Linear convergence of iterative soft-thresholding

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ABSTRACT. In this article a unified approach to iterative soft-thresholding algorithms for the solution of linear operator equations in infinite dimensional Hilbert spaces is presented. We formulate the algorithm in the framework of generalized gradient methods and present a new convergence analysis. As main result we show that the algorithm converges with linear rate as soon as the underlying operator satisfies the so-called finite basis injectivity property or the minimizer possesses a so-called strict sparsity pattern. Moreover it is shown that the constants can be calculated explicitly in special cases (i.e. for compact operators). Furthermore, the techniques also can be used to establish linear convergence for related methods such as the iterative thresholding algorithm for joint sparsity and the accelerated gradient projection method.

1. Introduction

This paper is concerned with the convergence analysis of numerical algorithms for the solution of linear inverse problems in the infinite-dimensional setting with so-called sparsity constraints. The background for this type of problem is, for example, the attempt to solve the linear operator equation $Ku = f$ in an infinite-dimensional Hilbert space which models the connection between some quantity of interest u and some measurements f . Often, the measurements f contain noise which makes the direct inversion ill-posed and practically impossible. Thus, instead of considering the linear equation, a regularized problem is posed for which the solution is stable with respect to noise. A common approach is to regularize by minimizing a Tikhonov functional [7, 15, 28]. A special class of these regularizations has been of recent interest, namely of the type

$$
\min_{u \in \ell^2} \frac{\|Ku - f\|^2}{2} + \sum_{k=1}^{\infty} \alpha_k |u_k| \tag{1.1}
$$

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These problems model the fact that the quantity of interest u is composed of a few elements, i.e. it is sparse in some given, countable basis. To make this precise, let $A:\mathcal{H}_{1}\rightarrow \mathcal{H}_{2}$ be a bounded operator between two Hilbert spaces and let $\{\psi_k\}$ be an orthonormal basis of \mathcal{H}_1 . Denote with $B: \ell^2 \to \mathcal{H}_1$ the synthesis operator $B(u_k) = \sum_k u_k \psi_k$. Then the problem

$$
\min_{u\in\mathcal{H}_1}\frac{\|Au-f\|^2}{2}+\sum_{k=1}^\infty\alpha_k|\langle u,\,\psi_k\rangle|
$$

can be rephrased as [\(1.1\)](#page-0-0) with $K = AB$. Indeed, solutions of this type of problem admit only finitely many non-zero coefficients and often coincide with the sparsest solution possible [10, 18, 20].

Unfortunately, the numerical solution of the above (non-smooth) minimization problem is not straightforward. There is a vast amount of literature dealing with efficient computational algorithms for equivalent formulations of the problem [8, 12, 14, 16, 21, 22, 27, 33], both in the infinite-dimensional setting as well as for finitely many dimensions, but mostly for the finitedimensional case. An often-used, simple but apparently slow algorithm is the iterative soft-thresholding (or thresholded Landweber) procedure which is known to converge in the strong sense in infinite dimensions [7]. The algorithm is simple: it just needs an initial value $u⁰$ and an operator with $\|K\|$ < 1. The iteration reads as

$$
u^{n+1} = \mathbf{S}_{\alpha}(u^n - K^*(Ku^n - f)), \quad (\mathbf{S}_{\alpha}(w))_k = \text{sgn}(w_k)[|w_k| - \alpha_k]_+.
$$

In practice it is important to know moreover convergence rates for the algorithms or at least an estimate for the distance to a minimizer to evaluate the fidelity of the outcome of the computations. The convergence proofs in the infinite-dimensional case presented in [7], and for generalizations in [5], however, do not imply a-priori estimates and do not inherently give any rate of convergence, although, in many cases, linear convergence can be deduced quite easily from the fact that iterative thresholding converges strongly and from the special structure of the algorithm. To the best knowledge of the authors, [3] contains the first results about the convergence of iterative algorithms for linear inverse problems with sparsity constraints in infinite dimensions for which the convergence rate is inherent in the respective proof. There, an iterative hard-thresholding procedure has been proposed for which, if K is injective, a convergence rate of $\mathcal{O}(n^{-1/2})$ could be established.

The main purpose of this paper is to develop a general and unified framework for the convergence analysis of algorithms for the problem [\(1.1\)](#page-0-0) and related problems, especially for the iterative soft-thresholding algorithm. We show that the iterative soft-thresholding algorithm converges linearly in almost every case and point out how to obtain a-priori estimates. To this end, we formulate the iterative soft-thresholding as a generalized gradient projection method which leads to a new proof for the strong convergence which is independent of the proof given in [7]. The techniques used for our approach may shed new light on the known properties of the iterative soft-thresholding related methods.

We distinguish two key properties which lead to linear convergence. The first is called *finite basis injectivity* (FBI) and is a property of the operator K only, while the second is called a *strict sparsity pattern* of a solution of the minimization problem [\(1.1\)](#page-0-0).

Definition 1. An operator $K: \ell^2 \to \mathcal{H}_2$ mapping into a Hilbert space has the *finite basis injectivity* property, if for all finite subsets $I \subset \mathbb{N}$ the operator $K|_I$ is injective, i.e. for all $u, v \in \ell^2$ with $Ku = Kv$ and $u_k = v_k = 0$ for all $k \notin I$ it follows $u = v$.

Definition 2. A solution u^* of (1.1) possesses a *strict sparsity pattern* if whenever $u_k^* = 0$ for some k there follows $|K^*(Ku^* - f)|_k < \alpha_k$.

The main result can be summarized by the following:

Theorem 1. Let $K: \ell^2 \to \mathcal{H}_2$, $K \neq 0$ be a linear and continuous operator as well as $f \in H_2$. Consider the sequence $\{u^n\}$ given by the iterative soft*thresholding procedure*

$$
u^{n+1} = \mathbf{S}_{s_n\alpha}(u^n - s_n K^*(Ku^n - f)), \quad \left(\mathbf{S}_{s_n\alpha}(w)\right)_k = \text{sgn}(w_k) \left[|w_k| - s_n \alpha_k\right]_+\tag{1.2}
$$

with step size

$$
0 < \underline{s} \le s_n \le \overline{s} < 2 / \|K\|^2 \tag{1.3}
$$

and a $u^0 \in \ell^2$ such that $\sum_{k=1}^{\infty} \alpha_k |u_k^0| < \infty$. Then, there is a minimizer u^* such that $u^n \to u^*$ in ℓ^2 .

Moreover, suppose that either

- 1. K *possesses the FBI property, or*
- 2. u [∗] *possesses a strict sparsity pattern.*

Then, $u^n \rightarrow u^*$ *with a linear rate, i.e. there exists a* $C > 0$ *and a* $0 \leq \lambda < 1$ *such that* $||u^n - u^*|| \leq C\lambda^n$.

Remark 1 (Examples for operators with the FBI property). In the context of inverse problems with sparsity constraints, the FBI property is natural, since the operators A are often injective. Prominent examples are the Radon transform [25], solution operators for partial differential equations, e.g. in heat conduction problems [6] or inverse boundary value problems like electrical impedance tomography [26]. The combination with a synthesis operator B for an orthonormal basis does not influence the injectivity.

Moreover, the restriction to orthonormal bases can be relaxed. The results presented in this paper also hold if the system $\{\psi_k\}$ is a frame or even a dictionary—as long as the FBI property is fulfilled. This is for example the case for a frame which consists of two orthonormal bases where no element of one basis can be written as a finite linear combination of elements of the other. This is typically the case, e.g. for a trigonometric basis and the Haar wavelet basis on a compact interval. One could speak of FBI frames or FBI dictionaries.

Remark 2 (Strict sparsity pattern). This condition can be interpreted as follows. We know that the weighted ℓ^1 -regularization imposes sparsity on a solution u^* in the sense that $u^*_k = 0$ for all but finitely many k , hence the name sparsity constraint. For the remaining indices, the equations $(K^*Ku^*)_k = K^*f - \alpha_k \operatorname{sgn}(u_k^*)$ are satisfied which corresponds to an approximate solution of the generally ill-posed equation $Ku = f$ in a certain way. Now the condition that the solutions of (1.1) possess a strict sparsity pattern says that $u_k^* = 0$ for some index k can occur only because of the sparsity constraint but never for the solution of the linear equation. We emphasize that Theorem [1](#page-2-0) states that whenever $\{u^n\}$ converges to a solution u^* with strict sparsity pattern, then the speed of convergence has to be linear for all bounded linear operators K.

The proof of Theorem [1](#page-2-0) will be divided into three sections. First, in Section [2,](#page-3-0) we introduce a framework in which iterative soft-thresholding according to [\(1.2\)](#page-2-1) can be interpreted as a generalized gradient projection method. We derive descent properties for generalized gradient methods and show under which conditions we can obtain linear convergence in Section [3.](#page-5-0) We show in Section [4](#page-11-0) that a Bregman-distance estimate for problems of the type [\(1.1\)](#page-0-0) gives a new convergence proof for the iterative soft-thresholding. In Section [5](#page-16-0) we illustrate the broad range of applicability of the results with two more examples. Finally, some conclusions about the implications of the results are drawn in Section [6.](#page-20-0)

2. Iterative soft-thresholding and a generalized gradient projection method

A common approach to solve smooth unconstrained minimization problems are methods based on moving in the direction of steepest descent, i.e. the negative gradient. In constrained optimization, the gradient is often projected back to the feasible set, yielding the well-known gradient projection algorithm method $[11, 19, 23]$. In the following, a step of generalization is introduced: The method is extended to deal with sums of smooth and nonsmooth functionals, and covers in particular constrained smooth minimization problems. The gain is that the iteration [\(1.2\)](#page-2-1) fits into this generalized framework.

Similar to the generalization performed in [4], its main idea is to replace the constraint by a general proper, convex and lower semi-continuous functional Φ which leads, for the gradient projection method, to the successive application of the associated proximity operators, i.e.

$$
J_s: w \mapsto \underset{v \in \mathcal{H}}{\text{argmin}} \ \frac{\|v - w\|^2}{2} + s\Phi(v) \ . \tag{2.1}
$$

The generalized gradient projection method for minimization problems of type

$$
\min_{u \in \mathcal{H}} F(u) + \Phi(u) \tag{2.2}
$$

then read as follows.

Algorithm 1.

- 1. *Choose a* $u^0 \in \mathcal{H}$ with $\Phi(u^0) < \infty$ and set $n = 0$.
- **2.** *Compute the next iterate* u^{n+1} *according to*

$$
u^{n+1} = J_{s_n}\left(u^n - s_n F'(u^n)\right) .
$$

where s_n *satisfies* an appropriate step-size rule and J_s from [\(2.1\)](#page-4-0).

3. *Set* $n := n + 1$ *and continue with Step 2.*

Note that the solutions of the minimization problem are exactly the fixed points of the algorithm. Moreover, the case $\Phi = I_{\Omega}$, where Ω is a closed and convex constraint, yields the classical gradient projection method which is known to converge provided that certain assumptions are fulfilled and a suitable step-size rule has been chosen [9, 11].

In the following, we assume that F is differentiable, F' is Lipschitz continuous with constant L and usually choose the step-sizes such that

$$
0 < \underline{s} \le s_n \le \overline{s} < 2/L. \tag{2.3}
$$

Note that form the trivial case $L = 0$ we agree that $2/L = \infty$.

Remark 3 (Forward-backward splitting). The generalization of the gradient projection method leads to a special case of the so-called *proximal forward-backward splitting method* which amounts to the iteration

$$
u^{n+1} = u^n + t_n \Big(J_{s_n} \big(u - s_n (F'(u^n) + b^n) \big) + a^n - u^n \Big)
$$

where $t_n \in [0,1]$ and $\{a^n\}, \{b^n\}$ are absolutely summable sequences in \mathcal{H} . In [5], it is shown that this method converges strongly to a minimizer under appropriate conditions. There exist, however, no general statements about convergence rates so far. Here, we restrict ourselves to the special case of the generalized gradient projection method.

Finally, it is easy to see that the iterative soft-thresholding algorithm [\(1.2\)](#page-2-1) is a special case of this generalized gradient projection method in case the functionals $F: \ell^2 \to \mathbb{R}$ and $\Phi: \ell^2 \to]-\infty, \infty]$ are chosen according to

$$
F(u) = \frac{\|Ku - f\|^2}{2} \quad , \quad \Phi(u) = \begin{cases} \sum_{k=1}^{\infty} \alpha_k |u_k|, & \text{if the sum converges} \\ \infty, & \text{else} \end{cases}
$$
 (2.4)

where $K: \ell^2 \to \mathcal{H}_2$ is linear and continuous between the Hilbert spaces ℓ^2 and \mathcal{H}_2 , $f \in \mathcal{H}_2$ and $\{\alpha_k\}$ is sequence satisfying $\alpha_k \geq \alpha > 0$ for all k.

Here, $F'(u) = K^*(Ku - f)$, so in each iteration step of Algorithm [1](#page-4-1) we have to solve

$$
\min_{v \in \ell^2} \frac{\|u^n - s_n K^*(Ku^n - f) - v\|^2}{2} + s_n \sum_{k=1}^{\infty} \alpha_k |v_k|
$$

for which the solution is given by soft-thresholding, i.e.

$$
v = \mathbf{S}_{s_n\alpha}(u^n - s_n K^*(Ku^n - f)),
$$

with $\mathbf{S}_{s_n\alpha}$ according to [\(1.2\)](#page-2-1), see [7], for example.

Since the Lipschitz constant associated with F' does not exceed $||K||^2$, this result can be summarized as follows:

Proposition 1. Let $K : \ell^2 \to \mathcal{H}_2$ be a bounded linear operator, $f \in$ \mathcal{H}_2 *and* $0 < \underline{\alpha} \leq \alpha_k$ *. Let* F *and* Φ *be chosen according to* [\(2.4\)](#page-5-1)*. Then Algorithm* [1](#page-4-1) with step-size $\{s_n\}$ according to [\(1.3\)](#page-2-2) coincides with the iterative *soft-thresholding procedure* [\(1.2\)](#page-2-1)*.*

Here and in the following, we also agree to set $2/\Vert K \Vert^2 = \infty$ in [\(1.3\)](#page-2-2) for the trivial case $K = 0$.

3. Convergence of the generalized gradient projection method

In the following, conditions which ensure convergence of the generalized gradient projection method are derived. The key is the descent of the functional $F + \Phi$ in each iteration step. The following lemma states some basic properties of one iteration.

Lemma 1. *Let* F *be differentiable with* F ′ *Lipschitz continuous with associated constant* L *and* Φ *be proper, convex and lower semi-continuous. Set* $v = J_s(u - sF'(u))$ as in [\(2.1\)](#page-4-0) for some $s > 0$ and denote by

$$
D_s(u) = \Phi(u) - \Phi(v) + \langle F'(u), u - v \rangle \tag{3.1}
$$

Then it holds:

$$
\forall w \in \mathcal{H}: \ \Phi(w) - \Phi(v) + \langle F'(u), w - v \rangle \ge \frac{\langle u - v, w - v \rangle}{s}.
$$
 (3.2)

$$
D_s(u) \ge \frac{\|v - u\|^2}{s} \tag{3.3}
$$

$$
(F + \Phi)(v) \le (F + \Phi)(u) - \left(1 - \frac{sL}{2}\right)D_s(u). \tag{3.4}
$$

Proof. Since v solves the problem

$$
\min_{v \in \mathcal{H}} \frac{\|v - u + sF'(u)\|^2}{2} + s\Phi(v)
$$

it immediately follows that the subdifferential inclusion $u - sF'(u) - v \in$ $s\partial\Phi(v)$ is satisfied, see [13, 29] for an introduction to convex analysis and subdifferential calculus. This can be rewritten to

$$
\langle u - sF'(u) - v, w - v \rangle \le s(\Phi(w) - \Phi(v)) \quad \text{for all } w \in \mathcal{H},
$$

while rearranging and dividing by s proves the inequality (3.2) . The in-equality [\(3.3\)](#page-6-1) follows by setting $w = u$ in [\(3.2\)](#page-6-0).

To show inequality [\(3.4\)](#page-6-2), we observe

$$
(F + \Phi)(v) - (F + \Phi)(u) + D_s(u) = F(v) - F(u) + \langle F'(u), u - v \rangle
$$

=
$$
\int_0^1 \langle F'(u + t(v - u)) - F'(u), v - u \rangle dt.
$$

Using the Cauchy-Schwarz inequality and the Lipschitz continuity we obtain

$$
(F + \Phi)(v) - (F + \Phi)(u) + D_s(u) \le \int_0^1 tL ||v - u||^2 dt = \frac{L}{2} ||v - u||^2.
$$

Finally, applying the estimate [\(3.3\)](#page-6-1) leads to [\(3.4\)](#page-6-2).

Remark 4 (A weaker step-size condition). If the step-size in the generalized gradient projection method is chosen such that $s_n \leq \overline{s} < 2/L$, then we can conclude from [\(3.4\)](#page-6-2) that

$$
(F + \Phi)(u^{n+1}) \le (F + \Phi)(u^n) - \delta D_{s_n}(u^n)
$$
\n(3.5)

where $\delta = 1 - \frac{\overline{s}L}{2}$ $\frac{2L}{2}$. Of course, the constraint on the step size is only sufficient to guarantee such a decrease. A weaker condition is the following:

$$
\left| \int_0^1 \langle F'(u^n + t(u^{n+1} - u^n)) - F'(u^n), u^{n+1} - u^n \rangle \, \mathrm{d}t \right| \le (1 - \delta) D_{s_n}(u^n) \tag{3.6}
$$

 \Box

for some $\delta > 0$. Regarding the proof of Lemma [1,](#page-5-2) it is easy to see that this condition also leads to the estimate (3.5) . Unfortunately, (3.6) can only be verified a-posteriori, i.e. with the knowledge of the next iterate u^{n+1} . So one has to guess an s_n and check if [\(3.6\)](#page-6-4) is satisfied, otherwise a different s_n has to be chosen. In practice, this means that one iteration step is lost and consequently more computation time is needed, reducing the advantages of a more flexible step size.

While the descent property [\(3.5\)](#page-6-3) can be proven without convexity assumptions on F , we need such a property to estimate the distance of the functional values to the global minimum of $F + \Phi$ in the following. We introduce for any sequence $\{u^n\} \subset \mathcal{H}$ according to Algorithm [1](#page-4-1) the values

$$
r_n = (F + \Phi)(u^n) - \left(\min_{u \in \mathcal{H}} (F + \Phi)(u)\right). \tag{3.7}
$$

Proposition 2. *Let* F *be convex and continuously differentiable with Lipschitz continuous derivative. Let* {u ⁿ} *be a sequence generated by Algorithm* [1](#page-4-1) *such that the step-sizes are bounded from below, i.e.* $s_n \geq s > 0$, *and that we have*

$$
(F + \Phi)(u^{n+1}) \le (F + \Phi)(u^n) - \delta D_{s_n}(u^n)
$$

for a $\delta > 0$ *with* $D_{s_n}(u^n)$ *according to* [\(3.1\)](#page-5-3)*.*

1. If $F + \Phi$ *is coercive, then the values* r_n *according to* [\(3.7\)](#page-7-0) *satisfy* $r_n \to 0$ with rate $\mathcal{O}(n^{-1})$, *i.e.* there exists a $C > 0$ such that

$$
r_n \leq Cn^{-1}
$$

2. If for a minimizer u^* and some $c > 0$ the values r_n from (3.7) satisfy

$$
||u^n - u^*||^2 \le cr_n , \qquad (3.8)
$$

.

then $\{r_n\}$ *vanishes exponentially and* $\{u^n\}$ *converges linearly to* u^* , *i.e. there exists a* $C > 0$ *and a* $\lambda \in [0, 1]$ *such that*

$$
||u^n - u^*|| \leq C\lambda^n.
$$

Proof. We first prove an estimate for r_n and then treat the cases separately. For this purpose, pick an optimal $u^* \in \mathcal{H}$ and observe that the decrease in each iteration step can be estimated by

$$
r_n - r_{n+1} = (F + \Phi)(u^n) - (F + \Phi)(u^{n+1}) \geq \delta D_{s_n}(u^n) ,
$$

according to the assumptions. Note that $D_{s_n}(u^n) \geq 0$ by [\(3.3\)](#page-6-1), so $\{r_n\}$ is non-increasing.

Use the convexity of F to deduce

$$
r_n \leq \Phi(u^n) - \Phi(u^*) + \langle F'(u^n), u^n - u^* \rangle
$$

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$$
= D_{s_n}(u^n) + \langle F'(u^n), u^{n+1} - u^* \rangle + \Phi(u^{n+1}) - \Phi(u^*)
$$

\n
$$
\leq D_{s_n}(u^n) + \frac{\langle u^n - u^{n+1}, u^{n+1} - u^* \rangle}{s_n}
$$

\n
$$
\leq D_{s_n}(u^n) + \frac{||u^{n+1} - u^*||}{\sqrt{s_n}} \sqrt{D_{s_n}(u^n)}
$$

by applying the Cauchy-Schwarz inequality as well as [\(3.2\)](#page-6-0) and [\(3.3\)](#page-6-1). With the above estimate on $r_n - r_{n+1}$ and $0 < \underline{s} < s_n$ we get

$$
\delta r_n \le (r_n - r_{n+1}) + \frac{\sqrt{\delta} \|u^{n+1} - u^*\|}{\sqrt{\underline{s}}} \sqrt{r_n - r_{n+1}} . \tag{3.9}
$$

We now turn to prove the first statement of the proposition. Assume that $F + \Phi$ is coercive, so from the fact that $\{r_n\}$ is non-increasing follows that $||u^n - u^*||$ has to be bounded by a $C_1 > 0$. Furthermore, $0 \leq r_n$ – $r_{n+1} \leq r_0 < \infty$, implying

$$
\delta r_n \le (\sqrt{r_0} + \sqrt{\delta \underline{s}^{-1}} C_1) \sqrt{r_n - r_{n+1}}
$$

and consequently

$$
qr_n^2 \le r_n - r_{n+1}
$$
, $q = \left(\frac{\delta}{\sqrt{r_0} + \sqrt{\delta s^{-1}}C_1}\right)^2 > 0$.

Standard arguments then give the rate $r_n = \mathcal{O}(n^{-1})$, we repeat them here for convenience. The above estimate on $r_n - r_{n+1}$ as well the property that ${r_n}$ is non-increasing yields

$$
\frac{1}{r_{n+1}} - \frac{1}{r_n} = \frac{r_n - r_{n+1}}{r_n r_{n+1}} \ge q \frac{r_n^2}{r_n r_{n+1}} \ge q
$$

which, summed up, leads to

$$
\frac{1}{r_n} - \frac{1}{r_0} = \sum_{i=0}^{n-1} \frac{1}{r_{i+1}} - \frac{1}{r_i} \ge nq \quad \Rightarrow \quad r_n^{-1} \ge nq + r_0^{-1}
$$

and consequently, since $q > 0$, to the desired rate $r_n \leq (nq + r_0^{-1})^{-1} \leq Cn^{-1}$.

Regarding the second statement, assume that there is a $c > 0$ such that $||u^n - u^*||^2 \leq c r_n$ for some optimal u^* and each n. Starting again at [\(3.9\)](#page-8-0) and applying Young's inequality yields, for each $\varepsilon > 0$,

$$
\delta r_n \le (r_n - r_{n+1}) + \frac{\delta \varepsilon ||u^{n+1} - u^*||^2}{2\underline{s}} + \frac{r_n - r_{n+1}}{2\varepsilon}.
$$

Choosing $\varepsilon = \underline{sc}^{-1}$ and exploiting the assumption $||u^{n+1} - u^*||^2 \le cr_{n+1}$ as well as the fact $r_{n+1} \leq r_n$ then imply

$$
\delta r_n \le (r_n - r_{n+1}) + \frac{\delta}{2} r_n + \frac{r_n - r_{n+1}}{2g^{c-1}} \quad \Rightarrow \quad r_n - r_{n+1} \ge \frac{\delta g^{c-1}}{2g^{c-1} + 1} r_n
$$

which in turn establishes the exponential decay rate

$$
r_{n+1} \le \left(1 - \frac{\delta \underline{s}c^{-1}}{2\underline{s}c^{-1} + 1}\right) r_n \le \lambda^2 r_n , \quad \text{with } \lambda = \left(1 - \frac{\delta \underline{s}c^{-1}}{2\underline{s}c^{-1} + 1}\right)^{1/2} \in [0, 1].
$$
\n(3.10)

Using $||u^n - u^*||^2 \le cr_n$ again finishes the proof:

$$
||u^n - u^*|| \le (c r_n)^{1/2} \le (c r_0)^{1/2} \lambda^n.
$$

 \Box

Proposition [2](#page-7-1) tells us that we only have to establish [\(3.8\)](#page-7-2) to obtain strong convergence with linear convergence rate. This can be done with determining how fast the functionals F and Φ vanish at some minimizer. This can be made precise by introducing the following notions which also turn out to be the essential ingredients to show [\(3.8\)](#page-7-2): First, define for a minimizer $u^* \in \mathcal{H}$ the functional

$$
R(v) = \langle F'(u^*), v - u^* \rangle + \Phi(v) - \Phi(u^*).
$$
 (3.11)

Note that if the subgradient of Φ in u^* is unique, R is the Bregman distance of Φ in u^* , a notion which is extensively used in the analysis of descent algorithms [2, 30]. Moreover, we make use of the remainder of the Taylor expansion of F ,

$$
T(v) = F(v) - F(u^*) - \langle F'(u^*), v - u^* \rangle . \tag{3.12}
$$

Remark 5 (On the Bregman distance). In many cases the Bregmanlike distance R is enough to estimate the descent properties, see [3,30]. For example, in case that Φ is the p-th power of a norm of a 2-convex Banach space X, i.e. $\Phi(u) = ||u||_X^p$ with $p \in]1,2]$, which is moreover continuously embedded in H , one can show that

$$
||v - u^*||_X^2 \le C_1 R(v)
$$

holds on each bounded set of X, see [34]. Consequently, with $j_p = \partial \frac{1}{p}$ $\frac{1}{p}$ || \cdot || $\frac{p}{X}$ denoting the duality mapping with gauge $t \mapsto t^{p-1}$,

$$
||v - u^*||^2 \le C_2 ||v - u^*||_X^2
$$

\n
$$
\le C_1 C_2 (||v||_X^p - ||u^*||_X^p - p\langle j_p(u^*), v - u^* \rangle) = cR(v)
$$

observing that R is in this case the Bregman distance. Often, Tikhonov functionals for inverse problems admit such a structure, e.g.

$$
\min_{u \in \ell^2} \frac{\|Ku - f\|^2}{2\alpha} + \sum_{k=1}^{\infty} |u_k|^p,
$$

FIGURE 1: Illustration of the Bregman-like distance R and the Taylor distance T for a convex Φ and a smooth F. Note that for the optimal value u^* it holds $-F'(u^*) \in \partial \Phi(u^*).$

a regularization which is also topic in [7]. As one can see in complete analogy to Proposition [1,](#page-5-4) the generalized gradient projection method also amounts to the iteration proposed there, so as a by-product and after verifying that the prerequisites of Proposition [2](#page-7-1) indeed hold, one immediately gets a linearly convergent method.

However, in the case that Φ is not sufficiently convex, the Bregman distance alone is not sufficient to obtain the required estimate on the r_n . This is the case for F and Φ according [\(2.4\)](#page-5-1). In this situation we also have to take the "Taylor distance" T into account. Figure [1](#page-10-0) shows an illustration of the values R and T . One could say that the Bregman distance measures the error corresponding to the Φ part while the Taylor distance does the same for the F part.

The functionals R and T possess the following properties:

Lemma 2. *Consider the problem* [\(2.2\)](#page-4-2) *where* F *is convex, differentiable* and Φ *is proper, convex and lower semi-continuous.* If $u^* \in \mathcal{H}$ *is a solution of* [\(2.2\)](#page-4-2) and $v \in H$ *is arbitrary, then the functionals* R and T *according to* [\(3.11\)](#page-9-0) *and* [\(3.12\)](#page-9-1)*, respectively, are non-negative and satisfy*

$$
R(v) + T(v) = (F + \Phi)(v) - (F + \Phi)(u^*).
$$

Proof. The identity is obvious from the definition of R and T . For the non-negativity of R, note that since u^* is a solution, it holds that $-F'(u^*) \in$ $\partial \Phi(u^*)$. Hence, the subgradient inequality reads as

$$
\Phi(u^*) - \langle F'(u^*), v - u^* \rangle \leq \Phi(v) \quad \Rightarrow \quad R(v) \geq 0.
$$

Likewise, the property $T(v) \geq 0$ is a consequence of the convexity of F. \Box

Now it follows immediately that $R(v) = T(v) = 0$ whenever v is a minimizer. To conclude this section, the main statement about the convergence of the generalized gradient projection method reads as:

Theorem 2. *Let* F *be a convex, differentiable functional with Lipschitzcontinuous derivative (with associated Lipschitz constant* L*),* Φ *be proper, convex and lower semi-continuous and* $\{u^n\}$ *be a sequence generated by Algorithm* [1](#page-4-1) *with step-size according to* [\(2.3\)](#page-4-3)*. Moreover, suppose that* $u^* \in H$ *is a solution of the minimization problem* [\(2.2\)](#page-4-2)*.*

If, for each $M \in \mathbb{R}$ *there exists a constant* $c(M) > 0$ *such that*

$$
||v - u^*||^2 \le c(M)\big(R(v) + T(v)\big) \tag{3.13}
$$

for each v *satisfying* $(F + \Phi)(v) \leq M$ *and* $R(v)$ *and* $T(v)$ *defined by* [\(3.11\)](#page-9-0) and (3.12) *, respectively, then* $\{u^n\}$ *converges linearly to the unique minimizer* u ∗ *.*

Proof. A step-size chosen according to (2.3) implies, by Lemma [1,](#page-5-2) the descent property [\(3.5\)](#page-6-3) with $\delta = 1 - \overline{s}L/2$. In particular, from (3.5) follows that ${r_n}$ is non-increasing (also remember [\(3.3\)](#page-6-1) means in particular that $D_{s_n}(u^n) \ge 0$). Now choose $M = (F + \Phi)(u^0) < \infty$ for which, by assumption, a $c > 0$ exists such that

$$
||u^{n} - u^{*}||^{2} \le c(R(u^{n}) + T(u^{n})) = cr_{n}.
$$

Hence, the prerequisites for Proposition [2](#page-7-1) are fulfilled and consequently, $u^n \to u^*$ with a linear rate. Finally, the minimizer has to be unique: If u^{**} is also a minimizer, then u^{**} plugged into [\(3.13\)](#page-11-1) gives $||u^{**} - u^*||^2 = 0$ and consequently $u^* = u^{**}$. \Box

4. Convergence rates for the iterative soft-thresholding method

We now turn to the proof of the main result, Theorem [1,](#page-2-0) which collects the results of Sections [2](#page-3-0) and [3.](#page-5-0) Within this section, we consider the regularized inverse problem [\(1.1\)](#page-0-0) under the prerequisites of Proposition [1.](#page-5-4) It is known that at least one minimizer for (1.1) exists $[7]$.

We have already seen in Proposition [1](#page-5-4) that the iterative thresholding procedure [\(1.2\)](#page-2-1) is equivalent to a generalized gradient projection method. Our aim is, on the one hand, to apply Proposition [2](#page-7-1) in order to get strong convergence from the descent rate $\mathcal{O}(n^{-1})$. On the other hand, we will show the applicability of Theorem [2](#page-11-2) for K possessing the FBI property which implies the desired convergence speed. Observe that F and Φ meet the requirements of Theorem [2](#page-11-2) and that the step-size rule [\(1.3\)](#page-2-2) immediately implies [\(2.3\)](#page-4-3), so we only have to verify [\(3.13\)](#page-11-1). This will be done, among a Bregman-distance estimate, in the following lemma, which is also serving as the crucial prerequisite for showing convergence.

Lemma 3. For each minimizer u^* of [\(1.1\)](#page-0-0) and each $M \in \mathbb{R}$, there exists *a* $c_1(M, u^*)$ and a subspace $U \subset \ell^2$ with finite-dimensional complement such *that for the Bregman-like distance* [\(3.11\)](#page-9-0) *it holds that*

$$
R(v) \ge c_1(M, u^*) \| P_U(v - u^*) \|^2 \tag{4.1}
$$

whenever $(F + \Phi)(v) \leq M$ *with* F and Φ *defined by* [\(2.4\)](#page-5-1)*.*

If K *moreover satisfies the FBI property, there is a* $c_2(M, u^*, K)$ 0 such that, whenever $(F + \Phi)(v) \leq M$, the associated Bregman-Taylor *distance according to* [\(3.11\)](#page-9-0) *and* [\(3.12\)](#page-9-1) *satisfies*

$$
R(v) + T(v) \ge c_2(M, u^*, K) \|v - u^*\|^2.
$$
 (4.2)

Proof. Let u^* be a minimizer of [\(1.1\)](#page-0-0) and assume that $v \in \ell^2$ satisfies $\Phi(v) \leq M$ for a $M \geq 0$. Then,

$$
R(v) = \sum_{k=1}^{\infty} \alpha_k |v_k| - \sum_{k=1}^{\infty} \alpha_k |u_k^*| + \sum_{k=1}^{\infty} w_k^*(v_k - u_k^*)
$$
(4.3)

where $w^* = -F'(u^*) = -K^*(Ku^* - f)$. Now since $w^* \in \partial \Phi(u^*)$ we have $w_k^* \in \alpha_k \operatorname{sgn}(u_k^*)$ for each k (note that $\partial |\cdot| = \operatorname{sgn}(\cdot)$ with $\operatorname{sgn}(0) = [-1, 1]),$ meaning that

$$
\alpha_k(|v_k| - |u_k^*|) + w_k^*(v_k - u_k^*) \ge 0
$$

for each k. Denote by $I = \{k \ge 1 : |w_k^*| = \alpha_k\}$ which has to be finite since $w^* \in \ell^2$ implies

$$
\infty > \sum_{k \in I} |w_k^*|^2 = \sum_{k \in I} \alpha_k^2 \ge \sum_{k \in I} \underline{\alpha}^2 = |I|\underline{\alpha}^2.
$$

Moreover, $w_k^* \to 0$ as $k \to \infty$, so there has to be a $\rho < 1$ such that $|w_k^*|/\alpha_k \le$ ρ for each $k \in \mathbb{N} \backslash I$. Also, if $k \in \mathbb{N} \backslash I$, then $|w_k^*| \leq \rho \alpha_k$ which means in particular that $u_k^* = 0$ since the opposite contradicts $w_k^* \in \alpha_k \operatorname{sgn}(u_k^*)$. So, one can estimate [\(4.3\)](#page-12-0):

$$
R(v) \ge \sum_{k \notin I} \alpha_k |v_k| + w_k^* v_k \ge \sum_{k \notin I} \alpha_k (1 - \rho) |v_k|
$$

$$
\ge (1 - \rho) \alpha \sum_{k \notin I} |v_k - u_k^*| \ge (1 - \rho) \alpha \left(\sum_{k \notin I} |v_k - u_k^*|^2\right)^{1/2}
$$

using the fact that one can estimate the ℓ^2 -sequence norm with the ℓ^1 sequence norm, see [3] for example. With $U = \{v \in \ell^2 : v_k = 0 \text{ for } k \in I\},\$ the above also reads as $R(v) \geq (1 - \rho) \underline{\alpha} || P_U(v - u^*) ||$ with P_U being the orthogonal projection onto U in ℓ^2 .

Next, observe that $\underline{\alpha}||P_U v|| \leq \underline{\alpha}||v||_1 \leq \Phi(v) \leq M + 1$, hence we have $||P_U v||^{-1} \ge \underline{\alpha}/(M+1)$. Consequently,

$$
R(v) \ge \frac{(1-\rho)\alpha^2}{M+1} ||P_U(v-u^*)||^2 = c_1(M, u^*) ||P_U(v-u^*)||^2
$$

which corresponds to the estimate [\(4.1\)](#page-12-1). Finally, $\Phi(v) \leq M$ whenever $(F +$ $\Phi(v) \leq M$ and there is no v such that $(F + \Phi)(v) \leq 0$. Hence, for each $M \in \mathbb{R}$ there is a constant for which [\(4.1\)](#page-12-1) holds whenever $(F + \Phi)(v) \leq M$ which is the desired statement for R.

To prove [\(4.2\)](#page-12-2), suppose K possesses the FBI property. Recall that $T(v)$ can be expressed by

$$
T(v) = F(v) - F(u^*) - \langle F'(u^*), v - u^* \rangle = \frac{\|K(v - u^*)\|^2}{2} \ . \tag{4.4}
$$

The claim now is that there is a $C(M, u^*, K)$ such that

$$
||u||^2 \leq C(M, u^*, K) \big(c_1(M, u^*) ||P_U u||^2 + \frac{1}{2} ||K u||^2 \big)
$$

for each $u \in \ell^2$. We will derive this constant directly. First split $u =$ $P_U u + P_{U^{\perp}} u$, so we can estimate, with the help of the inequalities of Cauchy-Schwarz and Young $(ab \leq a^2/4 + b^2$ for $a, b \geq 0)$,

$$
\frac{\|Ku\|^2}{2} = \frac{\|KP_{U^{\perp}}u\|^2}{2} + \langle KP_{U^{\perp}}u, KP_{U}u \rangle + \frac{\|KP_{U}u\|^2}{2}
$$

$$
\geq \frac{\|KP_{U^{\perp}}u\|^2}{4} - \frac{\|KP_{U}u\|^2}{2} \geq \frac{\|KP_{U^{\perp}}u\|^2}{4} - \frac{\|K\|_{L}^2}{2} \|P_{U}u\|^2.
$$

Since K fulfills the FBI property, the operator restricted to U^{\perp} is injective on the finite-dimensional space U^{\perp} , so there exists a $\bar{c}(U,K) > 0$ such that $\bar{c}(U,K) \| P_{U^{\perp}} u \|^2 \leq \| K P_{U^{\perp}} u \|^2$ for all $u \in \ell^2$. Hence,

$$
||P_{U^{\perp}}u||^{2} \le 4\bar{c}(U,K)^{-1} \left(\frac{1}{2}||K||^{2}||P_{U}u||^{2} + \frac{1}{2}||Ku||^{2}\right)
$$

and consequently

$$
||u||^{2} = ||P_{U^{\perp}}u||^{2} + ||P_{U}u||^{2}
$$

\n
$$
\leq 4\bar{c}(U, K)^{-1} \left(\left(\frac{1}{2} ||K||^{2} + \frac{1}{4}\bar{c}(U, K) \right) ||P_{U}u||^{2} + \frac{1}{2} ||Ku||^{2} \right)
$$

\n
$$
\leq \frac{2||K||^{2} + \bar{c}(U, K) + 4c_{1}(M, u^{*})}{\bar{c}(U, K)c_{1}(M, u^{*})} \left(c_{1}(M, u^{*}) ||P_{U}u||^{2} + \frac{1}{2} ||Ku||^{2} \right)
$$

giving a constant $c(M, u^*, K) > 0$ since U depends on u^* .

This finally yields the statement

$$
||v - u^*||^2 \le c(M, u^*, K) (c_1(M, u^*) ||P_U(v - u^*)||^2 + \frac{1}{2} ||K(v - u^*)||^2)
$$

\n
$$
\le c(M, u^*, K) (R(v) + T(v)),
$$

consequently, [\(4.2\)](#page-12-2) holds for $c_2(M, u^*, K) = c(M, u^*, K)^{-1}$.

In the following, we will see that the estimate (4.1) considered in $R(u^n)$ already leads to strong convergence of the iterative soft-thresholding procedure. Nevertheless, we utilize [\(4.2\)](#page-12-2) later, when the linear convergence result will be proven.

Lemma 4. Let $K: \ell^2 \to \mathcal{H}_2$ be a linear and continuous operator as well as $f \in \mathcal{H}_2$. Consider the sequence $\{u^n\}$ which is generated by the iterative soft*thresholding procedure* [\(1.2\)](#page-2-1) *with step-sizes* $\{s_n\}$ *according to* [\(1.3\)](#page-2-2)*. Then,* {u ⁿ} *converges to a minimizer in the strong sense.*

Proof. Since the Lipschitz constant for F' satisfies $L \leq ||K||^2$, the step sizes are fulfilling [\(2.3\)](#page-4-3) which implies, by Lemma [1,](#page-5-2) the descent property [\(3.5\)](#page-6-3) with $\delta = 1 - \overline{s} ||K||^2 / 2$. This means in particular that the associated functional distances $\{r_n\}$ are non-increasing (since (3.3)) in particular gives that $D_{s_n}(u^n) \geq 0$. Moreover, the descent result in Proposition [2](#page-7-1) yields that the iterates $\{u^n\}$ satisfy $R(u^n) \leq r_n \leq \mathcal{O}(n^{-1})$. Since $(F + \Phi)(u^n) \leq$ $(F + \Phi)(u^0) = M$, we can apply Lemma [3](#page-12-3) and the estimate [\(4.1\)](#page-12-1) leads to strong convergence of $\{P_Uu^n\}$, i.e. $P_Uu^n \to P_Uu^*$.

Next, consider the complement parts $\{P_{U^{\perp}}u^n\}$ in the finite-dimensional space U^{\perp} . Since $||P_{U^{\perp}}u^n|| \leq ||u^n|| \leq \underline{\alpha}^{-1}\Phi(u^n) \leq r_0$, the sequence $\{P_{U^{\perp}}u^n\}$ is contained in a relative compact set in U^{\perp} , hence there is a (strong) accumulation point $u^{**} \in U^{\perp}$. Together with $P_U u^n \to P_U u^*$ we can conclude that there is a subsequence satisfying $u^{n_l} \to P_U u^* + u^{**} = u^{***}$. Moreover, { $uⁿ$ } is a minimizing sequence, so u^{***} has to be a minimizer.

Finally, the whole sequence has to converge to u^{***} : The mappings $T_n(u) = J_{s_n}\big(u - s_n F'(u)\big)$ satisfy

$$
||T_n(u) - T_n(v)|| \le ||(I - s_n K^* K)(u - v)|| \le ||u - v||
$$

for all $u, v \in \ell^2$, since all proximal mappings J_{s_n} are non-expansive and $s_n \leq$ 2 $\frac{2}{\|K\|^2}$. So if, for an arbitrary $\varepsilon > 0$ there exists a n such that $\|u^n - u^{***}\| \leq \varepsilon$, then

$$
||u^{n+1} - u^{***}|| = ||T_n(u^n) - T_n(u^{***})|| \le ||u^n - u^{***}|| \le \varepsilon
$$

since u^{***} is minimizer and hence a fixed point of each T_n (see Section [2\)](#page-3-0). By induction, $u^n \to u^{***}$ strongly in ℓ^2 . \Box

With the notions of FBI property and strict sparsity pattern from Definitions [1](#page-2-3) resp. [2,](#page-2-4) one is able to show linear convergence as soon as one of this two situations is given.

 \Box

Proof of Theorem [1.](#page-2-0) Observe that the prerequisites of Lemma [4](#page-14-0) are fulfilled, so there exists a minimizer u^* such that $u^n \to u^*$ in ℓ^2 . Thus, we have to show that each of the two cases stated in the theorem leads to a linear convergence rate.

Consider the first case, i.e. K possesses the FBI property. We utilize that, by Lemma [3,](#page-12-3) the Bregman-Taylor distance according to [\(3.11\)](#page-9-0) and (3.12) can be estimated such that (3.13) is satisfied for some $c > 0$. This implies, by Theorem [2,](#page-11-2) the linear convergence rate.

 \Box

For the proof for the second case, we refer to Appendix [A.](#page-21-0)

Corollary 1. In particular, choosing the step-size constant, i.e. $s_n = s$ $with s \in]0,2||K||^{-2}[$ also leads to linear convergence under the prerequisites *of Theorem [1,](#page-2-0) for example the step-size* $s_n = 1$ *works for* $||K|| < \sqrt{2}$.

Remark 6 (Descent and Bregman-Taylor implies linear rate). With Theorem [2,](#page-11-2) the linear convergence follows directly from the estimate of the Bregman-Taylor distance

$$
||v - u^*||^2 \le c(M, u^*, K)\big(R(v) + T(v)\big) \quad \text{whenever} \quad (F + \Phi)(v) \le M
$$

which can be established if K satisfies the FBI property. Since the proof of Theorem [2](#page-11-2) relies essentially on Proposition [2,](#page-7-1) one can easily convince oneself that the applicability of this proposition is sufficient for linear convergence, which is already the case if [\(3.5\)](#page-6-3) and $0 < \underline{s} \leq s_n$ is satisfied.

Remark 7 (The weak step-size condition as accelerated method). As already mentioned in Remark [4,](#page-6-5) the condition on the step-size can be relaxed. In the particular setting that $F(u) = \frac{1}{2} || Ku - f||^2$, the estimate [\(3.6\)](#page-6-4) reads as

$$
\left| \int_0^1 \langle K^* K(u^n + t(u^{n+1} - u^n)) - K^* K u^n, u^{n+1} - u^n \rangle \, \mathrm{d}t \right|
$$

=
$$
\frac{\|K(u^{n+1} - u^n)\|^2}{2} \le (1 - \delta) D_{s_n}(u^n)
$$

Now, the choice s_n according to

$$
s_n \|K(u^{n+1} - u^n)\|^2 \le 2(1 - \delta) \|u^{n+1} - u^n\|^2 , \qquad (4.5)
$$

is sufficient for the above, since one has the estimate [\(3.3\)](#page-6-1). Together with the boundedness $0 < \underline{s} \leq s_n$, this is exactly the step-size 'Condition (B)' in [8].

Hence, as can be easily seen, the choice gives sufficient descent in order to apply Proposition [2.](#page-7-1) Consequently, linear convergence remains valid for such an 'accelerated' iterative soft-thresholding procedure if K possesses the FBI property, see Remark [6.](#page-15-0)

Remark 8 (Relaxation of the FBI property). It is possible to relax the FBI property. Suppose that K fulfills the FBI property *of order* $S = |I|$ (with the set I defined in the proof of Lemma [3\)](#page-12-3), i.e., that $K|_I$ is injective for every finite subset $I \subset \mathbb{N}$ of size less or equal to S. This immediately yields the existence of $\bar{c}(U, K) > 0$ such that $\bar{c}(U, K) ||u||^2 \le ||Ku||^2$ for each $u \in U^{\perp}$ where U^{\perp} is the finite-coefficient subspace as defined in the proof of Lemma [3.](#page-12-3) One can easily check that the remaining arguments also remain true and consequently, Theorem [1](#page-2-0) still holds.

The constants in the estimates of Lemma [3](#page-12-3) and Theorem [1](#page-2-0) are in general not computable unless the solution is determined. Nonetheless there are some situations in which prior knowledge about the operator K can be used to estimate the decay rate.

Theorem 3. Let $K : \ell^2 \to \mathcal{H}_2$, $K \neq 0$ be a compact, linear operator *fulfilling the FBI property and define*

$$
\sigma_k = \inf \left\{ \frac{\|Ku\|^2}{\|u\|^2} \; : \; u \neq 0 \; , \; u_l = 0 \; \text{for all } l \geq k \right\} \, ,
$$

$$
\mu_k = \sup \left\{ \frac{\|Ku\|^2}{\|u\|^2} \; : \; u \neq 0 \; , \; u_l = 0 \; \text{for all } l < k \right\} \, .
$$

Furthermore, choose k_0 *such that* $\mu_{k_0} \leq \frac{\alpha^2}{4||f||^2}$ *(with* ∞ *allowed on the right-hand side).* Let $\{u^n\}$ be a sequence generated by the iterative soft*thresholding algorithm* (1.2) *with initial value* $u^0 = 0$ *and constant step-size* $s = ||K||^{-2}$ for the minimization of (1.1) and let u^* denote a minimizer. *Then it holds that* $||u^n - u^*|| \leq C\lambda^n$ *with*

$$
\lambda = \max \left(1 - \frac{\sigma_{k_0}}{4\sigma_{k_0} + 8\|K\|^2}, 1 - \frac{\sigma_{k_0} \alpha^2}{4\sigma_{k_0} \alpha^2 + 2(\sigma_{k_0} + 2\|K\|^2)\|K\|^2\|f\|^2} \right)^{1/2}
$$

for some $C \geq 0$ *.*

The proof is given in Appendix [B.](#page-23-0)

5. Convergence of related methods

In this section, we show how linear convergence can be obtained for some related methods. In particular, iterative thresholding methods for minimization problems with joint sparsity constraints as well as an accelerated gradient projection method are considered. Both algorithms can be written as a generalized gradient projection method, hence the analysis carried out in Sections [2](#page-3-0) and [3](#page-5-0) can be applied, demonstrating the broad range of applications.

5.1 Joint sparsity constraints

First, we consider the situation of so-called joint sparsity for vector-valued problems, see [1,17,32]. The problems considered are set in the Hilbert space $({\ell}^2)^N$ for some $N \geq 1$ which is interpreted such that for $u \in ({\ell}^2)^N$ the k-th component u_k is a vector in \mathbb{R}^N . Given a linear and continuous operator $K: (\ell^2)^N \to \mathcal{H}_2$, some data $f \in \mathcal{H}_2$, a norm | · | of \mathbb{R}^N and a sequence $\alpha_k \geq \alpha > 0$, the typical inverse problem with joint sparsity constraints reads as

$$
\min_{u \in (\ell^2)^N} \; \frac{\|Ku - f\|^2}{2} + \sum_{k=1}^{\infty} \alpha_k |u_k| \; . \tag{5.1}
$$

In many applications, $|\cdot| = ||\cdot||_q$ for some $1 \leq q \leq \infty$.

To apply the generalized gradient projection method for [\(5.1\)](#page-17-0), we split the functional into

$$
F(u) = \frac{\|Ku - f\|^2}{2} \quad , \quad \Phi(u) = \sum_{k=1}^{\infty} \alpha_k |u_k| \; .
$$

Analogously to Proposition [1,](#page-5-4) one needs to know the associated proximal mappings J_s which can be reduced to the computation of the proximal mappings for ∂ | \cdot | on \mathbb{R}^N . These are known to be

$$
(I + s\partial |\cdot|)^{-1}(x) = (I - P_{\{|\cdot|_* \le s\}})(x)
$$

where $P_{\{| \cdot |_* \leq s\}}$ denotes the projection to the closed s-ball associated with the dual norm $|\cdot|_*$. Again, as can be seen in analogy to Proposition [1,](#page-5-4) the generalized gradient projection method for [\(5.1\)](#page-17-0) is given by the iteration

$$
u^{n+1} = \mathbf{S}_{s_n\alpha}(u^n - s_n K^*(Ku^n - f)) \quad , \quad (\mathbf{S}_{s_n\alpha}(w))_k = (I - P_{\{|\cdot|_* \le s_n\alpha_k\}})(w_k)
$$
\n(5.2)

where $\{s_n\}$ satisfies a suitable step-size rule, e.g. according to [\(1.3\)](#page-2-2) or [\(4.5\)](#page-15-1).

Let us examine this method with respect to convergence. First, fix a minimizer u^* which satisfies the optimality condition $w^* = -K^*(Ku^*$ f) $\in \partial \Phi(u^*)$. As one knows from convex analysis, this can also be formulated pointwise, and Asplund's characterization of ∂| · | (see [31], Proposition II.8.6) leads to

$$
|w_k^*|_* \le \alpha_k \quad \text{if} \quad u_k^* = 0
$$

$$
|w_k^*|_* = \alpha_k \text{ and } w_k^* \cdot u_k^* = \alpha_k |u_k^*| \quad \text{if} \quad u_k^* \neq 0
$$

where $w_k^* \cdot u_k^*$ denotes the usual inner product of w_k^* and u_k^* in \mathbb{R}^N . Now, one can proceed in complete analogy to the proof of Lemma [3](#page-12-3) in order to get an estimate of the associated Bregman distance: One constructs $I =$ ${k \in \mathbb{N}} : |w_k^*|_* = \alpha_k}$ as well as the closed subspace $U = \{v \in (\ell^2)^N :$

 $v_k = 0$ if $k \notin I$ for which U^{\perp} is finite-dimensional. Furthermore, we have $\rho = \sup_{k \notin I} |w_k^*|_{\star}/\alpha_k < 1$ and, by equivalence of norms in \mathbb{R}^N , one gets $C_0, c_0 > 0$ such that $c_0|x|_2 \leq |x| \leq C_0|x|_2$ (with $|x|_2^2 = x \cdot x$) for all $x \in \mathbb{R}^N$. Then, for a given $M \in \mathbb{R}$, whenever $(F + \Phi)(v) \leq M$,

$$
R(v) \geq \frac{(1-\rho)\alpha^2 c_0}{(M+1)C_0} \left(\sum_{k \notin I} |v_k - u_k^*|^2 \right)^{1/2} = c_1(M, u^*) ||P_U(v - u^*)||^2,
$$

establishing an analogon of (4.1) . If K moreover satisfies the FBI property, then one also gets an analogon to [\(4.2\)](#page-12-2), i.e.

$$
R(v) + T(v) \ge c_2(M, u^*, K) ||v - u^*||^2
$$

whenever $(F + \Phi)(v) \leq M$, by arguing analogously to Lemma [3.](#page-12-3)

Since these two inequalities are the essential ingredient for proving convergence as well as the linear rate, cf. Lemma [4](#page-14-0) and Theorem [1,](#page-2-0) it holds:

Theorem 4. *The iterative soft-thresholding procedure* [\(5.2\)](#page-17-1) *for the minimization problem* [\(5.1\)](#page-17-0) *converges to a minimizer in the strong sense in* $(\ell^2)^N$ *if the step-size rule* [\(1.3\)](#page-2-2) *is satisfied.*

Furthermore, the convergence will be at linear rate if K *possesses the FBI property and the step-size rule* [\(4.5\)](#page-15-1) *as well as* $0 < \underline{s} \leq s_n$ *is satisfied. In particular, this is the case when* $0 < \underline{s} \leq s_n \leq \overline{s} < 2/\Vert K \Vert^2$.

5.2 Accelerated gradient projection methods

An alternative approach to implement sparsity constraints for linear inverse problems is based on minimizing the discrepancy within a weighted ℓ^1 -ball [8]. With the notation used in Section [4,](#page-11-0) the problem can be generally formulated as

$$
\min_{u \in \Omega} \frac{\|Ku - f\|^2}{2} , \quad \Omega = \left\{ u \in \ell^2 : \sum_{k=1}^{\infty} \alpha_k |u_k| \le 1 \right\} .
$$
 (5.3)

For this classical situation of constrained minimization, one finds that the generalized gradient projection method and the gradient projection method coincide (for $F(u) = \frac{1}{2} || Ku - f||^2$ and $\Phi = I_{\Omega}$), see Section [2,](#page-3-0) and yield the iteration proposed in [8]. Consequently, classical convergence results hold for a variety of step-size rules [11], including the 'Condition (B)' introduced in [8], see also Remark [7.](#page-15-2)

Let us note that linear convergence results can be obtained with the same techniques which have been used to prove Theorem [1:](#page-2-0) First, consider the Bregman distance R associated with $\Phi = I_{\Omega}$ in a minimizer $u^* \in \Omega$. With $w^* = -K^*(Ku^* - f)$, the optimality condition reads as

$$
\langle w^*, v - u^* \rangle \le 0 \quad \text{for all } v \in \Omega \quad \Leftrightarrow \quad \|\alpha^{-1}w^*\|_{\infty} = \langle w^*, u^* \rangle
$$

where $(\alpha^{-1}w^*)_k = \alpha_k^{-1}w_k^*$ which is in ℓ^{∞} since $\alpha_k \geq \underline{\alpha} > 0$. Introduce $I = \{k : |\alpha_k^{-1} w_k^*| = ||\alpha^{-1} w^*||_{\infty}\}\$ which has to be finite since otherwise $w^* \notin \ell^2$, see the proof of Lemma [3.](#page-12-3) Suppose that $w^* \neq 0$ (which corresponds to $Ku^* \neq f$), so $\sup_{k \notin I} |\alpha_k^{-1}w_k^*|/||\alpha^{-1}w^*||_{\infty} = \rho < 1$. Moreover,

$$
\sum_{k \in I} \alpha_k |u_k^*| = 1 \quad \text{and} \quad \operatorname{sgn}(u_k^*) = \operatorname{sgn}(w_k^*) \text{ for all } k \text{ with } u_k^* \neq 0 \tag{5.4}
$$

since $\sum_{k \in I} \alpha_k |u_k^*| < 1$ leads to the contradiction

$$
\|\alpha^{-1}w^*\|_{\infty} = \langle w^*, u^* \rangle = \sum_{k=1}^{\infty} \alpha_k^{-1} w_k^* \alpha_k u_k^*
$$

\n
$$
\leq \sum_{k \notin I} |\alpha_k^{-1} w_k^*| |\alpha_k u_k^*| + \sum_{k \in I} \|\alpha^{-1}w^*\|_{\infty} \alpha_k |u_k^*|
$$

\n
$$
\leq \left(\rho \sum_{k \notin I} \alpha_k |u_k^*| + \sum_{k \in I} \alpha_k |u_k^*| \right) \|\alpha^{-1}w^*\|_{\infty} < \|\alpha^{-1}w^*\|_{\infty}
$$

while $sgn(u_k^*) \neq sgn(w_k^*)$ for some k with $u_k^* \neq 0$ implies the contradiction

$$
\|\alpha^{-1}w^*\|_{\infty} = \sum_{k=1}^{\infty} \alpha_k^{-1} w_k^* \alpha_k u_k^* < \sum_{k=1}^{\infty} |\alpha_k^{-1}w_k^*| |\alpha_k u_k^*| \leq \|\alpha^{-1}w^*\|_{\infty}.
$$

Moreover, $\sum_{k\in I} \alpha_k |u_k^*| = 1$ also yields $u_k^* = 0$ for all $k \notin I$. Further-more, observe that the equation for the signs in [\(5.4\)](#page-19-0) gives $\sum_{k\in I} w_k^* u_k^* =$ $\|\alpha^{-1}w^*\|_{\infty}$. For $v \notin \Omega$ we have $R(v) = \infty$, so estimate the Bregman distance for $v\in\Omega$ as follows:

$$
R(v) = -\langle w^*, v - u^* \rangle = \sum_{k \in I} \alpha_k^{-1} w_k^* \alpha_k (u_k^* - v_k) - \sum_{k \notin I} \alpha_k^{-1} w_k^* \alpha_k v_k
$$

\n
$$
\ge ||\alpha^{-1} w^*||_{\infty} - ||\alpha^{-1} w^*||_{\infty} \sum_{k \in I} \alpha_k |v_k| - \rho ||\alpha^{-1} w^*||_{\infty} \sum_{k \notin I} \alpha_k |v_k|
$$

\n
$$
\ge (1 - \rho) \sum_{k \notin I} \alpha_k |v_k| \ge (1 - \rho) \underline{\alpha} ||P_{U}v||_1
$$

where $U = \{u \in \ell^2 : u_k = 0 \text{ for } k \in I\}$. Using that $||v|| \le ||v||_1$ as well as $\underline{\alpha}||v|| \leq \underline{\alpha}||v||_1 \leq 1$ for all $v \in \Omega$ finally gives, together with $P_U u^* = 0$,

$$
R(v) \ge (1 - \rho)\underline{\alpha}^2 ||P_U(v - u^*)||^2
$$
 for all $v \in \ell^2$.

If K possesses the FBI property, one can, analogously to the argumentation presented in the proof of Lemma [3,](#page-12-3) estimate the Bregman-Taylor distance such that, for some $c(u^*, K) > 0$,

$$
R(v) + T(v) \ge c(u^*, K) \|v - u^*\|^2 \text{ for all } v \in \ell^2
$$

.

By Theorem [2,](#page-11-2) the gradient projection method for [\(5.3\)](#page-18-0) converges linearly. This remains true for each 'accelerated' step-size choice according to 'Condition (B)' in [8], see Remark [7.](#page-15-2) This result can be summarized in the following theorem.

Theorem 5. Assume that $K: \ell^2 \to \mathcal{H}_2$ satisfies the FBI property, $\alpha_k \geq$ $\underline{\alpha} > 0$ *and* $f \in \mathcal{H}_2 \setminus K(\Omega)$ *where* $K(\Omega) = \{Ku : ||\alpha u||_1 \leq 1\}.$

Then, the gradient projection method for the minimization problem [\(5.3\)](#page-18-0) *converges linearly, whenever the step-sizes rule* [\(4.5\)](#page-15-1) *as well as* $0 < \underline{s} \le$ s_n is fulfilled. This is in particular the case for $0 < \underline{s} \leq s_n \leq \overline{s} < 2/\Vert K \Vert^2$.

6. Conclusions

We conclude this article with a few remarks on the implications of our results. We showed that, in many cases, iterative soft-thresholding algorithms converge with linear rate and moreover that there are situations in which the constants can be calculated explicitly, see Theorem [3.](#page-16-1) In general, however, the factor λ , which determines the speed within the class of linearly-convergent algorithms, always depends on the operator K but in the considered cases also on the initial value u^0 and a solution u^* . Unfortunately, the dependence on a solution can cause λ to be arbitrarily close to 1, meaning that the iterative soft-thresholding converges arbitrarily slow in some sense, which is also often observed in practice.

One key ingredient for proving the convergence result is the FBI property. This property also plays a role in the performance analysis of Newton methods applied to minimization problems with sparsity constraints [21] and error estimates for ℓ^1 -regularization [24]. As we have moreover seen, linear convergence can also be obtained whenever we have convergence a solution with strict sparsity pattern. This result is closely connected with the fact that [\(1.1\)](#page-0-0), considered on a fixed sign pattern, is a quadratic problem, and hence the iteration becomes linear from some index on. The latter observation is also basis of a couple of different algorithms [12, 16, 27].

At last we want to remark that Theorem [2](#page-11-2) on linear convergence of the generalized gradient projection method holds in general and has been applied in a special case in order to prove Theorem [1.](#page-2-0) This generality also allowed for a unified treatment of the similar algorithms presented in Section [5](#page-16-0) as well as other penalty terms such as powers of certain 2-convex norms, see Remark [5.](#page-9-2) In all of these situations, linear convergence follows from descent properties on the one hand and Bregman (-Taylor) estimates on the other hand.

A. Proof of Theorem [1](#page-2-0) (continued)

For the second case, let u^* possess a strict sparsity pattern. Define, analogously to the above, the subspace $U = \{v \in \ell^2 : v_k = 0 \text{ if } u_k^* \neq 0\}.$ The desired result then is implied by the fact that there is an n_0 such that each u^{n+1} with $n \geq n_0$ can be written as

$$
u^{n+1} = (I - s_n P_{U^{\perp}} K^* K P_{U^{\perp}}) (u^n - u^*) + u^*.
$$

For this purpose, we introduce the notations $w^n = -K^*(Ku^n - f)$, $w^* =$ $-K^*(Ku^* - f)$ and recall the optimality condition $w^* \in \partial \Phi(u^*)$ which can be written as

$$
w_k^* \in [-\alpha_k, \alpha_k]
$$

\n
$$
w_k^* = \alpha_k
$$

\n
$$
w_k^* = -\alpha_k
$$

\nif $u_k^* > 0$
\nif $u_k^* > 0$
\nif $u_k^* < 0$.

Due to assumption that u^* has a strict sparsity pattern, $w_k^* \in [-\alpha_k, \alpha_k]$ if $u_k^* = 0$, and hence there is a $\rho > 0$ such that

$$
w_k^* \in [-(1-\rho)\alpha_k, (1-\rho)\alpha_k]
$$
 if $u_k^* = 0$

since $w_k^* \to 0$ for $k \to \infty$. Also note that $u^n \to u^*$ implies $w^n \to w^*$ and especially pointwise convergence.

We will treat each of the cases $u_k^* = 0$, $u_k^* > 0$ and $u_k^* < 0$ separately. **The case** $u_k^* = 0$: First, we find an index n_1 such that, for $n \geq n_1$,

$$
||u^n-u^*||\leq \frac{\rho}{2}\underline{\alpha}\,\underline{s}\quad,\quad ||w^n-w^*||\leq \frac{\rho}{2}\underline{\alpha}.
$$

So, if $k \in I_0$ with $I_0 = \{k : u_k^* = 0\}$, we have

$$
|u_k^n|\leq \frac{\rho}{2}s_n\alpha_k\quad,\quad |w_k^n|\leq |w_k^*|+|w_k^n-w_k^*|\leq (1-\rho)\alpha_k+\frac{\rho}{2}\alpha_k
$$

for each $n \geq n_1$. Consequently, for all of these k and n,

$$
|u^n - s_n K^*(Ku^n - f)|_k \le s_n \alpha_k
$$

hence the thresholding operation according to [\(1.2\)](#page-2-1) gives $u_k^{n+1} = 0$ for all $n \geq n_1$ and all $k \in I_0$. Thus, the iteration for $P_U u^n$ can be expressed by

$$
P_U u^{n+1} = (I - s_n P_{U^{\perp}} K^* K P_{U^{\perp}}) (u^n - u^*) + P_U u^* \tag{A.1}
$$

for all $n \ge n_1 + 1$ since $P_U u^n = P_U u^* = 0$.

The case $u_k^* > 0$: Next, investigate all $k \in I_+$ with $I_+ = \{k : u_k^* > 0\}.$ This has to be a finite set, so there is a $\delta_+ \in]0, \underline{\alpha}[$ such that $u_k^* \geq \delta_+$ for each of such k. So, choose n_{+} according to the requirements that for all $n \geq n_+$

$$
||u^n - u^*|| \le \frac{\delta_+}{2}
$$
, $||w^n - w^*|| \le \frac{\delta_+}{2\overline{s}}$.

Then, remembering that $w_k^* = \alpha_k$,

$$
u_k^n + s_n w_k^n = u_k^* + u_k^n - u_k^* + s_n (w_k^n - w_k^*) + s_n w_k^*
$$

\n
$$
\ge u_k^* - |u_k^n - u_k^*| - s_n |w_k^n - w_k^*| + s_n \alpha_k
$$

\n
$$
\ge \delta_+ - \frac{\delta_+}{2} - \frac{s_n \delta_+}{2\overline{s}} + s_n \alpha_k \ge s_n \alpha_k
$$

and hence the iteration gives, by $(w^n - w^*) = -K^*K(u^n - u^*),$

$$
u_k^{n+1} = u_k^n + s_n w_k^n - s_n \alpha_k
$$

= $u_k^n - u_k^* + s_n (w_k^n - w_k^*) + u_k^*$
= $((I - s_n K^* K)(u^n - u^*))_k + u_k^*$ (A.2)

for all $n \geq n_+$ and all $k \in I_+$.

The case $u_k^* < 0$: Analogously, considering the indices $k \in I$ with $I =$ ${k : u_k^* < 0}$, one can find an $n_-\text{ such that}$

$$
u_k^{n+1} = ((I - s_n K^* K)(u^n - u^*))_k + u_k^* \tag{A.3}
$$

also holds for all $n \geq n_-$ and all $k \in I_-$.

Choosing $n_0 = \max(n_1 + 1, n_+, n_-)$ and considering $(A.1)$ – $(A.3)$ as well as remembering that $P_U u^n = 0$ for $n > n_0$ yields that indeed

$$
u^{n+1} = (I - s_n P_{U^{\perp}} K^* K P_{U^{\perp}}) (u^n - u^*) + u^* . \tag{A.4}
$$

Eventually, we can split the iteration into the subspaces $V = \ker(KP_{U^{\perp}})$ and V^{\perp} , where V^{\perp} is taken with respect to U^{\perp} . For $n \geq n_0$,

$$
P_V u^{n+1} = (P_V - s_n P_V P_{U^{\perp}} K^* K P_{U^{\perp}}) (u^n - u^*) + P_V u^* = P_V u^n
$$

due to the fact that $V = \ker(KP_{U^{\perp}}) = \text{rg}(P_{U^{\perp}}K^*)^{\perp}$. Consequently, $P_V u^n =$ $P_V u^*$ since there would not hold that $u^n \to u^*$ otherwise. Note that V^{\perp} is finite dimensional, hence there is a $c > 0$ such that $c||P_{V^{\perp}}u||^2 \leq$ $||KP_{U^{\perp}}P_{V^{\perp}}u||^2 = ||KP_{V^{\perp}}u||^2$ for all $u \in \ell^2$. Consequently, each of the self-adjoint mappings $P_{V^{\perp}} - s_n P_{V^{\perp}} K^* K P_{V^{\perp}}$ is a strict contraction on V^{\perp} :

$$
\sup_{\|P_{V^{\perp}}u\|=1} |\langle (P_{V^{\perp}} - s_n P_{V^{\perp}} K^* K P_{V^{\perp}})u, P_{V^{\perp}}u \rangle|
$$
\n
$$
= \sup_{\|P_{V^{\perp}}u\|=1} \left| \|P_{V^{\perp}}u\|^2 - s_n \|K P_{V^{\perp}}\|^2 \right|
$$
\n
$$
\leq \max \left(\frac{\|K\|^2 - 1}{\|P_{V^{\perp}}u\|=1} \|P_{V^{\perp}}u\|^2 - s_n c \|P_{V^{\perp}}u\|^2 \right)
$$
\n
$$
\leq \max \left(\frac{\|K\|^2 - 1}{1 - s} \right) = \lambda < 1.
$$

Using that $u^n - u^* = P_{V} (u^n - u^*)$ for $n \ge n_0$ gives, plugged into [\(A.4\)](#page-22-1),

$$
u^{n+1} - u^* = (P_{V^{\perp}} - s_n P_{V^{\perp}} K^* K P_{V^{\perp}}) (u^n - u^*)
$$

so

$$
||u^{n+1} - u^*||^2 = ||P_{V^{\perp}}(u^{n+1} - u^*)||^2 \le \lambda^2 ||P_{V^{\perp}}(u^n - u^*)||^2 = \lambda^2 ||u^n - u^*||^2,
$$

meaning $||u^n - u^*|| \le \lambda^{n-n_0} ||u^{n_0} - u^*||$ for $n \ge n_0$. Finally, it is easy to find
a $C > 0$ such that $||u^n - u^*|| \le C\lambda^n$ for all n.

B. Proof of Theorem [3](#page-16-1)

Proof of Theorem [3.](#page-16-1) Note that $\sigma_k > 0$ because of the FBI property and that $\mu_k \to 0$ as $k \to \infty$ since K is compact (otherwise there would be a bounded sequence which converges weakly to zero with images not converging in the strong sense).

Our aim is to compute a constant $c_1 > 0$ such that $c_1 || P_k(v - u^*) ||^2 \le$ $R(v)$ on a suitable bounded set and for a suitable k. Here, P_k denotes the orthogonal projection onto the subspace $\{u \in \ell^2 : u_l = 0 \text{ for } l < k\}.$ We can assume without loss of generality that $f \neq 0$ and thus estimate the norm of $Ku^* - f$:

$$
\frac{\|Ku^* - f\|^2}{2} \le (F + \Phi)(u^*) \le (F + \Phi)(0) = \frac{\|f\|^2}{2} \quad \Rightarrow \quad \|Ku^* - f\| \le \|f\|
$$

Because the index k_0 is chosen such that $\mu_{k_0} \leq \underline{\alpha}^2/(4||f||^2)$ we can estimate

$$
||P_{k_0}K^*(Ku^* - f)|| = \sup_{||v|| \le 1} \langle Ku^* - f, KP_{k_0}v \rangle
$$

$$
\le \sup_{||v|| \le 1} ||Ku^* - f|| ||KP_{k_0}v|| \le \mu_{k_0}^{1/2} ||f|| \le \alpha/2
$$

and consequently, $w^* = -K^*(Ku^* - f)$ satisfies $|w_k^*| \le \underline{\alpha}/2$ for each $k \ge k_0$. Recall from the proof of Lemma [3](#page-12-3) that this in particular means that $u_k^* = 0$, so one obtains the estimate

$$
R(v) \geq \sum_{k \geq k_0} \alpha_k (|v_k| - |u_k^*|) + w_k^* v_k \geq \frac{1}{2} \alpha \sum_{k \geq k_0} |v_k - u_k^*| \geq \frac{1}{2} \alpha ||P_{k_0}(v - u^*)||.
$$

We assumed that the first iterate is $u^0 = 0$, so $(F + \Phi)(u^n) \le \frac{||f||^2}{2}$ $rac{\pi}{2}$. Consequently, $||P_{k_0}v||^{-1} \geq 2\underline{\alpha}||f||^{-2}$ whenever $\Phi(v) \leq (F + \Phi)(0)$, so

$$
R(v) \ge \underline{\alpha}^2 ||f||^{-2} ||P_{k_0}(v - u^*)||^2
$$

.

An estimate for the Taylor-distance T is found with the help of σ_{k_0} :

$$
T(v) = \frac{\|K(v - u^*)\|^2}{2} \ge \frac{\|KP_{k_0}^{\perp}(v - u^*)\|^2}{4} - \frac{\|K\|^2 \|P_{k_0}(v - u^*)\|^2}{2}
$$

$$
\geq \frac{\sigma_{k_0}}{4} \left(\| P_{k_0}^\perp (v - u^*) \|^2 + \| P_{k_0} (v - u^*) \|^2 \right) - \frac{(\sigma_{k_0} + 2 \| K \|^2) \| f \|^2}{4 \alpha^2} R(v) ,
$$

where $P_{k_0}^{\perp} = I - P_{k_0}$. Rearranging terms gives:

$$
\frac{\sigma_{k_0}}{4} \|v - u^*\|^2 \le \max\left(1, \frac{(\sigma_{k_0} + 2\|K\|^2)\|f\|^2}{4\underline{\alpha}^2}\right) \left(R(v) + T(v)\right)
$$

leading to the desired constant c in Proposition [2:](#page-7-1)

$$
||v - u^*||^2 \le \max\left(\frac{4}{\sigma_{k_0}}, \frac{(\sigma_{k_0} + 2||K||^2)||f||^2}{\sigma_{k_0}\underline{\alpha}^2}\right) r_n
$$
,

namely $c = \max \left(\frac{4}{\sigma} \right)$ $\frac{4}{\sigma_{k_0}}, \frac{(\sigma_{k_0}+2\|K\|^2)\|f\|^2}{\sigma_{k_0}\underline{\alpha}^2}$ $\frac{2||K||^2||f||^2}{\sigma_{k_0}\underline{\alpha}^2}$. Estimating λ according to [\(3.10\)](#page-9-3) with constant step-size $s = ||K||^{-2}$ and $\delta = 1 - s||K||^2/2 = 1/2$ yields

$$
\lambda^{2} \leq 1 - \frac{1}{4 + 2c||K||^{2}}
$$

= max $\left(1 - \frac{\sigma_{k_{0}}}{4\sigma_{k_{0}} + 8||K||^{2}}, 1 - \frac{\sigma_{k_{0}}\alpha^{2}}{4\sigma_{k_{0}}\alpha^{2} + 2(\sigma_{k_{0}} + 2||K||^{2})||K||^{2}||f||^{2}}\right)$.

Remark B.1. The proof of Proposition [2](#page-7-1) also establishes $||u^n - u^*|| \le$ $(c_{r_0})^{1/2}\lambda^n$ which implies in turn, by estimating $r_0 \leq (F + \Phi)(0) = ||f||^2/2$ and the maximum by the sum, the a-priori estimate

$$
||u^{n} - u^{*}|| \leq \sqrt{\frac{4\alpha^{2}||f||^{2} + (\sigma_{k_{0}} + 2||K||^{2})||f||^{4}}{2\sigma_{k_{0}}\alpha^{2}}} \cdot \max \left(1 - \frac{\sigma_{k_{0}}}{4\sigma_{k_{0}} + 8||K||^{2}}, 1 - \frac{\sigma_{k_{0}}\alpha^{2}}{4\sigma_{k_{0}}\alpha^{2} + 2(\sigma_{k_{0}} + 2||K||^{2})||K||^{2}||f||^{2}}\right)^{n/2}.
$$

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