Formulas of F-thresholds and F-jumping coefficients on toric rings *

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Abstract

Mustaţă, Takagi and Watanabe define F-thresholds, which are invariants of a pair of ideals in a ring of characteristic $p > 0$. In their paper, it is proved that F-thresholds are equal to jumping numbers of test ideals on regular local rings. In this note, we give formulas of F-thresholds and F-jumping coefficients on toric rings. By these formulas, we prove that there exists an inequality between F-jumping coefficients and F-thresholds. In particular, we observe a comparison between F-pure thresholds and F-thresholds in some cases. As applications, we give a characterization of regularity for toric rings defined by simplicial cones, and we prove the rationality of F-thresholds in some cases.

1 Introduction

Let R be a commutative Noetherian ring of characteristic $p > 0$. In [HY], Hara and Yoshida defined a generalized test ideal $\tau(\mathfrak{a}^c)$ of an ideal $\mathfrak{a} \subseteq R$ and a positive real number $c \in \mathbb{R}_{>0}$. This is a generalization of the test ideal $\tau(R)$, which appeared in the theory of tight closure (cf. [HH]). On the other hand, this ideal is a characteristic p analogue of a multiplier ideal (cf. [Laz]). Similarly, one can define a characteristic p analogue of a jumping coefficient of a multiplier ideal, which is called the F-jumping coefficient. In other words, $c \in \mathbb{R}_{>0}$ is an F-jumping coefficient of an ideal $\mathfrak{a} \subseteq R$ if $\tau(\mathfrak{a}^c) \neq \tau(\mathfrak{a}^{c-\varepsilon})$ for all $\varepsilon > 0$.

Mustaţă, Takagi and Watanabe studied an F-jumping coefficient. In [MTW], they defined another invariant of singularities, which is called the F-threshold. They proved that an F-threshold coincides with an F-jumping coefficient on a regular local ring of characteristic $p > 0$. Using this relation,

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they proved basic properties of F-jumping coefficients. Blickle, Mustaţă and Smith studied F-jumping coefficients or F-thresholds on F-finite regular rings. In particular, they proved the rationality and discreteness of F-thresholds for F-finite regular rings under some assumptions (cf. [BMS1] and [BMS2] for details), which partially solves an open problem in [MTW].

However, if rings have singularities, F-thresholds do not coincide with F-jumping coefficients in general. In [HMTW], Huneke, Mustaţă, Takagi and Watanabe studied various topics of F-thresholds for general settings. For example, they defined a new invariant called the F-threshold of a module, which coincides with an F-jumping coefficient for F-finite and F-regular local normal $\mathbb{Q}$-Gorenstein rings. As a corollary, they proved an inequality between the F-threshold and the F-pure threshold, which is the smallest F-jumping coefficient for a fixed ideal. They also gave examples of non-regular rings and ideals whose F-thresholds coincide with their F-pure thresholds.

In this paper, we consider F-thresholds and F-jumping coefficients of monomial ideals for toric rings, which are not necessarily regular. We give the explicit formula of F-thresholds in section 3, which is written in terms of cones corresponding to toric rings and Newton polyhedrons corresponding to monomial ideals. Using this formula, we compare F-thresholds with F-jumping coefficients in section 4. As applications, we give a characterization of regularity of toric rings defined by simplicial cones in Theorem 5.3. We also prove the rationality of F-thresholds of monomial ideals for toric rings defined by simplicial cones in Theorem 5.5.

2 The definition of F-thresholds

Throughout this paper, we assume that every ring R is reduced, and contains a perfect field k whose characteristic is $p > 0$. Let $F : R \to R$ be the Frobenius map which sends an element x of R to x^p. For a positive integer e, the ring R viewed as an R-module via the e-times iterated Frobenius map is denoted by ${}^eR$. We assume that a ring R is F-finite, that is, ${}^1R$ is a finitely generated R-module. We also assume that a ring R is F-pure, that is, the Frobenius map F is pure. For an ideal J and a positive integer e, $J^{[p^e]}$ is the ideal generated by p^e-th power elements of J. For example, if J is $(X_1, X_2^2) \subset k[X_1, X_2]$, then $J^{[p^e]}$ is $(X_1^{p^e}, X_2^{2p^e})$. We recall the definition and some remarks of F-thresholds which are defined by Mustaţă, Takagi and Watanabe in [MTW]. These are invariants of a pair of ideals.

Definition 2.1 (F-threshold, cf. [MTW, §1]). Let $\mathfrak{a}$ and J be nonzero proper ideals of a ring R such that $\mathfrak{a} \subseteq \sqrt{J}$. The p^e-th threshold $\nu_{\mathfrak{a}}^J(p^e)$ of $\mathfrak{a}$ with respect to J is defined as

$$\nu_{\mathfrak{a}}^J(p^e) := \max\{r \in \mathbb{N} | \mathfrak{a}^r \not\subseteq J^{[p^e]}\}.$$

Then we define the F-threshold $\mathrm{c}^J(\mathfrak{a})$ of $\mathfrak{a}$ with respect to J as

$$\mathrm{c}^J(\mathfrak{a}) := \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}.$$

Remark. Since R is F-pure, if $u \notin J^{[p^e]}$, then $u^p \notin J^{[p^{e+1}]}$. This implies that $\nu_{\mathfrak{a}}^J(p^e)/p^e \leq \nu_{\mathfrak{a}}^J(p^{e+1})/p^{e+1}$, and hence $\mathrm{c}^J(\mathfrak{a})$ exists under our assumption. Furthermore, the assumption that $\mathfrak{a} \subseteq \sqrt{J}$ implies that $\mathrm{c}^J(\mathfrak{a}) < \infty$. However, in general, this limit does not necessarily exist. In [HMTW], Huneke, Mustaţă, Takagi and Watanabe defined $\mathrm{c}_-^J(\mathfrak{a})$ and $\mathrm{c}_+^J(\mathfrak{a})$ as

$$\mathrm{c}_-^J(\mathfrak{a}) := \liminf \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}, \ \mathrm{c}_+^J(\mathfrak{a}) := \limsup \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e},$$

for ideals $\mathfrak{a}$ and J with $\mathfrak{a} \subseteq \sqrt{J}$. When $\mathrm{c}_-^J(\mathfrak{a}) = \mathrm{c}_+^J(\mathfrak{a})$, they call it the F-threshold of $\mathfrak{a}$ with respect to J, which is denoted by $\mathrm{c}^J(\mathfrak{a})$. They give a sufficient condition when $\mathrm{c}^J(\mathfrak{a})$ exists (cf. [HMTW, Lemma 2.3]).

Let R° be the set of elements of R which are not contained in any minimal prime ideals of R. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let c be a positive real number. For an R-module D, we define the $\mathfrak{a}^c$-tight closure of the zero submodule in D as the following, which is denoted by $0_D^{*\mathfrak{a}^c}$. For $z \in D$, an element z is contained in $0_D^{*\mathfrak{a}^c}$ if there exists $x \in R^\circ$ such that

$$x\mathfrak{a}^{\lceil cp^e \rceil}(1 \otimes z) = 0 \in {}^eR \otimes D,$$

where e runs all sufficiently large positive integers.

Definition 2.2 (test ideal)**.** Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and c a positive real number. Let $E := \oplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$, where $\mathfrak{m}$ runs all maximal ideals of R and $E_R(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. The test ideal $\tau(\mathfrak{a}^c)$ of $\mathfrak{a}$ and c is defined as

$$\tau(\mathfrak{a}^c) := \bigcap_{D \subseteq E} \mathrm{Ann}_R 0_D^{*\mathfrak{a}^c},$$

where D runs all finitely generated R-submodules of E.

In [MTW], they also proved the connection between F-thresholds and test ideals for regular local rings. Moreover, in [BMS2], they generalized it for regular rings.

Theorem 2.3 ([MTW, Proposition 2.7] and [BMS2, Proposition 2.23])**.** Let $\mathfrak{a}$ and J be proper ideals on a regular ring R such that $\mathfrak{a} \subseteq \sqrt{J}$. Then

$$\tau(\mathfrak{a}^{\mathrm{c}^J(\mathfrak{a})}) \subseteq J.$$

On the other hand, for a positive real number c, we have $\mathfrak{a} \subseteq \sqrt{\tau(\mathfrak{a}^c)}$, and also

$$\mathrm{c}^{\tau(\mathfrak{a}^c)}(\mathfrak{a}) \leq c.$$

In addition, there exists a map from the set of F-thresholds of $\mathfrak{a}$ to the set of test ideals of $\mathfrak{a}$ which sends the test ideal J to $\mathrm{c}^J(\mathfrak{a})$. Moreover, this map is bijective. The inverse map sends an F-threshold c of $\mathfrak{a}$ to $\tau(\mathfrak{a}^c)$.

By the two inequalities in Theorem 2.3, F-thresholds on a regular ring are equal to F-jumping coefficients. They are analogues of jumping coefficients of a multiplier ideal.

Corollary 2.4. For a fixed nonzero proper ideal $\mathfrak{a}$ on a regular ring R, the set of F-thresholds of $\mathfrak{a}$ is equal to the set of F-jumping coefficients of $\mathfrak{a}$.

3 A formula of F-thresholds on toric rings

Let us begin with fixing the notation about toric geometries. Let $N \cong \mathbb{Z}^d$ and $M \cong \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ which is isomorphic to $\mathbb{Z}^d$. The duality pair of $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ is denoted by

$$\langle\ ,\ \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}.$$

For a strongly convex rational polyhedral cone σ of $N_{\mathbb{R}}$, we define

$$\sigma^\vee := \{u \in M_{\mathbb{R}} | \langle u, v\rangle \geq 0, \forall v \in \sigma\}.$$

Let R be a toric ring defined by σ, that is, the subalgebra of Laurent polynomial $k[X_1^{\pm 1}, \cdots, X_d^{\pm 1}]$ generated by sets $\{X^u | u \in \sigma^\vee \cap M\}$, where X^u expresses $X_1^{u_1} \cdots X_d^{u_d}$ for $u = (u_1, \cdots, u_d) \in M$. Since we always assume that k is a perfect field, a toric ring is F-finite under our assumption. A proper ideal $\mathfrak{a}$ of R is said to be a monomial ideal if $\mathfrak{a}$ is generated by monomials of $R \subset k[X_1^{\pm 1}, \cdots, X_d^{\pm 1}]$. For a monomial ideal $\mathfrak{a}$, we define two types of sets in $\sigma^\vee$.

Definition 3.1. The Newton polyhedron $\mathrm{P}(\mathfrak{a})$ of $\mathfrak{a}$ is defined as

$$\mathrm{P}(\mathfrak{a}) := \mathrm{conv}\{u \in M | X^u \in \mathfrak{a}\}.$$

Moreover, we define

$$\mathrm{Q}(\mathfrak{a}) := \bigcup_{X^u \in \mathfrak{a}} u + \sigma^\vee.$$

In addition, for $\mathrm{P}(\mathfrak{a})$ and a positive real number λ, the sets $\lambda\mathrm{P}(\mathfrak{a})$ is defined as

$$\lambda\mathrm{P}(\mathfrak{a}) := \{\lambda u \in M_\mathbb{R} | u \in \mathrm{P}(\mathfrak{a})\}.$$

We can define $\lambda\mathrm{Q}(\mathfrak{a})$ by the same way.

The following proposition is basic properties of $\mathrm{Q}(\mathfrak{a})$ and $\mathrm{P}(\mathfrak{a})$, which follows immediately.

Proposition 3.2. Let $\mathfrak{a}$ be a monomial ideal of a toric ring R defined by a cone $\sigma \subseteq N_\mathbb{R}$.

(i). For $e \in \mathbb{Z}_{>0}$, it holds that $\mathrm{Q}(\mathfrak{a}) = (1/p^e)\mathrm{Q}(\mathfrak{a}^{[p^e]})$.

(ii). $\mathrm{P}(\mathfrak{a}) + \sigma^\vee \subseteq \mathrm{P}(\mathfrak{a})$.

(iii). If $\mathfrak{a} = (X^{\mathbf{a}_1}, \cdots, X^{\mathbf{a}_s})$, then $\mathrm{P}(\mathfrak{a}) = \mathrm{conv}\{\mathbf{a}_1, \cdots, \mathbf{a}_s\} + \sigma^\vee$.

Using this notation, we give a computation of F-thresholds in real affine geometries. This formula is a generalization of [HMTW, Eample 2.7]. Let R be a toric ring defined by a cone $\sigma \subseteq N_\mathbb{R}$. Let $\mathfrak{a}$ be a monomial ideal. For $u \in \sigma^\vee$, we define $\lambda_\mathfrak{a}(u)$ as

$$\lambda_\mathfrak{a}(u) := \begin{cases} \sup\{\lambda \in \mathbb{R}_{\geq 0} | u \in \lambda\mathrm{P}(\mathfrak{a})\} & (\exists \lambda \in \mathbb{R}_{>0} \ s.t. \ u \in \lambda\mathrm{P}(\mathfrak{a})), \\ 0 & (\forall \lambda \in \mathbb{R}_{>0}, \ u \notin \lambda\mathrm{P}(\mathfrak{a})). \end{cases}$$

Theorem 3.3. Let R and $\mathfrak{a}$ be as the above. Let J be a monomial ideal such that $\mathfrak{a} \subseteq \sqrt{J}$. Then

$$\mathrm{c}^J(\mathfrak{a}) = \sup_{u \in \sigma^\vee \setminus \mathrm{Q}(J)} \lambda_\mathfrak{a}(u).$$

Proof. We assume that $\mathfrak{a} = (X^{\mathbf{a}_1}, \cdots, X^{\mathbf{a}_s})$ where $\mathbf{a}_i \in M$ for $i = 1, \cdots, s$. To prove the theorem, we need the following two claims.

Claim 1. For any positive integers e, there exists $u \in \sigma^\vee \setminus \mathrm{Q}(J)$ such that $\nu_\mathfrak{a}^J(p^e)/p^e \leq \lambda_\mathfrak{a}(u)$.

Claim 2. For every element $u \in \sigma^\vee \setminus \mathrm{Q}(J)$, there exists a positive integer e such that $\nu_{\mathfrak{a}}^J(p^e)/p^e \geq \lambda_{\mathfrak{a}}(u)$.

We note that Claim 1 implies $\mathrm{c}^J(\mathfrak{a}) \leq \sup \lambda_{\mathfrak{a}}(u)$. Since the definition of right-hand side supremum, $\nu_{\mathfrak{a}}^J(p^e)/p^e \leq \sup \lambda_{\mathfrak{a}}(u)$. Thus $\mathrm{c}^J(\mathfrak{a}) \leq \sup \lambda_{\mathfrak{a}}(u)$ by the definition of F-thresholds and the fact that a supremum is the minimum number in upper bounds. By the similar argument, Claim 2 implies $\mathrm{c}^J(\mathfrak{a}) \geq \sup \lambda_{\mathfrak{a}}(u)$.

Proof of claim 1. We fix a positive integer e. Since the definition of the p^e-th threshold, for every $i = 1, \cdots, s$, there are nonnegative integers r_i such that $\sum r_i = \nu_{\mathfrak{a}}^J(p^e)$ and $X^{\sum r_i \mathbf{a}_i} \notin J^{[p^e]}$. In particular, $\sum r_i \mathbf{a}_i \notin \mathrm{Q}(J^{[p^e]})$. This is equivalent to $(1/p^e) \sum r_i \mathbf{a}_i \notin (1/p^e)\mathrm{Q}(J^{[p^e]})$. By Proposition 3.2 (i), we have $(1/p^e) \sum r_i \mathbf{a}_i \notin \mathrm{Q}(J)$. Hence

$$\frac{1}{p^e} \sum r_i \mathbf{a}_i = \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} \sum \frac{r_i}{\nu_{\mathfrak{a}}^J(p^e)} \mathbf{a}_i$$

which is an element of $(\nu_{\mathfrak{a}}^J(p^e)/p^e)\mathrm{P}(\mathfrak{a})$. Thus $\nu_{\mathfrak{a}}^J(p^e)/p^e \leq \lambda_{\mathfrak{a}}((1/p^e) \sum r_i \mathbf{a}_i)$. $\square$

Proof of Claim 2. We fix $u \in \sigma^\vee \setminus \mathrm{Q}(J)$, which satisfies $\lambda_{\mathfrak{a}}(u) \neq 0$. We find an integer e which satisfies the assertion of Claim 2 by three steps.
Step 1. We prove that there exists an element u' on the boundary $(\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil / p^e)\mathrm{P}(\mathfrak{a})$ such that $u' \notin \mathrm{Q}(J)$ for sufficiently large e. The following sequence of real numbers

$$\lambda_{\mathfrak{a}}(u) \leq \cdots \leq \frac{\lceil p^{e+1} \lambda_{\mathfrak{a}}(u) \rceil}{p^{e+1}} \leq \frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil}{p^e} \leq \cdots \leq \frac{\lceil p \lambda_{\mathfrak{a}}(u) \rceil}{p}$$

induces the sequence of Newton polyhedrons

$$\frac{\lceil p \lambda_{\mathfrak{a}}(u) \rceil}{p}\mathrm{P}(\mathfrak{a}) \subseteq \cdots \subseteq \frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil}{p^e}\mathrm{P}(\mathfrak{a}) \subseteq \frac{\lceil p^{e+1} \lambda_{\mathfrak{a}}(u) \rceil}{p^{e+1}}\mathrm{P}(\mathfrak{a}) \subseteq \cdots \subseteq \lambda_{\mathfrak{a}}(u)\mathrm{P}(\mathfrak{a}).$$

In particular, the above sequences are strict if $\lambda_{\mathfrak{a}}(u) \notin (1/p^e)\mathbb{Z}$ for all e. Since $u \notin \mathrm{Q}(J)$, we can find such u' by taking e sufficiently large.
Step 2. We prove that there exist nonnegative integers r_i for every $i = 1, \cdots, s$ such that $\sum r_i/p^e \geq \lambda_{\mathfrak{a}}(u)$ and $u'' := \sum r_i \mathbf{a}_i / p^e \notin \mathrm{Q}(J)$. Since u' is contained in $(\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil / p^e)\mathrm{P}(\mathfrak{a})$, u' can be written

$$\frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil}{p^e}(\sum c_i \mathbf{a}_i + \omega),$$

where c_i are nonnegative real numbers with $\sum c_i = 1$ and $\omega \in \sigma^\vee$ by Proposition 3.2 (iii). Let

$$r_i := \lceil \lceil p^e \lambda_{\mathfrak{a}}(u) \rceil c_i \rceil.$$

Then

$$\sum \frac{r_i}{p^e} \geq \frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil}{p^e} \sum c_i \geq \lambda_{\mathfrak{a}}(u).$$

Moreover,

$$|u'' + \frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil}{p^e} \omega - u'| \leq \sum | \frac{\lceil \lceil p^e \lambda_{\mathfrak{a}}(u) \rceil c_i \rceil}{p^e} - \frac{\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil c_i}{p^e} | \cdot |\mathbf{a}_i| < \frac{1}{p^e} \sum |\mathbf{a}_i|.$$

Since $u' \notin \mathrm{Q}(J)$, we have $u'' + (\lceil p^e \lambda_{\mathfrak{a}}(u) \rceil / p^e)\omega \notin \mathrm{Q}(J)$ if we choose e sufficiently large. By the definition of $\mathrm{Q}(J)$, we have $u'' \notin \mathrm{Q}(J)$.
STEP 3. Since $u'' \notin \mathrm{Q}(J)$,

$$p^e u'' \notin p^e \mathrm{Q}(J) = \mathrm{Q}(J^{[p^e]}).$$

Therefore $X^{p^e u''} \notin J^{[p^e]}$. On the other hand, $X^{p^e u''} \in \mathfrak{a}^{\sum r_i}$ by the construction of u''. Therefore $\sum r_i \leq \nu_{\mathfrak{a}}^J(p^e)$. This implies $\lambda_{\mathfrak{a}}(u) \leq \nu_{\mathfrak{a}}^J(p^e)/p^e$. □

We complete the proof of Theorem 3.3. □

4 A comparison between F-jumping coefficients and F-thresholds

F-pure thresholds are defined via F-singularities of the pair $(R, \mathfrak{a}^c)$ where c is a positive real number. See [TW, Definition 1.3, Definition 2.1] for details. Since F-finite toric rings are strongly F-regular, the F-pure thresholds can be defined as follows (See [TW, Proposition 2.2]).

Definition 4.1 (F-pure thresholds)**.** Let R be a toric ring, and $\mathfrak{a}$ a monomial ideal. The F-pure threshold $\mathrm{c}(\mathfrak{a})$ of $\mathfrak{a}$ is defined as

$$\mathrm{c}(\mathfrak{a}) := \sup\{c \in \mathbb{R}_{\geq 0} | \tau(\mathfrak{a}^c) = R\}.$$

Hence the F-pure threshold of $\mathfrak{a}$ is the smallest F-jumping coefficient of $\mathfrak{a}$. In [HMTW], the inequality between an F-pure threshold and an F-threshold on a local ring was given in terms of the F-threshold of a module ([HMTW, Section 4.]). In this section, we consider the inequality on toric rings, by a combinatorial method. Furthermore, we consider the connection between arbitrary F-jumping coefficients and F-thresholds with respect to some monomial ideals. To compute F-pure thresholds and F-jumping coefficients of monomial ideals, we introduce the following theorem given by Blickle.

Theorem 4.2 ([B, Theorem 3])**.** Let R be the toric ring defined by $\sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_n \subseteq N_{\mathbb{R}} := \mathbb{R}^d$, where $v_j \in N$ are primitive. Then the test ideal $\tau(\mathfrak{a}^c)$ of a monomial ideal $\mathfrak{a}$ is also a monomial ideal. Moreover, $X^u \in \tau(\mathfrak{a}^c)$ for $u \in M$ if and only if there exists $\omega \in M_{\mathbb{R}}$ such that

$$\langle \omega, v_j \rangle \leq 1,\ j = 1, \cdots, n,$$
$$u + \omega \in \mathrm{Int}(c\mathrm{P}(\mathfrak{a})).$$

By this theorem, the F-pure threshold of a monomial ideal on a toric ring can be described as the following corollary.

Corollary 4.3. Let R and $\mathfrak{a}$ be as in Theorem 4.2. Then the F-pure threshold $\mathrm{c}(\mathfrak{a})$ of $\mathfrak{a}$ is described as

$$\mathrm{c}(\mathfrak{a}) = \sup_{u \in \sigma^{\vee} \setminus \mathrm{O}} \lambda_{\mathfrak{a}}(u),$$

where

$$\mathrm{O} := \{u \in \sigma^{\vee} | \exists j,\ \langle u, v_j \rangle \geq 1\}.$$

Proof. First, we assume that $\mathrm{c}(\mathfrak{a}) < \sup \lambda_{\mathfrak{a}}(u)$. Then there exists $\alpha \in \mathbb{R}_{\geq 0}$ such that

$$\mathrm{c}(\mathfrak{a}) < \alpha < \sup \lambda_{\mathfrak{a}}(u).$$

By the definition of F-pure thresholds, $\tau(\mathfrak{a}^{\alpha}) \subsetneq R$. Then there exists $\beta \in \mathbb{R}_{\geq 0}$ such that

$$\alpha < \beta < \sup \lambda_{\mathfrak{a}}(u)$$

and $\beta = \lambda_{\mathfrak{a}}(u')$ for $u' \in \sigma^{\vee} \setminus \mathrm{O}$. This implies that $u' \in \beta\mathrm{P}(\mathfrak{a})$. In particular, $u' \in \mathrm{Int}(\alpha\mathrm{P}(\mathfrak{a}))$. In addition, $\langle u', v_j \rangle < 1$ for all j. By Theorem 4.2, it contradicts that $\tau(\mathfrak{a}^{\alpha}) \subsetneq R$. Therefore $\mathrm{c}(\mathfrak{a}) \geq \sup \lambda_{\mathfrak{a}}(u)$. Second, we assume $\mathrm{c}(\mathfrak{a}) > \sup \lambda_{\mathfrak{a}}(u)$. There exists $\alpha \in \mathbb{R}_{\geq 0}$ such that

$$\sup \lambda_{\mathfrak{a}}(u) < \alpha < \mathrm{c}(\mathfrak{a})$$

and $\tau(\mathfrak{a}^{\alpha}) = R$. This implies that there exists $\omega \in \sigma^{\vee}$ such that $\langle \omega, v_j \rangle \leq 1$ for all j and

$$\omega \in \mathrm{Int}(\alpha\mathrm{P}(\mathfrak{a})).$$

For $1 > \varepsilon > 0$, we have $\langle (1-\varepsilon)\omega, v_j \rangle = 1 - \varepsilon < 1$. Thus $(1-\varepsilon)\omega \in \sigma^\vee \setminus \mathrm{O}$. On the other hand, since $\omega \in \mathrm{Int}(\alpha \mathrm{P}(\mathfrak{a}))$, it holds that

$$(1-\varepsilon)\omega \in \alpha \mathrm{P}(\mathfrak{a}),$$

for sufficiently small ε. Therefore

$$\sup_{u \in \sigma^\vee \setminus \mathrm{O}} \lambda_{\mathfrak{a}}(u) < \lambda_{\mathfrak{a}}((1-\varepsilon)\omega),$$

which is a contradiction. Thus $\mathrm{c}(\mathfrak{a}) \geq \sup \lambda_{\mathfrak{a}}(u)$, which completes the proof of the corollary. □

Using this presentation, we compare an F-pure threshold with an F-threshold with respect to the maximal monomial ideal on a toric ring.

Proposition 4.4. Let R, σ and $\mathfrak{a}$ be as in Corollary 4.3, and $\mathfrak{m}$ the maximal monomial ideal of R. Then

$$\mathrm{c}(\mathfrak{a}) \leq \mathrm{c}^{\mathfrak{m}}(\mathfrak{a}).$$

Proof. By the definitions, it is enough to show that $\mathrm{Q}(\mathfrak{m}) \subseteq \mathrm{O}$. In particular, it is enough to show $\mathrm{Q}(\mathfrak{m}) \cap M \subseteq \mathrm{O}$. It follows immediately. □

Remark. In general, for an ideal $\mathfrak{a}$, we have $\mathrm{c}^{J'}(\mathfrak{a}) \leq \mathrm{c}^{J}(\mathfrak{a})$, where J and J' are ideals with $J \subseteq J'$ and $\mathfrak{a} \subseteq \sqrt{J}$. Therefore the F-pure threshold of $\mathfrak{a}$ is less than or equal to all F-thresholds of $\mathfrak{a}$.

Now we generalize this comparison to arbitrary F-jumping coefficients and F-thresholds.

Lemma 4.5. Let R, σ and $\mathfrak{a}$ be as in Theorem 4.2 and $\omega,\ \omega' \in \sigma^\vee$. For all $j = 1, \cdots, n$, we assume that

$$\langle \omega, v_j \rangle \leq \langle \omega', v_j \rangle.$$

Then $\lambda_{\mathfrak{a}}(\omega) \leq \lambda_{\mathfrak{a}}(\omega')$.

Proof. If $\lambda_{\mathfrak{a}}(\omega) = 0$, it is trivial. We prove this lemma in the case $\lambda_{\mathfrak{a}}(\omega) \neq 0$. By the assumption, there exists $\omega'' \in \sigma^\vee$ such that $\omega' = \omega + \omega''$. Let $\lambda := \lambda_{\mathfrak{a}}(\omega)$. Since $\omega/\lambda \in \mathrm{P}(\mathfrak{a})$,

$$\frac{\omega'}{\lambda} = \frac{\omega}{\lambda} + \frac{\omega''}{\lambda} \in \mathrm{P}(\mathfrak{a}) + \sigma^\vee.$$

By Proposition 3.2 (ii), we have $\omega'/\lambda \in \mathrm{P}(\mathfrak{a})$. Hence $\lambda \leq \lambda_{\mathfrak{a}}(\omega')$. □

Proposition 4.6. Let R, σ and $\mathfrak{a}$ be as in Theorem 4.2. For $u \in \sigma^\vee \cap M$, we define the nonnegative number $\mu_{\mathfrak{a}}(u)$ as

$$\mu_{\mathfrak{a}}(u) := \sup_{\omega \in \sigma^\vee \setminus \mathrm{O}} \lambda_{\mathfrak{a}}(u + \omega),$$

and the nonnegative number $\mathrm{c}^i(\mathfrak{a})$ as

$$\mathrm{c}^i(\mathfrak{a}) = \inf_{X^u \in \tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})})} \mu_{\mathfrak{a}}(u),$$

where $\mathrm{c}^0(\mathfrak{a}) := 0$. Then $\mathrm{c}^i(\mathfrak{a})$ is the i-th F-jumping coefficient of $\mathfrak{a}$.

Proof. We show that $\mathrm{c}^i(\mathfrak{a})$ is a jumping number of the test ideal. We assume that

$$\tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})}) = (X^{\mathbf{b}_1}, \cdots, X^{\mathbf{b}_t}).$$

By Lemma 4.5,

$$\mathrm{c}^i(\mathfrak{a}) = \inf_{j=1,\cdots,t} \mu_{\mathfrak{a}}(\mathbf{b}_j).$$

Since $\{\mathbf{b}_j\}$ is a finite set, there exists j' such that $\mathrm{c}^i(\mathfrak{a}) = \mu_{\mathfrak{a}}(\mathbf{b}_{j'})$. By the definition of $\mathrm{c}^i(\mathfrak{a})$, for all $\omega \in \sigma^\vee \setminus \mathrm{O}$,

$$\mathbf{b}_{j'} + \omega \notin \mathrm{Int}(c^i(\mathfrak{a})\mathrm{P}(\mathfrak{a})).$$

This implies that $X^{\mathbf{b}_{j'}} \notin \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})})$ by Theorem 4.2. On the other hand, there exists $\omega' \in \sigma^\vee \setminus \mathrm{O}$ such that

$$\mathbf{b}_{j'} + \omega' \in \mathrm{Int}((\mathrm{c}^i(\mathfrak{a}) - \varepsilon)\mathrm{P}(\mathfrak{a})),$$

for all $\varepsilon > 0$. This also implies that $X^{\mathbf{b}_{j'}} \in \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})-\varepsilon})$. Therefore $\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})}) \subsetneq \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})-\varepsilon})$ and hence $\mathrm{c}^i(\mathfrak{a})$ is a jumping number.

We show that $\mathrm{c}^i(\mathfrak{a})$ is the i-th F-jumping coefficient of $\mathfrak{a}$. In other words, $\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})-\varepsilon}) = \tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})})$ for all $\varepsilon > 0$ with $\mathrm{c}^{i-1}(\mathfrak{a}) \leq \mathrm{c}^i(\mathfrak{a}) - \varepsilon$. The inclusion $\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})-\varepsilon}) \subseteq \tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})})$ follows immediately from Theorem 4.2. The opposite inclusion follows from the definition of $\mathrm{c}^i(\mathfrak{a})$. In fact, if $X^u \in \tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})})$, then $\mathrm{c}^i(\mathfrak{a}) - \varepsilon < \mathrm{c}^i(\mathfrak{a}) \leq \mu_{\mathfrak{a}}(u)$, by definition of $\mathrm{c}^i(\mathfrak{a})$. Hence there exists $\omega \in \sigma^\vee \setminus \mathrm{O}$ such that

$$u + \omega \in \mathrm{Int}((\mathrm{c}^i(\mathfrak{a}) - \varepsilon)\mathrm{P}(\mathfrak{a})).$$

This implies that $X^u \in \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})-\varepsilon})$ by Theorem 4.2. We complete the proof of the proposition. □

Proposition 4.7. We have the following inequality:

$$\mathrm{c}^i(\mathfrak{a}) \leq \mathrm{c}^{\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})})}(\mathfrak{a}).$$

Proof. Since $\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})}) \subsetneq \tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})})$, there exists $u \in \sigma^\vee \cap M$ such that $X^u \in \tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})})$ and $X^u \notin \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})})$. By Proposition 4.6,

$$\mathrm{c}^i(\mathfrak{a}) \leq \mu_{\mathfrak{a}}(u). \tag{1}$$

We claim that for all $\omega \in \sigma^\vee \setminus \mathrm{O}$,

$$\omega + u \in \sigma^\vee \setminus \mathrm{Q}(\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})})).$$

By Theorem 3.3, this claim implies that

$$\mu_{\mathfrak{a}}(u) \leq \mathrm{c}^{\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})})}(\mathfrak{a}). \tag{2}$$

The proof of the proposition is completed from inequalities (1) and (2). Now we prove that claim. We assume that there exists $\omega \in \sigma^\vee \setminus \mathrm{O}$ such that $u + \omega \in \mathrm{Q}(\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})}))$. There exist $u' \in M$ and $\omega' \in \sigma^\vee$ such that $X^{u'} \in \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})})$ and $u + \omega = u' + \omega'$. Thus $u - u' = \omega' - \omega \in M$. On the other hand, since $u = u' + \omega' - \omega \in M$ and $X^u \notin \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})})$, we have $\omega' - \omega \notin \sigma^\vee$. That is, there exists j such that $\langle(\omega' - \omega), v_j\rangle < 0$. Therefore

$$0 \leq \langle \omega', v_j\rangle < \langle \omega, v_j\rangle < 1.$$

It contradicts that $\omega' - \omega \in M$. Hence we have the claim, and then we complete the proof of the proposition. □

Remark. Since a toric ring is strongly F-regular, $\mathfrak{a} \subseteq \tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})})$. Hence $\mathrm{c}^{\tau(\mathfrak{a}^{\mathrm{c}^i(\mathfrak{a})})}(\mathfrak{a})$ exists and is a finite number.

5 Applications

Let us give some applications of results in previous sections. As we see in Corollary 2.4, for an arbitrary ideal $\mathfrak{a}$, F-thresholds of $\mathfrak{a}$ are equal to F-jumping coefficients of $\mathfrak{a}$ on regular rings. By the formula of F-thresholds, we see that if R is a toric ring which has at most Gorenstein singularities, then there exists a monomial ideal $\mathfrak{a} \subseteq R$ such that $\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$.

Proposition 5.1. Let R be a Gorenstein toric ring defined by a cone $\sigma \subseteq N_{\mathbb{R}} = \mathbb{R}^d$ and $\mathfrak{m}$ the maximal monomial ideal. There exist a monomial ideal $\mathfrak{a} \subseteq R$ such that $\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$.

Proof. We assume that $\sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_n$, where $v_j \in N$ are primitive numbers. For a Gorenstein toric ring R, there exists an element $\omega \in \sigma^\vee \cap M$ such that $\langle \omega, v_j \rangle = 1$ for all $j = 1, \cdots, n$. By Lemma 4.5, for a monomial ideal $\mathfrak{a} \subseteq R$, we can describe

$$\mathrm{c}(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\omega).$$

Let $\mathfrak{a} = (X^\omega)$. We have $\mathrm{P}(\mathfrak{a}) = \omega + \sigma^\vee$, and clearly $\mathrm{c}(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\omega) = 1$. Since $\omega \in M \setminus \mathbf{0}$, we have $\omega \in \mathrm{Q}(\mathfrak{m})$. Hence $\mathrm{P}(\mathfrak{a}) \subseteq \mathrm{Q}(\mathfrak{m})$. By Theorem 3.3, that implies $\mathrm{c}^{\mathfrak{m}}(\mathfrak{a}) \leq 1 = \mathrm{c}(\mathfrak{a})$. On the other hand, $\mathrm{c}(\mathfrak{a}) \leq \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$ follows by Proposition 4.4. We complete the proof of the proposition. □

For 2-dimensional toric rings, we see that the opposite assertion of Proposition 5.1 is true. However, it is false in general toric rings whose dimension are greater than 3.

Proposition 5.2. Let R be a 2-dimensional toric ring, and $\mathfrak{m}$ the maximal monomial ideal of R. If there exists a monomial ideal $\mathfrak{a} \subseteq R$ such that $\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$, then R has at most Gorenstein singularities.

Proof. Suppose that R is defined by a cone σ. By taking a suitable change of coordinates, it suffices to consider cones $\sigma := \mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(a, b) \subseteq \mathbb{R}^2$, where $b > 0$ and the greatest common divisor of a and b is 1. The following three cases are trivial. If $a = 0$, then R is the polynomial ring. If $a = 1, b = 1$, then $R = k[X_1,\ X_1^{-1}X_2]$, which is a regular ring. If $a = 1$, $b > 1$, then $R = k[X_1, X_2, X_1^b X_2^{-1}] \cong k[x, y, z]/(xz - y^b)$. Note that $\mathrm{Spec}R$ has an A_{b-1} singularity. Hence R is a Gorenstein ring. Assume that $a > 1$. We have $\sigma^\vee = \mathbb{R}_{\geq 0}(0, 1) + \mathbb{R}_{\geq 0}(b, -a)$, and the point $\omega = (1, (1-a)/b)$ which satisfies

$$\langle \omega, (1, 0) \rangle = \langle \omega, (a, b) \rangle = 1.$$

If $\omega \notin \mathrm{Q}(\mathfrak{m})$, then for all monomial ideals $\mathfrak{a}$, we have $\mathrm{c}(\mathfrak{a}) < \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$. In fact, by taking $\varepsilon > 0$ with $(1 + \varepsilon)\omega \notin \mathrm{Q}(\mathfrak{m})$, we have a strict inequality;

$$\mathrm{c}(\mathfrak{a}) < \lambda_{\mathfrak{a}}((1 + \varepsilon)) \leq \mathrm{c}^{\mathfrak{m}}(\mathfrak{a}).$$

By the assumption of the proposition, $\omega \in \mathrm{Q}(\mathfrak{m})$. Thus it is enough to prove that $\omega \in M$ under the assumption $\omega \in \mathrm{Q}(\mathfrak{m})$. By the definition of $\mathrm{Q}(\mathfrak{m})$, if $\omega \in \mathrm{Q}(\mathfrak{m})$, then there exists a lattice point $u \in \sigma^\vee \cap M \setminus \{\mathbf{0}\}$ such that $\omega - u \in \sigma^\vee$. Since $u \in \sigma^\vee$, the lattice point u is written as $u = \lambda_1(0, 1) + \lambda_2(b, -a) \in M$, where λ_1 and λ_2 are positive. Since $\omega - u \in \sigma^\vee$, we have $(1/b) - \lambda_1 \geq 0$ and $(1/b) - \lambda_2 \geq 0$. Since $\mathbf{0} \neq u \in M$ and $b \in \mathbb{Z}_{>0}$, we have $\lambda_2 = 1/b$. Hence $u = (1, \lambda_1 - (a/b))$. Since $u \in M$, there exists an integer l such that $l = \lambda_1 - (a/b)$ and

$$-\frac{a}{b} \le l \le \frac{1-a}{b}.$$

Since $a,\ b \in \mathbb{Z}$ and the greatest common divisor of a and b is 1, we have $bl = 1 - a$. Thus $b|(1-a)$. This implies that $\omega \in M$. The remaining case when $a < 0$ follows by the same argument. We complete the proof of the proposition. □

Example 1. Suppose $N = \mathbb{R}^3$. We define generators $\{v_i\}$ of a cone $\sigma \subseteq N_{\mathbb{R}}$ as

$$v_1 := (1,0,0),\ v_2 := (1,1,0),\ v_3 := (0,1,r).$$

Let $\omega := (1, 0, 1/r) \in \sigma^\vee$. Since $\langle \omega,\ v_i \rangle = 1$ for all i, the toric ring R defined by σ has an r-Gorenstein singularity. We choose a set of generators of $\sigma^\vee$ as

$$u_1 := (r, -r, 1),\ u_2 := (0, r, -1),\ u_3 := (0,0,1).$$

Then

$$\omega = \frac{1}{r}u_1 + \frac{1}{r}u_2 + \frac{1}{r}u_3.$$

Since $\omega - (1/r)u_3 \in \sigma^\vee \cap M$, we have $\omega \in \mathrm{Q}(\mathfrak{m})$, where $\mathfrak{m} \subseteq R$ is the maximal monomial ideal. Let $\mathfrak{a}$ be a monomial ideal generated by $X^{r\omega}$. Then $(1/r)\mathrm{P}(\mathfrak{a}) = \omega + \sigma^\vee \subseteq \mathrm{Q}(\mathfrak{m})$. The same argument in the proof of Proposition 5.1 implies $\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a}) = 1/r$.

Example 2. Suppose $N = \mathbb{R}^d$, where $d > 3$. We consider the cone σ generated by

$$\begin{aligned}
v_1 &:= (1,0,0,0, && \cdots && ,0) \\
v_2 &:= (1,1,0,0, && \cdots && ,0) \\
v_3 &:= (0,1,r,0, && \cdots && ,0) \\
v_i &:= (0,0,0,0, && \cdots, 0, \overset{\overset{i}{\vee}}{1}, 0, \cdots && ,0),\ 3 < i \le d.
\end{aligned}$$

By the same argument in Example 1, we have a monomial ideal $\mathfrak{a}$ of a d-dimensional r-Gorenstein ring $R = k[\sigma^\vee \cap M]$ such that $\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$.

Using F-thresholds and F-pure thresholds, we give a criterion of regularities for a toric ring defined by a simplicial cone.

Theorem 5.3. Let R be a toric ring defined by a simplicial cone σ, and $\mathfrak{m}$ the maximal monomial ideal. If there exists a monomial ideal $\mathfrak{a}$ such that $\sqrt{\mathfrak{a}} = \mathfrak{m}$ and

$$\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a}),$$

then R is a regular ring.

Proof. Let $\sigma \subseteq N_{\mathbb{R}} := \mathbb{R}^d$. Since σ is simplicial, we may assume that

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots \mathbb{R}_{\geq 0} v_d,$$

where $v_j \in N$ and $\{v_1, \cdots, v_d\}$ are $\mathbb{R}$-linearly independent. Hence there exist $u_i \in M$ and $l_i \in \mathbb{Z}_{>0}$ such that

$$\sigma^\vee = \mathbb{R}_{\geq 0} u_1 + \cdots + \mathbb{R}_{\geq 0} u_d,$$

and $\langle u_i, v_j \rangle = l_i \delta_{ij}$. Moreover, for all $i, j = 1, \cdots, d$, we assume v_j and u_i are primitive. Since σ is simplicial, R is $\mathbb{Q}$-Gorenstein. Hence there exists $\omega \in M \otimes \mathbb{Q}$ such that

$$\mathrm{c}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\omega).$$

By Theorem 3.3,

$$\lambda_{\mathfrak{a}}(\omega) \mathrm{P}(\mathfrak{a}) \subseteq \mathrm{Q}(\mathfrak{m}). \tag{3}$$

To prove the theorem, it is enough to show that $l_i = 1$ for every $i = 1, \cdots, d$. We derive a contradiction assuming $l_i > 1$ for some i. Since $\sqrt{\mathfrak{a}} = \mathfrak{m}$, for a sufficiently large nonnegative integer l, we have $X^{lu_i} \in \mathfrak{a}$. In particular, $\lambda_{\mathfrak{a}}(\omega) l u_i \in \lambda_{\mathfrak{a}}(\omega) \mathrm{P}(\mathfrak{a})$. If we choose sufficiently large l, then we have

$$0 < \frac{l_i - 1}{\lambda_{\mathfrak{a}}(\omega) l l_i - 1} < 1.$$

Let $\alpha \in \mathbb{R}_{>0}$ such that $0 < \alpha < (l_i - 1)/(\lambda_{\mathfrak{a}}(\omega) l l_i - 1)$. By the definition of $\mathrm{P}(\mathfrak{a})$ and (3),

$$\alpha \lambda_{\mathfrak{a}}(\omega) l u_i + (1 - \alpha)\omega \in \mathrm{Q}(\mathfrak{m}).$$

On the other hand, for all j,

$$\langle \alpha \lambda_{\mathfrak{a}}(\omega) l u_i + (1 - \alpha)\omega, v_j \rangle = \begin{cases} 1 - \alpha < 1 & (j \neq i), \\ \alpha \lambda_{\mathfrak{a}}(\omega) l l_i + 1 - \alpha < l_i & (j = i). \end{cases}$$

By the definition of $\mathrm{Q}(\mathfrak{m})$, there exist $l_i' \in \mathbb{Z}_{>0}$, $u \in M \cap \mathrm{Q}(\mathfrak{m})$ and $u' \in \sigma^\vee$ such that

$$\langle u, v_j \rangle = \begin{cases} 0 & (j \neq i) \\ l_i' < l_i & (j = i), \end{cases}$$

and

$$\alpha\lambda_{\mathfrak{a}}(\omega)lu_i + (1-\alpha)\omega = u + u'.$$

However, the existence of u contradicts the primitiveness of u_i. Thus $l_i = 1$. Eventually, for every $i = 1, \cdots, d$, we have $l_i = 1$. Therefore we complete the proof of the theorem. □

On the other hand, there exist a toric R defined by a non-simplicial cone and a maximal ideal $\mathfrak{m}$ such that $\mathrm{c}(\mathfrak{m}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{m})$.

Example 3 ([HMTW, Remark 2.5]). If $R = k[X_1X_3, X_2X_3, X_3, X_1X_2X_3]$ and $\mathfrak{m} = (X_1X_3, X_2X_3, X_3, X_1X_2X_3)$, then R is a toric ring whose defining cone is

$$\sigma = \mathbb{R}_{\geq 0}(1,0,0) + \mathbb{R}_{\geq 0}(0,1,0) + \mathbb{R}_{\geq 0}(-1,0,1) + \mathbb{R}_{\geq 0}(0,-1,1).$$

There exists $\omega = (1,1,2) \in \sigma^{\vee}$ which entails

$$\langle \omega, (1,0,0)\rangle = \langle \omega, (0,1,0)\rangle = \langle \omega, (-1,0,1)\rangle = \langle \omega, (0,-1,1)\rangle = 1.$$

By Corollary 4.3 and Lemma 4.5, for every monomial ideal $\mathfrak{a}$, we have $\mathrm{c}(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\omega)$. Hence $\mathrm{c}(\mathfrak{m}) = 2$. On the other hand, we can compute $\mathrm{c}^{\mathfrak{m}}(\mathfrak{m}) = 2$.

Finally, we discuss about the rationality of F-thresholds. This was given as an open problem in [MTW]. For some regular rings, Blickle, Mustaţă and Smith give the affirmative answer. In [BMS2], they prove the rationality of F-thresholds of all proper ideals $\mathfrak{a}$ with respect to ideals J which entail $\mathfrak{a} \subseteq \sqrt{J}$ on an F-finite regular ring essentially of finite type over k ([BMS2, Theorem 3.1]). In addition, they also prove in cases that $\mathfrak{a} = (f)$ is principal on an F-finite regular ring ([BMS1, Theorem 1.2]). On the other hand, Katzman, Lyubeznik and Zhang prove in cases that $\mathfrak{a} = (f)$ is principal on an excellent regular local ring, that is not necessarily F-finite ([KLZ]). We will prove rationality of an F-threshold of a monomial ideal $\mathfrak{a}$ with respect to an $\mathfrak{m}$-primary monomial ideal J on a toric ring. This argument is described in terms of real affine geometries. We define the affine half space $\mathrm{H}^+(v;\lambda)$ as

$$\mathrm{H}^+(v;\lambda) := \{u \in M_{\mathbb{R}} | \langle u, v\rangle \geq \lambda\},$$

where $v \in N_{\mathbb{R}}$ and $\lambda \in \mathbb{R}$. We also define the hyperplane $\partial\mathrm{H}^+(v;\lambda)$ as

$$\partial\mathrm{H}^+(v;\lambda) := \{u \in M_{\mathbb{R}} | \langle u, v\rangle = \lambda\}.$$

Assume that $\mathfrak{a}$ is a monomial ideal of a toric ring. Since $\mathrm{P}(\mathfrak{a})$ is a convex polyhedral set, it is written as an intersection of finite affine half spaces. we observe the form of $\mathrm{P}(\mathfrak{a})$.

Lemma 5.4. Let R be a toric ring defined by a cone $\sigma \subseteq N_{\mathbb{R}} = \mathbb{R}^d$, and $\mathfrak{a}$ a monomial ideal of R. Then there exist $v'_l \in N_{\mathbb{Q}} := N \otimes \mathbb{Q}$ and $\lambda'_l \in \mathbb{Q}$ for $l = 1, \cdots, t$ such that $\mathrm{P}(\mathfrak{a}) = \cap_{l=1}^{t} \mathrm{H}^+(v'_l; \lambda'_l)$.

Proof. Since σ is a rational polyhedral cone, so is $\sigma^\vee$. Hence there exists $u_i \in M$ such that

$$\sigma^\vee = \mathbb{R}_{\geq 0} u_1 + \cdots + \mathbb{R}_{\geq 0} u_m.$$

We assume that $\mathfrak{a} = (X^{\mathbf{a}_1}, \cdots, X^{\mathbf{a}_s})$. We consider the rational polyhedral cone τ of $M_{\mathbb{R}} \times \mathbb{R}$ as

$$\tau := \mathbb{R}_{\geq 0}(\mathbf{a}_1, 1) + \cdots + \mathbb{R}_{\geq 0}(\mathbf{a}_s, 1) + \mathbb{R}_{\geq 0}(u_1, 0) + \cdots + \mathbb{R}_{\geq 0}(u_m, 0).$$

For such τ and $\mathrm{P}(\mathfrak{a})$,

$$\tau \cap (M_{\mathbb{R}} \times \{1\}) = \mathrm{P}(\mathfrak{a}) \times \{1\}. \tag{4}$$

In fact, let $(u, 1)$ be an element of the left-hand side. Then

$$(u, 1) = \sum_{i=1}^{s} a_i(\mathbf{a}_i, 1) + \sum_{j=1}^{m} b_j(u_j, 0),$$

where $a_i, b_j \geq 0$. By the definition, $\sum a_i = 1$. By Proposition 3.2 (iii), $u \in \mathrm{P}(\mathfrak{a})$. The similar argument implies the opposite inclusion. Since τ is the rational polyhedral convex cone, for $l = 1, \cdots, t$, there exists $(v'_l, \mu_l) \in N_{\mathbb{Q}} \times \mathbb{Q}$ such that

$$\tau = \bigcap_{l=1}^{t} \mathrm{H}^+((v'_l, \mu_l); 0), \tag{5}$$

where $\mathrm{H}^+((v'_l, \mu_l); 0)$ is the affine half space of $M_{\mathbb{R}} \times \mathbb{R}$. The duality pair of $M_{\mathbb{R}} \times \mathbb{R}$ and $N_{\mathbb{R}} \times \mathbb{R}$ is defined as

$$\langle (u, \lambda), (v, \mu) \rangle := \langle u, v \rangle + \lambda\mu,$$

for every $u \in M_{\mathbb{R}},\ v \in N_{\mathbb{R}}$ and $\lambda,\ \mu \in \mathbb{R}$. Under this duality,

$$\mathrm{H}^+((v, \mu); 0) \cap (M_{\mathbb{R}} \times \{1\}) = \mathrm{H}^+(v; -\mu) \times \{1\}.$$

Therefore if we set $\lambda'_l := -u_l$ for each $l = 1, \cdots, t$, the assertion of the theorem follows by (4) and (5). $\square$

Theorem 5.5. Let R, σ and $\mathfrak{a}$ be as in Lemma 5.4. Furthermore, we assume that σ is a d-dimensional simplicial cone. Let J be an $\mathfrak{m}$-primary monomial ideal, where $\mathfrak{m}$ is the maximal monomial ideal of R. Then the F-threshold $\mathrm{c}^J(\mathfrak{a})$ of $\mathfrak{a}$ with respect to J is a rational number.

Proof. We denote by $\partial \mathrm{Q}(J)$ the boundary of $\mathrm{Q}(J)$ in $\sigma^\vee$. By Lemma 5.4, if there exists a finite set $B \subseteq M_{\mathbb{Q}} \cap \partial \mathrm{Q}(J)$ such that

$$\mathrm{c}^J(\mathfrak{a}) = \max_{\omega \in B} \lambda_{\mathfrak{a}}(\omega),$$

then we have $\mathrm{c}^J(\mathfrak{a}) \in \mathbb{Q}$.
First, we prove that

$$\mathrm{c}^J(\mathfrak{a}) = \sup_{\omega \in \partial \mathrm{Q}(J)} \lambda_{\mathfrak{a}}(\omega).$$

By Theorem 3.3, if there exists an element $\omega \in \sigma^\vee$ such that $\mathrm{c}^J(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\omega)$, then $\omega \in \partial \mathrm{Q}(J)$. In fact, if such ω is in $\sigma^\vee \setminus \mathrm{Q}(J)$, there exists $\varepsilon > 0$ such that $(1+\varepsilon)\omega \in \sigma^\vee \setminus \mathrm{Q}(J)$. This implies that $\mathrm{c}^J(\mathfrak{a}) \geq (1+\varepsilon)\lambda_{\mathfrak{a}}(\omega)$. It is a contradiction, thus we are done.
Second, we prove the existence of $B \subseteq M_{\mathbb{Q}} \cap \partial \mathrm{Q}(J)$. We assume that $\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_d$, where v_j are primitive lattice points. Since σ is simplicial, for every j, there exists $u_j \in M_{\mathbb{Q}}$ such that

$$\langle u_j, v_l \rangle = \delta_{jl}, \ \ l \in \{1, \cdots, d\}.$$

Since J is $\mathfrak{m}$-primary, there exists a nonnegative integer r_j such that $r_j u_j \in \mathrm{Q}(J)$. That implies $\partial \mathrm{Q}(J)$ is bounded. We define the order $\leq_\sigma$ over $\partial \mathrm{Q}(J)$ as $u \leq_\sigma u'$ if

$$\langle u, v_j \rangle \leq \langle u', v_j \rangle, \ \ \forall j = 1, \cdots, d.$$

Then $\partial \mathrm{Q}(J)$ has maximal elements with respect to this order. Let $B \subseteq \partial \mathrm{Q}(J)$ be the set of maximal elements with respect to the order $\leq_\sigma$. By Lemma 4.5, we conclude

$$\mathrm{c}^J(\mathfrak{a}) = \sup_{\omega \in \partial \mathrm{Q}(J)} \lambda_{\mathfrak{a}}(\omega) = \sup_{\omega \in B} \lambda_{\mathfrak{a}}(\omega).$$

To show that B is a finite set of $M_{\mathbb{Q}}$, we prove the following claim.

Claim. Let $J = (X^{\mathbf{b}_1}, \cdots, X^{\mathbf{b}_t})$. We assume that $u \in B$, that is,

(i). $u \in \partial \mathrm{Q}(J)$,

(ii). u is a maximal element with respect to the order $\leq_\sigma$ in $\partial \mathrm{Q}(J)$.

Then for every $j = 1, \cdots, d$, there exists i_j such that

$$u \in \bigcap_{j=1}^{n} \left(\mathbf{b}_{i_j} + (\partial \mathrm{H}^+(v_j; 0) \cap \sigma^\vee) \right). \tag{6}$$

In particular, B is a finite set and $u \in M_\mathbb{Q}$.

Proof of Claim. We suppose that u does not satisfy (6). Then there exists $j' \in \{1, \cdots, d\}$ such that

$$u \notin \mathbf{b}_i + (\partial \mathrm{H}^+(v_{j'}; 0) \cap \sigma^\vee), \tag{7}$$

for all $i = 1, \cdots, t$. We choose $u' \in \sigma^\vee$ as

$$\langle u', v_j \rangle = \langle u, v_j \rangle, \ (j \neq j'),$$

$$\langle u', v_{j'} \rangle = \lfloor \langle u, v_{j'} \rangle \rfloor + 1.$$

Since σ is simplicial, u' uniquely exists. We will show that the existence of u' contradicts the assumption (ii). By the construction of u', we have $u' \in \mathrm{Q}(J)$. To see $u' \notin \mathrm{IntQ}(J)$, we rephrase the assumption (i). Since $u \notin \mathrm{IntQ}(J)$, we have $u \notin \mathbf{b}_i + \mathrm{Int}(\sigma^\vee)$ for all $i = 1, \cdots, t$. Furthermore, this is equivalent to the existence of l_i such that

$$\langle u, v_{l_i} \rangle \leq \langle \mathbf{b}_i, v_{l_i} \rangle, \tag{8}$$

for each $i = 1, \cdots, t$. If $l_i \neq j'$, we have directly

$$\langle u', v_{l_i} \rangle = \langle u, v_{l_i} \rangle \leq \langle \mathbf{b}_i, v_{l_i} \rangle,$$

by the construction of u' and the relation (8). On the other hand, if $l_i = j'$, then the relation (8) and (7) implies

$$\lfloor \langle u, v_{j'} \rangle \rfloor \leq \langle \mathbf{b}_i, v_{j'} \rangle - 1,$$

because $\mathbf{b}_i \in M$. Hence $\langle u', v_{l_i} \rangle \leq \langle \mathbf{b}_i, v_{l_i} \rangle$. Eventually, in both cases, $u' \notin \mathrm{IntQ}(J)$. Therefore $u' \in \partial \mathrm{Q}(J)$. By the construction of u', the element u is not a maximal element in $\partial \mathrm{Q}(J)$. It contradicts the assumption (ii). We complete the proof of Claim.

□

We complete the proof of the theorem. □

Now we consider the rationality of F-jumping coefficients on $\mathbb{Q}$-Gorenstein toric rings. The rationality of F-jumping coefficients is the consequence of the fact that test ideals are equal to multiplier ideals ([HY, Theorem 4.8] and [B, Theorem 1]). However, we also give its proof by the combinatorial method.

Proposition 5.6. Let R, σ, $\mathfrak{a}$ be as in Lemma 5.4. Moreover, we assume R is an r-Gorenstein toric ring. Then for all i, the i-th F-jumping coefficient $\mathrm{c}^i(\mathfrak{a})$ of $\mathfrak{a}$ is a rational number.

Proof. In the proof of Proposition 4.6, we have seen that there exists $\mathbf{b} \in M$ such that $\mathrm{c}^i(\mathfrak{a}) = \mu_{\mathfrak{a}}(\mathbf{b})$, where $X^{\mathbf{b}}$ is one of generators of $\tau(\mathfrak{a}^{\mathrm{c}^{i-1}(\mathfrak{a})})$. By the similar argument to that of the proof of Proposition 5.1, there exists $\omega \in \sigma^\vee$ such that $\mathrm{c}^i(\mathfrak{a}) = \lambda_{\mathfrak{a}}(\mathbf{b} + \omega/r)$. Since ω corresponds to the generator of $\omega_R^{(r)}$, where ω_R is the canonical module of R, we see $\omega \in M$. Hence $\mathbf{b} + \omega/r \in M_{\mathbb{Q}}$. Therefore $\mathrm{c}^i(\mathfrak{a})$ is a rational number. ☐

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