

On the Configuration Spaces of Homogeneous Loop Quantum Cosmology and Loop Quantum Gravity

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Abstract

The set of homogeneous isotropic connections, as used in loop quantum cosmology, forms a line l in the space of all connections \mathcal{A} . This embedding, however, does not continuously extend to an embedding of the configuration space \bar{l} of homogeneous isotropic loop quantum cosmology into that of loop quantum gravity, $\bar{\mathcal{A}}$. This follows from the fact that the parallel transports for general, non-straight paths in the base manifold do not depend almost periodically on l . Analogous results are given for the anisotropic case.

1 Introduction

In loop quantum cosmology (LQC), highly symmetric cosmological models are quantized in analogy to loop quantum gravity (LQG). [5, 2, 3] One of the key features is the similar compactifications of their configuration spaces: in LQG from \mathcal{A} to $\bar{\mathcal{A}}$ – in LQC from \mathbb{R} to $\bar{\mathbb{R}}_{\text{Bohr}}$, the Bohr compactification of \mathbb{R} . In both cases, “distributional” objects are added to smooth ones: added to smooth connections in LQG, to real numbers in LQC. In fact, for homogeneous isotropic models (with $k = 0$), the classical configuration space is spanned by cA_* , where c runs over \mathbb{R} and A_* is a fixed homogeneous and isotropic connection. As in LQG, one does not consider these connections themselves, but their parallel transports along certain edges. Usually, only straight edges have been taken into account. For such edges γ the parallel transports can be written down explicitly; they equal

$$h_{cA_*}(\gamma) = e^{-cA_*(\dot{\gamma})\ell(\gamma)},$$

where $\ell(\gamma)$ denotes the length of γ . In particular, they are periodic in c (hence, almost periodic, as well) and can be extended from \mathbb{R} to $\bar{\mathbb{R}}_{\text{Bohr}}$. Another advantage of straight edges is that they separate the points in \mathbb{R} . At the same time, however, the notion of straightness

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requires a background metric. Therefore, it seems appropriate to consider general, non-straight edges. But, as we will prove in this note, these edges, in general, do not lead to almost periodic parallel transports, whence they cannot be extended continuously to $\overline{\mathbb{R}}_{\text{Bohr}}$. This will directly show that the configuration space of LQC is not continuously embedded into that of LQG. The importance of the question of the existence of this embedding was highlighted in [6]. General arguments extend our results to the anisotropic case.

The paper is organized as follows: First we derive the differential equations for the parallel-transport matrix elements along arbitrary curves. These are always second-order ODEs with a constant coefficient in order 2, but generally non-constant and c -dependent coefficients in orders 1 and 0. The coefficients encode, in particular, the underlying curve; they are all constant w.r.t. the curve parameter iff this curve is a spiral arc. Next, we introduce an “invariant” \mathcal{E} for the ODEs that (up to a constant) reduces to the standard energy invariant in the case of a harmonic oscillator. This generalized energy is constant along the path up to $\mathcal{O}(\frac{1}{c})$. In the almost periodic case, \mathcal{E} is fully invariant along the path. Together with analyticity, this imposes restrictions to the coefficients of the ODE implying that the path is a straight line. Finally, we find that the embedding $\mathbb{R} \rightarrow \mathcal{A}$ is not extendable to an embedding $\overline{\mathbb{R}}_{\text{Bohr}} \rightarrow \overline{\mathcal{A}}$ since this required the almost periodicity of all parallel transports. All the results are given for the homogeneous isotropic case with $k = 0$. Since the almost periodicity on \mathbb{R}^3 implies the almost periodicity on \mathbb{R} being the diagonal in \mathbb{R}^3 , the results extend to the general homogeneous case with $k = 0$. The paper concludes with an appendix summarizing the main facts on almost periodic functions used here.

2 Preliminaries

Let P be a principal fibre bundle over a manifold M with compact structure Lie group \mathbf{G} , and let A_* be a connection in P . As we are aiming at homogeneous isotropic cosmology with $k = 0$, we assume the base manifold to be contractible; hence P is trivial. Thus we may regard A_* as a \mathfrak{g} -valued 1-form on M . Moreover, again by triviality of the bundle, cA_* is a connection for every $c \in \mathbb{R}$. Given an analytic edge $\gamma : I \rightarrow M$ over an interval $I \subseteq \mathbb{R}$ containing 0 and using the standard trivialization of P , we denote the parallel transport along γ from 0 to t w.r.t. cA_* by $g_c(t) \in \mathbf{G}$. The differential equation determining g_c is

$$\begin{aligned} \dot{g}_c(t) &= -cA_*(\dot{\gamma}(t)) g_c(t) \\ g_c(0) &= e_{\mathbf{G}}. \end{aligned}$$

The configuration space $\overline{\mathcal{A}}$ of loop quantum gravity (or, more general, quantum geometry) equals $\text{Hom}(\mathcal{P}, \mathbf{G})$, where \mathcal{P} is the groupoid of all analytic paths in M (after modding out reparametrizations and immediate retracings). It is given the initial topology induced by the projections $\pi_\gamma : \overline{\mathcal{A}} \rightarrow \mathbf{G}$ with $\gamma \in \mathcal{P}$ and $\pi_\gamma(\overline{A}) := \overline{A}(\gamma)$ assigning to $\overline{A} \in \overline{\mathcal{A}}$ its parallel transport along γ . In an obvious manner, \mathcal{A} is densely embedded into $\overline{\mathcal{A}}$. For further reading we refer to [3, 7]. In loop quantum gravity, specifically, M is a three-dimensional Cauchy slice and $\mathbf{G} = SU(2)$.

3 Differential Equations for $SU(2)$

From now on, we consider the case of $M = \mathbb{R}^3$ and $\mathbf{G} = SU(2)$ only. Moreover, A_* is a homogeneous and isotropic connection. Let us write elements of $SU(2)$ as

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

with $a, b \in \mathbb{C}$ fulfilling $|a|^2 + |b|^2 = 1$. In the following, w.l.o.g., we assume that

$$A_* = \tau_1 dx + \tau_2 dy + \tau_3 dz$$

with

$$\tau_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Consequently, using $\gamma(t) = (x(t), y(t), z(t))$, we get

$$A_*(\dot{\gamma}(t)) = \dot{x}\tau_1 + \dot{y}\tau_2 + \dot{z}\tau_3 = \begin{pmatrix} -i\dot{z} & -i\dot{x} - \dot{y} \\ -i\dot{x} + \dot{y} & i\dot{z} \end{pmatrix} = -i \begin{pmatrix} n & m \\ \bar{m} & -n \end{pmatrix}$$

with analytic

$$\begin{aligned} m &:= \dot{x} - i\dot{y} \\ n &:= \dot{z}. \end{aligned}$$

In the following, we always assume that γ is parametrized w.r.t. arc length, i.e.,

$$|\bar{m}|^2 + n^2 = \|\dot{\gamma}\|^2 \equiv 1.$$

The equation for the parallel transport reads now for each c

$$\begin{pmatrix} \dot{a} & \dot{b} \\ -\dot{\bar{b}} & \dot{\bar{a}} \end{pmatrix} = ic \begin{pmatrix} n & m \\ \bar{m} & -n \end{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad (1)$$

or

$$\dot{a} = ic(na - m\bar{b}) \quad (2)$$

$$\dot{b} = ic(nb + m\bar{a}) \quad (3)$$

with the initial conditions

$$a(0) = 1$$

$$b(0) = 0.$$

From that we get for $m \neq 0$

$$\begin{aligned} \ddot{a} &= ic(\dot{n}a + n\dot{a} - \dot{m}\bar{b} - m\dot{\bar{b}}) \\ &= ic(\dot{n}a + n\dot{a} - \dot{m}\bar{b}) - c^2 m(\bar{m}a + n\bar{b}) \\ &= ic(\dot{n}a + n\dot{a}) - c^2 m\bar{m}a - (ic\dot{m} + c^2 mn)\frac{1}{m}(na - \frac{1}{ic}\dot{a}) \\ &= ic(\dot{n} - Mn)a - c^2 a + M\dot{a} \quad (\text{since } m\bar{m} + nn = 1) \end{aligned} \quad (4)$$

with

$$M := \frac{\dot{m}}{m}.$$

Analogously, we get

$$\begin{aligned} \ddot{b} &= ic(\dot{n}b + n\dot{b} + \dot{m}\bar{a} + m\dot{\bar{a}}) \\ &= ic(\dot{n}b + n\dot{b} + \dot{m}\bar{a}) - c^2 m(\bar{m}b - n\bar{a}) \\ &= ic(\dot{n}b + n\dot{b}) - c^2 m\bar{m}b + (ic\dot{m} + c^2 mn)\frac{1}{m}(\frac{1}{ic}\dot{b} - nb) \\ &= ic(\dot{n} - Mn)b - c^2 b + M\dot{b}. \end{aligned} \quad (5)$$

4 An Energy-like Invariant

The central object we will exploit to prove the non-almost periodicity result is given by

Definition 4.1 $\mathcal{E}_a(c, t) := (\bar{m}a^2 + 2na\bar{b} - m\bar{b}^2)|_0^t$

It generalizes the energy conservation functional known from the special case of the differential equation $\ddot{x} + \omega^2 x = 0$ describing an harmonic oscillator. In fact, again assuming $m \neq 0$, we have by (2) and $m\bar{m} + nn = 1$

$$\mathcal{E}_a(c, t) = (\bar{m}a^2 + 2nab\bar{b} - m\bar{b}^2)|_0^t = \frac{1}{m} \left(-(na - m\bar{b})^2 + a^2 \right) \Big|_0^t = \frac{1}{m} \left(\frac{\dot{a}^2}{c^2} + a^2 \right) \Big|_0^t.$$

In order to eliminate the disturbing $\frac{1}{m}$ -term and to remove the first-order term in our differential equations (4) and (5), we decompose a as usual into¹

$$a = \sqrt{m} \alpha.$$

This implies

$$\begin{aligned} \sqrt{m} a &= m\alpha, \\ \sqrt{m} \dot{a} &= \frac{1}{2}\dot{m}\alpha + m\dot{\alpha}, \\ \sqrt{m^3} \ddot{a} &= \frac{1}{2}m\ddot{m}\alpha - \frac{1}{4}\dot{m}^2\alpha + m\dot{m}\dot{\alpha} + m^2\ddot{\alpha}. \end{aligned}$$

Now, the differential equation for a transforms into

$$\begin{aligned} 0 &= \sqrt{m^3} (\ddot{a} - M\dot{a} + c^2a - ic(\dot{n} - Mn)a) \\ &= \frac{1}{2}Mm^2\alpha - \frac{1}{4}\dot{m}^2\alpha + m^2\ddot{\alpha} + c^2m^2\alpha - ic(\dot{n} - Mn)m^2\alpha \end{aligned}$$

and

$$\ddot{\alpha} + c^2\alpha = \left(\frac{1}{4}M^2 - \frac{1}{2}\dot{M} + ic(\dot{n} - Mn) \right) \alpha.$$

With $\rho := \frac{1}{4}M^2 - \frac{1}{2}\dot{M}$ and $\sigma := i(\dot{n} - Mn)$, we have

$$\frac{d}{dt} \left(\frac{\dot{\alpha}^2}{c^2} + \alpha^2 \right) = \frac{2}{c^2} (\ddot{\alpha} + c^2\alpha) \dot{\alpha} = \frac{1}{c^2} (\rho + c\sigma) \frac{d}{dt} \alpha^2,$$

hence

$$\begin{aligned} \mathcal{E}_\alpha(c, t) &:= \left(\frac{\dot{\alpha}^2}{c^2} + \alpha^2 \right) \Big|_0^t = \frac{1}{c^2} \int_0^t (\rho + c\sigma) \frac{d}{d\tau} \alpha^2 d\tau \\ &= -\frac{1}{c^2} \int_0^t (\dot{\rho} + c\dot{\sigma}) \alpha^2 d\tau + \frac{1}{c^2} (\rho + c\sigma) \alpha^2 \Big|_0^t. \end{aligned}$$

For each t , the functions ρ and σ , as well as their derivatives, are bounded on $[0, t]$. Since m is independent of c and does, moreover, nowhere vanish, we have $\sup_{[0, t]} |\alpha| \leq C$ with a c -independent constant C . (Recall $\|a\|_\infty = 1$.) This immediately implies that

$$\lim_{c \rightarrow \infty} \mathcal{E}_\alpha(c, t) = 0$$

for each parameter value t . Rewriting \dot{a} in terms of a and \dot{a} , we get

$$\dot{a}^2 = \frac{1}{m^2} (\sqrt{m} \dot{a} - \frac{1}{2}\dot{m}\alpha)^2 = \frac{1}{m} (\dot{a}^2 - Ma\dot{a} + \frac{1}{4}M^2a^2),$$

and we have

$$\mathcal{E}_a(c, t) \equiv \frac{1}{m} \left(\frac{\dot{a}^2}{c^2} + a^2 \right) \Big|_0^t = \mathcal{E}_\alpha(c, t) + \frac{Ma}{mc^2} \left(\dot{a} - \frac{1}{4}Ma \right) \Big|_0^t.$$

Due to

$$\|\dot{a}\|_\infty \leq c(\|n\|_\infty + \|m\|_\infty)$$

we have

Lemma 4.1

$$\lim_{c \rightarrow \infty} \mathcal{E}_a(c, t) = 0$$

Corollary 4.2 If, for some parameter value t , the functions $a(t)$ and $b(t)$ are almost periodic w.r.t. c , then $\mathcal{E}_a(c, t) = 0$ for all c .

¹The choice of the square-root leaf is made continuously w.r.t. t . Moreover, throughout this section we continue to assume $m \neq 0$.

For the definition of almost periodicity, see Appendix A.

Proof If $a(t)$ and $b(t)$ are almost periodic, then $\mathcal{E}_a(\cdot, t)$ is almost periodic as well. Since (cf. Lemma A.1) any almost periodic function is already constant, provided it converges while the argument is approaching ∞ , we get the proof. **qed**

Of course, the preceding corollary remains correct if we replace a by b .

From now on, we call t sloppily almost periodic iff $a(t)$ and $b(t)$ depend almost periodically on c . This yields

Lemma 4.3 If the set of almost periodic parameter values contains an accumulation point, then $\mathcal{E}_a \equiv 0$.

Proof For general reasons [1], \mathcal{E}_a depends locally (real) analytically on $(c$ and) t . Now, for each c , the function $t \mapsto \mathcal{E}_a(c, t)$ has an accumulation point of zeros by the preceding corollary. Consequently, this function vanishes identically for each c . **qed**

Next, we would like to investigate the derivative of \mathcal{E}_a w.r.t. t . For that, we consider

$$F := f_2 a^2 + f_1 a \bar{b} + f_0 \bar{b}^2, \quad (6)$$

where f_0, f_1, f_2 may depend on t , but not on c . We have

$$\dot{F} = (\dot{f}_2 + ic(2f_2 n - f_1 \bar{m})) a^2 + (\dot{f}_1 - 2ic(f_2 m + f_0 \bar{m})) a \bar{b} + (\dot{f}_0 - ic(2f_0 n + f_1 m)) \bar{b}^2.$$

If a and b are almost periodic, it follows that

$$\dot{f}_2 a^2 + \dot{f}_1 a \bar{b} + \dot{f}_0 \bar{b}^2$$

is almost periodic, implying for almost periodic \dot{F} that

$$ic((2f_2 n - f_1 \bar{m}) a^2 - 2(f_2 m + f_0 \bar{m}) a \bar{b} - (2f_0 n + f_1 m) \bar{b}^2)$$

is almost periodic as well. Consequently (cf. Lemma A.2), the almost periodic function

$$(2f_2 n - f_1 \bar{m}) a^2 - 2(f_2 m + f_0 \bar{m}) a \bar{b} - (2f_0 n + f_1 m) \bar{b}^2$$

is even identically zero. This, finally, implies that

$$\dot{F} = \dot{f}_2 a^2 + \dot{f}_1 a \bar{b} + \dot{f}_0 \bar{b}^2, \quad (7)$$

for almost periodic a, b and \dot{F} .

Lemma 4.4 If the set of almost periodic parameter values contains an accumulation point, then we have

$$\dot{\bar{m}} a^2 + 2\dot{n} a \bar{b} - \dot{m} \bar{b}^2 = 0.$$

Proof Setting

$$\begin{aligned} f_2 &:= \bar{m} \\ f_1 &:= 2n \\ f_0 &:= -m \end{aligned}$$

into (6), we get $\mathcal{E}_a(c, t) = F(c, t) - F(c, 0)$. The assertion now follows from (7) and Lemma 4.3. **qed**

We note that, by induction, one gets $\bar{m}^{(r)} a^2 + 2n^{(r)} a \bar{b} - m^{(r)} \bar{b}^2 = 0$ for all non-zero $r \in \mathbb{N}$. Moreover, the equation for $r = 1$ stated in the lemma above could also have been obtained directly, as for that choice of f_0, f_1, f_2 the ic -term already vanishes without the almost-periodicity assumption.

Corollary 4.5 Any curve having an accumulation point of almost periodic parameter values is a straight line.

Proof Assume there is an almost periodic t with $|\dot{m}(t)| \neq 0$. As \dot{m} is continuous, we have $\dot{m} \neq 0$ on some open U containing t . Moreover, Lemma 4.3 and Lemma 4.4 show

$$\begin{aligned}\bar{m}a^2 + 2na\bar{b} - m\bar{b}^2 &= \bar{m}(0) \\ \dot{\bar{m}}a^2 + 2\dot{n}a\bar{b} - \dot{m}\bar{b}^2 &= 0.\end{aligned}$$

Next, a short calculation shows that for either sign

$$\left(2\frac{\dot{n}^2}{\dot{m}^2} - \frac{m}{\dot{m}} + \frac{\dot{m}}{\dot{m}} - 2\frac{n\dot{n}}{m\dot{m}} \pm 2\left(\frac{n}{\dot{m}} - \frac{\dot{n}}{\dot{m}}\right)\sqrt{\frac{\dot{n}^2}{\dot{m}^2} + \frac{\dot{m}}{\dot{m}}}\right)\bar{b}^2 = \frac{\bar{m}(0)}{\dot{m}},$$

i.e., \bar{b}^2 does not depend on c all over U —and neither does b . (The large bracket term cannot be zero, because this contradicts the assumption that $m \neq 0$ everywhere.) The same is true for a . Consequently, $\partial_c a$ and $\partial_c b$ vanish on U . Using now the original differential equations, we get on U

$$\begin{aligned}0 &= \partial_t \partial_c a \equiv \partial_c \dot{a} = i(na - m\bar{b}) + ic(n\partial_c a - m\partial_c \bar{b}) = i(na - m\bar{b}) \\ 0 &= \partial_t \partial_c b \equiv \partial_c \dot{b} = i(nb + m\bar{a}) + ic(n\partial_c b + m\partial_c \bar{a}) = i(nb + m\bar{a}).\end{aligned}$$

This implies

$$a = (n^2 + \bar{m}m)a = nm\bar{b} + \bar{m}ma = 0$$

and thus $b = 0$ on U . This, of course, is a contradiction. Consequently, $\dot{m} \equiv 0$, whence also $\dot{n} \equiv 0$, and γ is a line. **qed**

5 Non-Almost Periodicity

Theorem 5.1 Let γ be an analytic curve that is not part of a straight line.

Then there is a $T > 0$, such that the parallel transport $g_c(t)$ along γ is not almost periodic w.r.t. c for any $0 < t < T$.

Proof The case $m(0) \neq 0$ is proven in Corollary 4.5. If $m(0) = 0$, choose some $h \in SU(2)$, such that none of its matrix elements vanishes. Conjugating (1) by h , we get a new differential equation of the same type; however, m is replaced by some function of m , n and h , not vanishing at 0. Since the property of being almost periodic does not change under conjugation with a fixed element, we get the proof. **qed**

Remark At present, only accumulation points of almost periodic parameter values can be excluded for nonlinear curves. It is still unclear whether there might exist isolated almost periodic parameter values. Nevertheless we do not expect this to be case; for paths that are spiral arcs, this will indeed be proven in the next section. If we were right in general, the non-almost periodicity in the theorem above would be given for all $0 < t \leq \ell(\gamma)$.

6 Special Cases: Spiral Arcs

The differential equation (4) for a has, in general, t -dependent coefficients. Nevertheless, there are special cases where the coefficients are independent of t : lines and circles, or more generally, spirals. They, on the other hand, will turn out to be the only possible examples for $m \neq 0$. Indeed, constancy of coefficients implies $M \equiv \kappa$ for some $\kappa \in \mathbb{C}$. Therefore $\dot{m} = \kappa m$ or $m = Ce^{\kappa t}$ for some non-zero $C \in \mathbb{C}$. There remain two cases:

Case 1: $\kappa = 0$

Then m is constant, whence n^2 is constant implying that n is constant as well. Altogether $\dot{\gamma}$ is constant, whence γ is a line.

Case 2: $\kappa \neq 0$

Then $\dot{n} - Mn = \dot{n} - \kappa n$ is constant, say Λ . Thus, $n = -\frac{\Lambda}{\kappa} + De^{\kappa t}$ for some $D \in \mathbb{C}$. Since n and its derivatives $\dot{n}(0) = \kappa D$ and $\ddot{n}(0) = \kappa^2 D$ are real, we deduce D is real.

Subcase 2a: $D \neq 0$.

Then κ is real (since κD is real) and therefore Λ as well (recall that D and n are real). We get for all parameter values t

$$\begin{aligned} 1 &= |m|^2 + n^2 = |C|^2 e^{2\kappa t} + \left(-\frac{\Lambda}{\kappa} + De^{\kappa t}\right)^2 \\ &= (|C|^2 + D^2)e^{2\kappa t} - 2D\frac{\Lambda}{\kappa}e^{\kappa t} + \frac{\Lambda^2}{\kappa^2}. \end{aligned}$$

This requires $|C| = D = 0$, implying $n = \pm 1$ and $m \equiv 0$, in contradiction to the assumption above.

Subcase 2b: $D = 0$.

Then n is constant. From

$$1 = |m|^2 + n^2 = |C|^2 e^{2(\operatorname{Re} \kappa)t} + \frac{\Lambda^2}{\kappa^2}$$

we deduce that $\operatorname{Re} \kappa = 0$. Thus $\kappa = i\lambda$ for some non-zero $\lambda \in \mathbb{R}$, and $m = Ce^{i\lambda t}$ with $|C| \leq 1$. Therefore

$$x - iy = \frac{C}{i\lambda} e^{i\lambda t} + E$$

with some $E \in \mathbb{C}$. n is constant with $n^2 = 1 - |C|^2$. Geometrically, γ is (part of) a spiral, constantly moving in z -direction which projects down to (part of) a circle in the x - y -plane.

Altogether, we see that the only possible cases for constant coefficients are edges whose tangents are constant in z -direction and project down to a circle or a line in the x - y -plane. In these cases, the differential equation looks like

$$\ddot{a} + i\lambda\dot{a} + (c^2 - c\kappa)a = 0 \tag{8}$$

with $\lambda, \kappa \in \mathbb{R}$, subject to the condition $|\kappa| \leq |\lambda|$. The case $\lambda = 0$ (and, consequently, $\kappa = 0$, as well) corresponds to the case of a line. In general, the case $\kappa = 0$ represents edges lying in a plane perpendicular to the z -axis, unless $\lambda = 0$. The equation for b reduces in the constant-coefficient case to

$$\ddot{b} + i\lambda\dot{b} + (c^2 - c\kappa)b = 0. \tag{9}$$

The general solution for Eq. (8) is

$$a(t) = e^{-i\frac{\lambda}{2}t} \left(a(0) \cos \Delta t + \frac{1}{\Delta} (\dot{a}(0) + i\frac{\lambda}{2}a(0)) \sin \Delta t \right)$$

with

$$\Delta = \sqrt{\frac{\lambda^2}{4} + c(c - \kappa)} = \sqrt{\left(c - \frac{\kappa}{2}\right)^2 + \frac{\lambda^2 - \kappa^2}{4}}.$$

Hence, we get for $\Delta \neq 0$ with

$$\begin{aligned}\dot{a}(0) &= icn(0) \\ \dot{b}(0) &= icm(0)\end{aligned}$$

the solutions

$$\begin{aligned}a(t) &= e^{-i\frac{\lambda}{2}t} \left(\cos \Delta t + \frac{i}{\Delta} (cn(0) + \frac{\lambda}{2}) \sin \Delta t \right) \\ b(t) &= e^{-i\frac{\lambda}{2}t} \frac{i}{\Delta} cm(0) \sin \Delta t.\end{aligned}$$

In the case that γ is a line, λ and \varkappa are zero. With $\Delta = |c|$, we get

Lemma 6.1 If γ is a line, then

$$\begin{aligned}a(t) &= \cos ct + in(0) \sin ct \\ b(t) &= im(0) \sin ct.\end{aligned}$$

In particular, these functions are periodic.

In the remaining part of this section we are going to show that the lines are the only type of spiral arcs having almost periodic parallel transports. Others spiral arcs cannot even have isolated almost periodic parallel transports answering the open question in the remark above, at least for this type of paths.

We start with a criterion for non-almost periodicity.

Lemma 6.2 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Assume there is an interval I in \mathbb{R} and some $r, \varepsilon > 0$, such that

$$\sup_J |f| - \sup_I |f| \geq \varepsilon \quad (10)$$

for all intervals J with $\inf J > r$ and $|J| = |I|$. Then f is not almost periodic.

Proof Assume f is almost periodic. Then there is some $L > 0$ (w.l.o.g., we may choose $L > r + |\inf I|$), such that there is some $\omega \in [L, 2L]$ with $|f(c) - f(c - \omega)| < \varepsilon$ for all $c \in \mathbb{R}$. Define $J := I + \omega$. By assumption, $\inf J = \inf I + \omega \geq -|\inf I| + L > r$. Choose now $C \in J$ with $|f(C)| = \sup_J |f|$. Since $C - \omega \in I$, we have

$$|f(C) - f(C - \omega)| \geq |f(C)| - |f(C - \omega)| \geq \sup_J |f| - \sup_I |f| \geq \varepsilon$$

by assumption (10). Contradiction. **qed**

Proposition 6.3 Let $t, \lambda, \varkappa \in \mathbb{R}$ with $|\varkappa| \leq |\lambda|$. Then the function $f : \mathbb{R} \rightarrow \mathbb{R}$, given by

$$f(c) := \frac{c}{\sqrt{\frac{\lambda^2}{4} + c(c - \varkappa)}} \sin \sqrt{\frac{\lambda^2}{4} + c(c - \varkappa)} t$$

is almost periodic iff $\lambda = 0$ or $t = 0$.

Here, in the case of $|\varkappa| = |\lambda|$, we set $f(\frac{\varkappa}{2}) := ct$ to get a continuous function.

Proof • W.l.o.g., we may assume that $\varkappa \leq 0$ and $\lambda, t \geq 0$. Since the cases $\lambda = 0$ (see Lemma 6.1) and $t = 0$ are obvious, we even may assume $\lambda, t > 0$.
• Possibly ignoring $c = 0$, the zeros of f are given by

$$c_{\pm k} := \frac{\varkappa}{2} \pm \sqrt{\frac{\pi^2 k^2}{t^2} - \frac{\lambda^2 - \varkappa^2}{4}} \quad (11)$$

for positive integers $k \geq k_0$ with

$$k_0 := \left\lceil \frac{t}{2\pi} \sqrt{\lambda^2 - \varkappa^2} \right\rceil.$$

We have $c_k < c_{k+1}$ for all k , and one easily checks that

$$\lim_{k \rightarrow \infty} (c_{k+1} - c_k) = \frac{\pi}{t}.$$

- Observe that the function

$$\varphi(c) := \frac{c}{\sqrt{\frac{\lambda^2}{4} + c(c - \varkappa)}}$$

is strictly monotonously increasing on \mathbb{R}_+ with $\lim_{c \rightarrow \infty} \varphi(c) = 1$. Defining $I_k := [c_k, c_{k+1}]$ for $k \geq k_0$, we have from the sine properties

$$\sup_{I_k} |f| \geq |f(c_{k+\frac{1}{2}})| = \varphi(c_{k+\frac{1}{2}}) \geq \inf_{I_k} \varphi = \varphi(c_k)$$

for $k \geq k_0$, where we have naturally extended (11) to half-integer indices. Moreover, for each interval I in \mathbb{R}_+

$$\sup_I |f| \leq \sup_I \varphi = \varphi(\sup I).$$

- Now, let $I := [c_{k_0}, c_{k_1}]$ with $k_1 \in \mathbb{N}_+$ and $|I| \geq 3\frac{\pi}{t}$. Next, choose some $k_\varrho \in \mathbb{N}_+$ with $c_{k+1} - c_k < \frac{3}{2}\frac{\pi}{t}$ for all integer $k \geq k_\varrho$. Choose, finally, $r > \max\{c_{k_1}, c_{k_\varrho}\}$ and let $\varepsilon := \varphi(r) - \varphi(c_{k_1}) > 0$.

Then each interval $J = [c, c + |I|]$ with $c > r$ contains at least one complete interval $I_k = [c_k, c_{k+1}]$ with $k \geq k_\varrho$. In fact, the half-interval $[c_{k_\varrho}, \infty) \supset [r, \infty)$ containing J is covered by the intervals I_i , $i \geq k_\varrho$, each having length less than $\frac{3}{2}\frac{\pi}{t}$. On the other hand, the length of J equals the length of I being at least $3\frac{\pi}{t}$. This implies (recall $c_k > r$)

$$\begin{aligned} \sup_J |f| - \sup_I |f| &\geq \sup_{I_k} |f| - \varphi(c_{k_1}) \geq \varphi(c_k) - \varphi(c_{k_1}) \\ &> \varphi(r) - \varphi(c_{k_1}) = \varepsilon. \end{aligned}$$

Lemma 6.2 gives the proof. qed

Corollary 6.4 The parallel transport along a path γ being (part of) a line or a spiral arc is almost periodic as a function of \mathbb{R} into $\mathbf{G} = SU(2)$ iff γ is (part of) a line.

7 Relating the Configuration Spaces of LQC and LQG

Now we are prepared to answer the question, whether the configuration space of loop quantum cosmology is continuously embedded into the configuration space of loop quantum gravity. More precisely, can the canonical embedding

$$\begin{aligned} \iota: \mathbb{R} &\longrightarrow \mathcal{A} \\ c &\longmapsto cA_* \end{aligned}$$

with a fixed homogeneous isotropic A_* be continuously extended to an embedding

$$\bar{\iota}: \overline{\mathbb{R}}_{\text{Bohr}} \longrightarrow \overline{\mathcal{A}}?$$

If this were the case, the mapping

$$\iota_{\overline{\mathcal{A}}} \circ \bar{\iota}: \overline{\mathbb{R}}_{\text{Bohr}} \supseteq \mathbb{R} \longrightarrow \mathcal{A} \longrightarrow \overline{\mathcal{A}}$$

had to be continuous (with \mathbb{R} given the topology inherited from $\overline{\mathbb{R}}_{\text{Bohr}}$ and $\iota_{\overline{\mathcal{A}}}$ being the standard embedding of \mathcal{A} into $\overline{\mathcal{A}}$). From basic theorems on projective limits [7] it follows that this is equivalent to the continuity of

$$\pi_e \circ \iota_{\overline{\mathcal{A}}} \circ \iota : \overline{\mathbb{R}}_{\text{Bohr}} \supseteq \mathbb{R} \longrightarrow \mathcal{A} \longrightarrow \overline{\mathcal{A}} \longrightarrow \mathbf{G}$$

for all edges e in M . A map from \mathbb{R} to \mathbf{G} , on the other hand, is continuous w.r.t. the Bohr topology iff it is almost periodic [9].² However, since, as seen above, there are edges e that yield non-almost periodic functions

$$\begin{aligned} \pi_e \circ \iota_{\overline{\mathcal{A}}} \circ \iota : \mathbb{R} &\longrightarrow \mathbf{G}, \\ c &\longmapsto h_e(cA_*) \end{aligned}$$

we get a contradiction.

Consequently, the embedding ι of \mathbb{R} into \mathcal{A} cannot be continuously extended to an embedding of $\overline{\mathbb{R}}_{\text{Bohr}}$ into $\overline{\mathcal{A}}$, i.e., of the configuration space of loop quantum cosmology into that of loop quantum gravity.

8 Anisotropic Case

In the anisotropic case one does not consider the connections cA_* with A_* being homogeneous isotropic, but the connections

$$A_{\mathbf{c}} = c_1\tau_1 dx + c_2\tau_2 dy + c_3\tau_3 dz$$

with $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^3$. Of course, $A_{\mathbf{c}}$ is isotropic iff all components of \mathbf{c} coincide. This observation allows us to extend the results above to the anisotropic case thanks to

Lemma 8.1 Let $f : \mathbb{R}^3 \longrightarrow \mathbb{C}$ be an almost periodic function.
Then $f^{\text{diag}} : \mathbb{R} \longrightarrow \mathbb{C}$ with $f^{\text{diag}}(x) := (x, x, x)$ is almost periodic as well.

Proof f is almost periodic iff [9] it is a uniform limit $\lim_{n \rightarrow \infty} f_n$ of trigonometric polynomials. Since each $(f_n)^{\text{diag}}$ is, of course, a trigonometric polynomial again and since

$$\sup_{\mathbb{R}} |f^{\text{diag}} - (f_n)^{\text{diag}}| \equiv \sup_{(x,x,x) \in \mathbb{R}^3} |f - f_n| \leq \sup_{\mathbb{R}^3} |f - f_n| \rightarrow 0,$$

f^{diag} is a uniform limit of trigonometric polynomials, hence almost periodic as well. **qed**

Now we have from Theorem 5.1

Theorem 8.2 Let γ be an analytic curve that is not part of a straight line.
Then there is a $T > 0$, such that

$$g_{\mathbf{c}}(t) := \text{parallel transport w.r.t. } A_{\mathbf{c}} \text{ along } \gamma \text{ from } \gamma(0) \text{ to } \gamma(t)$$

is not almost periodic w.r.t. \mathbf{c} for any $0 < t < T$.

Completely analogously to Section 7, we see that the embedding

$$\begin{aligned} \iota_{\text{aniso}} : \mathbb{R}^3 &\longrightarrow \mathcal{A} \\ \mathbf{c} &\longmapsto A_{\mathbf{c}} \end{aligned}$$

cannot be continuously extended to an embedding

$$\overline{\iota}_{\text{aniso}} : \overline{\mathbb{R}}_{\text{Bohr}}^3 \longrightarrow \overline{\mathcal{A}}.$$

Remark Both in the isotropic and in the anisotropic case, we expect all main results to remain true if considering non-zero k , i.e. spherical or hyperbolic universes. Of course, then, straight lines have to be replaced by geodesics.

²In [9] the case of complex-valued functions has been considered. The general statement for \mathbf{G} -valued functions follows from Appendix A together with the fact that a map to \mathbf{G} is continuous iff all of its corresponding matrix-element functions are so. Recall that any compact Lie group is (embedded into) a matrix group.

9 Notes Added in Proof

As communicated to us by Martin Bojowald, Tim Koslowski (for details, see [8]) has given power series expansions for the solutions in the case of planar edges ($n = 0$). These indicate that each of these parallel transports can be written as a sum of a function being (almost) periodic in c and a function vanishing for large $|c|$. This is confirmed by our explicit solutions in the constant-coefficient case.

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Appendix

A Basics on Almost Periodic Functions

Definition A.1 A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is called **almost periodic** iff it is continuous and for any $\varepsilon > 0$ there is an $L > 0$, such that each open interval in \mathbb{R} of length L contains a ξ , such that $|f(x + \xi) - f(x)| \leq \varepsilon$ for all $x \in \mathbb{R}$. [4]

Constant functions are almost periodic; sum and products of almost periodic functions are almost periodic as well. Almost periodic functions are always bounded.

Lemma A.1 Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be an almost periodic function, such that $\lim_{c \rightarrow \infty} f(c)$ exists and equals $f_\infty \in \mathbb{C}$. Then f is constantly equal f_∞ .

Proof Let $\varepsilon > 0$.

By assumption, there is an $L > 0$, such that $|f(c) - f_\infty| < \frac{1}{2}\varepsilon$ for all $c > L$. Assume there is some $C \in \mathbb{R}$, such that $|f(C) - f_\infty| > \varepsilon$. Since f is, moreover, almost periodic, there is some $\omega > L - C$, such that $|f(c) - f(c + \omega)| < \frac{1}{2}\varepsilon$ for all $c \in \mathbb{R}$. In particular, we have with $C + \omega > L$

$$\frac{1}{2}\varepsilon > |f(C) - f(C + \omega)| \geq |f(C) - f_\infty| - |f(C + \omega) - f_\infty| > \varepsilon - \frac{1}{2}\varepsilon = \frac{1}{2}\varepsilon,$$

contradicting our assumption. Hence $\|f - f_\infty\|_\infty \leq \varepsilon$. Consequently, $f \equiv f_\infty$.

qed

Lemma A.2 Let $f : \mathbb{R} \rightarrow \mathbb{C}$ as well as $c \mapsto cf(c)$ be almost periodic.

Then f vanishes identically.

Proof Since almost periodic functions are bounded, $cf(c)$ is bounded. Hence, $f(c) \rightarrow 0$ for $c \rightarrow \infty$. Since f is almost periodic, we have $f \equiv 0$ from Lemma A.1. **qed**

The definition of almost periodicity can immediately be extended to functions mapping to Banach spaces X : Just replace modulus by norm. If X is, e.g., the space of $n \times n$ matrices, then the almost periodicity of $f : \mathbb{R} \rightarrow X$ is equivalent to the almost periodicity of all matrix elements of f . Again, constant functions are almost periodic; sums and products of almost periodic functions are almost periodic as well.

The substitution of \mathbb{R} by an arbitrary locally compact abelian group G is less direct. [9] Here, one considers the dual group Γ of G and then the dual group $\overline{G}_{\text{Bohr}}$ of Γ , where Γ is given the discrete topology. Via Gelfand transform, $\overline{G}_{\text{Bohr}}$ is compact abelian and contains G as a dense subset. It is called Bohr compactification of G . Almost periodic functions on G are now the G -restrictions of continuous functions on $\overline{G}_{\text{Bohr}}$.

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³Previous title: On the physical interpretation of states in loop quantum cosmology.