, we first let $Df = \widetilde{D_E f}$ for a section f of E, and $Dg = \widetilde{D_{E^*} g}$ for a section g of E^*. We then extend the definition by

$$D(\xi_1 \wedge \xi_2) = D\xi_1 \wedge \xi_2 + (-1)^{\deg \xi_1} \xi_1 \wedge D\xi_2,$$

where $D\xi_i = d\xi_i$ if ξ_i happens to be a differential form, and $\deg \xi_1$ is the total degree of ξ_1. For example, $\deg(\alpha \wedge e_1 \wedge e_1^*) = \deg \alpha + 2$, where $\deg \alpha$ is the degree of α as a differential form. We let

$$\mathcal{L}^m = \bigoplus_p \Gamma(X \times X, \Lambda^p E^* \wedge \Lambda^{p+m} T_{0,1}^*(X \times X));$$

note that $\mathcal{L}^m$ is a subspace of $\Gamma(X \times X, G_E)$. The operator ∇_η will act in a natural way as $\nabla_\eta \colon \mathcal{L}^m \to \mathcal{L}^{m+1}$. If $f \in \mathcal{L}^m$ and $g \in \mathcal{L}^k$, then $f \wedge g \in \mathcal{L}^{m+k}$. We also see that ∇_η obeys Leibniz' rule, and that $\nabla_\eta^2 = 0$.

Definition 2. For a form $f(z, \zeta)$ on $X \times X$, we define

$$\int_E f(z, \zeta) \wedge e_1 \wedge e_1^* \wedge \ldots \wedge e_n \wedge e_n^* = f(z, \zeta).$$

Note that if I is the identity on E, then $\tilde{I} = e \wedge e^* = e_1 \wedge e_1^* + \ldots + e_n \wedge e_n^*$. It follows that $\tilde{I}_n = e_1 \wedge e_1^* \wedge \ldots \wedge e_n \wedge e_n^*$ (with the notation $a_n = a^n/n!$), so the definition above is independent of the choice of frame. Our derivation D and $\int_E$ interact in the following way:

Proposition 3. *If $F \in \Gamma(X \times X, G_E)$ then*

$$d \int_E F = \int_E DF.$$

We will now construct integral formulas on $X \times X$. As a first step, we find a section σ of E^* such that $\delta_\eta \sigma = 1$ outside Δ. For reasons that will become apparent, we choose σ to have minimal pointwise norm with respect to the metric h, which means that $\sigma = \sum_{ij} h_{ij} \bar{\eta}_j e_i^* / |\eta|^2$. Close to Δ, it is obvious that $|\sigma| \lesssim 1/|\eta|$, and a calculation shows that we also have $|\bar{\partial}\sigma| \lesssim 1/|\eta|^2$. Next, we construct a section u with the property that $\nabla_\eta u = 1 - R$ where R is a current with support on Δ. We set

$$(7) \qquad\qquad u = \frac{\sigma}{\nabla_\eta \sigma} = \sum_{k=0}^\infty \sigma \wedge (\bar{\partial}\sigma)^k,$$

and note that $u \in \mathcal{L}^{-1}$. By $u_{k,k-1}$ we will mean the term in u of degree k in E^* and degree $k-1$ in $T^*_{0,1}(X \times X)$. It is easily checked that $\nabla_\eta u = 1$ outside Δ.

The following theorem yields a Koppelman formula by Theorem 1, with the trivial line bundle as H:

Theorem 4. *Let $E \to X \times X$ be a vector bundle with a section η which defines the diagonal Δ of $X \times X$. We have*

$$\bar{\partial} K = [\Delta] - P,$$

where

$$(8) \qquad K = \int_E u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n \quad and \quad P = \int_E \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n,$$

and u is defined by (7).

Note that since $D\eta$ contains no e_i^*'s, we have

$$P = \int_E (\frac{i\tilde{\Theta}}{2\pi})_n = \det \frac{i\Theta}{2\pi} = c_n(E),$$

i.e., the nth Chern class of E. The factor

$$\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi}$$

is actually the supercurvature associated with the operator ∇_η if we view ∇_η as a superconnection in the sense of Quillen, [14]. In fact, we have the following Bianchi identity:

$$(9) \qquad \nabla_\eta \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right) = 0,$$

for a direct proof see, e.g., [8].

The idea behind the proof of Theorem 4 is that by (9) and Proposition 3 we have

$$\bar{\partial} \int_E u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n = \int_E \bar{\partial} \left[u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n \right] =$$

$$= -\int_E \nabla_\eta \left[u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n \right] =$$

$$(10) \qquad = -\int_E \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n + \frac{1}{(2\pi i)^n} \int_E R \wedge (D\eta)_n.$$

The left hand term in (10) is P. The rest of the proof consists of proving that

$$(11) \qquad \frac{1}{(2\pi i)^n} \int_E R \wedge (D\eta)_n = [\Delta],$$

which is proved by choosing local coordinates on X, and reducing the problem to the $\mathbb{C}^n$-case. For details of the proof, see, e.g., [8].

As explained in the introduction, we will obtain more flexible formulas if we use weights.

Definition 5. A section g with values in $\mathcal{L}_0$ is a weight if $\nabla_\eta g = 0$ and $g_{0,0}(z, z) = 1$.

Theorem 4 goes through with essentially the same proof if we take

$$(12) \quad K_g = \int_E u \wedge g \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n \quad \text{and} \quad P_g = \int_E g \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n,$$

as shown by the following calculation:

$$\bar{\partial}K_g = -\int_E \nabla_\eta u \wedge g \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n =$$

$$(13) \qquad = -\int_E (g - R) \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n = [\Delta] - P_g,$$

which follows from the proof of Theorem 4 and the properties of weights.

Finally, we will use weights taking values in $\mathrm{Hom}(H_\zeta, H_z)$ to construct Koppelman formulas for differential forms with values in the vector bundle $H \to X$. We define

$$G_{E,H} = \mathrm{Hom}(H_\zeta, H_z) \otimes \Lambda[T^*(X \times X) \oplus E \oplus E^*] \to X \times X$$

and

$$(14) \quad \mathcal{L}_H^m := \bigoplus_p \Gamma(X \times X, \mathrm{Hom}(H_\zeta, H_z) \otimes [\Lambda^p E^* \wedge \Lambda^{p+m} T_{0,1}^*(X \times X)]).$$

We define δ_η on $\Gamma(X \times X, G_{E,H})$ as $\mathrm{Id} \otimes \delta_\eta$, where Id acts on the factors in $\mathrm{Hom}(H_\zeta, H_z)$ and δ_η on the factors in $\Lambda[T^*(X \times X) \oplus E \oplus E^*]$. We also need to extend the derivation D to $\Gamma(X \times X, G_{E,H})$. If a_1 is a differential form taking values in $\mathrm{Hom}(H_\zeta, H_z)$, and $a_2 \in \Gamma(X \times X, G_E)$, then we define

$$D(a_1 \wedge a_2) = D_{\mathrm{Hom}(H_\zeta, H_z)} a_1 \wedge a_2 + (-1)^{\deg a_1} a_1 \wedge Da_2,$$

where $D_{\mathrm{Hom}(H_\zeta, H_z)}$ is the Chern connection on $\mathrm{Hom}(H_\zeta, H_z)$. It is obvious that Leibniz' rule holds for both δ_η and the extended D, with the degree taken as the total degree in E, E^* and $T^*(X \times X)$.

If $F \in \mathcal{L}_H^0$, then in analogy with Proposition 3 we have

$$D_{\mathrm{Hom}(H_\zeta, H_z)} \int_E F = \int_E DF.$$

It follows that we also have $\bar{\partial} \int_E F = \int_E \bar{\partial}F$.

Let $g \in \mathcal{L}_H^0$ be such that $\nabla_\eta g = 0$ and $g_{0,0}(z,z) = \mathrm{Id}$. In that case we can use g as a weight just as in (12) and get

$$(15) \qquad \bar{\partial} K_g = [\Delta]_{g_{0,0}} - P_g$$

by a calculation similar to (13), and then we get a Koppelman formula by Theorem 1.

Remark 6. To obtain more general formulas, one can find forms K and P such that

$$(16) \qquad D_{\mathrm{Hom}(H_\zeta, H_z)} K_g = [\Delta]_{g_{0,0}} - P_g$$

by setting $\nabla_\eta^{\mathrm{full}} = \delta_\eta - D$ and checking that the corresponding equation (9) and Theorem 4 are still valid. See for example [8] for details. This will give the same formulas as in [4], if H is the trivial line bundle. We can use weights just as before, if we require that a weight g has the property $\nabla_\eta^{\mathrm{full}} g = 0$ instead of $\nabla_\eta g = 0$.

3. Algebraic properties of weights

In this section we investigate some general constructions of weights with the purpose of generating weights for a wide class of derived bundles from two given vector bundles and weights for these. This method will be useful later when we focus on line bundles over Grassmannians.

To be more precise, we let H and H' be holomorphic vector bundles over the complex manifold X and assume that X fulfills the requirements of our general setup for constructing Koppelman formulas, i.e., $X \times X$ admits a holomorphic vector bundle E with a holomorphic section defining the diagonal. Assume also that $g \in \Gamma(X \times X, G_{E,H})$ and $g' \in \Gamma(X \times X, G_{E,H'})$ are weights for H and H' respectively. We shall see that one can naturally define weights $g \otimes g'$ and $g \wedge g'$ (when $H = H'$), as well as g^* for the bundles $H \otimes H', H \wedge H$ and H^* respectively. This generalizes the fact, mentioned in the introduction, that the product of weights for the trivial bundle is again a weight.

3.1. Tensor products and exterior products of weights.
For operators $A \in H_z \otimes H_\zeta^*$ and $B \in H_z \otimes (H_\zeta')^*$ the tensor product $A \otimes B$ defined by

$$(17) \qquad A \otimes B(u \otimes v) := A(u) \otimes B(v), u \in H_\zeta, v \in H_\zeta'$$

is a linear operator in $\mathrm{Hom}(H_\zeta \otimes H_\zeta', H_z \otimes H_z')$. We can therefore extend the exterior multiplication on the vector space G_E to a linear map (which we still denote by $\otimes$)

$$\otimes : (G_{E,H})_{(z,\zeta)} \otimes (G_{E,H'})_{(z,\zeta)} \to (G_{E,H \otimes H'})_{(z,\zeta)}$$

given by

$$(A \otimes \omega) \otimes (B \otimes \omega') \mapsto (A \otimes B) \otimes (\omega \wedge \omega'),$$

for $\omega, \omega' \in (G_E)_{(z,\zeta)}$. This operation defines a natural fiberwise multiplication on sections.

Lemma 7. *The operator ∇_η acts as a graded derivation with respect to the multiplication, $\otimes$, of sections, i.e.,*

$$\nabla_\eta\big((A \otimes \omega) \otimes (B \otimes \omega')\big) = \nabla_\eta(A \otimes \omega) \otimes (B \otimes \omega')$$
$$+ (-1)^{deg\,\omega}(A \otimes \omega) \otimes \nabla_\eta(B \otimes \omega'),$$

where A and B are local smooth sections of $H_z \otimes H_\zeta^*$ and $H_z' \otimes (H_\zeta')^*$ respectively, and ω and ω' are local smooth sections of G_E.

Proof. We first observe that

$$\nabla_\eta(A \otimes \omega) = -\bar{\partial}A \otimes \omega + A \otimes \nabla_\eta \omega,$$

and likewise for $B \otimes \omega'$. Hence,

$$\nabla_\eta\big((A \otimes B) \otimes (\omega \wedge \omega')\big)$$

$$= -\bar{\partial}(A \otimes B) \otimes (\omega \wedge \omega') + (A \otimes B) \otimes \nabla_\eta(\omega \wedge \omega')$$
$$= -\bar{\partial}A \otimes (B \otimes (\omega \wedge \omega')) + (A \otimes B) \otimes (\nabla_\eta \omega \wedge \omega') -$$
$$A \otimes (\bar{\partial}B \otimes \omega \wedge \omega') + (-1)^{deg\,\omega}(A \otimes B) \otimes (\omega \wedge \nabla_\eta \omega')$$
$$= (-\bar{\partial}A \otimes \omega + A \otimes \nabla_\eta \omega) \otimes (B \otimes \omega') +$$
$$(A \otimes \omega) \otimes (-\bar{\partial}B \otimes \omega' + (-1)^{deg\,\omega}B \otimes \nabla_\eta \omega')$$
$$= \nabla_\eta(A \otimes \omega) \otimes (B \otimes \omega') + (-1)^{deg\,\omega}(A \otimes \omega) \otimes \nabla_\eta(B \otimes \omega').$$

$\square$

Corollary 8. *Given weights g and g' for H and H' respectively, the section*

$$g \otimes g' \in \Gamma(X \times X, G_{E, H \otimes H'})$$

is a weight for $H \otimes H'$.

We next turn to exterior products of a vector bundle. Recall that when A and A' are operators in $\mathrm{Hom}(H_\zeta, H_z)$, $A \wedge A'$ is the operator in $\mathrm{Hom}(\Lambda^2 H_\zeta, \Lambda^2 H_z)$ given by

$$A \wedge A'(u \wedge u') = A(u) \wedge A'(u') - A(u') \wedge A'(u).$$

We can then form the exterior product

$$\wedge \colon (G_{E,H})_{(z,\zeta)} \otimes (G_{E,H})_{(z,\zeta)} \to (G_{E, H \wedge H})_{(z,\zeta)}$$

given by

$$(A \otimes \omega) \otimes (A' \otimes \omega') \mapsto (A \wedge A') \otimes (\omega \wedge \omega').$$

It induces a natural exterior product on sections of $G_{E,H}$. Using the Leibniz identity

$$\bar{\partial}(A \wedge A') = \bar{\partial}A \wedge A' + A \wedge \bar{\partial}A',$$

the following lemma can be proved in the same manner as Lemma 7.

Lemma 9. *The operator ∇_η acts as a graded derivation with respect to the exterior multiplication of sections, i.e.,*

$$
\begin{aligned}
\nabla_\eta\big((A \otimes \omega) \wedge (A' \otimes \omega')\big) \; &= \; \nabla_\eta(A \otimes \omega) \wedge (A' \otimes \omega') + \\
&\quad (-1)^{deg\,\omega}(A \otimes \omega) \wedge \nabla_\eta(A' \otimes \omega'),
\end{aligned}
$$

where A and A' are local smooth sections of $H_z \otimes H_\zeta^*$, and ω and ω' are local smooth sections of G_E.

In analogy with Corollary 8, we have

Corollary 10. *Given weights g_1 and g_2 for H, the section*

$$g_1 \wedge g_2 \in \Gamma(X \times X, G_{E, H \wedge H})$$

is a weight for $H \wedge H$.

3.2. **Dual weights.** For a local section $A \otimes \omega$ of the bundle $G_{E,H}$, we define the adjoint section

$$(A \otimes \omega)^* := A^* \otimes \omega,$$

where $A^*(z, \zeta) \colon H_z^* \to H_\zeta^*$ is the standard dual operator to $A(z, \zeta)$ given by composing functionals with $A(z, \zeta)$. The relations

$$
\begin{aligned}
\nabla_\eta(A^* \otimes \omega) \; &= \; -\bar{\partial}A^* \otimes \omega + A^* \otimes \nabla_\eta \omega \\
&= \; -(\bar{\partial}A)^* \otimes \omega + (A \otimes \nabla_\eta \omega)^* \\
&= \; (\nabla_\eta(A \otimes \omega))^*
\end{aligned}
$$

prove the following lemma.

Lemma 11. *Given a weight g for the bundle H, the section g^* is a weight for the dual bundle H^*.*

4. The necessary constructions on Grassmannians

In this section we construct the ingredients necessary to generate weighted integral formulas on Grassmannians according to the recipe in Section 2. We start by reviewing some elementary facts and introducing some notation. Hereafter, X will denote the Grassmannian $Gr(k, N)$ of complex k-planes in $\mathbb{C}^N$. Just as $\mathbb{CP}^n$, $(= Gr(1, n + 1))$, has its tautological line bundle, X has a tautological rank k-vector bundle, which will be denoted by $H \to X$ from now on. We consider H as a subbundle of the trivial rank N-bundle, $\mathbb{C}^N \to X$, and the fiber of H above $p \in X$ is the k-plane in $\mathbb{C}^N$ corresponding to the point p. We will take the standard metric on $\mathbb{C}^N$ and this gives us a Hermitian metric on $H \subset \mathbb{C}^N$. From H we get a natural Hermitian line bundle $L = \det H$, which actually generates the Picard group; see Subsection 5.4. We also get the quotient bundle, $F := \mathbb{C}^N/H$, which is a holomorphic

vector bundle of rank $N - k$. As a C^∞-bundle, it is isomorphic to the bundle of orthogonal complements $H^\perp \subset \mathbb{C}^N$ via the mapping $\varphi \colon F \to H^\perp$ defined fiberwise by $\varphi(v + H_z) = v - \pi_{H_z} v$, where π_{H_z} is the orthogonal projection from $\mathbb{C}^N$ onto H_z. (If w is a $\mathbb{C}^N$-valued form we will, for simplicity, also write $\pi_{H_z} w$ for $(\pi_{H_z} \otimes \mathrm{Id}) w$.) The mapping φ and the metric on $H^\perp \subset \mathbb{C}^N$ gives us a metric on F.

Let $e = (e_1, \dots, e_N)$ be the standard basis for $\mathbb{C}^N$. The point on X corresponding to the k-plane $\mathrm{Span}\{e_1, \dots, e_k\}$ will be the reference point and denoted by p_0. A local holomorphic chart centered at p_0 can be defined as follows: Let z be a point in $\mathbb{C}^n := \mathbb{C}^{k(N-k)}$ and organize z as an $(N-k) \times k$-matrix, i.e.,

$$z = \begin{pmatrix} z_{11} & \cdots & z_{1k} \\ \vdots & & \vdots \\ z_{N-k,1} & \cdots & z_{N-k,k} \end{pmatrix} \in \mathbb{C}^n.$$

Associate to z the point on X corresponding to the k-plane spanned by the columns of the $N \times k$-matrix

$$(18) \qquad \begin{pmatrix} I \\ z \end{pmatrix}, \quad I = I_{k \times k},$$

with respect to the basis e. This actually gives us an injective map from $\mathbb{C}^n$ onto a dense subset $U \subset X$. We also get natural local holomorphic frames for the bundles H, L, and F over this chart. For $j = 1, \dots, k$, let $\mathfrak{h}_j(z)$ be the jth column of (18), i.e., $\mathfrak{h}_j(z) = e_j + \sum_{i=1}^{N-k} z_{ij} e_{k+i}$. Then $\mathfrak{h}_1, \dots, \mathfrak{h}_k$ are k pointwise linearly independent holomorphic sections of H over U. A natural holomorphic frame for L is thus $\mathfrak{l} = \mathfrak{h}_1 \wedge \cdots \wedge \mathfrak{h}_k$. Also, for $1 \le j \le N - k$, let $\mathfrak{f}_j(z)$ be the equivalence class defined by e_{k+j} in $F = \mathbb{C}^N / H$, in the fiber over z. Then $(\mathfrak{f}_1, \dots, \mathfrak{f}_{N-k})$ is a local holomorphic frame for F over U. The projection $\mathbb{C}^N \to F$, expressed in the e-basis for $\mathbb{C}^N$ and the frame $\mathfrak{f}$ for F, can then be written as the $(N-k) \times N$-matrix

$$(19) \qquad \begin{pmatrix} -z & I \end{pmatrix}, \quad I = I_{(N-k) \times (N-k)}.$$

For reference we note some more explicit expressions: As a mapping $\mathbb{C}^N_e \to \mathbb{C}^N_e$ expressed in the e-basis we have

$$\pi_H = \begin{pmatrix} I \\ z \end{pmatrix} (I + z^* z)^{-1} \begin{pmatrix} I & z^* \end{pmatrix}$$

and as a mapping $\mathbb{C}^N_e \to H_{\mathfrak{h}}$,

$$\pi_H = (I + z^* z)^{-1} \begin{pmatrix} I & z^* \end{pmatrix}.$$

The mapping $\varphi \colon F_{\mathfrak{f}} \to \mathbb{C}^N_e$ looks like

$$\varphi = \begin{pmatrix} -(I + z^* z)^{-1} z^* \\ I - z(I + z^* z)^{-1} z^* \end{pmatrix}.$$

We have defined the metric, $\langle \cdot, \cdot \rangle_F$, on F via φ so the Hermitian metric-matrix, h_F, expressed in the frame $\mathfrak{f}$ satisfies $(h_F)_{i,j} = \langle \varphi(\mathfrak{f}_i), \varphi(\mathfrak{f}_j) \rangle_{\mathbb{C}^N}$, (with the convention that $\langle v, w \rangle_F = v^t h_F \bar{w}$). Using the explicit expression for φ, a computation then gives

$$h_F^t(z) = (I + zz^*)^{-1},$$

and so the Chern curvature-matrix of F is

$$\Theta_F = \bar{\partial}(\bar{h}_F^{-1} \partial \bar{h}_F) = \partial \bar{\partial} \log(I + zz^*),$$

where the last expression should be interpreted in the functional calculus sense. For the bundle H we get

$$h_H^t = I + z^* z, \text{ and } \Theta_H = \partial \bar{\partial} \log(I + z^* z)^{-1},$$

expressed in the frame $\mathfrak{h}$.

4.1. **The bundle E and the section η.** We will construct a holomorphic vector bundle $E \to X_z \times X_\zeta$ of rank $n \, (= k(N-k))$ and a global holomorphic section η of it defining the diagonal. As in Section 2, we let H_z and H_ζ denote the pull-back of the tautological bundle under the projections $X_z \times X_\zeta \to X_z$ and $X_z \times X_\zeta \to X_\zeta$ respectively and we define F_z similarly. However, for convenience we will occasionally abuse this notation and also write, e.g., H_z for the fiber of the bundle $H_z \to X_z \times X_\zeta$ above a point (z, ζ). This ambiguity is (partly) justified since one can identify fibers of $H_z \to X_z \times X_\zeta$ above points (z, ζ) for any ζ. This means also that, e.g, $\{\mathfrak{h}_j(z)\}$ is a local holomorphic frame for $H_z \to X_z \times X_\zeta$ over $U_z \times X_\zeta$.

The bundle E is simply $E = F_z \otimes H_\zeta^*$ and then $\mathfrak{e}_{ij} := \mathfrak{f}_i(z) \otimes \mathfrak{h}_j^*(\zeta)$, $1 \le i \le N-k, 1 \le j \le k$, is a holomorphic frame for E over $U \times U \subset X \times X$. To define η we start with a vector $v \in H_\zeta$ and via $H_\zeta \subset \mathbb{C}_\zeta^N \cong \mathbb{C}_z^N$ we can identify v with a vector $\tilde{v} \in \mathbb{C}_z^N$. We then let $\eta(v)$ be the projection of $\tilde{v}$ on $F_z = \mathbb{C}_z^N / H_z$.

Proposition 12. *The section η of E is holomorphic and defines the diagonal in $X \times X$.*

Proof. It is clear that $\eta(v)$ vanishes if and only if v belongs to the fiber above a point in the diagonal $\Delta \subset X \times X$. Hence, η is a global section of $\mathrm{Hom}(H_\zeta, F_z) \cong E$ and vanishes precisely on Δ. In the coordinates and frames described above, η has the form

$$\eta = \zeta - z.$$

In fact, if $v = \sum_1^k v_j \mathfrak{h}_j(\zeta) \in H_\zeta$ then $\eta(v)$ is the image in F_z of $\sum_1^k v_j e_j + \sum_{i=1}^{N-k} \sum_{j=1}^k \zeta_{ij} v_j e_{k+i}$. By (19) this is equal to $\sum_{i=1}^{N-k} \sum_{j=1}^k (\zeta_{ij} - z_{ij}) v_j e_{k+i}$. We thus see that η is holomorphic and vanishes to the first order on Δ. $\square$

4.2. Bundles and weights.

The bundle $L = \det H$ actually generates the Picard group of holomorphic line bundles; cf. Section 5.3, and [18]. We will construct weights for the line bundles $L^r := L^{\otimes r} \to X$, and for the vector bundle $H \to X$. We start by defining two fundamental sections γ_0 and γ_1 of $\operatorname{Hom}(H_\zeta, H_z)$ and $\operatorname{Hom}(H_\zeta, H_z) \otimes E^* \wedge T_{0,1}^*(X \times X)$ respectively. For $v \in H_\zeta$ we first identify v with the vector $\tilde{v}$ in the trivial bundle $\mathbb{C}_z^N \to X_z \times X_\zeta$ via $H_\zeta \subset \mathbb{C}_\zeta^N \cong \mathbb{C}_z^N$. We then put $\gamma_0(v) = \pi_{H_z} \tilde{v}$. In the $\mathfrak{h}$-frames described above, γ_0 is simply the $k \times k$-matrix

$$(20) \qquad \gamma_0 = (I + z^* z)^{-1} (I + z^* \zeta).$$

It is a little bit more complicated to describe γ_1: Let ξ and v be (germs of) smooth sections of E and H_ζ respectively. Since $E = F_z \otimes H_\zeta^*$, $\xi(v)$ defines naturally a smooth section of F_z and hence, $\varphi(\xi(v))$ is a smooth section of $H_z^\perp \subset \mathbb{C}_z^N$. We then put $-\gamma_1(\xi \otimes v) = \pi_{H_z}(\bar{\partial}\varphi(\xi(v)))$, which is a smooth section of $H_z \otimes T_{0,1}^*(X \times X)$. We check that γ_1 so defined actually is tensorial. Let f be (a germ of) a smooth function. We then get

$$
\begin{aligned}
\gamma_1(f\xi \otimes v) &= -\pi_{H_z}\big(\bar{\partial}\,\varphi(f\xi(v))\big) \\
&= -\pi_{H_z}\big(\varphi(\xi(v)) \otimes \bar{\partial}f + f\bar{\partial}\varphi(\xi(v))\big) \\
&= -\pi_{H_z}\big(\varphi(\xi(v))\big) \otimes \bar{\partial}f + f\gamma_1(\xi \otimes v).
\end{aligned}
$$

But $\pi_{H_z}\big(\varphi(\xi(v))\big) = 0$ since $\varphi(\xi(v)) \in H_z^\perp$, and so $\gamma_1(f\xi \otimes v) = f\gamma_1(\xi \otimes v)$. (One could also note that $\gamma_1(\xi \otimes v) = -[\pi_{H_z}, \bar{\partial}]\varphi(\xi(v))$, where $[\pi_{H_z}, \bar{\partial}]$ is the commutator.) Hence, γ_1 defines a section of $H_z \otimes T_{0,1}^*(X_z \times X_\zeta) \otimes E^* \otimes H_\zeta^* \cong \operatorname{Hom}(H_\zeta, H_z) \otimes E^* \wedge T_{0,1}^*(X \times X)$. A computation in the local coordinates shows that

$$(21) \qquad \gamma_1 = \sum_{i,j=1}^{k} \mathfrak{h}_i(z) \otimes \mathfrak{h}_j^*(\zeta) \otimes M_{ij},$$

where M is the $k \times k$-matrix of E^*-valued $(0,1)$-forms

$$(22) \qquad M = \bar{\partial}\big((I + z^* z)^{-1} z^*\big) \wedge \mathfrak{e}^*.$$

Here, $\mathfrak{e}^*$ is the matrix with entries $(\mathfrak{e}_{ij})^*$.

Proposition 13. *The section $G := \gamma_0 + \gamma_1 \in \mathcal{L}_H^0$, (cf. (14)), is a weight for the tautological bundle H.*

Proof. We need to check that $\gamma_0(z, z) = \operatorname{Id}$ and that $\nabla_\eta G = 0$. The first equality is obvious from the definition. For the second one we have to verify the two equations $\bar{\partial}\gamma_0 = \delta_\eta \gamma_1$ and $\bar{\partial}\gamma_1 = 0$. Let v be a germ of a holomorphic section of H_ζ. Via $H_\zeta \subset \mathbb{C}_\zeta^N \cong \mathbb{C}_z^N$ we may view v as a holomorphic section of $\mathbb{C}_z^N$ and then we can write

$$
\begin{aligned}
(\delta_\eta \gamma_1)(v) &= -\pi_{H_z}\big(\bar{\partial}(\varphi(\eta(v)))\big) = -\pi_{H_z}\big(\bar{\partial}(\pi_{H_z^{\perp}} v)\big) \\
&= -\pi_{H_z}\big(\bar{\partial}(v - \pi_{H_z} v)\big) = \bar{\partial}_{H_z}(\pi_{H_z} v) \\
&= \bar{\partial}_{H_z}(\gamma_0(v)).
\end{aligned}
$$

Hence, $\bar{\partial}_{H_z}(\gamma_0(v)) = \delta_\eta \gamma_1(v)$ for any germ of holomorphic section v of H_ζ. It follows that $\bar{\partial}\gamma_0 = \delta_\eta \gamma_1$. Now, let ξ be a germ of a holomorphic section of E. Then $\xi(v)$ is a germ of a holomorphic section of F_z. One can (locally) lift $\xi(v)$ to a germ of a holomorphic section, $\widetilde{\xi(v)}$, of $\mathbb{C}^N$ that projects to $\xi(v)$. We then get

$$
\begin{aligned}
\bar{\partial}_{H_z}\gamma_1(\xi \otimes v) &= \bar{\partial}_{H_z}\big(\pi_{H_z}\bar{\partial}(\varphi(\xi(v)))\big) = \bar{\partial}_{H_z}\big(\pi_{H_z}\bar{\partial}(\widetilde{\xi(v)} - \pi_{H_z}\widetilde{\xi(v)})\big) \\
&= -\bar{\partial}_{H_z}\big(\pi_{H_z}\bar{\partial}(\pi_{H_z}\widetilde{\xi(v)})\big) = -\bar{\partial}_{H_z}^2(\pi_{H_z}\widetilde{\xi(v)}) \\
&= 0.
\end{aligned}
$$

Hence, $\bar{\partial}_{H_z}\gamma_1(\xi \otimes v) = 0$ for any holomorphic ξ and v, and this finishes the proof. $\qquad\square$

By the algebraic properties of weights established in Section 3 we now get that $g := G \wedge \cdots \wedge G$ (the exterior product of G with itself k times) is a weight for L. It is easy to check that

$$
g_{0,0} = \gamma_0 \wedge \cdots \wedge \gamma_0 = k! \frac{\det(I + z^*\zeta)}{\det(I + z^*z)}
$$

in the frame $\mathfrak{l}$ for L. Weights for positive powers of L are then obtained by taking powers of g. By the results at the end of Section 3 we can also get weights for H^* and $L^{-r} = (L^*)^{\otimes r}$ from G. If one wants to construct weights for H^* geometrically, as we have done in this section, it is easier to take $F_\zeta \otimes H_z^*$ as the bundle E. However, our Koppelman formulas have an inherent duality and this gives us weighted formulas for forms with values in H^* and L^{-r} from the weighted formulas for H and L^r.

5. Representation-theoretic interpretations

In this section we describe X and its line bundles in terms of group actions and representations. The purpose of this is threefold. First of all, this point of view gives an easy description of the Picard group of X. Secondly, and more importantly, we prove that the weights we have constructed earlier will all be invariant under a certain group action; a property which will turn out be highly useful in the last section with applications to Bergman kernels. Finally, in this setup, we can fairly easily prove that the restriction of the bundle E to the diagonal is equivalent to the holomorphic cotangent bundle $T_{1,0}^*$ of X.

5.1. **The Grassmannian as a homogeneous space.** The linear action of the group $GL(N,\mathbb{C})$ on $\mathbb{C}^N$ induces an action as holomorphic automorphisms of X, and this action is clearly transitive. Hence, we can describe X as a homogeneous space $X \cong GL(N,\mathbb{C})/P$, where

$$P := \left\{ \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \middle| \det A \det D \neq 0 \right\}$$

is the stabilizer of p_0. One can also restrict the action to the subgroup $SL(N, \mathbb{C})$ and still have a transitive group action; this time exhibiting X as the homogeneous space $SL(N, \mathbb{C})/P'$, where

$$P' := \left\{ \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \middle| \det A \det D = 1 \right\}$$

is the stabilizer of p_0 in $SL(N, \mathbb{C})$. The reason that we mention this realization is that some of the results we refer to later hold only for quotients of semisimple Lie groups. A third realization is given by restricting the $GL(N, \mathbb{C})$-action to the unitary group $U(N)$. The stabilizer of p_0 in this subgroup is

$$\left\{ \left(\begin{array}{cc} A & 0 \\ 0 & D \end{array} \right) \middle| A \in U(k), D \in U(N-k) \right\} \cong U(k) \times U(N-k),$$

and hence we have a third description of X as the quotient space $U(N)/(U(k) \times U(N-k))$.

5.2. **The bundles H, F, and E.** We recall that a vector bundle $\mathcal{V} \to X$ is said to be *homogeneous* under a group G if G acts on it by bundle automorphisms in such a way that the corresponding action on X is transitive. As a consequence, the stabilizer, G_{p_0}, of p_0 in G acts linearly on the fiber $\mathcal{V}_{p_0}$, i.e., $\mathcal{V}_{p_0}$ carries a representation, τ, of G_{p_0}. The vector bundle $\mathcal{V}$ can then be reconstructed from the representation τ as the set of equivalence classes

$$G \times_{G_{p_0}} \mathcal{V}_{p_0} := G \times \mathcal{V}_{p_0}/\sim,$$

where the equivalence relation $\sim$ is defined as $(g, v) \sim (g', v')$ if and only if $(g', v') = (gx^{-1}, \tau(x)v)$ for some x in G_{p_0}. The G-action is then given by $[(g', v)] \overset{g}{\mapsto} [(gg', v)]$, where the brackets denote the equivalence classes of the respective pairs. The holomorphic vector bundles are those associated with holomorphic representations, τ, of G_{p_0}, i.e., $\tau : G_{p_0} \to \text{End}(\mathcal{V}_{p_0})$ is a holomorphic group homomorphism.

Suppose now that $H \subset G$ is a closed subgroup of G which also acts transitively on X. Then we can describe X as a quotient $H/(H \cap G_{p_0})$ and form the H-homogeneous vector bundle $\mathcal{V}^H := H \times_{H \cap G_{p_0}} \mathcal{V}_{p_0}$. This latter bundle is in fact equivalent to the former one via the bundle mapping

$$\begin{aligned} \Psi^G_H : H \times_{H \cap G_{p_0}} \mathcal{V}_{p_0} &\to G \times_{G_{p_0}} \mathcal{V}_{p_0}, \\ [(h, v)]_H &\mapsto [(h, v)]_G, \end{aligned}$$

where the brackets denote the respective equivalence classes.

For our purposes, this means that we can choose to view $GL(N, \mathbb{C})$-homogeneous vector bundles as $SL(N, \mathbb{C})$-homogeneous ones without any

loss of information as long as the corresponding representations of P' are restrictions of P-representations. Moreover, since P is the complexification of $U(k) \times U(N-k)$ (i.e., $U(k) \times U(N-k)$ is a totally real submanifold of P), a holomorphic representation of P is uniquely determined by its restriction to $U(k) \times U(N-k)$. Hence we can also view the vector bundle as only $U(N)$-homogeneous.

The group $GL(N,\mathbb{C})$ acts naturally on the trivial bundle $X \times \mathbb{C}^N$ by $(p,v) \overset{g}{\mapsto} (g(p), gv)$. The tautological bundle H is invariant under this action, and is therefore a $GL(N,\mathbb{C})$-homogeneous vector bundle. We let $\tau : P \to \mathrm{End}(\mathbb{C}^k)$ denote the corresponding representation of P on $H_{p_0} \cong \mathbb{C}^k$, namely

$$\tau \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} v = Av, \; v \in \mathbb{C}^k.$$

Since the subbundle H of $\mathbb{C}^N$ is $GL(N,\mathbb{C})$-invariant, there is a well-defined action on the quotient bundle $F = \mathbb{C}^N/H$; i.e., F is also a homogeneous bundle. We can identify the fiber F_{p_0} with $\mathbb{C}^{N-k}$, and we let ρ denote the corresponding P-representation given by

$$\rho \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} v = Dv, \; v \in \mathbb{C}^{N-k}.$$

The bundle $E \to X \times X$ is homogeneous under the product group $GL(N,\mathbb{C}) \times GL(N,\mathbb{C})$, and the representation of $P \times P$ on the fiber $(F_z \otimes H_\zeta^*)_{(p_0,p_0)} \cong \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{N-k})$ is the tensor product representation $\rho \otimes \tau^*$ given by

$$
\begin{aligned}
\rho \otimes \tau^*(g_z, g_\zeta) Z = D_z Z A_\zeta^{-1}, \quad g_\zeta &= \begin{pmatrix} A_\zeta & B_\zeta \\ 0 & D_\zeta \end{pmatrix}, \\
g_z &= \begin{pmatrix} A_z & B_z \\ 0 & D_z \end{pmatrix}, \\
Z &\in M_{N-k,k}(\mathbb{C}).
\end{aligned}
$$

The trivial bundle $\mathbb{C}^N$ is equipped with the standard Euclidean metric which is $U(N)$-invariant; and the tautological bundle H inherits this metric, thus admitting an isometric action of $U(N)$. Moreover, we recall that the quotient bundle F is smoothly equivalent to the orthogonal complement, $H^\perp$, to the tautological bundle. It should be pointed out that $H^\perp$ is not a holomorphic vector bundle, whereas F is. Since the metric on F is induced from that on $H^\perp$, the $U(N)$-action on F is also isometric. Moreover, the bundle E is equipped with a tensor product metric, and therefore the Cartesian product $U(N) \times U(N)$ acts isometrically on E.

The Chern connections and curvatures of the three bundles H, F, and E are invariant under the respective group actions since they are associated with invariant metrics. We recall that the invariance of a curvature, $\Theta_\mathcal{V}$, of a holomorphic homogeneous vector bundle $\mathcal{V}$ means the invariance as a section of the bundle $\mathrm{End}(\mathcal{V}) \otimes T_{1,1}^*$ with respect to the natural action on sections of this bundle. Concretely, this means that

$$\Theta_{\mathcal{V}}(gp)(u,v)w = g\Theta_{\mathcal{V}}(p)(dg^{-1}(gp)u, dg^{-1}(gp)v)g^{-1}w,$$
$$u \in T^*_{(1,0),gp}, \quad v \in T^*_{(0,1),gp}, \quad w \in \mathcal{V}_{gp}.$$

In particular, it follows that the curvature is determined by its value at a fixed reference point. We shall return to the Chern curvature of E below, and give an explicit formula for it at the point p_0. First, however, we shall undertake a closer study of the restriction of E to the diagonal.

The action of the group $U(N)$ on X defines a fibration $q : U(N) \to X$ given by $q(g) = g(p_0)$ which is $U(N)$-equivariant with respect to left multiplication, $L_g : x \mapsto gx$, on the group itself, and the action on X, i.e., $q(gx) = g(q(x))$ holds for $g, x \in U(N)$. Moreover, the right action $R_l : x \mapsto xl^{-1}$ of the subgroup $U(k) \times U(N-k)$ on $U(N)$ preserves each fiber $q^{-1}(p)$ for $p \in X$, and yields a diffeomorphism $U(k) \times U(N-k) \cong q^{-1}(p)$. This equips $U(N)$ with the structure of a principal $U(k) \times U(N-k)$-bundle over X. Since the right action of $U(k) \times U(N-k)$ commutes with left multiplication, the group $U(N)$ acts equivariantly with respect to the action of $U(k) \times U(N-k)$. Moreover, the embedding of $U(N)$ into $M_N(\mathbb{C})$ induces an Riemannian structure on $U(N)$ by restriction of the trace inner product $(A, B) \mapsto \operatorname{tr}(AB^*)$, and the left multiplication is isometric with respect to this inner product. For any $g \in U(N)$ with $q(g) = p$, we have an orthogonal decomposition

$$(23) \qquad T_g(U(N)) = T_g(q^{-1}(p)) \oplus T_g(q^{-1}(p))^{\perp},$$

and this decomposition is invariant under left multiplication. The restriction of the differential of q to the orthogonal complement $T_g(q^{-1}(p))^{\perp}$ yields an isomorphism

$$dq(g)|_{T_g(q^{-1}(p))^{\perp}} : T_g(q^{-1}(p))^{\perp} \to T_{q(g)}(X).$$

For any $p \in X$ we thus have a family of subspaces parametrized by the set $q^{-1}(p)$ to which the tangent space at p is isomorphic. We therefore define an equivalence relation on the tangent bundle $T(U(N))$ by

$$(24) \qquad (g, v) \sim (g', v') \quad \text{iff} \quad (g', v') = (R_l(g), dR_l(g)v),$$

for some $l \in U(k) \times U(N-k)$. By the isometry of the left multiplication, the orthogonal complement bundle $\cup_p T(q^{-1}(p))^{\perp}$ is a $U(N)$-homogeneous vector bundle. Moreover, for any vector in this subbundle, the whole equivalence class lies in the subbundle since also the right action is isometric. It follows that $S := \cup_p T(q^{-1}(p))^{\perp}/ \sim$ is a well-defined $U(N)$-homogeneous vector bundle over X. Clearly, S is equivalent to the tangent bundle $T(X)$, and thus it inherits a complex structure.

Proposition 14. *The restriction of E to the diagonal $\Delta(X \times X)$ is equivalent to the holomorphic cotangent bundle $T^*_{1,0}(X)$.*

Proof. We prove that E^* is equivalent to S. Since S is $U(N)$-homogeneous, it is uniquely determined by the corresponding representation of $U(k) \times U(N - k)$ on the fiber S_{p_0}. For the identity element $e \in U(N)$, the tangent space $T_e(U(N))$ is isomorphic to the Lie algebra

$$\mathfrak{u}(N) := \{X \in M_N(\mathbb{C}) | X^* = -X\},$$

and the subspaces in the decomposition (23) are explicitly given by

$$(25) \qquad T_e(q^{-1}(p_0)) \;=\; \left\{ \left(\begin{array}{cc} Y & 0 \\ 0 & Z \end{array} \right) \middle| Y^* = -Y, Z^* = -Z \right\},$$

$$(26) \qquad T_e(q^{-1}(p_0))^\perp \;=\; \left\{ \left(\begin{array}{cc} 0 & B \\ -B^* & 0 \end{array} \right) \middle| B \in M_{k,N-k}(\mathbb{C}) \right\}.$$

For $v = \left(\begin{array}{cc} 0 & B \\ -B^* & 0 \end{array} \right) \in T_e(q^{-1}(p_0))^\perp$, and $l = \left(\begin{array}{cc} A & 0 \\ 0 & D \end{array} \right) \in U(k) \times U(N - k)$,

$$
\begin{aligned}
dL_l(e)v \;&=\; \left(\begin{array}{cc} A & 0 \\ 0 & D \end{array} \right) \left(\begin{array}{cc} 0 & B \\ -B^* & 0 \end{array} \right) \\
&=\; \left(\begin{array}{cc} 0 & AB \\ -DB^* & 0 \end{array} \right).
\end{aligned}
$$

We can represent the equivalence class of this tangent vector by a tangent vector at the identity, namely by

$$
\begin{aligned}
dR_l(l)dL_l(e)v \;&=\; \left(\begin{array}{cc} 0 & AB \\ -DB^* & 0 \end{array} \right) \left(\begin{array}{cc} A^{-1} & 0 \\ 0 & D^{-1} \end{array} \right) \\
&=\; \left(\begin{array}{cc} 0 & ABD^{-1} \\ -(ABD^{-1})^* & 0 \end{array} \right).
\end{aligned}
$$

Hence, we can identify the representation of $U(k) \times U(N - k)$ on S_{p_0} with the representation on $M_{k,N}(\mathbb{C})$ given by $B \mapsto ABD^{-1}$, i.e., with the representation $\tau \otimes \rho^*$ on $\mathrm{Hom}(\mathbb{C}^{N-k}, \mathbb{C}^k)$, which is precisely the $U(k) \times U(N - k)$ representation associated to the restriction of E^* to the diagonal. $\square$

Remark 15. In [4], Berndtsson proves that any appropriate bundle $E \to X \times X$ has to coincide with the holomorphic cotangent bundle on the diagonal. In the case of $\mathbb{CP}^n$, an independent proof of Proposition 14 can be found in the book [7] by Demailly; Proposition 15.7 in Chapter V.

By the identification $T_{p_0}(X)$ with the subspace $T_e(q^{-1}(p_0))^\perp$ in (26), we have an explicit realization of its complexification

$$T_{p_0}(X)^{\mathbb{C}} \cong \left\{ \left(\begin{array}{cc} 0 & B \\ C & 0 \end{array} \right) \middle| B \in M_{k,N-k}(\mathbb{C}), C \in M_{N-k,k}(\mathbb{C}) \right\}.$$

Consider now the element $\left(\begin{array}{cc} i\frac{N-k}{N}I_k & 0 \\ 0 & -i\frac{k}{N}I_{N-k} \end{array} \right) \in T_e(q^{-1}(p_0)) \cong \mathfrak{u}(k) \times \mathfrak{u}(N - k)$. Its adjoint action determines the complex structure, J_{p_0}, at p_0 by

$$J_{p_0} \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} := \left[\begin{pmatrix} i\frac{N-k}{N}I_k & 0 \\ 0 & -i\frac{k}{N}I_{N-k} \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \right]$$
$$= \begin{pmatrix} 0 & iB \\ -(iB)^* & 0 \end{pmatrix}.$$

The splitting of $T_{p_0}^{\mathbb{C}}$ into the the $\pm i$-eigenspaces is given by

$$T_{(1,0),p_0}(X) \cong \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \Big| Y \in M_{k,N-k}(\mathbb{C}) \right\},$$
$$T_{(0,1),p_0}(X) \cong \left\{ \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} \Big| Z \in M_{N-k,k}(\mathbb{C}) \right\}.$$

We recall that the curvature Θ_E at the point p_0 is given by the formula

$$(27) \qquad \Theta_E(p_0)(Y,Z)W = (\rho \otimes \tau^*)'([Y,Z])(W),$$

where $(\rho \otimes \tau^*)'$ denotes the differentiated representation of the Lie algebra $\mathfrak{u}(k) \times \mathfrak{u}(N-k)$ given by $(\rho \otimes \tau^*)'(X) := \frac{d}{dt}(\rho \otimes \tau^*)(\exp tX)|_{t=0}$. The explicit expression for (27) is

$$\Theta_E(p_0)(Y,Z)W = (\rho \otimes \tau^*)' \begin{pmatrix} YZ & 0 \\ 0 & ZY \end{pmatrix}(W)$$
$$= ZYW - WYZ, \quad W \in M_{N-k,k}(\mathbb{C}).$$

5.3. **Invariance of weights.** In this section we study a natural action of $U(N)$ on sections of the bundles $\mathrm{Hom}(L_\zeta^r, L_z^r) \otimes G_E$, and prove that the corresponding weights are invariant under that action.

Recall that for an action of a group, G, on a vector bundle $\mathcal{V} \to M$, a natural action is induced on the space of sections by

$$(28) \qquad\qquad\qquad (gs)(z) := gs(g^{-1}z),$$

where the second action on the right hand side refers to the action on the total space of the bundle. The bundles $\mathrm{Hom}(L_\zeta^r, L_z^r) \otimes G_E$ are equipped with the natural $U(N) \times U(N)$ actions given as tensor (and exterior) products of the actions described in the previous section and their duals. In what follows, we will consider the action of $U(N)$ (embedded as the diagonal subgroup of $U(N) \times U(N)$) given by restriction. The actions on the respective total spaces are the obvious ones, and we will therefore use the simple notation from (28) for such an action.

We let $g^r := g^{\otimes r}$ for $r \geq 0$ and $g^r := (g*)^{\otimes r}$ for $r \leq 0$ denote the weight for the line bundle L^r.

Proposition 16. *The weight g^r is a $U(N)$-invariant section of the vector bundle $\mathrm{Hom}(L_\zeta^r, L_z^r) \otimes G_E$.*

Proof. It clearly suffices to prove that the section $G = \gamma_0 + \gamma_1$ is an invariant section of $\mathrm{Hom}(H_\zeta, H_\zeta)$; and for this, we prove that γ_0 and γ_1 are invariant separately. We now fix an orthonormal basis, $\{h_1, \ldots, h_k\}$, for H_z. For any $u \in H_\zeta$ and $l \in U(N)$, we have

$$
\begin{aligned}
(l\gamma_0)(u) = l\gamma_0(l^{-1}u) &= l \sum_{i=1}^{k} \langle l^{-1}u, l^{-1}h_i \rangle l^{-1}h_i \\
&= \sum_{i=1}^{k} \langle u, h_i \rangle h_i \\
&= \gamma_0(u),
\end{aligned}
$$

which shows the invariance of γ_0. We now consider γ_1, and therefore choose a local section f of F near the point $z \in X$. Then, we have

$$
\begin{aligned}
(l\gamma_1)(f \otimes u) &= -l(\pi_{H_{l^{-1}z}}(\bar{\partial}\varphi(l^{-1}f)) \otimes l^{-1}u) \\
&= -l(\pi_{H_{l^{-1}z}}(l^{-1}\bar{\partial}\varphi(f)) \otimes l^{-1}u) \\
&= -\pi_{H_z}(\bar{\partial}\varphi(f) \otimes u) \\
&= \gamma_1(f \otimes u),
\end{aligned}
$$

where the third equality is completely analogous to the invariance of γ_0. This concludes the proof. $\qquad\square$

We now turn our attention to the form P_{g^r} defined in (12) again.

Corollary 17. *The form P_{g^r} is $U(N)$-invariant.*

Proof. First of all, an argument similar to the proof of Proposition 16 shows that the section η is $U(N)$-invariant. Secondly, the Chern connection D_E on E commutes with the $U(N)$-action, and hence $D\eta$ is also $U(N)$-invariant. The curvature Θ is even $U(N) \times U(N)$-invariant; and hence it follows that the form $g \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n$ is $U(N)$-invariant. We now claim that the operator $\int_E$ is $U(N)$-equivariant. Indeed, the identity section $I \in \mathrm{End}(E)$ is obviously $U(N)$-invariant, and so is therefore also the section $\tilde{I}_n$ defined in connection with Definition 2. Hence, $\int_E$ is an equivariant operator, and this also finishes the proof. $\qquad\square$

The canonical splitting $T^*(X \times X) \cong T_z^*(X) \oplus T_\zeta^*(X)$ of the cotangent bundle of $X \otimes X$ is $U(N) \times U(N)$-invariant, and hence (P_{g^r}) can be decomposed as

$$
\tag{29}
(P_{g^r}) = \sum_{\substack{p'+p''=n \\ q'+q''=n}} (P_{g^r})_{p',p'',q',q''},
$$

where $(P_{g^r})_{p',p'',q',q''}$ is a section of $\mathrm{Hom}(H_\zeta, H_z) \otimes \Lambda^{p',q'}(T_z^{\mathbb{C}})^* \wedge \Lambda^{p'',q''}(T_\zeta^{\mathbb{C}})^*$, i.e., it is of bidegree (p', q') in the z-variable, and of bidegree (p'', q'') in the

ζ-variable according to the splitting. By the invariance of the splitting, we also have

Corollary 18. *The terms $(P_{g^r})_{p',p'',q',q''}$ in the decomposition (29) are $U(N)$-invariant.*

Only the term $(P_{g^r})_{n,0,n,0}$ which has bidegree (n,n) in the z-variable will contribute to the integral in the Koppelman formula. Later we will examine this term more closely.

Corollary 19. *The current K_{g^r} in (12) is $U(N)$-invariant.*

Proof. It clearly suffices to prove that u in (7) is $U(N)$-invariant; and since the group action commutes with the $\bar{\partial}$-operator and exterior powers, it only remains to prove the invariance of σ. Note that σ can be described by the equation

$$\sigma(v) = \frac{\langle v, \eta \rangle_E}{|\eta|_E^2}, \quad v \in E.$$

The invariance of σ now follows immediately from the invariance of η and from the fact that the action of $U(N)$ preserves the metric. $\qquad\square$

5.4. **Line bundles on X.** In this subsection we recapitulate how the Picard group of X can be described in terms of holomorphic characters. All of this is classical theory and well-known, even though the results in their explicit form can be hard to find in the literature. The reason for including it in the paper is rather to give an overview for readers who are not familiar with representation theory of Lie groups.

Suppose now that $\mathcal{L} \to X$ is a $SL(N, \mathbb{C})$-homogenous holomorphic line bundle. The corresponding P'-representation then amounts to a holomorphic character $\chi_{\mathcal{L}} : P' \to \mathbb{C}^*$. Moreover, it is well-known that all holomorphic line bundles over X are in fact $SL(N, \mathbb{C})$-homogeneous (cf. [18]), and hence the Picard group $H^1(X, \mathcal{O}^*)$ is isomorphic to the multiplicative group of holomorphic characters of P'.

Suppose now first that $\chi \colon P \to \mathbb{C}^*$ is a holomorphic character. (This is no restriction, as we shall later see that all holomorphic characters of P' are restrictions of P-characters.) It is well-known that it is then uniquely determined by its restriction to the Levi-subgroup $GL(k) \times GL(N-k)$ realized as

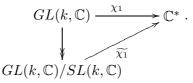$$\left\{ \left(\begin{array}{cc} A & 0 \\ 0 & D \end{array} \right) \middle| \det A \det D \neq 0 \right\}.$$

By restricting to the respective factors, we can uniquely express χ as a product $\chi = \chi_1 \chi_2$, where χ_1 and χ_2 are characters of $GL(k, \mathbb{C})$ and $GL(N-k, \mathbb{C})$ respectively. Let $\chi_1' : \mathfrak{gl}(k, \mathbb{C}) \to \mathbb{C}$ denote the differential at the identity of χ_1. Then χ_1' annihilates the commutator ideal in the decomposition

$$\mathfrak{gl}(k, \mathbb{C}) = \mathfrak{z}(\mathfrak{gl}(k, \mathbb{C})) \oplus [\mathfrak{gl}(k, \mathbb{C}), \mathfrak{gl}(k, \mathbb{C})]$$
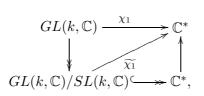
of $\mathfrak{gl}(k,\mathbb{C})$ as the direct sum of the center and the commutator. More specifically, we have the identity

$$[\mathfrak{gl}(k,\mathbb{C}),\mathfrak{gl}(k,\mathbb{C})] = \mathfrak{sl}(k,\mathbb{C}),$$

from which it follows that the normal subgroup $SL(k,\mathbb{C})$ lies in the kernel of the character χ_1. Hence, χ_1 descends to a character, $\widetilde{\chi_1}$, of the quotient group $GL(k,\mathbb{C})/SL(k,\mathbb{C})$, yielding the commuting diagram

$$
\begin{array}{ccc}
GL(k,\mathbb{C}) & \xrightarrow{\ \chi_1\ } & \mathbb{C}^* \ . \\
\downarrow & \nearrow{\scriptstyle \widetilde{\chi_1}} & \\
GL(k,\mathbb{C})/SL(k,\mathbb{C}) & &
\end{array}
$$

Moreover, the quotient group is isomorphic to $\mathbb{C}^*$ via the mapping $gSL(k,\mathbb{C}) \mapsto \det g$, and hence we have the diagram

$$
\begin{array}{ccc}
GL(k,\mathbb{C}) & \xrightarrow{\ \chi_1\ } & \mathbb{C}^* \\
\downarrow & \nearrow{\scriptstyle \widetilde{\chi_1}} & \uparrow \\
GL(k,\mathbb{C})/SL(k,\mathbb{C}) & \hookrightarrow & \mathbb{C}^*,
\end{array}
$$

which allows us to identify $\widetilde{\chi_1}$ with a holomorphic character $\mathbb{C}^* \to \mathbb{C}^*$. The latter ones are easily described. Indeed, by holomorphy, any such character is uniquely determined by its restriction to the totally real subgroup $S^1 \subset \mathbb{C}^*$, on which it gives a character $S^1 \to S^1$. Hence, it is of the form $\zeta \mapsto \zeta^m$, for some integer m. The analogous result holds of course for χ_2. Summing up, we have thus found that

$$\chi \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) = \det A^m \det D^n,$$

for some $m,n \in \mathbb{Z}$.

The line bundle corresponding to the choice $m = 1, n = 0$ is the determinant of the tautological vector bundle. To study the line bundle corresponding to the parameters $m = 0, n = 1$, we consider it as a $SL(N,\mathbb{C})$-homogeneous line bundle, which amounts to restricting the corresponding character to the subgroup P' of P. We let χ' denote the differential at the identity of this character. The Lie algebra $\mathfrak{p}'$ admits a decomposition

$$\mathfrak{p}' = \mathfrak{z}(\mathfrak{p}') \oplus \left[\mathfrak{p}', \mathfrak{p}' \right]$$

as the direct sum of its center and its commutator ideal. These two ideals are given by

$$\mathfrak{z}(\mathfrak{p}') = \left\{ \left(\begin{array}{cc} c(N-k)I_k & 0 \\ 0 & -ckI_{N-k} \end{array} \right) \middle| c \in \mathbb{C} \right\},$$

$$[\mathfrak{p}', \mathfrak{p}'] = \left\{ \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \middle| \mathrm{tr}A = \mathrm{tr}D = 0 \right\}.$$
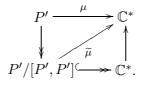
On the group level, we have the commutator subgroup

$$[P', P'] = \left\{ \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \middle| \det A = \det D = 1 \right\},$$

and the quotient group $P'/[P', P']$ has complex dimension one. In fact, an isomorphism $\Phi : P'/[P', P'] \to \mathbb{C}^*$ is given by

$$\Phi(g\,[P', P']) = \det A,$$

for $g = \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right)$.

If $\mu : P' \to \mathbb{C}^*$ is a holomorphic character, it factors through the projection onto the quotient group just as above, yielding a holomorphic character $\tilde{\mu} : P'/[P', P'] \to \mathbb{C}^*$. Using the isomorphism Φ above, we obtain the commuting diagram

$$
\begin{array}{ccc}
P' & \xrightarrow{\mu} & \mathbb{C}^* \\
\downarrow & \nearrow_{\tilde{\mu}} & \uparrow \\
P'/[P', P'] & \hookrightarrow & \mathbb{C}^*.
\end{array}
$$

From this, we conclude that $\mu \left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) = \det A^j$, for some $j \in \mathbb{Z}$. In particular, it follows that μ can naturally be extended to a holomorphic character $P \to \mathbb{C}^*$. Moreover, the dual bundle to the determinant of the tautological vector bundle corresponds to the P'-character $\left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \mapsto \det A^{-1} = \det D$, which can be extended to the P-character $\left(\begin{array}{cc} A & B \\ 0 & D \end{array} \right) \mapsto \det D$. It is easy to see that the $GL(N, \mathbb{C})$-homogeneous line bundle associated with this holomorphic character is isomorphic to the determinant of the quotient bundle $F = \mathbb{C}^N/H$.

5.5. **The Bott-Borel-Weil theorem.** In this subsection we briefly describe some group representations associated with homogeneous vector bundles.

Suppose now that G is a complex Lie group acting transitively and holomorphically on a complex manifold M, so that we can write $M \cong G/T$ for some closed subgroup $T \subseteq G$. Let $\mathcal{V} \to M$ be a G-homogeneous holomorphic vector bundle. Recall that the action of G on $\mathcal{V}$ induces the action on smooth sections given by (28). Since G acts holomorphically on M, there is a natural action on $\mathcal{V}$-valued (p, q)-forms (by taking the pullback composed

with inversion). Moreover, the action commutes with the $\overline{\partial}$-operator on $\mathcal{V}$, from which it follows that the action preserves closed forms and exact form; thus inducing an action on the Dolbeault cohomology groups $H^{p,q}(M, \mathcal{V})$. In the case when G is a complexification of some semisimple compact Lie group, $G_{\mathbb{R}}$, the Bott-Borel-Weil theorem (cf. [2], Theorem. 5.0.1) gives a realization of all irreducible representations of $G_{\mathbb{R}}$ as $H^{0,q}(M, \mathcal{L})$ for some homogeneous line bundle, $\mathcal{L}$, over M, and also states the vanishing of the other sheaf cohomology groups associated with $\mathcal{L}$. We shall see examples of it in the context of the vanishing theorems of the next section.

6. Applications

6.1. **Vanishing theorems.** We would like to find vanishing theorems for the bundles L^r and L^{-r} over X by means of the Koppelman formula. This will yield explicit solutions to the $\bar{\partial}$-equation in the cohomology groups which are trivial.

Let D in Theorem 1 be the whole of X, and let $\phi(\zeta)$ be a $\bar{\partial}$-closed form of bidegree (p,q) taking values in L_ζ^r, with $r > 0$. The only obstruction to solving the $\bar{\partial}$-equation is then the term $\int_\zeta \phi(\zeta) \wedge P_{g^r}(\zeta, z)$. We have

$$
(30) \qquad P_{g^r} = \int_E g^r \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}}{2\pi} \right)_n =
$$
$$
= \int_E \sum_{j=1}^{\min(kr,n)} C_j(g^r)_{j,j} \wedge (D\eta)^j \wedge (\tilde{\Theta}_\zeta + \tilde{\Theta}_z)^{n-j}
$$

where $(g^r)_{j,j}$ is the term in g^r which has bidegree $(0,j)$ and takes values in $\Lambda^j E^*$. Note that all the differentials in g are in the z variable; this is because $\bar{\partial}_\zeta$ commutes with π_{H_z}.

Theorem 20. *The cohomology groups $H^{p,q}(X, L^r)$ are trivial in the following cases:*

    a) *$p \neq q$ and $r = 0$.*
    b) *$p > q$ and $r > 0$.*
    c) *$p < q$, $rk < q - p$, and $r > 0$.*
    d) *$p < q$ and $r < 0$.*
    e) *$p > q$, $rk < p - q$, and $r < 0$.*

Proof. a) If $r = 0$ we do not need a weight, and in that case

$$
P = \int_E \left(\frac{i\tilde{\Theta}}{2\pi} \right)_n = c_n(E),
$$

or the n:th Chern form of E. It is obvious that P consists of terms with bidegree (k,k) in z and $(n-k, n-k)$ in ζ, and thus $\int \phi \wedge P = 0$ if ϕ has bidegree (p,q) with $p \neq q$.

b) Since the only source of antiholomorphic differentials in ζ is $\tilde{\Theta}_\zeta$, which is a $(1,1)$-form, we can never get more $d\bar{\zeta}_i$:s than $d\zeta_i$:s. This means that

$\int_\zeta \phi(\zeta) \wedge P_{g^r} = 0$ if ϕ has bidegree (p,q) where $p > q$ (since then P_{g^r} would need to have bidegree $(n-p, n-q)$ in ζ with $n-q > n-p$).

c) If $\phi(\zeta)$ has bidegree (p,q), then P_{g^r} needs to have bidegree (p,q) in z. We can take at most p of the $\tilde{\Theta}_z$:s. We will then need at least $q - p$ more $d\bar{z}_i$:s, and these have to come from the factor g^r. But g^r has maximal bidegree $(0, rk)$, so if $rk < q - p$ the obstruction will vanish.

d) By duality, if we have a (p,q)-form ϕ taking values in L^r with $r < 0$, the obstruction is given by $\int_z \phi(z) \wedge P_{g^{-r}}(\zeta, z)$. This is zero unless there is a term in $P_{g^{-r}}$ of bidegree (p,q) in ζ. By the same argument as in the proof of b), the obstruction vanishes if $q > p$.

e) If $\phi(z)$ has bidegree (p,q), then $P_{g^{-r}}$ needs to have bidegree $(n-p, n-q)$ in z, where $n - q > n - p$. The rest follows as in the proof of c).   □

Remark 21. In $\mathbb{CP}^n$, we can get rid of the obstruction in more cases, either by proving that P_{g^r} is $\bar{\partial}_\zeta$-exact (since then Stokes' theorem can be applied), or by proving that it is $\bar{\partial}_z$-exact (since then $\int_\zeta \phi \wedge P_{g^r}$ will be $\bar{\partial}_z$-exact as well). See [8] for details.

Part d) of the above theorem is the special case of the Bott-Borel-Weil theorem for the parabolic quotient $GL(N, \mathbb{C})/P$. For $r = -1$, all vanishing theorems were proved by le Potier in [13]. He also proved vanishing theorems for exterior and symmetric powers of the tautological bundle and its dual. In [17], Snow gives an algorithm for computing all Dolbeault cohomology groups for all line bundles over Grassmannians. Implementing the algorithm in a computer, Snow obtains various vanishing theorems including ours. It is worth noting that both le Potier and Snow obtain their results by reduction to the Bott-Borel-Weil theorem.

6.2. **Bergman kernels.** We will see that the projection part, P_{g^r}, of our Koppelman formula for L^r basically is the Bergman kernel associated with the space of holomorphic sections of L^{-r}. We begin by examining P_{g^r}. Recall that

$$g^r = \left((\gamma_0 + \gamma_1)^k\right)^{\otimes r} = \Big(\sum_{j=0}^{k} \binom{k}{j} \gamma_0^{k-j} \wedge \gamma_1^j \Big)^{\otimes r} =: (\gamma_0^k)^{\otimes r} + \tilde{g}^r,$$

where γ_0^k of course is the kth exterior power of γ_0, is our weight for L^r. The projection kernel in our Koppelman formula for L^r is thus

$$
\begin{aligned}
P_{g^r} &= \int_E g^r \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}_E}{2\pi} \right)_n \\
&= (\gamma_0^k)^{\otimes r} \otimes \int_E \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}_E}{2\pi} \right)_n + \int_E \tilde{g}^r \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\tilde{\Theta}_E}{2\pi} \right)_n \\
&=: P_{g^r}^0 + \tilde{P}_{g^r}.
\end{aligned}
$$

Let $\mathcal{P}_{g^r}$ and $\mathcal{P}_{g^r}^0$ be the parts of P_{g^r} and $P_{g^r}^0$ respectivly, which have bidegree (n, n) in the z-variables. Let us examine $\mathcal{P}_{g^r}$ and $\mathcal{P}_{g^r}^0$ more closely on the set $Z := \{p_0\} \times X_\zeta$. In our local coordinates and frames over $U_z \times U_\zeta$ we have by (20) that $\gamma_0 = (I + z^*z)^{-1}(I + z^*\zeta)$. On Z intersected with $\{p_0\} \times U_\zeta$, denoted Z' below, we thus have $\gamma_0 = I$ expressed in our frames. According to (21) and (22), we see that, as a matrix in our frames for H_z and H_ζ, $\gamma_1 = dz^* \wedge \mathfrak{e}^*$ on Z'. Moreover, a straightforward computation shows that the part of $D\eta$, which does not contain any differentials in the ζ-variables, equals $-\sum_{i,j} dz_{ij} \wedge \mathfrak{e}_{ij}$ on Z'. Also, the part of $\tilde{\Theta}_E$, which does not contain any differentials in the ζ-variables, is $\widetilde{\Theta_{F_z} \otimes \operatorname{Id}}_{H_\zeta^*}$. We thus see that the building blocks for $\mathcal{P}_{g^r}$ and $\mathcal{P}_{g^r}^0$ are independent of ζ on Z' when expressed in our frames. Since both $\mathcal{P}_{g^r}$ and $\mathcal{P}_{g^r}^0$ take values in a line bundle we must have $\mathcal{P}_{g^r} = C\mathcal{P}_{g^r}^0$ on Z'. But Z' is dense in Z and so this equality holds on Z by continuity. Now, by Corollary 18 in Subsection 5.3, it follows that both $\mathcal{P}_{g^r}$ and $\mathcal{P}_{g^r}^0$ are invariant under the diagonal group in $U(N) \times U(N)$ and since Z intersects each orbit under this group we can conclude that $\mathcal{P}_{g^r} = C\mathcal{P}_{g^r}^0$ on all of $X \times X$.

Given any holomorphic section f of L^{-r}, $r > 0$, and any vector v_p in the fiber of L^r above an arbitrary point p, our Koppelman formula now gives

$$(31) \quad f(p).v_p = \int_{X_z} \mathcal{P}_{g^r}(z, p) \wedge v_p \wedge f(z) = C \int_{X_z} \mathcal{P}_{g^r}^0(z, p) \wedge v_p \wedge f(z).$$

It is easy to compute $\mathcal{P}_{g^r}^0$ explicitly, and one gets

$$\mathcal{P}_{g^r}^0 = \left(\frac{i}{2\pi}\right)^n (\gamma_0^k)^{\otimes r} \otimes \int_E \left(\widetilde{\Theta_{F_z} \otimes \operatorname{Id}}_{H_\zeta^*}\right)_n = \left(\frac{i}{2\pi}\right)^n (\gamma_0^k)^{\otimes r} \otimes c_{N-k}(\Theta_{F_z})^k.$$

Moreover, Θ_{F_z} is the $U(N)$-invariant curvature of F_z, so it follows that $c_{N-k}(\Theta_{F_z})^k$ is a $U(N)$-invariant (n, n)-form and hence equal to a constant times the invariant volume form dV. We have thus obtained

$$(32) \quad f(p).v_p = C \int_{X_z} f(z).(\gamma_0^k)^{\otimes r} v_p dV(z)$$

for any holomorphic section f of L^{-r}. Modulo a multiplicative constant, one also has that $dV = (c_1(L))^n$, and then the above formula assumes the following form expressed in the frames and coordinates discussed above.

$$f(\zeta) = C \int_{\mathbb{C}^n} f(z) \frac{\det(I + z^*\zeta)^r}{\det(I + z^*z)^r} \left((\partial\bar{\partial} \log \det(I + z^*z))\right)^n.$$

We will now describe what will be the Bergman kernel. Let $\rho^r : L_z^r \to L_z^{-r}$ be the antilinear identification induced by the metric, i.e., $\rho^r(v) = \langle \cdot, v \rangle_{L_z^r}$, and define $K_r(z, \zeta) : L_\zeta^r \to L_z^{-r}$ by $K_r(z, \zeta) = \rho^r \circ (\gamma_0^k)^{\otimes r}$. Then one easily checks that $K_r(z, \zeta)$ is a fiberwise antilinear map, which depends antiholomorphically on ζ. To show that it actually depends holomorphically on z we consider the adjoint operator $K_r(z, \zeta)^* : L_z^r \to L_\zeta^{-r}$ and the operator

$K_r(\zeta, z) \colon L_z^r \to L_\zeta^{-r}$. We know that the latter operator depends antiholo-
morphically on z. Note also that since $K_r(z, \zeta)$ is fiberwise antilinear, the
adjoint should be defined by $(K_r(z, \zeta)^* u).v = \overline{u.(K_r(z, \zeta)v)}$ for $u \in L_z^r$ and
$v \in L_\zeta^r$. It is then straightforward to check that $K_r(z, \zeta)^* = K_r(\zeta, z)$, and
so $K_r(z, \zeta)^*$ must depend antiholomorphically on z. It follows that $K_r(z, \zeta)$
depends holomorphically on z. In particular, for any non-zero vector $v \in L_p^r$,
the mapping $z \mapsto K_r(z, p)v$ defines a global non-zero holomorphic section of
L^{-r}. In fact, these sections generate $H^0(X, L^{-r})$ as we now show. Consider
the Bergman space A_r^2 defined as $H^0(X, L^{-r})$ equipped with the norm

$$\|f\|_{A_r^2}^2 := \int_X \|f\|_{L^{-r}}^2 dV, \ \ f \in H^0(X, L^{-r}).$$

We claim that, modulo a multiplicative constant, $K_r(z, \zeta)$ is the Bergman
kernel for A_r^2, i.e., that $K_r(z, \zeta)$ is the fiberwise antilinear map $L_\zeta^r \to L_z^{-r}$,
which depends holomorphically on z and antiholomorphically on ζ, and has
the property that for any $f \in A_r^2$ and any vector $v \in L_\zeta^r$ (in the fiber above
ζ) one has

$$f(\zeta).v = \langle f, K_r(\cdot, \zeta)v \rangle_{A_r^2} = \int_X \langle f(z), K_r(z, \zeta)v \rangle_{L_z^{-r}} dV(z).$$

It only remains to verify this last property. But this reproducing property
follows directly from (32) after noting the following equality, which basically
is the definition of $K_r(z, \zeta)$:

$$u.((\gamma_0^k)^{\otimes r}v) = \langle u, K_r(z, \zeta)v \rangle_{L_z^{-r}}, \ \text{ for all } u \in L_z^{-r}, \text{ and all } v \in L_\zeta^r.$$

Remark 22. In the case of $\mathbb{CP}^n$ it is not too hard to compute $\mathcal{P}_{g^r}$ directly
from its definition. For instance, one can first verify in local or homogeneous
coordinates that the part of $\gamma_1 \wedge D\eta$ which contains no $d\zeta$ or $d\bar{\zeta}$ is equal to
$-\gamma_0 \otimes \widetilde{\Theta_{F_z} \otimes \mathrm{Id}}_{\mathcal{O}(1)_\zeta}$, cf. Proposition 4.1 and the weight α in [8]. Then, a
straightforward computation shows that $\mathcal{P}_{g^r}$ is equal to

$$\binom{n+r}{n} \left(\frac{i}{2\pi}\right)^n \gamma_0^r \otimes \det(\Theta_{F_z}).$$

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