A REMARK ON FROBENIUS DESCENT FOR VECTOR BUNDLES

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Abstract. We give a class of examples of vector bundles on a relative smooth projective curve over $\operatorname{Spec} \mathbb{Z}$ such that for infinitely many prime reductions the bundle has a Frobenius descent, but the restriction to the generic fiber in characteristic zero is not semistable. In the third section of the paper we prove for a large class of varieties (including abelian varieties) that any vector bundle with this Frobenius descent property is generically semistable.

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1. Introduction

Let X be a smooth projective variety defined over an algebraically closed field of characteristic $p > 0$ with a fixed very ample line bundle $\mathcal{O}_X(1)$. Further we denote by F the absolute Frobenius morphism $F : X \to X$ which is the identity on the topological space underlying X and the pth power map on the structure sheaf $\mathcal{O}_X$. A vector bundle $\mathcal{E}$ on X descends under F if there exists another vector bundle $\mathcal{F}$ such that $\mathcal{E} \cong F^*(\mathcal{F})$. This note is inspired by the recent preprint [6] of K. Joshi. In the relative situation, where a morphism $\mathcal{X} \to \operatorname{Spec} R$ with generic fiber $X := \mathcal{X}_0$ is given and R is a $\mathbb{Z}$-domain of finite type, Joshi asked the following interesting question: "assume X is a smooth projective variety and suppose V is a vector bundle which descends under Frobenius modulo an infinite set of primes then is it true that V is semistable (with respect to any ample line bundle on X)?" He gives a positive answer to this question for rank two vector bundles under the additional assumption that $\operatorname{Pic}(X) = \mathbb{Z}$.

In section 2 of this paper we provide a class of examples which give a negative answer to this question in general. We show that on the relative Fermat curve $C = V_+(X^d + Y^d + Z^d) \to \operatorname{Spec} \mathbb{Z}$, with $d \geq 5$ odd, there exists a vector bundle $\mathcal{E}$ of rank two such that for infinitely many prime numbers p the reduction $\mathcal{E}_p = \mathcal{E}|_{C_p}$ modulo p has a Frobenius descent, but $\mathcal{E}_0 = \mathcal{E}|_{C_0}$ is not semistable on the fiber over the generic point. In section 3 we give an affirmative answer to this question under the assumption that for every closed point $\mathfrak{m} \in \operatorname{Spec} R$ every semistable vector bundle on the fiber $\mathcal{X}_{\mathfrak{m}}$ is strongly semistable. We recall that a semistable vector bundle $\mathcal{E}$ is strongly semistable if $F^{e*}(\mathcal{E})$ is semistable for all integers $e \geq 0$. This provides further examples of varieties with $\operatorname{Pic}(X) \neq \mathbb{Z}$ (for example abelian varieties) for which the question of Joshi still has a positive answer.

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2. A counterexample for vector bundles on curves

In this section we give an example of a rank two vector bundle on a generically smooth projective relative curve over $\operatorname{Spec} \mathbb{Z}$ such that infinitely many prime reductions have a Frobenius descent but the bundle is not semistable on the generic fiber in characteristic zero.

Our example will use the syzygy bundle $\operatorname{Syz}(X^2, Y^2, Z^2)(m)$ on Fermat curves $C = V_+(X^d + Y^d + Z^d) \subset \mathbb{P}^2$ defined over a field K. This vector bundle is defined by the short exact sequence

$$0 \longrightarrow \operatorname{Syz}(X^2, Y^2, Z^2)(m) \longrightarrow \mathcal{O}_C(m-2)^3 \longrightarrow \mathcal{O}_C(m) \longrightarrow 0,$$

where the penultimate mapping is given by $(s_1, s_2, s_3) \mapsto s_1 X^2 + s_2 Y^2 + s_3 Z^2$. The bundle $\operatorname{Syz}(X^2, Y^2, Z^2)(m)$ is semistable for $d \geq 5$ by [2, Proposition 6.2]. In positive characteristic $p > 0$, since the presenting sequence only involves locally free sheaves, it is easy to see that the Frobenius pull-back $F^*(\operatorname{Syz}(X^2, Y^2, Z^2)(m)) \cong \operatorname{Syz}(X^{2p}, Y^{2p}, Z^{2p})(mp)$.

Lemma 2.1. *Let $d = 2\ell + 1$ with $\ell \geq 2$ and let $C := \operatorname{Proj} K[X, Y, Z]/(X^d + Y^d + Z^d)$ be the Fermat curve of degree d defined over a field K of characteristic $p \equiv \ell \mod d$. Then the Frobenius pull-back of $\operatorname{Syz}(X^2, Y^2, Z^2)(3)$ sits inside the short exact sequence*

$$0 \longrightarrow \mathcal{O}_C(\ell - 1) \longrightarrow \operatorname{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \longrightarrow \mathcal{O}_C(-\ell + 1) \longrightarrow 0.$$

In particular, the Frobenius pull-back is not semistable and this sequence constitutes its Harder-Narasimhan filtration.

Proof. We write $2p = dk + 2\ell$ with k even. The pull-back $\operatorname{Syz}(X^{2p}, Y^{2p}, Z^{2p})$ of $\operatorname{Syz}(X^2, Y^2, Z^2)$ has a non-trivial global section in total degree $d(k+1+k/2)$ by [3, Proof of Proposition 1.2]. From the presenting sequence of the pull-back one reads off the degree as follows:

$$\begin{aligned}
\deg(\operatorname{Syz}(X^{2p}, Y^{2p}, Z^{2p})(d(k+1+k/2)) &= d(2d(k+1+k/2) - 6p) \\
&= d(2d(k+1+k/2) - 3(dk + 2\ell)) \\
&= d(2d - 6\ell) \\
&= d(-2\ell + 2) < 0.
\end{aligned}$$

Since a semistable vector bundle of negative degree can not have non-trivial global sections, the Frobenius pull-back $\mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})$ is not semistable. We obtain a non-trivial mapping $\mathcal{O}_C(\ell - 1) \to \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$. We want to show that this mapping constitutes the Harder-Narasimhan filtration of the pull-back, meaning that this mapping has no zeros. Hence, assume that we have a factorization

$$\mathcal{O}_C(\ell - 1) \longrightarrow \mathcal{L} \longrightarrow \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p),$$

where $\mathcal{L}$ is a subbundle of the syzygy bundle and has degree $\deg(\mathcal{L}) := \alpha \geq (\ell - 1)d$. We have the short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \longrightarrow \mathcal{L}' \longrightarrow 0,$$

where $\mathcal{L}'$ is a line bundle of degree $-\alpha$. By [15, Corollary 2^p] (or [16, Theorem 3.1]) the inequality

$$\mu_{\max}(\mathcal{S}) - \mu_{\min}(\mathcal{S}) = \alpha - (-\alpha) = 2\alpha \leq 2g - 2$$

holds, where $\mathcal{S} := \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$ and g denotes the genus of C. The genus formula for plane curves yields

$$2g - 2 = (d - 1)(d - 2) - 2 = d(d - 3) = 2d(\ell - 1).$$

Therefore, we obtain $\alpha = d(\ell - 1)$. Hence, $\mathcal{O}_C(\ell - 1) \cong \mathcal{L}$ and the Harder-Narasimhan filtration is indeed $0 \subset \mathcal{O}_C(\ell - 1) \subset \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$. $\square$

Remark 2.2. Using Hilbert-Kunz theory and its geometric interpretation developed in [4] and [17] one can give an alternative (but more complicated) proof that the line bundle $\mathcal{O}_C(\ell - 1)$ is the maximal destabilizing subbundle of the syzygy bundle $\mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$. We recall that for a rank two vector bundle the Harder-Narasimhan filtration is already strong in the sense of [9, Paragraph 2.6]. By the formula given in [4, Theorem 3.6] we can compute from the short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \longrightarrow \mathcal{L}' \longrightarrow 0$$

the Hilbert-Kunz multiplicity $e_{HK}(I)$ (see [12]) of the ideal $I = (X^2, Y^2, Z^2)$ in the homogeneous coordinate ring $R := K[X, Y, Z]/(X^d + Y^d + Z^d)$ of the curve C and obtain $e_{HK}(I) = 3d + \frac{\alpha^2}{dp^2}$. But, by [13, Theorem 2.3] the Hilbert-Kunz multiplicity of I equals $e_{HK}(I) = 3d + \frac{d}{4}\frac{(d-3)^2}{p^2}$ which implies $\alpha = d(\ell - 1)$.

Remark 2.3. We briefly comment on the situation for $\ell = 0, 1$. For $\ell = 0$ (and $p \neq 2$) we have $\mathrm{Syz}(X^2, Y^2, Z^2)(3) \cong \mathcal{O}_{\mathbb{P}^1}^2$ and this is also true for its Frobenius pull-back. For $\ell = 1$, we get the Fermat cubic which is an elliptic curve. In this case we have an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathrm{Syz}(X^2, Y^2, Z^2)(3) \longrightarrow \mathcal{O}_C \longrightarrow 0\,,$$

where the (only) global non-trivial section is given by the curve equation. So the syzygy bundle is F_2 in Atiyah's classification [1] and is semistable, but not stable. Its Frobenius pull-back is either F_2 (for $p \equiv 1 \mod 3$, i.e. Hasse invariant one) or $\mathcal{O}_C^2$ (for $p \equiv 2 \mod 3$, i.e. Hasse invariant zero).

In the relative situation

$$C := \operatorname{Proj}(\mathbb{Z}_d[X,Y,Z]/(X^d+Y^d+Z^d)) \longrightarrow \operatorname{Spec}\mathbb{Z}_d$$

every fiber $C_p := C \times_{\operatorname{Spec}\mathbb{Z}_d} \operatorname{Spec}\mathbb{F}_p$ is a smooth projective curve, namely the Fermat curve, defined over the prime field $\mathbb{F}_p$ (and $\bar{C}_p := C \times_{\operatorname{Spec}\mathbb{Z}_d} \bar{\mathbb{F}}_p$ is a smooth projective curve over the algebraic closure of $\mathbb{F}_p$) for every prime number p such that $p \nmid d$. We remind that by the Theorem of Dirichlet [14, Chapitre VI, §4, Théorème and Corollaire] there exist infinitely many prime numbers $p \equiv \ell \mod d$.

Lemma 2.4. *Let $d = 2\ell+1$, $\ell \geq 2$, and consider the smooth projective relative curve $C := \operatorname{Proj}(\mathbb{Z}_d[X,Y,Z]/(X^d+Y^d+Z^d)) \longrightarrow \operatorname{Spec}\mathbb{Z}_d$. Then the sequence (from Lemma 2.1)*

$$0 \longrightarrow \mathcal{O}_{C_p}(\ell-1) \longrightarrow \operatorname{Syz}(X^{2p},Y^{2p},Z^{2p})(3p) \longrightarrow \mathcal{O}_{C_p}(-\ell+1) \longrightarrow 0$$

does not split for almost all primes $p \equiv \ell \mod d$.

Proof. Since $\operatorname{Syz}(X^{2p},Y^{2p},Z^{2p})(3p) \cong F^*(\operatorname{Syz}(X^2,Y^2,Z^2)(3))$ holds on every fiber C_p, the bundle $\operatorname{Syz}(X^{2p},Y^{2p},Z^{2p})(3p)$ carries an integrable connection ∇_p with p-curvature zero by the Cartier-correspondence [7, Theorem 5.1]. Assume that the sequence does split for some $p \equiv \ell \mod d$. Then $\mathcal{O}_{C_p}(\ell-1)$ is a direct summand of $\operatorname{Syz}(X^{2p},Y^{2p},Z^{2p})(3p)$. The summand $\mathcal{O}_{C_p}(\ell-1)$ carries also a connection with the same properties. Hence, again by the Cartier-correspondence it has a Frobenius descent and so its degree $d(\ell-1)$ is divisible by p. But this can only hold for finitely many p. □

Example 2.5. As above we consider the smooth relative curve

$$C := \operatorname{Proj}(\mathbb{Z}_d[X,Y,Z]/(X^d+Y^d+Z^d)) \longrightarrow \operatorname{Spec}\mathbb{Z}_d,$$

with $d = 2\ell+1$, $\ell \geq 2$. The Čech-cohomology class $c = Z^{d-1}/XY \in H^1(C, \mathcal{O}_C(d-3)) \cong \operatorname{Ext}^1(\mathcal{O}_C(-\ell+1), \mathcal{O}_C(\ell-1))$ defines an extension

$$0 \longrightarrow \mathcal{O}_C(\ell-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C(-\ell+1) \longrightarrow 0$$

with the corresponding restrictions to each fiber $C_{\mathfrak{p}}$, where $\mathfrak{p} = (0)$ or $\mathfrak{p} = (p)$, $p \nmid d$. Note that this extension is non-trivial on every fiber. This vector bundle $\mathcal{E}$ is our example. As $\ell \geq 2$ the bundle $\mathcal{E}_0 = \mathcal{E}|_{C_0}$ is not semistable on C_0. By Lemma 2.1 we have for $p \equiv \ell \mod d$ an extension

$$0 \longrightarrow \mathcal{O}_{C_p}(\ell-1) \longrightarrow \operatorname{Syz}(X^{2p},Y^{2p},Z^{2p})(3p) \longrightarrow \mathcal{O}_{C_p}(-\ell+1) \longrightarrow 0$$

corresponding to $c' \in H^1(C_p, \mathcal{O}_{C_p}(2\ell-2)) = H^1(C_p, \mathcal{O}_{C_p}(d-3))$, and by Lemma 2.4 we have $c' \neq 0$ for almost all $p \equiv \ell \mod d$. We claim that

$\mathcal{E}_p = \mathcal{E}|_{C_p} \cong F^*(\mathrm{Syz}(X^2, Y^2, Z^2)(3))$ holds for these prime numbers. Since $\omega_{C_p} = \mathcal{O}_{C_p}(d-3) = \mathcal{O}_{C_p}(2\ell-2)$ and $h^1(C_p, \omega_{C_p}) = 1$ we have $c = \lambda c'$ for some $\lambda \in \mathbb{F}_p^{\times}$. Moreover, multiplication with λ induces an automorphism $\omega_{C_p} \stackrel{\cdot\lambda}{\to} \omega_{C_p}$ of line bundles as well as an automorphism $H^1(C_p, \omega_{C_p}) \stackrel{\cdot\lambda}{\to} H^1(C_p, \omega_{C_p})$ of vector spaces. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{C_p}(2\ell-2) & \longrightarrow & \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p+\ell-1) & \longrightarrow & \mathcal{O}_{C_p} & \longrightarrow & 0 \\ & & \downarrow \cdot\lambda & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & \mathcal{O}_{C_p}(2\ell-2) & \longrightarrow & \mathcal{E}_p(\ell-1) & \longrightarrow & \mathcal{O}_{C_p} & \longrightarrow & 0, \end{array}$$

where the map in the middle is an isomorphism of vector bundles. Hence, $\mathcal{E}_p \cong \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \cong F^*(\mathrm{Syz}(X^2, Y^2, Z^2)(3))$ and therefore $\mathcal{E}_p$ admits a Frobenius descent on every fiber C_p.

Remark 2.6. Example 2.5 extends to all Fermat curves $C^d = V_+(X^d + Y^d + Z^d)$ where the degree d has an odd divisor $d' \geq 5$. To see this we write $d = d'n$ and look at the cover $f : C^d \to C^{d'}$ induced by the ring map which sends each variable to its nth power. Then the pull-back under f of the vector bundles considered in Example 2.5 provide also an example on C^d with the same properties.

3. A positive result

Let $\mathcal{X} \to \mathrm{Spec}\, R$ be a smooth projective morphism of relative dimension $d \geq 1$, where R is a domain of finite type over $\mathbb{Z}$. Typical examples for the base are $\mathrm{Spec}\, \mathbb{Z}$ or arithmetic schemes $\mathrm{Spec}\, D$, where D is the ring of integers in a number field. Let $\mathcal{E}$ be a vector bundle over $\mathcal{X}$. In [6, Theorem 4.2] K. Joshi proved under the assumptions $\mathrm{Pic}(X) = \mathbb{Z}$ ($X = \mathcal{X}_0$) and $\mathrm{rk}(\mathcal{E}) = 2$ that $\mathcal{E}_0 = \mathcal{E}|_X$ is semistable if for infinitely many closed points $\mathfrak{m} \in \mathrm{Spec}\, R$ of arbitrarily large residue characteristic the reduction $\mathcal{E}_{\mathfrak{m}}$ admits a Frobenius descent on the fiber $X_{\mathfrak{m}} = \mathcal{X}_{\mathfrak{m}}$. The aim of this section is to prove (using essentially the same methods) this result for vector bundles of arbitrary rank under the assumption that for every closed point $\mathfrak{m}$ every semistable vector bundle $\mathcal{F}$ on $X_{\mathfrak{m}}$ is strongly semistable, i.e. $F^{e*}(\mathcal{F})$ is semistable for all $e \geq 0$ (it is enough to assume this for infinitely many closed points $\mathfrak{m}$ of arbitrary large residue characteristic). It is interesting to note that Joshi used in [6, Theorem 2.1] the condition $\mathrm{Pic}(Y) = \mathbb{Z}$ on a smooth projective variety Y in positive characteristic and a further hypothesis on Y to prove that every semistable rank two vector bundle on Y is strongly semistable.

Theorem 3.1. *Let R be a $\mathbb{Z}$-domain of finite type and let $f : \mathcal{X} \to \mathrm{Spec}\, R$ be a smooth projective morphism of relative dimension $d \geq 1$ together with a fixed f-very ample line bundle $\mathcal{O}_{\mathcal{X}}(1)$ and let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$. Further* *assume that every semistable vector bundle is strongly semistable (with respect to $\mathcal{O}_{X_\mathfrak{m}}(1)$) for every fiber $X_\mathfrak{m}$, $\mathfrak{m}$ a closed point in $\operatorname{Spec} R$. Then the following holds: If $\mathcal{E}_\mathfrak{m} = \mathcal{E}|_{X_\mathfrak{m}}$ has a Frobenius descent for infinitely many closed points $\mathfrak{m} \in \operatorname{Spec} R$ of arbitrarily large residue characteristic, then $\mathcal{E}_0$ is semistable on the generic fiber $X = X_0 = \mathcal{X}_0$.*

Proof. One can show by induction over $\dim R$ that there exists a bound b such that $\mu_{\max}(\mathcal{E}_\mathfrak{m}) \leq b$ for all closed points $\mathfrak{m} \in \operatorname{Spec} R$ (see [5, Lemma 3.1] for an explicit proof). For a closed point $\mathfrak{m} \in \operatorname{Spec} R$ with descent data $\mathcal{E}_\mathfrak{m} \cong F^*(\mathcal{F}_\mathfrak{m})$, $\mathcal{F}_\mathfrak{m}$ locally free on the fiber $X_\mathfrak{m}$, we have

$$\mu_{\max}(\mathcal{E}_\mathfrak{m}) = \operatorname{char}(\kappa(\mathfrak{m}))\mu_{\max}(\mathcal{F}_\mathfrak{m})$$

because semistable vector bundles are strongly semistable on every fiber $X_\mathfrak{m}$ by assumption. Since we have $\mathcal{E}_\mathfrak{m} \cong F^*(\mathcal{F}_\mathfrak{m})$ for infinitely many closed points $\mathfrak{m}$ of arbitrarily large residue characteristics, this forces the similar equalities $\deg(\mathcal{E}_0) = \deg(\mathcal{E}_\mathfrak{m}) = \operatorname{char}(\kappa(\mathfrak{m}))\deg(\mathcal{F}_\mathfrak{m})$ (we take the degree always with respect to $\mathcal{O}_{X_\mathfrak{m}}(1)$) which implies $\deg(\mathcal{E}_\mathfrak{m}) = \deg(\mathcal{F}_\mathfrak{m}) = 0$. Assume the restriction $\mathcal{E}_0$ to the generic fiber X is not semistable. Then by the openness of semistability [11, Section 5] every restriction $\mathcal{E}_\mathfrak{m}$ on $X_\mathfrak{m}$ is not semistable. Again by our assumption, $\mathcal{F}_\mathfrak{m}$ is not semistable either and so $\mu_{\max}(\mathcal{F}_\mathfrak{m}) \geq 1/r$, $r = \operatorname{rk}(\mathcal{E})$. This gives

$$b \geq \mu_{\max}(\mathcal{E}_\mathfrak{m}) = \operatorname{char}(\kappa(\mathfrak{m}))\mu_{\max}(\mathcal{F}_\mathfrak{m}) \geq \frac{\operatorname{char}(\kappa(\mathfrak{m}))}{r}$$

which contradicts the assumption that we have Frobenius descent at closed points $\mathfrak{m} \in \operatorname{Spec} R$ of arbitrarily large residue characteristic. □

Corollary 3.2. *Let R be a $\mathbb{Z}$-domain of finite type and let $f : \mathcal{X} \to \operatorname{Spec} R$ be a smooth projective morphism of relative dimension $d \geq 1$ together with a fixed f-very ample line bundle $\mathcal{O}_\mathcal{X}(1)$ and let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$. Suppose that the fibers $X_\mathfrak{m}$, $\mathfrak{m} \in \operatorname{Spec} R$ closed, fulfill at least one of the following (not necessarily independent) properties:*

1. *$X_\mathfrak{m}$ is an abelian variety,*
2. *$X_\mathfrak{m}$ is a homogenous space of the form G/P where P is a reduced parabolic subgroup,*
3. *the cotangent bundle $\Omega_{X_\mathfrak{m}}$ fulfills $\mu_{\max}(\Omega_{X_\mathfrak{m}}) \leq 0$.*

Then the following holds: If $\mathcal{E}_\mathfrak{m}$ has a Frobenius descent for infinitely many closed points $\mathfrak{m} \in \operatorname{Spec} R$ of arbitrarily large residue characteristics, then $\mathcal{E}_0$ is semistable on $X = X_0$.

Proof. That every semistable vector bundle is strongly semistable in the case (3) is due to [10, Theorem 2.1]. Condition (3) holds in particular for the varieties occurring in (1) and (2). Other proofs for this property in cases (1) and (2) are given in [15, Corollary 3^p] and in case (3) in [9, Corollary 6.3]. Hence, the assertion follows from Theorem 3.1. □

Remark 3.3. On the one hand it is well-known that every semistable vector bundle on an elliptic curve is strongly semistable (cf. [18, Appendix]). So elliptic curves provide an important class of smooth projective varieties with $\mathrm{Pic}(X) \neq \mathbb{Z}$ for which Theorem 3.1 holds. On the other hand it is also known that for every smooth projective curve of genus $g \geq 2$ there exists a semistable vector bundle $\mathcal{F}$ such that $F^*(\mathcal{F})$ is not semistable (see [8, Theorem 1]). So we see that Theorem 3.1 is applicable in relative dimension one only for elliptic curves and the projective line $\mathbb{P}^1$.

References

1. M. F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc. **7** (1957), 414–452.
2. H. Brenner, *Computing the tight closure in dimension two*, Mathematics of Computations **74** (2005), no. 251, 1495–1518.
3. ———, *On a problem of Miyaoka*, Number Fields and Function Fields - Two Parallel Worlds (B. Moonen R. Schoof G, van der Geer, ed.), Progress in Mathematics, vol. 239, Birkhäuser, 2005, pp. 51–59.
4. ———, *The rationality of the Hilbert-Kunz multiplicity in graded dimension two*, Math. Ann. **334** (2006), no. 1, 91–110.
5. H. Brenner and A. Kaid, *On deep Frobenius descent and flat bundles*, Preprint (2007), arXiv:0712.1794.
6. K. Joshi, *Some remarks on vector bundles*, Preprint (2007), http://math.arizona.edu/kirti/homepage.
7. N. M. Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin*, Inst. Hautes Études Sci. Publ. Math. **39** (1970), 175–232.
8. H. Lange and C. Pauly, *On Frobenius-destabilized rank-2 vector bundles over curves*, Comm. Math. Helv. **83** (2008), 179–209.
9. A. Langer, *Semistable sheaves in positive characteristic*, Ann. Math. **159** (2004), 251–276.
10. V. B. Mehta and A. Ramanathan, *Homogeneous bundles in characteristic p*, Algebraic Geometry - open problems, Lect. Notes Math., vol. 997, 1982, pp. 315–320.
11. Y. Miyaoka, *The Chern class and Kodaira dimension of a minimal variety*, Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math., vol. 10, 1987, pp. 449–476.
12. P. Monsky, *The Hilbert-Kunz function*, Math. Ann. **263** (1983), 43–49.
13. ———, *The Hilbert-Kunz multiplicity of an irreducible trinomial*, J. Algebra **304** (2006), no. 3.
14. J. P. Serre, *Cours d'arithmétique*, Presses Universitaires de France, 1970.
15. N. I. Shepherd-Barron, *Semi-stability and reduction mod p*, Topology **37** (1997), no. 3, 659–664.
16. X. Sun, *Remarks on semistability of G-bundles in positive characteristic*, Comp. Math. **119** (1999), 41–52.
17. V. Trivedi, *Semistability and Hilbert-Kunz multiplicity for curves*, J. Algebra **284** (2005), no. 2, 627–644.
18. L. W. Tu, *Semistable bundles over an elliptic curve*, Adv. Math. **98** (1993), 1–26.

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