

ON PERMUTATION POLYTOPES

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ABSTRACT. A permutation polytope is the convex hull of a group of permutation matrices. In this paper we investigate the combinatorics of permutation polytopes and their faces. As applications we completely classify ≤ 4 -dimensional permutation polytopes and the corresponding permutation groups up to a suitable notion of equivalence. We also provide a list of combinatorial types of possibly occurring ≤ 4 -faces of permutation polytopes.

INTRODUCTION

One of the most intensively studied convex polytopes is the Birkhoff polytope, also known as the assignment polytope, also known as the polytope of doubly stochastic matrices [BR74, BG77, BL91, BS96, Zei99, CRY00, Pak00]. It is the convex hull in $\mathbb{R}^{n \times n}$ of the $n \times n$ permutation matrices. This polytope naturally appears in various contexts such as enumerative combinatorics [Sta86, Ath05], optimization [Tin86, Fie88, Pak00, BS03], and statistics [Pak00] (and references therein), as well as in representation theory [Onn93, BFL+02], and in the context of the van der Waerden conjecture for the permanent [BG77].

In the present article, we propose to systematically study general permutation polytopes. These are defined as the convex hull of a subgroup G of the group of $n \times n$ permutation matrices. This is a convex geometric invariant of a permutation representation, and it yields various numerical invariants like dimension, volume, diameter, f -vector, etc.

A number of authors have studied special classes of permutation polytopes different from the Birkhoff polytope. Brualdi and Liu [BL91] compute basic invariants of the polytope of the alternating group; for this polytope, Hood and Perkinson [HP04] describe exponentially many facets. Collins and Perkinson [CP04] observe that Frobenius polytopes have a particularly simple combinatorial structure, and Steinkamp [Ste99] adds results about dihedral groups. Most recently, Guralnick and Perkinson [GP06] investigate general permutation polytopes, their dimension, and their graph.

Main results. In Section 1, we introduce the main objects of our study, representation polytopes and permutation polytopes. We also add a note on 0/1-polytopes, Proposition 1.3, showing that there is only a finite number of lattice equivalence classes of d -dimensional 0/1-polytopes.

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In Section 2, we discuss notions of equivalence of representations and the associated representation polytopes, respectively permutation polytopes. In particular, we introduce stable and effective equivalence of representations.

In Section 3, we investigate combinatorial properties of permutation polytopes and their faces. In particular, we are interested in the question which polytopes can be realized as faces of permutation polytopes. The first main theorem, Theorem 3.5, says that, if a permutation polytope is combinatorially a product, then the permutation group has a natural product structure. As a second result, we give an explicit construction, Theorem 3.8, showing that pyramids over faces of permutation polytopes appear again as faces of permutation polytopes. Further, we are interested in centrally symmetric faces and polytopes. We show in Theorem 3.10 that free sums of crosspolytopes and cubes occur as faces of permutation polytopes. Finally, we construct in Theorem 3.15 the essentially unique permutation group where the permutation polytope is a crosspolytope.

In Section 4, we use the results from the previous section to classify in Theorem 4.1 up to effective equivalence all permutation representations whose polytopes have dimension ≤ 4 . We also start the more difficult classification of combinatorial types of low dimensional polytopes which appear as faces of permutation polytopes. Theorem 4.3 gives a complete answer for dimension ≤ 3 , and there remain only two 4-polytopes for which we could not decide whether or not they can be realized. The lists of permutation groups and polytopes can also be found on the webpage [BHNP07].

Those examples and classifications suggest a number of open questions and conjectures which we formulate in Section 5.

Remark: Some authors use the notion of permutation polytope differently: the convex hull of a G -orbit in \mathbb{R}^n . These are linear projections of our permutation polytopes. Examples include the permutahedron, the traveling salesman polytope, or any polytope with a vertex transitive group of automorphisms. They appear in combinatorial optimization problems of various computational complexities [Onn93]. Moreover, orbit polytopes have recently been used to construct resolutions in group cohomology [EHS06].

1. REPRESENTATION-, PERMUTATION-, AND 0/1-POLYTOPES

1.1. Polytopes. For a standard reference on polytopes we refer to [Zie95]. A polytope P is the convex hull $\text{conv}(S)$ of a finite set of points S in a real vector space V . The dimension $\dim P$ is the dimension of the affine hull $\text{aff } P$ as an affine space. We say P is a d -polytope, if $\dim P = d$. If V is equipped with a full dimensional lattice Λ and we can choose $S \subset \Lambda$, then we call P a lattice polytope.

A face F of P (denoted by $F \preceq P$) is a subset where some linear functional is maximized. Zero-dimensional faces are vertices, one-dimensional faces are edges, and faces of codimension one are facets. The poset of faces ordered by inclusion is called the face lattice. The vertex set of P is denoted by $\mathcal{V}(P)$. The degree of a vertex is the number of edges it is contained in.

There is a hierarchy of equivalence relations on (lattice) polytopes. Two polytopes $P \subset \mathbb{R}^m$ and $Q \subset \mathbb{R}^n$ are affinely equivalent if there is an affine isomorphism of the affine hulls $\phi: \text{aff } P \rightarrow \text{aff } Q$ that maps P onto Q . For lattice equivalence we additionally require that ϕ is an isomorphism of the affine lattices $\text{aff } P \cap \Lambda \rightarrow \text{aff } Q \cap \Lambda'$. Combinatorial equivalence is merely an equivalence of the face lattices as posets.

$$\begin{array}{ccccc} \text{lattice} & & \text{affinely} & & \text{combinatorially} \\ \text{equivalent} & \Rightarrow & \text{equivalent} & \Rightarrow & \text{equivalent} \end{array}$$

The converse implications do not hold, for examples see [Zie00, Prop. 7].

1.2. Representation polytopes. Let $\rho: G \rightarrow \text{GL}(V)$ be a real representation of the finite group G with identity element e . It induces an \mathbb{R} -algebra homomorphism from the group algebra $\mathbb{R}[G]$ to $\text{End}(V)$, which we also denote by ρ .

Definition 1.1. The *representation polytope* $P(\rho)$ of the representation ρ is defined as the convex hull of $\rho(G)$ in the vector space $\text{End}(V)$.

Notice, that the representation ρ splits as a G -representation over \mathbb{C} into irreducible components:

$$(1.1) \quad \rho \cong \sum_{\sigma \in \text{Irr}(G)} c_{\sigma} \sigma$$

with $c_{\sigma} \in \mathbb{N}$ for σ in $\text{Irr}(G)$, the set of pairwise non-isomorphic irreducible \mathbb{C} -representations. We define the set of *irreducible factors* of ρ ,

$$\text{Irr}(\rho) := \{\sigma \in \text{Irr}(G) : c_{\sigma} > 0\}.$$

The group G acts on the polytope $P(\rho)$ by left multiplication, inducing an affine automorphism of $P(\rho)$:

$$g(\rho(h)) = \rho(g)\rho(h) \text{ for all } g, h \in G.$$

Therefore, since any vertex of $P(\rho)$ has to be contained in $\rho(G)$, and left multiplication on G is regular, thus transitive, we get:

$$(1.2) \quad \mathcal{V}(P(\rho)) = \rho(G).$$

Here is one application (the case of equality is treated in Corollary 2.8):

$$(1.3) \quad \dim P(\rho) \leq |\mathcal{V}(P(\rho))| - 1 \leq |G| - 1.$$

In particular, though there are infinitely many representations of finite groups of fixed order, they give rise to only finitely many combinatorial types of representation polytopes. We are going to see a stronger statement in Corollary 1.4.

More implications: All vertices of $P(\rho)$ have the same degree. When considering the combinatorics of a face of $P(\rho)$ we can always assume that it has $\rho(e) = \text{id}$ as a vertex. If F is a face of $P(\rho)$ with vertex set $\mathcal{V}(F) = \rho(U)$ for $U \subseteq G$, then F is also a face of the representation polytope $P(\rho')$, where $\rho': \langle U \rangle \rightarrow \text{GL}(V)$. Here $\langle U \rangle$ denotes the smallest subgroup of G containing U .

1.3. Permutation polytopes. We identify the symmetric group S_n on $\{1, \dots, n\}$ via the usual permutation representation with the set of $n \times n$ permutation matrices, i.e., the set of matrices with entries 0 or 1 such that in any column and any row there is precisely one 1. Throughout, we use cycle notation: For instance $(123)(45) \in S_6$ denotes the permutation $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$, $4 \mapsto 5$, $5 \mapsto 4$, $6 \mapsto 6$. Note that for $g_1, g_2 \in S_n$ we have $g_1 g_2 \in S_n$, while $g_1 + g_2 \in \text{Mat}_n(\mathbb{N})$. Here, for a set $C \subseteq \mathbb{C}$, we define $\text{Mat}_n(C)$ as the set of $n \times n$ matrices with entries in C . We identify $\text{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, thus we have the usual scalar product, i.e., $\langle A, B \rangle = \sum_{i,j} A_{i,j} B_{i,j}$ for $A, B \in \text{Mat}_n(\mathbb{R})$.

A subgroup of S_n is called permutation group. A faithful representation $\rho: G \rightarrow S_n$ is called permutation representation, thus G can be identified with the permutation group $\rho(G)$. For both situations we often write in short $G \leq S_n$.

Definition 1.2. For $G \leq S_n$ we define $P(G) := \text{conv}(G) \subseteq \text{Mat}_n(\mathbb{R})$, the *permutation polytope* associated to G . The convex hull of all permutations $B_n := P(S_n)$ is called the n^{th} *Birkhoff polytope*.

In particular, any permutation polytope is a representation polytope, as well as a lattice polytope with respect to the lattice $\text{Mat}_n(\mathbb{Z})$.

1.4. 0/1-polytopes. An important property of permutation polytopes is that they belong to the class of 0/1-polytopes. A 0/1-polytope is the convex hull of points in $\{0, 1\}^d$. They have been classified up to dimension 6 [Aic00, Aic07]. For a survey on these well-studied polytopes see [Zie00], where also the following basic fact is shown: any d -dimensional 0/1-polytope is affinely equivalent to a lattice polytope in $[0, 1]^d$. This implies immediately that there are only finitely many affine types of d -dimensional 0/1-polytopes. Even more is true.

Proposition 1.3. *Every d -dimensional 0/1-polytope is lattice equivalent to a lattice polytope in the 2^{2^d} -dimensional unit cube. In particular, there are only finitely many lattice types of d -dimensional 0/1-polytopes.*

This bound is far from optimal. All we care about is that it is finite. We have not found this result in the literature, so we include the proof.

Proof. Every d -dimensional 0/1-polytope is lattice equivalent to a full-dimensional lattice polytope which will, in general, no longer have 0/1 coordinates. But it still has the property that its vertices are the only lattice points it contains. Such a polytope can have no more than 2^d vertices, as two vertices with the same parity would have an integral midpoint.

Now suppose $P \subset [0, 1]^N$ is a d -dimensional 0/1-polytope with $N > 2^{2^d}$. Then there are two of the N coordinates which agree for every vertex of P , say, $P \subset \{x_i = x_j\}$. Thus we can delete the j^{th} coordinate and obtain a lattice equivalent 0/1-polytope $\subset [0, 1]^{N-1}$. \square

Corollary 1.4. *There are up to lattice equivalence only finitely many permutation polytopes associated to finite groups of fixed order.*

This follows from Equation (1.3).

2. NOTIONS OF EQUIVALENCE

Throughout let $\rho: G \rightarrow \mathrm{GL}(V)$ be a real representation.

2.1. Stable equivalence of representations. When working with permutation polytopes, one would like to identify representations that define affinely equivalent polytopes. For instance, this holds for the following five permutation groups: $\langle(1234)\rangle \leq S_4$, $\langle(1234)(5)\rangle \leq S_5$, $\langle(1234)(56)\rangle \leq S_6$, $\langle(1234)(56)(78)\rangle \leq S_8$, $\langle(1234)(5678)\rangle \leq S_8$. We are now going to introduce a suitable notion of equivalence on the real representations of a finite group. The crucial point is the observation that representation polytopes do not care about multiplicities of irreducible factors in the defining representation. For this let us fix a finite group G .

Definition 2.1. For a representation $\rho: G \rightarrow \mathrm{GL}(V)$ define the affine kernel $\ker^\circ \rho$ as

$$\ker^\circ \rho := \left\{ \sum_{g \in G} \lambda_g g \in \mathbb{R}[G] : \sum_{g \in G} \lambda_g \rho(g) = 0 \text{ and } \sum_{g \in G} \lambda_g = 0 \right\}$$

Say that a real representation $\rho': G \rightarrow \mathrm{GL}(V')$ is an *affine quotient* of ρ if $\ker^\circ \rho \subseteq \ker^\circ \rho'$.

Then real representations ρ_1 and ρ_2 of G are *stably equivalent*, if there are affine quotients ρ'_1 of ρ_1 and ρ'_2 of ρ_2 such that $\rho_1 \oplus \rho'_1 \cong \rho_2 \oplus \rho'_2$ as G -representations. For instance, ρ_1 is stably equivalent to $\rho_1 \oplus \rho'_1$.

Example 2.2. Let 1_G be the trivial representation of G . We observe that $\ker^\circ \rho = \ker 1_G \cap \ker \rho$. Hence, by Equation (1.1)

$$\ker^\circ \rho = \ker 1_G \cap \bigcap_{1_G \neq \sigma \in \mathrm{Irr}(\rho)} \ker \sigma.$$

Therefore, any real representation $\rho': G \rightarrow \mathrm{GL}(V')$ with $\mathrm{Irr}(\rho') \setminus \{1_G\} \subseteq \mathrm{Irr}(\rho) \setminus \{1_G\}$ is an affine quotient of ρ . For instance, ρ' may be the restriction of ρ to an invariant subspace of V .

Proposition 2.3. *Suppose ρ and $\bar{\rho}$ are stably equivalent real representations of a finite group G . Then $P(\rho)$ and $P(\bar{\rho})$ are affinely equivalent.*

Proof. It is enough to show that $P(\rho)$ and $P(\rho \oplus \rho')$ are affinely equivalent for an affine quotient ρ' of ρ .

The projection yields an affine map $P(\rho \oplus \rho') \rightarrow P(\rho)$. In order to construct an inverse, we need a map $\mathrm{aff} P(\rho) \rightarrow \mathrm{aff} P(\rho')$. The obvious choice is to map a point $\sum_{g \in G} \lambda_g \rho(g) \in \mathrm{aff} P(\rho)$ to $\sum_{g \in G} \lambda_g \rho'(g) \in \mathrm{aff} P(\rho')$ ($\sum_{g \in G} \lambda_g = 1$). This is well defined if (and only if) $\ker^\circ \rho \subseteq \ker^\circ \rho'$. \square

A priori, it is often not clear whether two representations are stably equivalent. Here we provide an explicit criterion:

Theorem 2.4. *Two real representations are stably equivalent if and only if they contain the same non-trivial irreducible factors.*

The proof will be given in the next subsection. We note that we have already seen the if-direction in Example 2.2.

2.2. The dimension formula. To prove Theorem 2.4 we recall the dimension formula of a representation polytope in [GP06].

The following equation is Theorem 3.2 of [GP06] (recall that the degree of a representation is the dimension of the vector space the group is acting on).

Theorem 2.5 (Guralnick, Perkinson).

$$\dim P(\rho) = \sum_{1_G \neq \sigma \in \text{Irr}(\rho)} (\deg \sigma)^2.$$

The proof relies on the Theorem of Frobenius and Schur [CR62, (27.8-10)] to determine the dimension of $\rho(\mathbb{C}[G])$, and then relates $\dim P(\rho)$ to $\dim_{\mathbb{C}} \rho(\mathbb{C}[G])$ via the following observation which explains the special role of the trivial representation.

Lemma 2.6. *The affine hull of $P(\rho)$ does not contain 0 if and only if $1_G \in \text{Irr}(\rho)$.*

Now, we can give the proof of the characterization of stable equivalence:

Proof of Theorem 2.4. It is enough to show that ρ and $\rho \oplus \rho'$ have the same non-trivial irreducible factors for an affine quotient ρ' of ρ .

By Proposition 2.3 $P(\rho)$ and $P(\rho \oplus \rho')$ are affinely equivalent. In particular, they have the same dimension. Since any irreducible factor of ρ is an irreducible factor of $\rho \oplus \rho'$, the dimension formula Theorem 2.5 implies that any non-trivial irreducible factor of ρ' already appears as an irreducible factor of ρ . Therefore, ρ and ρ' have the same non-trivial irreducible factors. \square

For an application let us look at the regular representation of a group G . This is the permutation representation $\text{reg}: G \rightarrow S_{|G|}$ via right multiplication. We have $\text{Irr}(\text{reg}) = \text{Irr}(G)$.

Lemma 2.7. *$P(\text{reg})$ is a simplex of dimension $|G| - 1$, and the vertices form a lattice basis of the lattice $\text{lin } P(\text{reg}) \cap \text{Mat}_{|G|}(\mathbb{Z})$.*

Proof. For this we enumerate the elements of G as $g_1, \dots, g_{|G|}$ with $g_1 = e$. Then for $i \in \{1, \dots, |G|\}$ the permutation matrix $\rho(g_i)$ of size $|G| \times |G|$ has in the first row only zeros except one 1 in column i . Hence, the matrices $g_1, \dots, g_{|G|}$ are linearly independent. Moreover, this shows that they form a lattice basis of $\text{lin } P(\text{reg}) \cap \text{Mat}_{|G|}(\mathbb{Z})$. \square

The dimension formula and Lemma 2.7 imply another proof of the following well-known equation:

$$|G| - 1 = \dim P(\text{reg}) = \sum_{1_G \neq \sigma \in \text{Irr}(G)} (\deg \sigma)^2.$$

We see that in a special case there is indeed a correspondence between stable equivalence and affine equivalence, this is [GP06, Cor. 3.3].

Corollary 2.8. *Let ρ be a faithful representation. Then $P(\rho)$ is a simplex if and only if ρ is stably equivalent to reg .*

Here is an example showing that stably equivalent permutation representations do not necessarily have lattice equivalent permutation polytopes.

Example 2.9. Let $G := \langle (12), (34) \rangle \leq S_4$. We define the following permutation representation: $\rho: G \rightarrow S_6$, by $(12) \mapsto (12)(34)$ and $(34) \mapsto (12)(56)$. Then $P(\rho)$ is a tetrahedron, and ρ is stably equivalent to the regular representation. However, the vertices of $P(\rho)$ do *not* form an affine lattice basis of the lattice $\text{aff } P(\text{reg}) \cap \text{Mat}_G(\mathbb{Z})$, in contrast to $P(\text{reg})$ by Lemma 2.7.

Affine equivalence is the same as lattice equivalence for the sublattice generated by the vertices. The previous example shows that lattice equivalence for the whole lattice $\text{Mat}_n(\mathbb{Z})$ is a more subtle condition. This relation deserves further study.

2.3. Effective equivalence of representations. The following example illustrates that stable equivalence is too rigid.

Example 2.10. Let $G := \langle (12), (34) \rangle \leq S_4$. Then $\rho_1: G \hookrightarrow S_4$ is a permutation representation with $P(G) = P(\rho_1)$ a square. On the other hand, we define another permutation representation $\rho_2: G \rightarrow S_4$, by $(12) \mapsto (12)$ and $(34) \mapsto (12)(34)$. Then $P(\rho_2)$ is the same square. However, ρ_1 and ρ_2 are *not* stably equivalent, since they do not have the same irreducible factors.

We observe that these two representations $\rho_1, \rho_2: G \rightarrow \text{GL}(V)$ are conjugated, i.e., there exists an automorphism ψ of G such that $\rho_2 = \rho_1 \circ \psi$. Hence, since $\rho_1(G) = \rho_2(G)$, we have $P(\rho_1) = P(\rho_2)$. However, conjugation permutes the irreducible factors, thus does not respect stable equivalence. To avoid this ambiguity we propose the following notion.

Definition 2.11. Two real representations $\rho_i: G_i \rightarrow \text{GL}(V_i)$ (for $i = 1, 2$) of finite groups are *effectively equivalent*, if there exists an isomorphism $\phi: G_1 \rightarrow G_2$ such that ρ_1 and $\rho_2 \circ \phi$ are stably equivalent G_1 -representations.

Moreover, we say $G_1 \leq S_{n_1}$ and $G_2 \leq S_{n_2}$ are *effectively equivalent* permutation groups, if $G_1 \hookrightarrow S_{n_1}$ and $G_2 \hookrightarrow S_{n_2}$ are effectively equivalent permutation representations.

By Theorem 2.4 we may put this definition in a nutshell: Two permutation groups are effectively equivalent if they are isomorphic as abstract groups such that via this isomorphism the permutation representations contain the same non-trivial irreducible factors.

In particular, effectively equivalent representations have affinely equivalent representation polytopes by Proposition 2.3. Of course, in general the converse cannot hold, since by Lemma 2.7 permutation groups that are not even isomorphic as abstract groups still may have affinely equivalent permutation polytopes. However, the following question remains open.

Question 2.12. Are there permutation groups G_1, G_2 that are isomorphic as abstract groups and whose permutation polytopes $P(G_1)$ and $P(G_2)$ are affinely equivalent, while G_1 and G_2 are not effectively equivalent?

By Theorem 4.1 there are no such permutation groups, if their permutation polytopes have dimension ≤ 4 .

3. THE COMBINATORICS OF PERMUTATION POLYTOPES

Throughout, $G \leq S_n$ is a permutation group. By $G \cong H$ we denote an (abstract) group isomorphism.

3.1. The smallest face containing a pair of vertices.

In [BG77, BL91] the diameter of the edge-graph of B_n and $P(A_n)$ was bounded from above by 2. Later this could be generalized in [GP06] to permutation polytopes associated to transitive permutation groups. For this, Guralnick and Perkinson needed a crucial observation, that we are going to recall here.

Definition 3.1. Let $e \neq g \in G$.

- The support $\text{supp}(g)$ is the complement of the set of fixed points.
- We denote by F_g the smallest face of $P(G)$ containing e and g .
- We denote by $g = z_1 \circ \dots \circ z_r$ the unique disjoint cycle decomposition of g in S_n , i.e., z_1, \dots, z_r are cycles with pairwise disjoint support, and $g = z_1 \cdots z_r$.
- Let $g = z_1 \circ \dots \circ z_r$. For $h \in S_n$ we say h is a *subelement* of g , if there is a subset $I \subseteq \{1, \dots, r\}$ such that $h = \prod_{i \in I} z_i$.
- g is called *indecomposable* in G , if e and g are the only subelements of g in G .

The following result is Theorem 3.5 in [GP06]. We include the very instructive proof here.

Theorem 3.2 (Guralnick, Perkinson). *Let $g \in G$. The vertices of F_g are precisely the subelements of g in G . In particular, e and g form an edge of $P(G)$ if and only if g is indecomposable in G .*

Therefore, the number of indecomposable elements (different from e) in G equals the degree of any vertex of $P(G)$.

Remark 3.3. The proof of Guralnick and Perkinson uses a simple but effective way of defining certain faces of a permutation polytope $P(G)$ for $G \leq S_n$. These faces are the intersections of $P(G)$ with faces of the Birkhoff polytope $P(S_n)$. Since this method will also be used for several results of this paper, we give here the explicit description.

Let $S \subseteq S_n$ be a subset of permutation matrices. We define the $n \times n$ -matrix $M(S) := \max(\sigma : \sigma \in S)$, where the maximum is applied for any entry. Then $M(S)$ has only entries in $\{0, 1\}$, thus $\langle M(S), g \rangle \leq n$ for any $g \in G$. Therefore, $F(S) := \{g \in G : \langle M(S), g \rangle = n\}$ is a face of $P(G)$. If $S \subseteq G$, then $S \subseteq \mathcal{V}(F(S))$.

If $S \subseteq G$ and $|S| \leq 2$, then $F(S)$ is even the smallest face of $P(G)$ containing S . This is part of the proof of Theorem 3.2.

While in the case of Birkhoff polytopes this implication holds also for $|S| \geq 3$, it is important to note that in the case of general permutation

polytopes it usually fails. The following example illustrates this phenomenon. Let $z_1 := (12)$, $z_2 := (34)$, $z_3 := (56)$, $z_4 := (78)$. We define $G := \langle z_1 z_2, z_1 z_3, z_1 z_4 \rangle \leq S_8$, and $S := \{e, z_1 z_2, z_1 z_3\}$. Then $P(G)$ is a four-dimensional crosspolytope, i.e., the dual is a 4-cube, and $P(G)$ contains a face with vertices S . However, $F(S)$ also contains the vertex $z_2 z_3$, so it is not the smallest face of $P(G)$ containing S .

Proof of Theorem 3.2. Let $S := \{e, g\} \subseteq G$, and $F(S)$ the face of $P(G)$ as defined in the previous remark. Then the vertices of $F(S)$ are precisely the subelements of g in G . On the other hand, let $g = z_1 \circ \cdots \circ z_r$, and $h = z_1 \circ \cdots \circ z_s$ ($s \leq r$) be a subelement of g in G . Then $h' := gh^{-1} = z_{s+1} \circ \cdots \circ z_r$ is also a subelement of g in G . Now, the following identity of matrices holds:

$$e + g = h + h'.$$

Therefore, $F(S)$ is centrally symmetric with center $(e + g)/2$. Hence, $F(S)$ is the smallest face F_g of $P(G)$ containing $S = \{e, g\}$. \square

In particular we see from the proof that, if $g \in G$ and $h \in \mathcal{V}(F_g)$, then the antipodal vertex of h in the centrally symmetric face F_g is given by gh^{-1} with $\text{supp}(h) \cap \text{supp}(gh^{-1}) = \emptyset$. Let us note this important restriction on the combinatorics of a permutation polytope.

Corollary 3.4. *The smallest face containing a given pair of vertices of a permutation polytope is centrally symmetric.*

This generalizes the well-known fact (e.g., see [BS96, Thm. 2.5]) that the smallest face of the Birkhoff polytope containing a pair of vertices is a cube. This strong statement is not true for general permutation polytopes. For instance in Corollary 3.11 we show that crosspolytopes appear as faces of permutation polytopes.

3.2. Products. Products of permutation polytopes are again permutation polytopes, and therefore also products of faces of permutation polytopes appear as faces of permutation polytopes.

In many cases, given a permutation group G and its permutation polytope $P(G)$, we would like to know all the permutation groups H such that $P(H)$ is combinatorially equivalent to $P(G)$. In the case of products the following result shows that we can reduce this question to each factor.

Theorem 3.5. *$P(G)$ is a combinatorial product of two polytopes Δ_1 and Δ_2 if and only if there are subgroups H_1 and H_2 in G such that*

- (a) $P(H_i)$ is combinatorially equivalent to Δ_i for $i = 1, 2$,
- (b) $\text{supp}(H_1) \cap \text{supp}(H_2) = \emptyset$,
- (c) $G = H_1 \times H_2$.

Proof. The if-part is easy to see. We have to prove the only-if part.

Let $G \leq S_n$. By assumption, there is a map v from the vertex set $\mathcal{V}(P(G)) = G$ to $\mathcal{V}(\Delta_1 \times \Delta_2) = \mathcal{V}(\Delta_1) \times \mathcal{V}(\Delta_2)$, inducing an isomorphism between the face lattices of $P(G)$ and of $\Delta := \Delta_1 \times \Delta_2$, which we also denote by v . Hence, any element $g \in G$ can be labeled as $v(g) = (v_1(g), v_2(g)) \in \mathcal{V}(\Delta)$ for unique vertices $v_1(g) \in \mathcal{V}(\Delta_1)$ and $v_2(g) \in \mathcal{V}(\Delta_2)$.

We write $v(e) =: (e_1, e_2)$, and define $H_1 := \{g \in G : v_2(g) = e_2\}$, as well as $H_2 := \{g \in G : v_1(g) = e_1\}$.

We claim

$$(3.1) \quad \text{supp}(H_1) \cap \text{supp}(H_2) = \emptyset.$$

Let $h_1 \in H_1$ and $h_2 \in H_2$. We have $v(h_1) = (x_1, e_2)$ and $v(h_2) = (e_1, x_2)$ for $x_1 \in \mathcal{V}(\Delta_1)$ and $x_2 \in \mathcal{V}(\Delta_2)$. For $i = 1, 2$ let us denote by F_i the smallest face of Δ_i containing e_i and x_i . Let us define $g \in G$ with $v(g) = (x_1, x_2)$. Since $P(G)$ is via v combinatorially equivalent to Δ , the face $F_1 \times F_2 \prec \Delta$ is the smallest face of Δ containing $v(e)$ and $v(g)$.

By Corollary 3.4 the face $F_g \prec P(G)$, satisfying $v(F_g) = F_1 \times F_2$, is centrally symmetric, and $h'_1 := gh_1^{-1} \in G$ is the antipodal vertex to h_1 . Since $\text{supp}(h_1) \cap \text{supp}(h'_1) = \emptyset$, it suffices to show $h_2 = h'_1$.

Since $x_1 \times F_2$ is the smallest face of Δ containing $v(h_1)$ and $v(g)$, we get by central symmetry of F_g that the smallest face of Δ containing $v(h'_1)$ and $v(e)$ has also $|\mathcal{V}(F_2)|$ vertices, thus $|\mathcal{V}(F_{h'_1})| = |\mathcal{V}(F_2)| = |\mathcal{V}(F_{h_2})|$. Note that by Theorem 3.2 h_1 , h_2 and h'_1 are subelements of g , thus determined by their support.

Now, there are two cases, since $\text{supp}(g) = \text{supp}(h_1) \sqcup \text{supp}(h'_1)$ (here \sqcup denotes the disjoint union).

- (1) $\text{supp}(h'_1) \subseteq \text{supp}(h_2)$:

Then h'_1 is a subelement of h_2 . However, $|\mathcal{V}(F_{h'_1})| = |\mathcal{V}(F_{h_2})|$ implies $F_{h'_1} = F_{h_2}$. Therefore, h_2 is also a subelement of h'_1 , thus $h_2 = h'_1$, as desired.

- (2) $\text{supp}(h_1) \cup \text{supp}(h_2) \subsetneq \text{supp}(g)$:

As in Remark 3.3 for $S := \{e, h_1, h_2\}$ we define the matrix $M(S)$ and the face $F(S)$ of $P(G)$. Since $F_1 \times F_2$ is the smallest face of Δ containing $v(e), v(h_1), v(h_2)$, we get $F_g \subseteq F(S)$. However, by our assumption there exists some $i \in \text{supp}(g)$ with $i \notin \text{supp}(h_1) \cup \text{supp}(h_2)$. Therefore, the only non-zero entry in the i th-row of $M(S)$ is on the diagonal, while the i th diagonal entry of the permutation matrix g is zero. Hence, $\langle M(S), g \rangle < n$, a contradiction.

This proves the claim (3.1).

Hence, $|H_1 H_2| = |H_1| |H_2| = |\mathcal{V}(\Delta_1)| |\mathcal{V}(\Delta_2)| = |\mathcal{V}(P(G))| = |G|$. Therefore, $H_1 H_2 = G$. Moreover, this implies that H_1 consists precisely of all elements of G that have disjoint support from all elements in H_2 , hence is a subgroup. The analogous argument holds for H_2 . Finally, $P(G)$, $P(H_1) \times P(H_2)$, and $\Delta_1 \times \Delta_2$ are combinatorially equivalent. \square

As an application we classify those permutation groups whose d -dimensional permutation polytopes have the maximal number of vertices.

Corollary 3.6. *Let $d := \dim P(G)$. Then*

$$\log_2 |G| \leq d \leq |G| - 1,$$

or equivalently

$$d + 1 \leq |G| \leq 2^d.$$

Moreover the following statements are equivalent:

- (1) $|G| = 2^d$,
- (2) $P(G)$ is combinatorially a d -cube,
- (3) $P(G)$ is lattice equivalent to $[0, 1]^d$,
- (4) G is effectively equivalent to $\langle (12), \dots, (2d-1 \ 2d) \rangle \leq S_{2d}$.

Proof. As was noted before, $P(G)$ is as a d -dimensional 0/1-polytope that is combinatorially equivalent to a lattice subpolytope of $[0, 1]^d$. Hence we get the inequalities. Moreover, it holds (1) \Leftrightarrow (2), (3) \Rightarrow (2) and (4) \Rightarrow (2). From Theorem 3.5 (in particular, statement (b)) we deduce (2) \Rightarrow (3) and (2) \Rightarrow (4). \square

Here is another application of Theorem 3.5. For this note that by Theorem 3.2 a permutation polytope $P(G)$ is simple if and only if there are dimension many indecomposable elements in G . Now, the main result of [KW00] states that any simple 0/1-polytope is a product of simplices. Therefore we can deduce from this geometric statement using Corollary 2.8 a result in representation theory (since the dimension of $P(G)$ can be computed from the irreducible factors by Theorem 2.5).

Corollary 3.7. *Let ρ be a permutation representation of a group G . If G contains precisely $\dim P(\rho)$ many indecomposable elements, then G is the product of subgroups H_1, \dots, H_l with mutually disjoint support, where ρ restricted to any H_i ($i = 1, \dots, l$) is stably equivalent to the regular representation of H_i .*

3.3. Pyramids. In experiments one observes that most faces of permutation polytopes are actually pyramids over lower dimensional faces. Here, we prove that for any face F of a permutation polytope there exists a permutation polytope having a face that is combinatorially a pyramid over F .

Theorem 3.8. *Let $G \leq S_n$. Then there is a permutation group $E \leq S_{2n}$, with $E \cong G \times \mathbb{Z}_2$, such that there is a face of $P(E)$ which is combinatorially a pyramid over $P(G)$.*

Proof. Let $G \leq S_n$. Embedding the product of permutation groups $S_n \times S_n$ into S_{2n} , we define $H := \{(\sigma, \sigma) : \sigma \in G\} \leq S_{2n}$. Then H is a subgroup of S_{2n} , and effectively equivalent to G . We define an involution

$$p := (1 \ n+1) (2 \ n+2) \cdots (n \ 2n) \in S_{2n}.$$

Then p commutes with each element in H , moreover $H \cap \langle p \rangle = \{e\}$. Hence, $E := H \langle p \rangle$ is a subgroup of S_{2n} and isomorphic to $H \times \langle p \rangle$. For $S := H \sqcup \{p\} \subseteq E$ we define as in Remark 3.3 the $2n \times 2n$ -matrix $M(S)$ defining a face $F(S)$ of $P(E)$. We claim that $F(S)$ is a pyramid over $\text{conv}(H)$.

First let us show that $\mathcal{V}(F(S)) = S$. Assume that there is some $hp \in F(S)$ with $h \neq e$. Let $h = (\sigma, \sigma)$ for $\sigma \in G$, $\sigma \neq e$. Assume σ maps 1 to 2. Then hp maps 1 to $n+2$. However, this implies that $\langle M(S), hp \rangle < 2n$, a contradiction.

Now, it remains to show that H is the set of vertices of a face of $F(S)$. As in Remark 3.3 we define the face $F(H)$ of $P(E)$. By construction $p \notin F(H)$. Then $F(H) \cap F(S)$ is a face of $F(S)$ which contains H but not p . \square

Corollary 3.9. *Pyramids over faces of permutation polytopes appear as faces of permutation polytopes.*

3.4. Free sums. Recall that free sums are the combinatorially dual operation to products. For instance the free sum of d intervals is a d -crosspolytope, i.e., the centrally symmetric d -polytope with the minimal number $2d$ of vertices.

In general we cannot expect that free sums of arbitrary faces of permutation polytopes are again faces of permutation polytopes. Corollary 3.4 shows that already faces that are bipyramids have to be necessarily centrally symmetric. However, we can explicitly construct the following centrally symmetric polytopes as faces.

Theorem 3.10. *Let l, d be natural numbers. There exists a face of a permutation polytope that is combinatorially the free sum of an l -crosspolytope and a d -cube.*

Proof. Since for $l \geq 1$ an l -crosspolytope is the free sum of an $(l - 1)$ -crosspolytope and a 1-cube, we may assume $l, d \geq 1$. We set $n_0 := 3d$. Let $z_1, \dots, z_d \in S_{n_0}$ be disjoint 3-cycles. We define $G_0 := \langle z_1, \dots, z_d \rangle$, an elementary abelian 3-group of order 3^d . It contains $g_0 := z_1 \cdots z_d$. Let $V_0 := \{z_1^{k_1} \cdots z_d^{k_d} : k_i \in \{0, 1\}\}$. This is precisely the set of subelements of g_0 in G_0 , hence, by Theorem 3.2 V_0 is the vertex set of a face F_0 of $P(G_0)$. This face F_0 is combinatorially a d -cube.

Now, we proceed by induction for $i = 1, \dots, l$. We define

$$n_i := 2n_{i-1}, \quad H_i := \{(g, g) : g \in G_{i-1}\} \leq S_{n_i}, \quad g_i := (g_{i-1}, g_{i-1}) \in H_i, \\ p_i := (g_{i-1}, e) \in S_{n_i}, \quad p'_i := (e, g_{i-1}) \in S_{n_i}, \quad G_i := H_i \langle p_i \rangle \leq S_{n_i}.$$

Note that H_i, G_i are elementary abelian 3-groups. Moreover, $G_i \cong H_i \times \langle p_i \rangle$. Let V_i be the set of subelements of g_i in G_i , and F_i the smallest face of $P(G_i)$ containing e and g_i , thus V_i is the vertex set of F_i by Theorem 3.2. By induction hypothesis we know that F_{i-1} is combinatorially the free sum of an $i - 1$ -crosspolytope and a d -cube. We show that F_i is a bipyramid over F_{i-1} with apexes $\{p_i, p'_i\}$. We claim

$$(3.2) \quad V_i = \{(v, v) : v \in V_{i-1}\} \sqcup \{p_i, p'_i\}.$$

Let $(v, v') \in V_i$. Then v, v' are subelements of g_{i-1} in G_{i-1} . Assume $v \neq v'$, in particular $(v, v') \notin H_i$. Since $(v, v') \in G_i$ and p_i has order three, we have either (i) $(v, v') = hp_i$ or (ii) $(v, v') = hp_i^{-1}$ for some $h \in H_i$. Let $h = (g, g)$ for $g \in G_{i-1}$. This yields either $(v, v') = (gg_{i-1}, g)$ or $(v, v') = (gg_{i-1}^{-1}, g)$, in particular, $g = v'$ is a subelement of g_{i-1} . Now, the case (i) implies $g = e$, thus $(v, v') = p_i$, since otherwise gg_{i-1} cannot be a subelement of g_{i-1} , because g_{i-1} decomposes into disjoint 3-cycles by construction. In the case (ii) we see analogously $g = g_{i-1}$, thus $(v, v') = p'_i$. This proves the claim (3.2).

Since by Corollary 3.4 F_i is centrally symmetric with antipodal vertices p_i and p'_i , and since $p_i, p'_i \notin \text{aff } H_i$ by construction, the claim implies that the face F_i is a bipyramid over the convex hull of all elements (v, v) (for $v \in V_{i-1}$), which is affinely equivalent to F_{i-1} . \square

Corollary 3.11. *For any dimension d there is a face of a permutation polytope that is a d -crosspolytope.*

Example 3.12. As an illustration of the proof of Theorem 3.10, we show how to obtain the octahedron as a face of a permutation polytope. The octahedron is the free sum of an interval and a square, so $l = 1$ and $d = 2$. Therefore, we define $z_1 := (123)$, $z_2 := (456)$, and $G_0 := \langle z_1, z_2 \rangle \leq S_6$. Now, we set $z'_1 := (789)$, $z'_2 := (10\ 11\ 12)$, and $G_1 := \langle z_1 z'_1, z_2 z'_2, z_1^{-1} z_2 z'_1 \rangle \leq S_{12}$. Then for $g_1 := z_1 z_2 z'_1 z'_2$ the face $F_1 := F_{g_1}$ of $P(G)$ is an octahedron with the vertex set $V_1 = \{e, z_1 z'_1, z_2 z'_2, z_1 z_2 z'_1 z'_2, z_1 z_2, z'_1 z'_2\}$.

3.5. Centrally symmetric permutation polytopes. We will establish a one-to-one correspondence between centrally symmetric permutation polytopes on one hand and certain subspaces of \mathbb{F}_2^r on the other. We will liberally identify sets $I \subseteq [r] := \{1, \dots, r\}$ with their incidence vectors $I \in \mathbb{F}_2^r$.

Suppose $P(G)$ is centrally symmetric, and let $g_0 = z_1 \circ \dots \circ z_r \in G$ be the vertex opposite to e . Then $P(G) = F_{g_0}$, and by Theorem 3.2, every element of G is a subelement of g_0 . Applied to $g_0^{-1} = z_1^{-1} \circ \dots \circ z_r^{-1}$ this yields that for $i = 1, \dots, r$ we have $z_i^{-1} = z_i$, so that z_i is a cycle of length two. Hence, the elements of G have order two. They can be written as $z_I := \prod_{i \in I} z_i$ for certain $I \subseteq [r]$. (For example, $z_\emptyset = e$, and $z_{[r]} = g_0$.)

Multiplication in G corresponds to addition (symmetric difference) in \mathbb{F}_2^r . This means that the set of I such that $z_I \in G$ is a subspace of \mathbb{F}_2^r . Conversely, given such a subspace containing the all-ones-vector $[r]$, we obtain a centrally symmetric permutation polytope. We have just proved the second part of the following proposition. The first part follows from Corollary 3.4.

Proposition 3.13. *$P(G)$ is centrally symmetric if and only if there is a pair of vertices such that $P(G)$ is the smallest face containing them.*

In this case, G is an elementary abelian 2-group, in particular the number of vertices of $P(G)$ is a power of two.

If G is an elementary abelian 2-group, $P(G)$ does not have to be centrally symmetric, see Theorem 4.1.

Continuing our discussion of centrally symmetric permutation polytopes, we can reorder the elements of $[n]$ ($G \subseteq S_n$) so that the matrices in G are block diagonal with blocks $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (plus $n - 2r$ blocks (1)). Projecting to the r upper right entries of the 2×2 blocks is a lattice isomorphism to \mathbb{Z}^r . Reduction mod 2 yields, again, our subspace.

Proposition 3.14. *If $P(G)$ is centrally symmetric, then the free sum of $P(G)$ with itself is again a permutation polytope.*

Proof. Suppose $P(G)$ is centrally symmetric with corresponding subspace $V \subseteq \mathbb{F}_2^r$. Define a new subspace

$$\tilde{V} := \{(I, I) : I \in V\} \cup \{(I, [r] - I) : I \in V\} \subset \mathbb{F}_2^r \times \mathbb{F}_2^r$$

We claim that the permutation polytope $P(\tilde{G})$ of the corresponding permutation group \tilde{G} realizes the free sum of $P(G)$ with itself.

We work with the $2r$ upper right entries $(x, y) \in \mathbb{R}^r \times \mathbb{R}^r$. Both, the diagonal embedding $x \mapsto (x, x)$ and the “anti-diagonal” embedding $x \mapsto (x, 1 - x)$ include $P(G)$ into $P(\tilde{G})$ and exhaust all vertices. Their images intersect in the unique common interior point $\frac{1}{2}(1, 1)$. \square

While in any dimension d there exists a permutation polytope which is a d -cube, this is not true for its combinatorial dual, the d -dimensional crosspolytope.

Theorem 3.15. *There is a d -dimensional permutation polytope $P(G)$ that is combinatorially a crosspolytope if and only if d is a power of two. In this case, the effective equivalence class of G is uniquely determined.*

Proof. The fact that d must be a power of two follows from Proposition 3.13. In that case, existence follows from Proposition 3.14. So we only need to show uniqueness.

Let G be a permutation group such that $P(G)$ is a d -dimensional crosspolytope, $d = 2^k$. Then $|G| = 2^{k+1}$, and our subspace $V \subset \mathbb{F}_2^r$ has dimension $k + 1$. Choose generators g_0, g_1, \dots, g_k of G , i.e., a basis of V . Consider the $k \times r$ matrix with rows $g_1, \dots, g_k \in \mathbb{F}_2^r$. If there are two equal columns i, j then we can omit z_i without changing the effective equivalence class. On the other hand, there can be at most 2^k different columns. As remarked above, we can embed $P(G) \hookrightarrow \mathbb{R}^r$. So, in particular, $r \geq d = 2^k$. Hence, $r = d$, and this matrix simply lists all possible 0/1 vectors. It is, up to permutation of the columns, uniquely defined. \square

Example 3.16. From the proof we get an explicit description of the permutation groups defining d -dimensional crosspolytopes. For instance, let $d = 4$. Since $r = d = 4$, we have $g_0 = z_1 \circ z_2 \circ z_3 \circ z_4$, where we may choose $z_1 = (12)$, $z_2 = (34)$, $z_3 = (56)$, $z_4 = (78)$. Since $k = 2$, we get $G = \langle g_0, g_1, g_2 \rangle$, where g_1 and g_2 are given by the rows of the following matrix consisting of all possible vectors in $\{0, 1\}^2$:

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

So, $g_1 = z_2 z_4$, $g_2 = z_3 z_4$, and $G = \langle (12)(34)(56)(78), (34)(78), (56)(78) \rangle$.

4. CLASSIFICATION RESULTS IN LOW DIMENSIONS

4.1. Classification of ≤ 4 -dimensional permutation polytopes. We would like to classify all permutation polytopes of given small dimension d . For this we take a look at the list of Aichholzer [Aic07] of combinatorial types of 0/1-polytopes in small dimension. For any such 0/1-polytope we first check whether it has constant vertex degree and satisfies the condition of Corollary 3.4. Then we go through the list of groups of size equal to the given number of vertices. Now, using the theoretical results of the previous section we can deduce from the combinatorial structure of the polytope whether this polytope can be realized as a permutation polytope, and even determine all respective permutation groups up to effective equivalence.

Combin. type of $P(G)$	Isom. type of G	Effective equiv. type of G
triangle	$\mathbb{Z}/3\mathbb{Z}$	$\langle(123)\rangle$
square	$(\mathbb{Z}/2\mathbb{Z})^2$	$\langle(12), (34)\rangle$
tetrahedron	$\mathbb{Z}/4\mathbb{Z}$	$\langle(1234)\rangle$
tetrahedron	$(\mathbb{Z}/2\mathbb{Z})^2$	$\langle(12)(34), (13)(24)\rangle$
triangular prism	$\mathbb{Z}/6\mathbb{Z}$	$\langle(12), (345)\rangle$
cube	$(\mathbb{Z}/2\mathbb{Z})^3$	$\langle(12), (34), (56)\rangle$
4-simplex	$\mathbb{Z}/5\mathbb{Z}$	$\langle(12345)\rangle$
B_3	S_3	$\langle(12), (123)\rangle$
prism over tetrahedron	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$\langle(1234), (56)\rangle$
prism over tetrahedron	$(\mathbb{Z}/2\mathbb{Z})^3$	$\langle(12)(34), (13)(24), (56)\rangle$
4-crosspolytope	$(\mathbb{Z}/2\mathbb{Z})^3$	$\langle(12)(34), (34)(78), (56)(78)\rangle$
product of triangles	$(\mathbb{Z}/3\mathbb{Z})^2$	$\langle(123), (456)\rangle$
prism over triang. prism	$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\langle(12), (345), (67)\rangle$
4-cube	$(\mathbb{Z}/2\mathbb{Z})^4$	$\langle(12), (34), (56), (78)\rangle$

Table 1: Permutation polytopes in dimension ≤ 4

Theorem 4.1. *Table 1 contains the list of all permutation groups G with d -dimensional permutation polytope $P(G)$ for $d \leq 4$ up to effective equivalence.*

Proof. $d = 2$: The triangle and the square are the only two-dimensional 0/1-polytopes. If $P(G)$ is a triangle, then use Corollary 2.8. If $P(G)$ is a square, then use Corollary 3.6.

$d = 3$: There are 4 combinatorial types of three-dimensional 0/1-polytopes with constant vertex degree satisfying the condition of Corollary 3.4.

- (1) $P(G)$ is a tetrahedron: Then use Corollary 2.8.
- (2) $P(G)$ is a triangular prism: Then use Theorem 3.5.
- (3) $P(G)$ is an octahedron: Then $|G| = 6$, however by Proposition 3.13 $|G|$ has to be a power of two, a contradiction.
- (4) $P(G)$ is a cube: Then use Corollary 3.6.

$d = 4$: There are 9 combinatorial types of four-dimensional 0/1-polytopes with constant vertex degree satisfying the condition of Corollary 3.4.

- (1) $P(G)$ is a 4-simplex: Then use Corollary 2.8.
- (2) $P(G)$ is combinatorially equivalent to the Birkhoff polytope $B_3 = P(S_3)$. Then $|G| = 6$. There are two cases:
 If $G \cong \mathbb{Z}/6\mathbb{Z}$, then either G is generated by an element of order 6, thus G is effectively equivalent to $\langle(123456)\rangle \leq S_6$, or G is generated by two elements of orders 2 and 3. In the latter case, the two elements necessarily have disjoint support, since G is abelian. In the first case $P(G)$ is a 5-simplex, in the second case $P(G)$ is a prism. Both cases yield contradictions.

Hence, $G \cong S_3$. Now, $\text{Irr}(S_3) = \{1_{S_3}, \rho_1, \rho_2\}$ with $\deg \rho_1 = 1$ and $\deg \rho_2 = 2$. By the dimension formula the permutation representation ρ associated to the permutation group G can only have ρ_2 as an irreducible factor. Therefore, Theorem 2.4 implies that G is effectively equivalent to $\langle (12), (123) \rangle$.

- (3) $P(G)$ is a prism over a tetrahedron: Use Theorem 3.5 and the classification for $d = 3$.
- (4) $P(G)$ is a 4-crosspolytope: See Example 3.16.
- (5) $P(G)$ is a product of two triangles: Use Theorem 3.5.
- (6) $P(G)$ is a prism over the triangular prism: Use Theorem 3.5 and the classification for $d = 3$.
- (7) $P(G)$ is a 4-cube: Use Corollary 3.6.
- (8) $P(G)$ is a prism over the octahedron: Then Proposition 3.13 yields that the number of vertices has to be a power of two, but $P(G)$ has 12 vertices, a contradiction.
- (9) $P(G)$ is a hypersimplex: Any vertex is contained in precisely three facets that are octahedra. Since the inversion map on G is given by the transposition map on $P(G)$, it induces an automorphism of $P(G)$ of order two, and hence, since 3 is odd, there has to be an octahedron F that contains e and whose vertex set is invariant under inversion.

Let g be the unique vertex of F opposite to e . Hence, g is fixed by the inversion map, so $g = g^{-1}$. As already noted in the proof of Proposition 3.13, this yields that any vertex of $F = F_g$ (besides e) has order two, so there are at least five elements of order two in G . On the other hand, since 5 divides $|G| = 10$, there exists a subgroup of order 5, so we conclude that there are precisely four elements of G that have order 5.

Now take $F' (\neq F)$ as one of the other two octahedra that contain e . We denote by h the unique vertex $h (\neq g)$ in F' opposite to e . As just seen, the order of h has to be 5. Since h has to be a product of disjoint cycles of order 5, any subelement of $h (\neq e)$ also has order 5, so by Theorem 3.2 there are at least five vertices of $F' = F_h$ of order 5, a contradiction. \square

Remark 4.2. Another approach following Theorem 2.4 and Corollary 3.6 would be to determine all abstract groups G of order $\leq 2^d$ and to calculate the finite set $\text{Irr}(G)$. Then for any subset $S \subseteq \text{Irr}(G)$ it would be enough to find, if possible, some permutation representation ρ with $\text{Irr}(\rho) = S$. However, the last task seems to be neither practically nor theoretically easy to achieve, compare [BP98].

4.2. Classification of ≤ 4 -dimensional faces. Compared to the classification of permutation polytopes the question whether a given 0/1-polytope is combinatorially equivalent to the face of some permutation polytope is much more difficult. If the answer is supposed to be positive, then one has to construct an explicit permutation group, the dimension of whose permutation polytope might increase dramatically. A systematic way to perform this task is yet to be discovered. However, to show that the answer is negative is even more challenging, since we lack good combinatorial obstructions of the type given in Corollary 3.4.

Theorem 4.3. *The following list contains all combinatorial types of d -dimensional 0/1-polytopes for $d \leq 4$ that may possibly appear as faces F of some permutation polytope $P(G)$:*

- $d = 2$: *There are 2 combinatorial types realized as F : triangle and square. Both appear as faces of Birkhoff polytopes.*
- $d = 3$: *There are 5 combinatorial types realized as F : tetrahedron, square pyramid, triangular prism, cube, and octahedron. The first four appear as faces of Birkhoff polytopes.*
- $d = 4$: *There are 21 combinatorial types that may possibly appear as F :*
 - (a) *11 of these appear as faces of Birkhoff polytopes: 4-simplex, pyramid over square pyramid, Birkhoff polytope B_3 (free sum of two triangles), pyramid over prism over triangle, wedge W over base edge of square pyramid, pyramid over cube, prism over tetrahedron, product of two triangles, prism over square pyramid, product of triangle and square, 4-cube.*
 - (b) *8 of these can be realized as F : 4-crosspolytope, prism over octahedron, pyramid over octahedron, bipyramid over cube, wedge over the facet of an octahedron, dual of W (see (a)), hypersimplex (the combinatorial type of $\{x \in [0, 1]^5 : \sum_{i=1}^5 x_i = 2\}$), and one special 0/1-polytope P .*
 - (c) *2 of these are given by special 0/1-polytopes Q_1, Q_2 , where it is unknown, if they have a realization as F .*

The description of the combinatorial types of P, Q_1, Q_2 can be found in Table 2.

Proof. By [BS96] any d -dimensional face F of some Birkhoff polytope is already realized in B_{2d} . Hence, by looking at the faces of B_6, B_8 , we find all combinatorial types of ≤ 4 -dimensional faces of Birkhoff polytopes.

Table 2: Vertex-Facet-Incidences of P, Q_1, Q_2

P : f-vector (8, 21, 22, 9)	Q_1 : f-vector (7, 19, 23, 11)	Q_2 : f-vector (8, 25, 32, 15)
[0 2 5 6]	[0 1 2 3 4]	[0 1 2 3 4 5]
[0 2 4 6]	[1 2 3 4 5]	[2 4 5 6]
[0 1 2 3 4]	[0 1 2 5]	[1 3 5 6]
[0 1 2 3 5]	[1 3 5 6]	[1 2 5 6]
[1 3 4 7]	[0 1 3 6]	[0 1 2 6]
[1 3 5 7]	[0 1 5 6]	[0 1 3 7]
[2 3 4 6 7]	[0 2 5 6]	[0 1 6 7]
[2 3 5 6 7]	[2 4 5 6]	[0 2 6 7]
[0 1 4 5 6 7]	[0 2 4 6]	[0 2 4 7]
	[3 4 5 6]	[0 3 4 7]
	[0 3 4 6]	[2 4 6 7]
		[1 3 6 7]
		[3 5 6 7]
		[4 5 6 7]
		[3 4 5 7]

Let us now consider the general case of a d -dimensional face of a permutation polytope.

$d = 2$: The triangle and the square are the only two-dimensional 0/1-polytopes.

$d = 3$: There are 5 combinatorial types of three-dimensional 0/1-polytopes that satisfy the condition of Corollary 3.4. Since the first four are realized as Birkhoff polytopes, we only have to deal with the octahedron. This was done in Example 3.12.

$d = 4$: There are 21 combinatorial types of four-dimensional 0/1-polytopes that satisfy the condition of Corollary 3.4. Of these, 11 can be realized as faces of Birkhoff polytopes. Here are the remaining 10 cases:

- (1) The 4-crosspolytope: See Table 1.
- (2) The prism over an octahedron: The octahedron is a face of a permutation polytope, so also the prism is.
- (3) Pyramid over octahedron: See Corollary 3.9 and the classification for $d = 3$.
- (4) Bipyramid over cube: See Theorem 3.10.
- (5) The dual of W : Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ be eight 3-cycles with pairwise disjoint support, realized as elements in S_{24} . Let e_1 be an involution that exchanges a_1 and a_2 , i.e., for $a_1 = (123)$ and $a_2 = (456)$, we define $e_1 := (14)(25)(36)$. In the same way we define e_2 as the involution exchanging b_1 and b_2 , and d_1 exchanging a_3 and a_4 , and d_2 exchanging b_3 and b_4 . Then we define $v_1 := a_1a_2a_3a_4$, $v_2 := b_1b_2b_3b_4$, $v_3 := d_1d_2$, and $v_4 := e_1e_2$. Let $G := \langle v_1, v_2, v_3, v_4 \rangle \leq S_{24}$. Then $|G| = 36$. As in Remark 3.3 we define the face $F := F(\{e, v_1, v_2, v_3, v_4\})$ of $P(G)$. Now, we check using `GAP` and `polymake` that the combinatorial type of F is indeed the dual of W .
- (6) P : Let $a_1, a_2, b_1, b_2, c_1, c_2$ be six 3-cycles with pairwise disjoint support, realized as elements in S_{18} . Let e_1 be an involution that exchanges b_1 and b_2 , as before, and e_2 an involution that exchanges c_1 and c_2 . Then we define $v_1 := a_1a_2$, $v_2 := b_1b_2c_1c_2$, $v_3 := a_1b_1b_2$, $v_4 := e_1e_2$. Let $G := \langle v_1, v_2, v_3, v_4 \rangle \leq S_{18}$. Then $|G| = 54$. As in Remark 3.3 we define the face $F := F(\{e, v_1, v_2, v_4\})$ of $P(G)$. Now, we check that the combinatorial type of F is indeed P .
- (7) The wedge over the facet of an octahedron: Let $a_1, a_2, b_1, b_2, c_1, c_2, e_1, v_1, v_2, v_3$ be defined as for P . However $v_4 := e_1$. Then $G := \langle v_1, v_2, v_3, v_4 \rangle \leq S_{18}$ with $|G| = 54$. We define F as before. Now, we check that the combinatorial type of F is indeed as desired.
- (8) Hypersimplex: Let a_1, a_2, a_3, a_4, a_5 be five 3-cycles with pairwise disjoint support, realized as elements in S_{15} . We define $v_1 := a_1a_2$, $v_2 := a_3a_4$, $v_3 := a_1a_3$, $v_4 := a_1a_5$. Let $G := \langle v_1, v_2, v_3, v_4 \rangle \leq S_{15}$. Then $|G| = 81$. As in Remark 3.3 we define the face $F := F(\{e, v_1, v_2, v_4\})$ of $P(G)$. Then F has dimension five, and we check that F is a pyramid over the hypersimplex.
- (9) Q_1 : We could not find a permutation group G with a face F of $P(G)$ whose combinatorial type coincides with the one of Q_1 .
- (10) Q_2 : As for Q_1 . □

Conjecture 4.4. There are no four-dimensional faces of permutation polytopes having the combinatorial type of Q_1 or Q_2 (see Table 2).

Remark 4.5. The combinatorics of faces of permutation polytopes is in general much more complex than the one of faces of Birkhoff polytopes. For instance, any facet of a Birkhoff polytope is given as the set of matrices with entry 0 (resp. 1) at a fixed position i, j . Hence, any face of a Birkhoff polytope has the strong property that the vertices in the complement of a facet form a face. In dimension $d \leq 4$ all of the polytopes in Theorem 4.3 also possess this property, except the special 0/1-polytopes Q_1, Q_2 , which stresses the exceptional role of these polytopes. However, one should not jump to the wrong conclusion that this might be a necessary condition on a polytope to be a face of a permutation polytope. In dimension $d \geq 9$ there are examples of permutation polytopes that have facets whose complement is not even a subset of a proper face, see [BHNP07].

5. OPEN QUESTIONS AND CONJECTURES

5.1. Permutation polytopes. Inspired by an embedding result on Birkhoff faces in [BS96] we propose the following daring conjecture (in a weak and strong version), where the bound $2d$ would be sharp as the example of the d -cube shows, see Corollary 3.6. The existence of some bound follows from Proposition 1.3.

Conjecture 5.1. Let P be a d -dimensional permutation polytope. Then there exists a permutation group $G \leq S_{2d}$ such that $P(G)$ is combinatorially equivalent (or stronger, lattice equivalent) to P .

An even more natural formulation is given by the next conjecture, which was checked for $d \leq 4$ using Theorem 4.1. The statement may be phrased purely in terms of representation theory thanks to the dimension formula, see Theorem 2.5.

Conjecture 5.2. Let ρ be a permutation representation of a finite group G with $d := \dim P(\rho)$. Then there exists a stably equivalent permutation representation $\rho' : G \rightarrow S_n$ such that $n \leq 2d$.

The truth of this statement would imply the weak part of Conjecture 5.1.

The Birkhoff polytope B_3 is given by the full symmetric group S_3 . From Theorem 4.1 we observe that S_3 is essentially the only permutation group having B_3 as its permutation polytope.

Conjecture 5.3. Let $P(G)$ be a permutation polytope such that $P(G)$ is combinatorially equivalent to the Birkhoff polytope B_n for some n . Then the permutation group G is effectively equivalent to S_n .

Any element of a permutation group G induces by left, respectively, by right multiplication an affine automorphism of $P(G)$. If G is not abelian, this implies that there are more affine automorphisms of $P(G)$ than elements of G . We conjecture this to be true also in the abelian case, except if $|G| \leq 2$.

Conjecture 5.4. Let G be an abelian permutation group of order $|G| > 2$. Then the group of affine automorphisms of $P(G)$ contains more elements than G .

5.2. Faces of permutation polytopes. Observing the structure of centrally symmetric faces for $d \leq 4$ gives rise to the following question.

Question 5.5. Is there a centrally symmetric face of a permutation polytope that is not composable as products or free sums of lower dimensional centrally symmetric faces of permutation polytopes?

It should be true that bipyramids over centrally symmetric faces are again realizable as faces of permutation polytopes. Even more, we expect that it may be possible to generalize the construction of Theorem 3.10.

Conjecture 5.6. The free sum of centrally symmetric faces of permutation polytopes can be combinatorially realized as a face of a permutation polytope.

The next conjecture is based upon explicit checks in low dimensions.

Conjecture 5.7. Let F be a face of a permutation polytope. Then the wedge over a face F' of F can be combinatorially realized as a face of a permutation polytope, if (or only if) the complement of F' in F is a face of F .

5.3. Faces of permutation polytopes given by subgroups. It would be interesting to know which subgroups of a permutation group yield faces. One obvious class of such subgroups are stabilizers.

For this let us partition $[n] := \{1, \dots, n\} = \bigsqcup I_i$. Then the polytope of the stabilizer of this partition

$$\text{stab}(G; (I_i)_i) := \{\sigma \in G : \sigma(I_i) = I_i \text{ for all } i\} \leq G$$

is a face of $P(G)$.

Conjecture 5.8. Let $G \leq S_n$. Suppose $H \leq G$ is a subgroup such that $P(H) \preceq P(G)$ is a face. Then $H = \text{stab}(G; (I_i)_i)$ for a partition $[n] = \bigsqcup I_i$.

Proposition 5.9. *Conjecture 5.8 holds for $G = S_n$ and for $G \leq S_n$ cyclic.*

Proof. First, let $H \leq G = S_n$ with $P(H)$ a face of $P(G) = B_n$. Let $[n] = \bigsqcup I_i$ be the orbit partition of H . Then $H \leq \text{stab}(G; (I_i)_i) = \prod S_{I_i}$. We show that equality holds. The face $P(H) \preceq B_n$, is the intersection of the facets containing it. For $J = \{(i, j) : \sigma(i) \neq j \text{ for all } \sigma \in H\}$ that means, cf. Remark 4.5,

$$P(H) = \text{conv}\{A \in B_n : a_{ij} = 0 \text{ for all } (i, j) \in J\}.$$

Because H is transitive on I_i , we get $J \cap (I_i \times I_i) = \emptyset$ so that $H \geq \prod S_{I_i}$.

Second, let G be a cyclic subgroup of S_n , and let H be a subgroup of G . Let $G = \langle g \rangle$ and let $g = g_1 \circ \dots \circ g_r$ be the cycle decomposition of g . Then $H = \langle h \rangle$ for some $h = g^k = g_1^k \circ \dots \circ g_r^k$. If the length of g_i is z_i ,

then g_i^k splits into cycles of length $s_i := z_i / \gcd(z_i, k)$. Let O_1, \dots, O_l be the respective orbits of h . Then H stabilizes the partition

$$[n] = O_1 \sqcup \dots \sqcup O_l \sqcup ([n] \setminus (O_1 \cup \dots \cup O_l)).$$

Let $h' = g^t$ be another element in G which also stabilizes this partition. This implies $z_i / \gcd(z_i, t) \leq s_i$, thus $\gcd(z_i, k) \leq \gcd(z_i, t)$ for $i = 1, \dots, r$. Since k and t divide $|G| = o(g) = \text{lcm}(z_1, \dots, z_r)$, this yields that k divides t . Hence $h' = g^t \in H = \langle g^k \rangle$, and H is the full stabilizer in G of a partition of $[n]$. \square

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