

# Localizing the Elliott Conjecture at Strongly Self-absorbing $C^*$ -algebras, II —An Appendix

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## Abstract

This note provides some technical support to the proof of a result of W. Winter which shows that two unital separable simple amenable  $\mathcal{Z}$ -absorbing  $C^*$ -algebras with locally finite decomposition property satisfying the UCT whose projections separate the traces are isomorphic if their  $K$ -theory is finitely generated and their Elliott invariants are the same.

## 1 Introduction

In [11], W. Winter provided a fascinating method for the Elliott program of classification of amenable  $C^*$ -algebras. Let  $A$  and  $B$  be two unital separable simple amenable  $C^*$ -algebras which are  $\mathcal{Z}$ -absorbing. Winter showed that if  $A \otimes C$  is isomorphic to  $B \otimes C$ , for any unital UHF-algebra  $C$ , then there is a way to show that  $A$  is actually isomorphic to  $B$ . It is known that for many separable amenable simple  $C^*$ -algebras,  $A \otimes C$  has many known properties that  $A$  does not have. For example, it was proved (also by Winter) that, for every  $\mathcal{Z}$ -absorbing  $C^*$ -algebra  $A$  with locally finite decomposition rank whose projections separate the traces,  $A \otimes C$  has tracial rank zero for every UHF-algebra  $C$  (see [9]). However,  $A$  itself may not have finite tracial rank. Winter's method shows that two unital separable simple amenable  $\mathcal{Z}$ -absorbing  $C^*$ -algebras with locally finite decomposition rank and with finitely generated  $K$ -theory whose projections separate the traces are isomorphic if their Elliott invariants are isomorphic. The main purpose of this note is to provide some technical support to the proof of the above mentioned result of Winter.

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## 2 Some notation

**2.1.** Let  $A$  and  $B$  be two unital  $C^*$ -algebras. Suppose that  $\varphi, \psi : A \rightarrow B$  are two homomorphisms. Define the mapping torus of  $\varphi$  and  $\psi$  as follows:

$$M_{\varphi, \psi} = \{x \in C([0, 1], B) : x(0) = \varphi(a) \text{ and } x(1) = \psi(a) \text{ for some } a \in A\}. \quad (\text{e 2.1})$$

Thus one obtains an exact sequence:

$$0 \rightarrow SB \xrightarrow{\iota} M_{\varphi, \psi} \xrightarrow{\pi_0} A \rightarrow 0, \quad (\text{e 2.2})$$

where  $\pi_0 : M_{\varphi, \psi} \rightarrow A$  is the point-evaluation at  $t = 0$ .

Suppose that  $A$  is a separable amenable  $C^*$ -algebra. From (e 2.2), one obtains an element in  $Ext(A, SB)$ . In this case we identify  $Ext(A, SB)$  with  $KK^1(A, SB)$  and  $KK(A, B)$ .

Suppose that  $[\varphi] = [\psi]$  in  $KL(A, B)$ . The mapping torus  $M_{\varphi, \psi}$  corresponds to a trivial element in  $KL(A, B)$ . It follows that there are two exact sequences:

$$0 \rightarrow K_1(B) \xrightarrow{\iota_*} K_0(M_{\varphi, \psi}) \xrightarrow{(\pi_0)^*} K_0(A) \rightarrow 0 \text{ and} \quad (\text{e 2.3})$$

$$0 \rightarrow K_0(B) \xrightarrow{\iota_*} K_1(M_{\varphi, \psi}) \xrightarrow{(\pi_0)^*} K_1(A) \rightarrow 0. \quad (\text{e 2.4})$$

which are pure extensions of abelian groups.

**Definition 2.2.** Now let  $B$  be a unital  $C^*$ -algebra with non-empty tracial state space  $T(B)$ . Let  $u \in M_l(M_{\varphi, \psi})$  be a unitary which is a piecewise smooth function on  $[0, 1]$ . For each  $\tau \in T(B)$ , we denote by  $\tau$  the trace  $\tau \otimes Tr$  on  $M_l(B)$ , where  $Tr$  is the standard trace on  $M_l$ . Define

$$R_{\varphi, \psi}(u)(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt. \quad (\text{e 2.5})$$

When  $[\varphi] = [\psi]$  in  $KL(A, B)$  and  $\tau \circ \varphi = \tau \circ \psi$  for all  $\tau \in T(B)$ , there exists a homomorphism

$$R_{\varphi, \psi} : K_1(M_{\varphi, \psi}) \rightarrow Aff(T(B))$$

defined by

$$R_{\varphi, \psi}([u])(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt.$$

We will call  $R_{\varphi, \psi}$  the *rotation map* for the pair  $\varphi$  and  $\psi$ .

Moreover, the following diagram commutes:

$$\begin{array}{ccc} K_0(B) & \xrightarrow{\iota_*} & K_1(M_{\varphi, \psi}) \\ \rho_B \searrow & & \swarrow R_{\varphi, \psi} \\ & Aff(T(B)) & \end{array}$$

See section 3 of [6] for more information.

**Definition 2.3.** If furthermore,  $[\varphi] = [\psi]$  in  $KK(A, B)$  and  $A$  satisfies the Universal Coefficient Theorem, using Dadarlat-Loring's notation, one has the following splitting exact sequence:

$$0 \rightarrow \underline{K}(SB) \xrightarrow{[\iota]} \underline{K}(M_{\varphi_1, \varphi_2}) \xrightarrow{\begin{smallmatrix} [\pi_0] \\ \cong \\ \theta \end{smallmatrix}} \underline{K}(A) \rightarrow 0. \quad (\text{e 2.6})$$

In other words there is  $\theta \in \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(M_{\varphi_1, \varphi_2}))$  such that  $[\pi_0] \circ \theta = [\text{id}_A]$ . In particular, one has a monomorphism  $\theta|_{K_1(A)} : K_1(A) \rightarrow K_1(M_{\varphi, \psi})$  such that  $[\pi_0] \circ \theta|_{K_1(A)} = (\text{id}_A)_{*1}$ . Thus, one may write

$$K_1(M_{\varphi, \psi}) = K_0(B) \oplus K_1(A). \quad (\text{e 2.7})$$

Suppose also that  $\tau \circ \varphi_1 = \tau \circ \varphi_2$  for all  $\tau \in T(B)$ . Then one obtains the homomorphism

$$R_{\varphi, \psi} \circ \theta|_{K_1(A)} : K_1(A) \rightarrow \text{Aff}(T(B)). \quad (\text{e 2.8})$$

We write

$$\tilde{\eta}_{\varphi_1, \varphi_2} = 0,$$

if  $R_{\varphi, \psi} \circ \theta = 0$ , i.e.,  $\theta(K_1(A)) \in \ker R_{\varphi_1, \varphi_2}$  for some such  $\theta$ . Thus,  $\theta$  also gives the following:

$$\ker R_{\varphi, \psi} = \ker \rho_B \oplus K_1(A).$$

**Definition 2.4.** Let  $A$  and  $B$  be two unital  $C^*$ -algebras and let  $\varphi, \psi : A \rightarrow B$  be two homomorphisms. We say  $\varphi$  and  $\psi$  are *asymptotically unitarily equivalent* if there exists a continuous path of unitaries  $\{u(t) : t \in [0, 1]\}$  of  $B$  such that

$$\lim_{t \rightarrow 1} \text{ad } u(t) \circ \varphi(a) = \psi(a) \text{ for all } a \in A. \quad (\text{e 2.9})$$

We say  $\varphi$  and  $\psi$  are *strongly asymptotically unitarily equivalent* if  $u(t)$  can be so chosen that  $u(0) = 1$ .

We use the following result in the proof.

**Theorem 2.5.** (Theorem 9.1 of [6]) *Let  $A$  be a unital AH-algebra and let  $B$  be a unital simple  $C^*$ -algebra with tracial rank zero. Suppose that  $\varphi_1, \varphi_2 : A \rightarrow B$  are two monomorphisms. Then  $\varphi_1$  and  $\varphi_2$  are asymptotically unitarily equivalent if and only if*

$$[\varphi_1] = [\varphi_2] \text{ in } KK(A, B), \tau \circ \varphi_1 = \tau \circ \varphi_2 \text{ for all } \tau \in T(A) \text{ and } \tilde{\eta}_{\varphi_1, \varphi_2} = \{0\}. \quad (\text{e 2.10})$$

In what follows,  $Q$  denotes the UHF-algebra with  $K_0(Q) = \mathbb{Q}$  and  $[1_Q] = 1$ , and if  $\mathfrak{p}$  is a supernatural number  $M_{\mathfrak{p}}$  denotes the UHF-algebra associated with the supernatural number  $\mathfrak{p}$ .

### 3 The Main results

**Lemma 3.1.** *Let  $A$  be a unital  $C^*$ -algebra and let  $h_1, h_2, \dots, h_n$  be self-adjoint elements in  $A$ . Suppose that  $v$  is any unitary in  $A$  and*

$$u(t) = v \prod_{j=1}^n e^{ih_j t} \quad t \in [0, 1].$$

Then,

$$\int_0^1 \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt = \sum_{j=1}^n \tau(h_j)$$

for all  $\tau \in T(A)$ .

*Proof.* Note that for any unitary  $w \in A$  and any tracial state  $\tau \in T(A)$ ,

$$\tau(wh_jw^*) = \tau(h_j), \quad j = 1, 2, \dots, n.$$

It follows that

$$\tau\left(\frac{du(t)}{dt}u(t)^*\right) = \sum_{j=1}^n \tau(h_j)$$

for all  $\tau \in T(A)$ . Thus the lemma follows. □

**Lemma 3.2.** *Let  $B$  be a unital separable simple amenable  $C^*$ -algebra such that  $B \otimes Q$  has tracial rank zero. Let  $A$  be a unital separable amenable simple  $C^*$ -algebra with tracial rank zero satisfying the UCT such that  $K_i(A)$  is torsion free ( $i = 0, 1$ ).*

*Suppose that  $\varphi_1, \varphi_2 : A \rightarrow B \otimes Q$  are two unital homomorphisms with*

$$[\varphi_1] = [\varphi_2] \text{ in } KK(A, M \otimes Q).$$

*Suppose that  $\varphi_1$  induces an affine homeomorphism  $(\varphi_1)_\# : T(B \otimes Q) \rightarrow T(A)$  by*

$$(\varphi_1)_\#(\tau)(a) = \tau \circ \varphi_1(a)$$

*for all  $\tau \in T(B \otimes Q)$  and  $a \in A$ .*

*Then there exists an automorphism  $\alpha \in \text{Aut}(\varphi_1(A))$  with  $[\alpha] = [\text{id}_{\varphi_1(A)}]$  in  $KK(\varphi_1(A), \varphi_1(A))$  such that  $\alpha \circ \varphi_1$  and  $\varphi_2$  are strongly asymptotically unitarily equivalent.*

*Proof.* Let

$$M_{\varphi_1, \varphi_2} = \{f \in C([0, 1], B \otimes Q) : f(0) = \varphi_1(a) \text{ and } f(1) = \varphi_2(a) \text{ for some } a \in A\}.$$

Note that, since  $K_i(B \otimes Q)$  is torsion free,  $i = 0, 1$ ,

$$KK(A, B \otimes Q) \cong \text{Hom}(K_*(A), K_*(B \otimes Q)).$$

Since we assume that  $[\varphi_1] = [\varphi_2]$  in  $KK(A, B \otimes Q)$ , there exists a homomorphism  $\theta : K_i(A) \rightarrow K_i(M_{\varphi_1, \varphi_2})$  such that

$$0 \rightarrow K_{i-1}(B \otimes Q) \rightarrow K_i(M_{\varphi_1, \varphi_2}) \xrightarrow{[\pi_0]} K_i(A) \rightarrow 0$$

splits ( $i = 0, 1$ ).

Since  $A$  has real rank zero, we also have

$$\tau \circ \varphi_1(a) = \tau \circ \varphi_2(a) \text{ for all } a \in A \tag{e3.11}$$

and for all  $\tau \in T(B \otimes Q)$ .

Let  $R_{\varphi_1, \varphi_2} : K_1(M_{\varphi_1, \varphi_2}) \rightarrow \text{Aff}(T(B \otimes Q))$  be the rotation map. Put  $C = \varphi_1(A)$ . By the classification theorem of [5],  $C$  is a unital simple AH-algebra with no dimension growth and with real rank zero. Since  $K_i(C)$  is torsion free, it follows that  $C$  is a unital simple AT-algebra with real rank zero.

By 4.1 (see also Theorem 4.4) of [3], there exists an automorphism  $\alpha \in \text{Aut}(C)$  with  $[\alpha] = [\text{id}_C]$  in  $KK(C, C)$  satisfying the following: there is  $\theta_1 : K_1(C) \rightarrow K_1(M_\alpha)$  so that

$$(R_\alpha \circ \theta_1)((\varphi_1)_\#(\tau)) = -(R_{\varphi_1, \varphi_2} \circ \theta)(\tau) \text{ for all } \tau \in T(B \otimes Q),$$

where

$$M_\alpha = \{g \in C([0, 1], C) : g(0) = a \text{ and } g(1) = \alpha(a) \text{ for some } a \in C\}$$

and

$$R_\alpha(u)(\tau) = \frac{1}{2\pi} \int_0^1 \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt$$

for all unitaries  $u \in M_k(M_\alpha)$  ( $k = 1, 2, \dots$ ) and  $\tau \in T(C)$ . Note that  $[\alpha] = [\text{id}_C]$  in  $KK(C, C)$ . Therefore, one computes that

$$[\alpha \circ \varphi_1] = [\varphi_1] \text{ in } KK(A, B \otimes Q).$$

Now consider the mapping torus

$$M_{\alpha \circ \varphi_1, \varphi_2} = \{f \in C([0, 1], B \otimes Q) : f(0) = \alpha \circ \varphi_1(a) \text{ and } f(1) = \varphi_2(a) \text{ for some } a \in A\}.$$

Define  $\theta_2 : K_1(C) \rightarrow K_1(M_{\alpha \circ \varphi_1, \varphi_2})$  as follows:

Let  $k > 0$  be an integer and  $u \in M_k(M_{\alpha \circ \varphi_1, \varphi_2})$  be a unitary.

We may assume that there is a unitary  $w(t) \in M_k(M_{\varphi_1, \varphi_2})$  such that

$$w(0) = \varphi_1(u'), w(1) = \varphi_2(u'), [u'] = [u] \text{ in } K_1(A) \tag{e 3.12}$$

$$\text{and } \theta([u]) = [w(t)] \text{ in } K_1(M_{\varphi_1, \varphi_2}) \tag{e 3.13}$$

for some unitary  $u' \in M_k(A)$ . To simplify notation, without loss of generality, we may assume that there are  $h_1, h_2, \dots, h_n \in M_k(A)_{s.a.}$  such that

$$u^* u' = \prod_{j=1}^n \exp(ih_j).$$

Define  $z(t) = u \prod_{j=1}^n \exp(h_j t)$  ( $t \in [0, 1]$ ). Consider  $\{\varphi_1(z(t)) : t \in [0, 1]\}$ . Then

$$\varphi_1(z(0)) = \varphi_1(u) \text{ and } \varphi_1(z(1)) = \varphi_1(u').$$

Moreover, by 3.1,

$$\int_0^1 \tau\left(\frac{d\varphi_1(z(t))}{dt} \varphi_1(u(t))^*\right) dt = \sum_{j=1}^n \tau(\varphi_1(h_j))$$

for all  $\tau \in T(A)$ . Consider  $Z(t) = \varphi_2(z(1-t))$ . Then

$$Z(0) = \varphi_2(u') \text{ and } Z(1) = \varphi_2(u).$$

$$\int_0^1 \tau\left(\frac{d\varphi_2(z(1-t))}{dt} \varphi_2(z(1-t))^*\right) dt = - \sum_{j=1}^n \tau(\varphi_2(h_j))$$

for all  $\tau \in T(A)$ . Note that

$$\tau(\varphi_2(h_j)) = \tau(\varphi_1(h_j)), \quad j = 1, 2, \dots, n.$$

It follows that

$$\int_0^1 \tau\left(\frac{d\varphi_1(z(t))}{dt} \varphi_1(u(t))^*\right) dt + \int_0^1 \tau\left(\frac{d\varphi_2(z(1-t))}{dt} \varphi_2(z(1-t))^*\right) dt = 0$$

for all  $\tau \in T(A)$ .

Therefore, without loss of generality, we may assume that  $u = u'$  in (e 3.12). We may also assume that both paths are piecewise smooth.

We may also assume that there is a unitary  $s(t) \in M_k(M_{\alpha \circ \varphi_1, \varphi_1})$  such that

$$s(0) = \alpha \circ \varphi_1(u), s(1) = \varphi_1(u) \text{ in } K_1(A) \quad (\text{e 3.14})$$

$$\text{and } \theta_1([u]) = [s(t)] \text{ in } K_1(M_{\alpha \circ \varphi_1, \varphi_1}). \quad (\text{e 3.15})$$

Define  $\theta_2([u]) = [v]$ , where

$$v(t) = \begin{cases} s(2t) & \text{if } t \in [0, 1/2) \\ w(2(t - 1/2)) & \text{if } t \in [1/2, 1], \end{cases} \quad (\text{e 3.16})$$

Thus  $\theta_2$  gives a homomorphism from  $K_1(A)$  to  $K_1(M_{\alpha \circ \varphi_1, \varphi_2})$  such that  $(\pi_0)_{*1} \circ \theta_2 = \text{id}_{K_1(A)}$ . Since  $[\alpha \circ \varphi_1] = [\varphi_1]$  in  $KK(A, C)$ , we also have

$$[\alpha \circ \varphi_1] = [\varphi_2] \text{ in } KK(A, B \otimes Q).$$

There is also a homomorphism  $\theta'_2 : K_0(A) \rightarrow K_0(M_{\alpha \circ \varphi_1, \varphi_2})$  such that  $(\pi_0)_{*0} \circ \theta'_2 = [\text{id}_{K_0(A)}]$ .

Then

$$R_{\alpha \circ \varphi_1, \varphi_2}([u])(\tau) = \frac{1}{2\pi} \int_0^1 \tau\left(\frac{dv(t)}{dt} v(t)^*\right) dt \quad (\text{e 3.17})$$

$$= \frac{1}{2\pi} \int_0^{1/2} \tau\left(\frac{ds(2t)}{dt} s(2t)^*\right) dt + \quad (\text{e 3.18})$$

$$\frac{1}{2\pi} \int_{1/2}^1 \tau\left(\frac{dw(2(t - 1/2))}{dt} w(2(t - 1/2))^*\right) dt \quad (\text{e 3.19})$$

$$= R_{\alpha \circ \varphi_1, \varphi_1} \circ \theta_1([u])(\varphi_{\#}(\tau)) + R_{\varphi_1, \varphi_2} \circ \theta([u])(\tau) = 0 \quad (\text{e 3.20})$$

for all  $\tau \in T(B \otimes Q)$ .

Thus  $\tilde{\eta}_{\alpha \circ \varphi_1, \varphi_2} = 0$ . Note that, by [5],  $A$  is an AH-algebra. It follows from Theorem 2.5 that  $\alpha \circ \varphi_1$  and  $\varphi_2$  are asymptotically unitarily equivalent. Since  $K_1(B \otimes Q)$  is divisible,  $H_1(K_0(B \otimes Q), K_1(B \otimes Q)) = K_1(B \otimes Q)$ , and by 11.5 of [6], we conclude that  $\alpha \circ \varphi_1$  and  $\varphi_2$  are strongly asymptotically unitarily equivalent.  $\square$

**Lemma 3.3.** *Let  $B$  be a unital separable simple amenable  $C^*$ -algebra such that  $B \otimes Q$  has tracial rank zero. Let  $\mathfrak{p}$  be a supernatural number of infinite type and let  $A = C \otimes M_{\mathfrak{p}}$  be a unital separable amenable simple  $C^*$ -algebra with tracial rank zero satisfying the UCT such that  $K_i(A) = \text{Tor}(K_i(A)) \oplus G_i$ , where  $G_i$  is torsion free ( $i = 0, 1$ ). Suppose that  $\varphi_1, \varphi_2 : A \rightarrow B \otimes Q$  are two unital homomorphisms with*

$$[\varphi_1] = [\varphi_2] \text{ in } KK(A, M \otimes Q).$$

*Suppose also that  $\varphi_1$  induces an affine homeomorphism  $(\varphi_1)_{\#} : T(B \otimes Q) \rightarrow T(A)$  by*

$$(\varphi_1)_{\#}(\tau)(a) = \tau \circ \varphi_1(a)$$

*for all  $\tau \in T(B \otimes Q)$  and  $a \in A$ .*

*Then there exists an automorphism  $\alpha \in \text{Aut}(\varphi_1(A))$  with  $[\alpha] = [\text{id}_{\varphi_1(A)}]$  in  $KL(\varphi_1(A), \varphi_1(A))$  such that  $\alpha \circ \varphi_1$  and  $\varphi_2$  are strongly asymptotically unitarily equivalent.*

*Proof.* Let  $d : K_0(A) \rightarrow \text{Aff}(T(A))$  be the homomorphism induced by  $\tau([p])$  for all  $\tau \in T(A)$  and projections  $p \in A$ . There is a homomorphism  $h_1 : M_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}}$  such that  $h_1(1) = e$  for some projection  $e \in M_{\mathfrak{p}}$  with  $e \neq 0$  and  $e \neq 1$ . This  $h_1$  gives an injective homomorphism  $\gamma_1 : d(K_0(A)) \rightarrow d(K_0(C \otimes M_{\mathfrak{p}}))$  such that  $\gamma_1(r) = [e]r$  for  $r \in d(K_0(C \otimes M_{\mathfrak{p}}))$ . Put  $A_1 = \varphi_1(A)$ . Define  $h \in \text{Hom}(K_*(A_1), K_*(A_1))$  such that

$$h|_{K_0(A_1)} = \gamma_1 \circ d, \quad h|_{G_1} = \text{id}|_{G_1} \quad \text{and}$$

$$h|_{\text{Tor}(K_1(A_1))} = \{0\}.$$

There is a unital simple  $A\mathbb{T}$ -algebra  $D$  with real rank zero such that

$$(K_0(D), K_0(D)_+, [1_D], K_1(D)) = (h(K_0(A_1)), h((K_0(A_1))_+), h([1_{A_1}]), G_1).$$

So  $K_i(D) = G_i$  ( $i = 0, 1$ ). It follows from [5] that there exists a unital homomorphism  $\psi'_1 : A_1 \rightarrow D$  such that  $(\psi'_1)_{*i} = h$ . Moreover, there is a homomorphism  $\iota : D \rightarrow A_1$  so that  $(\iota)_{*i} = \text{id}_{G_i}$ ,  $i = 0, 1$ .

Put

$$\kappa = [\text{id}_{A_1}] - [\iota \circ \psi'_1] \in KL(A_1, A_1).$$

Note that  $\kappa \in KL(A_1, A_1)_+$  (see [5]). It follows from [5] that there is a homomorphism  $\psi_2 : A_1 \rightarrow A_1$  such that  $\psi_2(1_{A_1}) = 1_{A_1} - \iota \circ \psi'_1(1_{A_1}) = e_1$  (for some projection  $e_1 \in A_1$ ) and  $[\psi_2] = \kappa$ . Define

$$\Phi(a) = \iota \circ \psi'_1(a) + \psi_2(a) \quad \text{for all } a \in A_1.$$

It is clear that

$$[\Phi] = [\text{id}_{A_1}] \quad \text{in } KL(A_1, A_1).$$

It follows that

$$[\Phi \circ \varphi_2] = [\varphi_1] \quad \text{in } KK(A, B \otimes Q).$$

Let  $\theta_1 : K_i(A) \rightarrow K_i(M_{\Phi \circ \varphi_1, \varphi_2})$  ( $i = 0, 1$ ) be such that

$$0 \rightarrow K_{i-1}(B \otimes Q) \rightarrow K_i(M_{\Phi \circ \varphi_1, \varphi_2}) \xrightarrow{[\pi_0]}_{\theta_1} K_i(A) \rightarrow 0$$

splits, where

$$M_{\Phi \circ \varphi_1, \varphi_2} = \{f \in C([0, 1], B \otimes Q) : f(0) = \Phi \circ \varphi_1(a) \text{ and } f(1) = \varphi_2(a) \text{ for some } a \in C\}.$$

Let  $\eta_1 = R_{\Phi \circ \varphi_1, \varphi_2} \circ \theta_1$ . We will identify  $D$  with  $\iota(D)$  and we identify  $\text{Aff}(T(D))$  with  $\text{Aff}(T(A))$ . It follows from [3] that there exists an automorphism  $\alpha_1 \in \text{Aut}(D)$  with  $[\alpha_1] = [\text{id}_D]$  in  $KK(D, D)$  satisfying the following: there is  $\theta_2 : K_1(D) \rightarrow K_1(M_{\alpha_1})$  such that

$$R_{\alpha_1} \circ \theta_2((\varphi_1)_{\#}(\tau)) = -\eta_1(\tau) \quad \text{for all } \tau \in T(B \otimes Q),$$

where

$$M_{\alpha_1} = \{f \in C([0, 1], D) : f(0) = \alpha_1(c) \text{ and } f(1) = c \text{ for some } c \in D\}.$$

Now define  $\alpha' : D \oplus e_1\Phi(A_1)e_1 \rightarrow D \oplus e_1\Phi(A_1)e_1$  by

$$\alpha'(c) = \alpha_1(c) \quad \text{for all } c \in D \quad \text{and} \quad \alpha'(a) = a \quad \text{for all } a \in e_1\Phi(A_1)e_1.$$

Define  $\alpha = \alpha' \circ \Phi : A_1 \rightarrow A_1$ . Then

$$[\alpha] = [\text{id}_{A_1}] \quad \text{in } KL(A_1, A_1)$$

and

$$[\alpha \circ \varphi_1] = [\varphi_2] \text{ in } KK(A, B \otimes Q).$$

Let  $k > 0$  be an integer and  $u \in M_k(M_{\Phi \circ \varphi_1, \varphi_2})$  be a unitary.

We may assume that there is a unitary  $w(t) \in M_k(M_{\varphi_1, \varphi_2})$  such that

$$w(0) = \Phi \circ \varphi_1(u'), w(1) = \varphi_2(u'), [u'] = [u] \text{ in } K_1(A) \quad (\text{e 3.21})$$

$$\text{and } \theta_1([u]) = [w(t)] \text{ in } K_1(M_{\Phi \circ \varphi_1, \varphi_2}) \quad (\text{e 3.22})$$

for some unitary  $u' \in M_k(A)$ .

We may also assume that there is a unitary  $s'(t) \in M_k(M_{\alpha_1})$  such that

$$s'(0) = \alpha_1(h(u'')), s'(1) = u'', [h(u'')] = [h(u)] \text{ in } K_1(D) \quad (\text{e 3.23})$$

$$\text{and } \theta_2([u]) = [s] \text{ in } K_1(M_{\alpha_1}) \quad (\text{e 3.24})$$

for some unitary  $u'' \in M_k(A)$ . Define  $s(t) = s'(t) \oplus \psi'_2 \circ \varphi_1(u'')$  for  $t \in [0, 1]$ . As in the proof of 3.2, we may assume that  $u' = u'' = u$ . Now define  $\theta : K_1(A) \rightarrow K_1(M_{\alpha \circ \varphi_1, \varphi_2})$  as follows:  $\theta([u]) = [v]$ , where

$$v(t) = \begin{cases} s(2t) & \text{if } t \in [0, 1/2) \\ w(2(t - 1/2)) & \text{if } t \in [1/2, 1], \end{cases} \quad (\text{e 3.25})$$

Thus  $\theta$  gives a homomorphism from  $K_1(A)$  to  $K_1(M_{\alpha \circ \varphi_1, \varphi_2})$  such that  $(\pi_0)_* \circ \theta = \text{id}_{K_1(A)}$ . Since  $[\alpha \circ \varphi_1] = [\varphi_1]$  in  $KK(A, A)$ , we also have

$$[\alpha \circ \varphi_1] = [\varphi_2] \text{ in } KK(A, B \otimes Q).$$

Then

$$R_{\alpha \circ \varphi_1, \varphi_2}([u])(\tau) = \frac{1}{2\pi} \int_0^1 \tau\left(\frac{dv(t)}{dt} v(t)^*\right) dt \quad (\text{e 3.26})$$

$$= \frac{1}{2\pi} \int_0^{1/2} \tau\left(\frac{ds(2t)}{dt} s(2t)^*\right) dt + \quad (\text{e 3.27})$$

$$\frac{1}{2\pi} \int_{1/2}^1 \tau\left(\frac{dw(2(t - 1/2))}{dt} w(2(t - 1/2))^*\right) dt \quad (\text{e 3.28})$$

$$= R_{\alpha_1 \circ \Phi \circ \varphi_1, \Phi \circ \varphi_1} \circ \theta_2([u])((\varphi_1)_\#(\tau)) + R_{\Phi \circ \varphi_1, \varphi_2} \circ \theta_1([u])(\tau) = 0 \quad (\text{e 3.29})$$

for all  $\tau \in T(B \otimes Q)$ .

Thus  $\tilde{\eta}_{\alpha \circ \varphi_1, \varphi_2} = 0$ . It follows from Theorem 2.5 that  $\alpha \circ \varphi_1$  and  $\varphi_2$  are asymptotically unitarily equivalent. Again, since  $K_1(B \otimes Q)$  is divisible, as in the proof of 3.2,  $\alpha \circ \varphi_1$  and  $\varphi_2$  are strongly asymptotically unitarily equivalent. □

**Theorem 3.4.** *Let  $A$  and  $B$  be two unital separable amenable simple  $C^*$ -algebras satisfying the UCT. Let  $\mathfrak{p}$  and  $\mathfrak{q}$  be supernatural numbers of infinite type such that  $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \cong Q$ . Suppose that  $A \otimes M_{\mathfrak{p}}, A \otimes M_{\mathfrak{q}}, B \otimes M_{\mathfrak{p}}$  and  $B \otimes M_{\mathfrak{q}}$  have tracial rank zero.*

*Let  $\sigma_{\mathfrak{p}} : A \otimes M_{\mathfrak{p}} \rightarrow B \otimes M_{\mathfrak{p}}$  and  $\rho_{\mathfrak{q}} : A \otimes M_{\mathfrak{q}} \rightarrow B \otimes M_{\mathfrak{q}}$  be two unital isomorphisms. Suppose*

$$[\sigma] = [\rho] \text{ in } KK(A \otimes Q, B \otimes Q),$$

*where  $\sigma = \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}$  and  $\rho = \rho_{\mathfrak{q}} \otimes \text{id}_{M_{\mathfrak{p}}}$ .*



Then there is an automorphism  $\alpha \in \text{Aut}(\sigma_{\mathfrak{p}}(A \otimes M_{\mathfrak{p}}))$  such that

$$[\alpha \circ \sigma_{\mathfrak{p}}] = [\sigma_{\mathfrak{p}}] \text{ in } KL(A \otimes M_{\mathfrak{p}}, B \otimes M_{\mathfrak{p}})$$

and  $\alpha \circ \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}$  is strongly asymptotically unitarily equivalent to  $\rho$ , if one of the following holds:

- (i)  $K_i(A \otimes M_{\mathfrak{p}})$  is torsion free ( $i = 0, 1$ ),
- (ii)  $K_i(A \otimes M_{\mathfrak{p}}) = \text{Tor}(K_i(A \otimes M_{\mathfrak{p}})) \oplus G_i$ , where  $G_i$  is torsion free  $i = 0, 1$ .

*Proof.* It follows from 3.2 that there exists  $\beta \in \text{Aut}(B \otimes Q)$  such that  $\beta \circ \sigma$  is strongly asymptotically unitarily equivalent to  $\rho$ . Moreover,  $[\beta] = [\text{id}_{B \otimes Q}]$  in  $KK(B \otimes Q, B \otimes Q)$ . Now consider two homomorphisms  $\sigma_{\mathfrak{p}}$  and  $\beta \circ \sigma_{\mathfrak{p}}$ . One has

$$[\beta \circ \sigma_{\mathfrak{p}}] = [\sigma_{\mathfrak{p}}] \text{ in } KK(A \otimes M_{\mathfrak{p}}, B \otimes Q).$$

Since  $\sigma_{\mathfrak{p}}$  is an isomorphism, it is easy to see that  $\sigma_{\mathfrak{p}} : T(B \otimes Q) \rightarrow T(A \otimes M_{\mathfrak{p}})$  is an affine homeomorphism.

In case (i), since  $K_i(A \otimes M_{\mathfrak{p}})$  is torsion free, by applying 3.2 again, one obtains  $\alpha \in \text{Aut}(\sigma_{\mathfrak{p}}(A \otimes M_{\mathfrak{p}}))$  such that  $\alpha \circ \sigma_{\mathfrak{p}}$  is strongly asymptotically unitarily equivalent to  $\beta \circ \sigma_{\mathfrak{p}}$ . Note that  $\sigma(A \otimes M_{\mathfrak{p}}) = B \otimes M_{\mathfrak{p}}$ . Put  $\sigma'_{\mathfrak{p}} = \alpha \circ \sigma_{\mathfrak{p}}$  and let  $\sigma' = \alpha \circ \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}$ .

Define  $\beta \circ \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}} : A \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \rightarrow (B \otimes Q) \otimes M_{\mathfrak{q}}$ . Note that  $j : M_{\mathfrak{q}} \rightarrow M_{\mathfrak{q}} \otimes M_{\mathfrak{q}}$  defined by  $a \rightarrow a \otimes 1$  is (strongly) asymptotically unitarily equivalent to an isomorphism. It follows that  $\sigma'$  is strongly unitarily equivalent to  $\beta \circ \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}$ . Since  $\beta \circ \text{id}_{M_{\mathfrak{q}}} \otimes 1, 1 \otimes \text{id}_{M_{\mathfrak{q}}} : M_{\mathfrak{q}} \rightarrow \beta(1 \otimes M_{\mathfrak{q}}) \otimes M_{\mathfrak{q}}$  are strongly asymptotically unitarily equivalent (in  $\beta(1 \otimes M_{\mathfrak{q}}) \otimes M_{\mathfrak{q}}$ ),  $\beta \circ \sigma$  and  $\beta \circ \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}$  are strongly asymptotically unitarily equivalent.

It follows that  $\sigma'$  is strongly asymptotically unitarily equivalent to  $\beta \circ \sigma$ . Consequently  $\sigma'$  is strongly asymptotically unitarily to  $\rho$ .

The proof of part (ii) is exactly the same but we will apply 3.3 instead. □

**Theorem 3.5.** (8.1 of [11]) *Let  $A$  and  $B$  be two unital separable amenable simple  $C^*$ -algebras satisfying the UCT. Suppose that  $A \otimes C$  and  $B \otimes C$  are both of tracial rank zero for any UHF-algebras. Suppose also that*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Then  $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$ , if either

- (i)  $\text{Tor}(K_0(A))$  and  $\text{Tor}(K_1(A))$  miss at least one (the same) prime order,
- (ii) or  $K_i(A) = \text{Tor}(K_i(A)) \oplus G_i$  for some torsion free  $G_i, i = 0, 1$ .

*Proof.* For case (i), suppose that  $\text{Tor}(K_i(A))$  misses the prime order  $p'$ . Then it is easy to find a pair of relatively prime supernatural integers  $\mathfrak{p}$  and  $\mathfrak{q}$  such that  $K_i(A \otimes M_{\mathfrak{p}})$  is torsion free ( $i = 0, 1$ ). Let

$$\Gamma : (K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B))$$

be the isomorphism. Let  $\kappa \in KK(A, B)$  be an element which gives  $\Gamma$ .

Then there are isomorphisms  $\sigma_{\mathfrak{p}} : A \otimes M_{\mathfrak{p}} \rightarrow B \otimes M_{\mathfrak{p}}$  and  $\rho_{\mathfrak{q}} : A \otimes M_{\mathfrak{q}} \rightarrow B \otimes M_{\mathfrak{q}}$  given by  $\kappa \otimes [\text{id}_{M_{\mathfrak{p}}}]$  and  $\kappa \otimes [\text{id}_{M_{\mathfrak{q}}}]$ . Since  $B \otimes Q$  is divisible, it is easy to see that

$$[\sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}] = [\rho_{\mathfrak{q}} \otimes \text{id}_{M_{\mathfrak{p}}}] \text{ in } KK(A \otimes Q, B \otimes Q).$$

Put  $\sigma = \sigma_{\mathfrak{p}} \otimes \text{id}_{M_{\mathfrak{q}}}$  and  $\rho = \rho_{\mathfrak{q}} \otimes \text{id}_{M_{\mathfrak{p}}}$ . Then, by 3.4,  $\sigma$  and  $\rho$  are strongly asymptotically unitarily equivalent. Therefore  $\Gamma$  can be lifted along  $\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}$  (see 4.7 of [11]).

One then applies Theorem 7.1 of [11].

For case (ii), there exists a sequence of integers  $\{m_k\}$  such that

$$M_{\mathfrak{p}} = \lim_{n \rightarrow \infty} (M_{m_k}, h_k),$$

where  $h_k$  is standard amplification. Thus  $(h_k)_{*i}$  is a multiplication by  $m_{k+1}/m_k$ . Note also that

$$A \otimes M_{\mathfrak{p}} = \lim_{n \rightarrow \infty} (A \otimes M_{m_k}, \text{id}_A \otimes h_k).$$

It follows that

$$K_i(A \otimes M_{\mathfrak{p}}) = \text{Tor}(K_i(A \otimes M_{\mathfrak{p}})) \oplus G'_i,$$

where  $G'_i$  is torsion free.

One then applies part (ii) of 3.4 as in the proof of (i). □

Note that in the following statement  $A$  is not assumed to be of real rank zero, a priori.

**Corollary 3.6.** *Let  $A$  be a unital separable simple  $\mathcal{Z}$ -absorbing ASH-algebra  $A$  whose projections separate the traces. Suppose that  $K_0(A)$  has the Riesz interpolation property and  $K_0(A)/\text{Tor}(K_0(A)) \not\cong \mathbb{Z}$ .*

*Then  $A$  has tracial rank zero and  $A$  is (isomorphic to) an AH-algebra with no dimension growth and with real rank zero if either (i) or (ii) of 3.5 hold.*

*Proof.* For any UHF-algebra  $C$ ,  $A \otimes C$  is approximately divisible and its projections separate traces. It follows from [1] that  $A \otimes C$  has real rank zero, stable rank one and weakly unperforated  $K_0(A \otimes C)$ . Note that  $A \otimes C$  is also an ASH-algebra. It follows from a result of Winter ([10]) that  $A \otimes C$  has tracial rank zero. Since  $A$  is  $\mathcal{Z}$ -absorbing,  $K_0(A)$  is weakly unperforated (Prop. 1.2 of [8]). Now by [2], there exists a unital simple AH-algebra  $B$  with real rank zero and with no dimension growth such that

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

It follows from [10] that  $B \otimes \mathcal{Z}$  has tracial rank zero. By 3.5,  $A \cong B \otimes \mathcal{Z}$ . □

Please see section 8 of [11] for other consequences of 3.5 and further discussion.

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