ASYMPTOTIC INTEGRATION AND DISPERSION FOR HYPERBOLIC EQUATIONS

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Abstract. The aim of this paper is to establish time decay properties and dispersive estimates for strictly hyperbolic equations with homogeneous symbols and with time-dependent coefficients whose derivatives belong to $L^1(\mathbb{R})$. For this purpose, the method of asymptotic integration is developed for such equations and representation formulae for solutions are obtained. These formulae are analysed further to obtain time decay of L^p–L^q norms of propagators for the corresponding Cauchy problems. It turns out that the decay rates can be expressed in terms of certain geometric indices of the limiting equation and we carry out the thorough analysis of this relation. This provides a comprehensive view on asymptotic properties of solutions to time-perturbations of hyperbolic equations with constant coefficients. Moreover, we also obtain the time decay rate of the L^p–L^q estimates for equations of these kinds, so the time well-posedness of the corresponding nonlinear equations with additional semilinearity can be treated by standard Strichartz estimates.

1. Introduction

This paper is devoted to several aspects of strictly hyperbolic equations of higher orders or of strictly hyperbolic systems with time-dependent coefficients. In particular, we will investigate the following topics:

- representation of solutions of equations of higher order;
- dispersive estimates for solutions.

Equations of orders larger than two appear often in the analysis of large first order systems and in the analysis of coupled equations of higher orders. In the present paper we will restrict our attention to the investigation of equations with homogeneous symbols (at the same time making a suitable preparation for the further development of this topic for equations with low order terms). In fact, we will concentrate on scalar equations of some order $m \in \mathbb{N}$ keeping in mind that in the case of a system its dispersion relation (the determinant) will be of such form, so the information on solution to the Cauchy problem for the dispersion relation will imply the information on solutions to the Cauchy problem of the original system. On one hand, we will introduce several techniques allowing to deal with equations of higher orders. On the other hand, already for the second order equations the new method that we propose in this paper will yield certain improvements and extensions of known results. In particular, we will improve the result on the decay rates in the dispersive estimates already for the time-dependent wave equation, as well as the time decay rate for

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the standard Kirchhoff equation, thus also improving the corresponding Strichartz estimates. It will also allow the inclusion of mixed terms in second order equations (a question which is known to be very delicate if we want to treat problems outside the perturbation framework). We will allow time-dependent coefficients and will assume that their derivatives are in $L^1(\mathbb{R})$. It is known that this property is satisfied in many situations, for example in applications to Kirchhoff equations and systems, etc. This will also allow us to obtain a comprehensive view on time-perturbations of equations with constant coefficients, in which case the assumption of the integrability of derivatives of coefficients is quite natural. For the purposes of this paper, we will develop the *asymptotic integration method* for hyperbolic partial differential equations with time-dependent coefficients. While this method is relatively well-known in the theory of ordinary differential equations (see e.g. Hartman [7]), its use in partial differential equations appears to be new.

We note that equations with constant coefficients have been thoroughly studied by Sugimoto in a series of papers [24, 25, 26] who described several interesting geometric quantities responsible for the rate of the time decay of L^p–L^q norms of their propagators. In particular, one has to look at the level sets of the characteristic roots of the symbol and at the orders with which tangent lines touch these sets. These orders become responsible for the time decay rate in the corresponding dispersion estimates and for indices of the subsequent Strichartz estimates. In fact, the appearing indices are related to the oscillation indices of integral kernels of the propagators, viewed as oscillatory integrals, and their classification is well studied in the singularity theory (e.g. [1]).

The case of strictly hyperbolic equations with constant coefficients with lower order terms has been thoroughly investigated in [22]. In particular, properties of characteristic roots are crucial in determining exact decay rates and the complete analysis is quite lengthy and involved. For example, in the case of equations of dissipative types analysed in [21] the decay rate is determined by properties of characteristics for small frequencies. A general analysis of this type is necessary for application to large systems, such as Grad systems in gas dynamics, or to Fokker-Planck equations, in which case the Galerkin approximation produces a sequence of scalar equations with orders going to infinity, see e.g. [19]. Applications to such problems give a strong additional motivation to the investigation of equations of higher orders of the type of those treated in this paper.

To become more precise, we consider the Cauchy problem for an m^{th} order strictly hyperbolic equation with time-dependent coefficients, for function $u = u(t, x)$:

$$L(t, D_t, D_x)u \equiv D_t^m u + \sum_{\substack{|\nu|+j=m \\ j\le m-1}} a_{\nu,j}(t) D_x^\nu D_t^j u = 0, \quad t \neq 0, \tag{1.1}$$

with the initial condition

$$D_t^k u(0, x) = f_k(x) \in C_0^\infty(\mathbb{R}^n), \quad k = 0, 1, \cdots, m-1, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $D_t = -i\partial_t$ and $D_x^\nu = (-i\partial_{x_1})^{\nu_1} \cdots (-i\partial_{x_n})^{\nu_n}$, $i = \sqrt{-1}$, for $\nu = (\nu_1, \ldots, \nu_n)$. Denoting by $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ the space of all functions which are locally Lipschitz on $\mathbb{R}$, we assume that each $a_{\nu,j}(t)$ belongs to $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ and satisfies

$$a'_{\nu,j}(t) \in L^1(\mathbb{R}) \quad \text{with } |\nu| + j = m. \tag{1.3}$$

Moreover, following the standard definition of equations of the regularly hyperbolic type (e.g. Mizohata [15]), we will assume that the symbol of the differential operator $L(t, D_t, D_x)$ has real and distinct roots $\varphi_1(t;\xi), \ldots, \varphi_m(t;\xi)$ for $\xi \neq 0$, and that

$$L(t, \tau, \xi) = (\tau - \varphi_1(t;\xi)) \cdots (\tau - \varphi_m(t;\xi)), \tag{1.4}$$

$$\inf_{\substack{|\xi|=1, t\in\mathbb{R} \\ j\neq k}} |\varphi_j(t;\xi) - \varphi_k(t;\xi)| > 0. \tag{1.5}$$

Let us point out the main difficulties when trying to establish dispersive estimates (i.e. the time decay estimates for the L^p–L^q norms) for equation (1.1). Contrary to the energy methods, for dispersive estimates we need to have a good idea about the propagators for the Cauchy problem (1.1)–(1.2). Thus, we need to make advances in the following two problems:

- to derive representation formulae for propagators for the Cauchy problem (1.1)–(1.2) with time-dependent coefficients. Ideally these propagators would be in the form of oscillatory integrals;
- to analyse the obtained representation formulae for propagators taking into account the geometric properties of characteristics which we know should be responsible for the time decay rates of L^p–L^q norms of propagators.

Thus, the aim of the paper is twofold. First, we will present representation formulae for propagators for such equations. For this purpose we will develop the asymptotic integration method which is a parameter dependent version of the asymptotic integration of ordinary differential equations (see e.g. [7]). We will trace the dependence on the parameter (which is the frequency in this case) which is essential for further investigation. This method, however, will present somewhat surprising results. For example, the amplitudes of propagators expressed in this form will have symbolic behaviour of type $(0,0)$ rather than the usual $(1,0)$. Nevertheless, this will be enough to carry out the second aim of this part which is the further investigation of the time decay properties of the propagators. The price that we will have to pay is that we may have to assume additional regularity of the Cauchy data for high frequencies. However, this is not so bad because estimates for bounded times already will require similar regularity assumptions.

We will analyse the obtained representations to derive time asymptotics of L^p–L^q norms of the necessary oscillatory integrals. There are several important differences with the case of the wave equation, where level sets of characteristics are nothing else but spheres, so one can simply apply the stationary phase method to the obtained oscillatory integrals. Now the critical points may be degenerate so the stationary phase method (especially in the parameter depending setting that we have here) does not work. In fact, the non-degeneracy of critical points is a rather strong assumption for higher order equations, where degeneracy of higher order may easily happen (examples of this are e.g. in [25]). That is why we will allow them to be degenerate of a finite order, and the time decay rates will depend on this order. On the other hand, van der Corput type estimates that are normally used in place of the stationary phase in such problems are essentially one-dimensional, and they do not take into account the geometric properties of characteristics (phases and characteristics do come from a hyperbolic equation after all). So, we need to apply a parameter dependent version (now time is the parameter) of van der Corput's lemma uniformly in $n-1$ directions of non-vanishing higher order curvatures. Moreover, this has to be done uniformly with respect to the time dependence of the propagators. In fact, we will relate the time-decay rates to the Sugimoto's indices of levels sets of characteristic roots of the limiting equation, thus establishing a more or less complete picture of perturbation properties of dispersive estimates for strictly hyperbolic equations with homogeneous symbols. For example, in the case of convex level sets one introduces the convex index γ which is the largest order of tangency of tangent lines to the level sets of characteristics of the limiting equation and it turns out that the L^p–L^q norm of the corresponding propagator decays as $t^{-\frac{n-1}{\gamma}}$. In the case of the second order equations one has $\gamma = 2$ and so one recovers that standard rate of decay of the wave equation (see [4, 5, 10, 23]), and many other known results for the time independent wave type second order equations. We also note that such index γ does not play any role for L^p–L^p estimates, where singularities of the projection from the canonical relations to the base space start playing a role (see e.g. survey paper [18]). The inclusion of mixed derivatives in the symbols may influence the value of γ. Moreover, mixed terms may make the analysis more complicated. Already for the second order equations this was demonstrated by Hirosawa and Reissig in [8], for the problem of the influence of oscillations in coefficients.

In addition, methods introduced in this paper may be applied to the study of strictly hyperbolic systems. For example, let $A(t, D_x)$ be the first order $m \times m$ pseudo-differential system, with entries $a_{ij}(t, \xi)$ being homogeneous with respect to ξ of order one and such that $\partial_t a_{ij}(\cdot, \xi) \in L^1(\mathbb{R})$ for all $\xi \in \mathbb{R}^n$. We consider the evolution equation

$$\partial_t U = iA(t, D_x)U, \quad U(0, x) = f(x), \ x \in \mathbb{R}^n. \tag{1.6}$$

Let us assume that system (1.6) is uniformly strictly hyperbolic (see Mizohata [15]), i.e. that its characteristics $\varphi_k(t;\xi)$, $k = 1, \cdots, m$, are real, and satisfy condition (1.5). Then the strict hyperbolicity implies that we can diagonalise it similar to Lemma 2.1. Thus, system (1.6) splits into m scalar first order equations of the form

$$\partial_t v_k = i\varphi_k(t; D_x)v_k, \quad k = 1, \ldots, m,$$

for function v_k related to the original vector function U. The condition on the integrability of time-derivatives of A implies that there is a limiting system $A^{\pm}(t, \xi) = \lim_{t\to\pm\infty} A(t, \xi)$ with characteristics $\varphi_k^{\pm}(\xi) = \lim_{t\to\pm\infty} \varphi_k(t; \xi)$, which exist since we assume that $\partial_t \varphi_k(\cdot; \xi) \in L^1(\mathbb{R})$. Therefore, solutions $v_k(t, x)$ can be analysed using estimates for oscillatory integrals that we establish in §4. Details of this analysis are different from those for the scalar equation (1.1), especially in the way of keeping track of the representation form for the time derivatives of the solution, so we omit the analysis of systems from this paper and it will appear elsewhere, together with its specific applications and with refinements of the analysis of high frequencies. We will also not discuss the case of oscillations in this paper, but we refer to, for example, the survey [17], for the overview of the case of the wave equations. The case of oscillations in higher order equations will appear elsewhere.

Thus, in §2 we will discuss the asymptotic integrations of the ordinary differential equations corresponding to our problem. Using these implicit representations, we will succeed to obtain the asymptotic integrations of (1.1)–(1.2). The precise statement will be given in §3.

Let us now give an informal overview of this method. Writing equation (1.1) as a system for

$$U = \left(|D|^{m-1}u, |D|^{m-2}D_t u, \dots, D_t^{m-1}u\right)^T$$

and taking the Fourier transform with respect to x, we can reduce it to the first order Cauchy problem

$$D_t U = A(t;\xi)U, \quad U(0) = U_0. \tag{1.7}$$

If we denote

$$\vartheta_j(t;\xi) = \int_0^t \varphi_j(s;\xi)\,ds,\ j = 1, \dots, m,$$

a natural candidate for the fundamental matrix for (1.7) is

$$\Phi(t;\xi) = \operatorname{diag}\left(e^{i\vartheta_1(t;\xi)}, \cdots, e^{i\vartheta_m(t;\xi)}\right).$$

So, we look for the solution of (1.7) in the form

$$U = \Phi(t;\xi)V, \ \text{ with } \ V(t;\xi) = \boldsymbol{\alpha}(\xi) + \boldsymbol{\varepsilon}(t;\xi),$$

where we want $\boldsymbol{\varepsilon}(t;\xi)$ to decay as $t \to \pm\infty$. It can be checked that there is a matrix $A_0(t;\xi)$ such that $D_t\Phi = A_0\Phi$ and we get

$$D_t U = D_t(\Phi V) = (D_t\Phi)V + \Phi D_t V = A_0 U + \Phi D_t \boldsymbol{\varepsilon}.$$

Thus, U becomes the solution of (1.7) if we choose V and $\boldsymbol{\varepsilon}$ such that

$$D_t V \equiv D_t \boldsymbol{\varepsilon} = \Phi^{-1}(A - A_0)\Phi V. \tag{1.8}$$

In §2 we will show that, in fact, there exists a global-in-time solution V of equation (1.8) of the required form $V = \boldsymbol{\alpha} + \boldsymbol{\varepsilon}$. Moreover, $\boldsymbol{\varepsilon}$ satisfies the property that $\boldsymbol{\varepsilon}(t;\xi) \to 0$ as $t \to \pm\infty$ for all $\xi \neq 0$. In addition, we will show the decay orders of both $\boldsymbol{\alpha}(\cdot)$ and $\boldsymbol{\varepsilon}(t;\cdot)$ and their derivatives. This will lead to an oscillatory integral representation of solution $u(t,x)$ of (1.1) of the form

$$u(t,x) = \sum_{k=0}^{m-1}\sum_{j=1}^{m} \mathcal{F}^{-1}\left[e^{i\vartheta_j(t;\xi)}\left(\alpha^j_{k,\pm}(\xi) + \varepsilon^j_{k,\pm}(t;\xi)\right)\widehat{f}_k(\xi)\right](x), \quad t \gtrless 0, \tag{1.9}$$

with amplitudes $\alpha^j_{k,\pm}(\xi), \varepsilon^j_{k,\pm}(t;\xi)$ of the form of $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ above. In fact, Theorem 3.1 will also yield a similar representation for the derivatives of $u(t,x)$ with respect to time. The main difference with equations with time independent coefficients here is that the amplitudes $\alpha^j_{k,\pm}(\xi)$ and $\varepsilon^j_{k,\pm}(t;\xi)$ will have the symbolic behavior of the type $(0,0)$ rather than the type $(1,0)$ usual for equations with constant coefficients. Indeed, such choice of phases globally as $\vartheta_j(t;\xi)$ introduces low order errors in the equation if we formally substitute (1.9) into (1.1) and as we know the lower order terms may change the time decay properties in an essential way (this is especially apparent for Schrödinger equations, but is also true in the hyperbolic case). Thus, the error should be somehow accounted for and the behaviour of amplitudes takes care of this. In any case, since we know that the needed regularity of data comes from other parts of the time-frequency phase space, we are still able to get the same time decay rate under an additional regularity assumption in the high frequency zone. So this difference does not change the final result in a big way.

Thus, in the second part of the paper we will use representation (1.9) to derive the time decay of the L^p–L^q norms of u, which in turn leads to Strichartz estimates and to well-posedness results for the corresponding semilinear equations in a rather (by now) standard way, so we will derive the dispersive estimates and will omit the details of the further standard analysis. In addition, in §4 we will present estimates for more general oscillatory integrals. Such estimates may be used not only in the application to the problem we are considering in this paper but in a wider range of applications. The estimates will rely on estimates for parameter dependent oscillatory integrals developed in [20]. We state such result here in Theorem 4.5. However, the meaning of the parameter is different in our setting. Thus, in our problem here time acts as a parameter while in problems for hyperbolic equations with constant coefficients but with lower order terms considered in [22] the phase functions were not homogeneous and their non-homogeneous contributions were considered to be a parameter from the point of view of the perturbation theory. In principle, it should be possible to combine problems with time-dependent coefficients with those with lower order terms but this will be a subject of another paper – here we have an advantage of making more use of the homogeneity of the symbols and hence also of phases, considerably simplifying some arguments. The obtained results can be applied to the global in time well-posedness problems of Kirchhoff equations of high orders and of Kirchhoff systems. Such applications will be addressed elsewhere.

Let $\varphi_k^{\pm}(\xi) = \lim_{t\to\pm\infty} \varphi_k(t;\xi)$ be the limits of characteristic roots as will be shown to exist in Proposition 2.3. Let us introduce the convex and non-convex Sugimoto indices for the level sets of these functions. In the time independent setting these indices have been introduced by Sugimoto in [24, 25]. These indices will determine the decay rate of propagators for large frequencies.

Let $\varphi \in C^\infty(\mathbb{R}^n\backslash 0)$ be a homogeneous of order one function and let $\Sigma_\varphi = \{\xi \in \mathbb{R}^n : \varphi(\xi) = 1\}$ be its level set. Suppose first that Σ_φ is convex. We define the *convex Sugimoto index* $\gamma(\Sigma_\varphi)$ of Σ_φ by

$$\gamma(\Sigma_\varphi) := \sup_{\sigma\in\Sigma_\varphi} \sup_P \gamma(\Sigma_\varphi;\sigma,P)\,, \tag{1.10}$$

where P is a plane containing the normal to Σ_φ at σ and $\gamma(\Sigma_\varphi;\sigma,P)$ denotes the order of the contact between the line $T_\sigma \cap P$ (where T_σ is the tangent plane at σ), and the curve $\Sigma_\varphi \cap P$.

In the case when the level set Σ_φ is not convex, we get a weaker result based on the van der Corput lemma. In this case we use the *non-convex Sugimoto index* $\gamma_0(\Sigma_\varphi)$ of Σ_φ which we define as

$$\gamma_0(\Sigma_\varphi) := \sup_{\sigma\in\Sigma_\varphi} \inf_P \gamma(\Sigma_\varphi;\sigma,P)\,, \tag{1.11}$$

where P and σ are the same as in the convex case.

We note that for the second order equations we have $\gamma=\gamma_0=2$ and the following theorem covers the case of the wave equation as a special case, also improving the corresponding result in [12]. We use the notation $L^p_s(\mathbb{R}^n)$ for the standard Sobolev space with s derivatives over $L^p(\mathbb{R}^n)$, and by $\dot L^p_s(\mathbb{R}^n)$ we denote its homogeneous version. The result on the dispersive estimates that we will prove among other things, is as follows:

Theorem 1.1. *Assume* (1.3)–(1.5). *Then the solution $u(t,x)$ of* (1.1) *satisfies the following estimates*:

(i) *Suppose that the set*

$$\Sigma_{\varphi_k^\pm}=\{\xi\in\mathbb{R}^n:\varphi_k^\pm(\xi)=1\}$$

is convex for all $k=1,\dots,m$, and set $\gamma=\max_{k=1,\dots,m}\gamma(\Sigma_{\varphi_k^\pm})$. In addition, suppose that $(1+|t|)^r a'_{\nu,j}\in L^1(\mathbb{R})$ for $1\le r\le[(n-1)/\gamma]+1$, and for all ν,j with $|\nu|+j=m$. Let $1<p\le 2\le q<+\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then for all $t\in\mathbb{R}$ we have the estimate

$$\|D_t^l D_x^\alpha u(t,\cdot)\|_{L^q(\mathbb{R}^n)}\le C(1+|t|)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\sum_{k=0}^{m-1}\left(\|f_k\|_{\dot L^p_{N_p+l+|\alpha|-k}(\mathbb{R}^n)}+\|f_k\|_{\dot L^p_{l+|\alpha|-k}(\mathbb{R}^n)}\right), \tag{1.12}$$

where $N_p=\left(n-\frac{n-1}{\gamma}+\left[\frac{n-1}{\gamma}\right]+1\right)\left(\frac{1}{p}-\frac{1}{q}\right)$, $l=0,\dots,m-1$, and α is any multi-index.

(ii) *Suppose that $\Sigma_{\varphi_k^\pm}$ is non-convex for some $k=1,\dots,m$, and let us set $\gamma_0=\max_{k=1,\dots,m}\gamma_0(\Sigma_{\varphi_k^\pm})$. In addition, suppose that $(1+|t|)a'_{\nu,j}\in L^1(\mathbb{R})$ for all ν,j with $|\nu|+j=m$. Let $1<p\le 2\le q<+\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then for all $t\in\mathbb{R}$ we have the estimate*

$$\|D_t^l D_x^\alpha u(t,\cdot)\|_{L^q(\mathbb{R}^n)}\le C(1+|t|)^{-\frac{1}{\gamma_0}\left(\frac{1}{p}-\frac{1}{q}\right)}\sum_{k=0}^{m-1}\left(\|f_k\|_{\dot L^p_{N_p+l+|\alpha|-k}(\mathbb{R}^n)}+\|f_k\|_{\dot L^p_{l+|\alpha|-k}(\mathbb{R}^n)}\right),$$

where $N_p=\left(n-\frac{1}{\gamma_0}+1\right)\left(\frac{1}{p}-\frac{1}{q}\right)$, $l=0,\dots,m-1$, and α is any multi-index.

Remark 1.2. *The way we formulate the estimates in Theorem 1.1 is to unify different estimates for different parts of the solution. This may explain the appearance of two norms in the right hand side of* (1.12), *for example, to account for both small and large frequencies. The much more precise estimates are possible and they are stated in Theorem 4.10.*

Let us now make only a few short remarks to compare our results with what is known for $m=2$. In the constant coefficient case and $p=2$ the estimates coincide with those for constant coefficient equations considered in [24]. Also, in the case of constant coefficients, the Sobolev index N_p in the estimate (1.12) can be improved since there is no addition of the integer part in its definition in this case. For the detailed overview of constant coefficients case we refer to [22], and results in this direction for non-constant coefficients were announced in [14] and the detailed proofs will appear elsewhere.

Further, in the time-dependent case, it can be already noted that the statement of Theorem 1.1 goes beyond results available in certain energy classes. For example, in the often considered case of the time–dependent wave equation (so that $m = 2$, e.g. [8], [13], [16], [17], etc.) one obtains the estimate for $\|D_t u(t,\cdot)\|_{L^q(\mathbb{R}^n)}$ and $\|\nabla u(t,\cdot)\|_{L^q(\mathbb{R}^n)}$ only, and not for solution u itself. Moreover, the use of homogeneous Sobolev spaces in (1.12) allows to gain more information in the low frequency region. At the same time, also already for the case $m = 2$, we make the assumption on only one derivative of the coefficients $a_{\nu,j}$, which is another improvement compared with the known literature. This improvement will be crucial in dealing with applications to Kirchhoff equations.

We will denote $\langle x\rangle = \sqrt{1+|x|^2}$. Constants may change from formula to formula, although they are usually denoted by the same letter.

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2. Asymptotic integration of ODE

In this section we will construct the asymptotic integration of the ordinary differential equation. By applying the Fourier transform on $\mathbb{R}^n_x$ to (1.1), we get

$$D_t^m v + \sum_{j=1}^m h_j(t;\xi) D_t^{m-j} v = 0, \tag{2.1}$$

where

$$h_j(t;\xi) = \sum_{|\nu|=j} a_{\nu,m-j}(t)\xi^\nu, \quad \xi \in \mathbb{R}^n$$

(note that there is a slight change of the meaning of j here compared to (1.1)). This is the ordinary differential equation of homogeneous m^{th} order with the parameter $\xi = (\xi_1,\dots,\xi_n)$. As usual, the strict hyperbolicity (1.4)–(1.5) means that the characteristic roots of (2.1) are real and distinct. We denote them by $\varphi_1(t;\xi),\dots,\varphi_m(t;\xi)$. Notice that each $\varphi_\ell(t;\xi)$ has a homogeneous degree of order one with respect to ξ. In this section we will develop an asymptotic integration of the equation (2.1). Let us start by writing (2.1) as the first order system. In (2.1) we put for brevity

$$H_j(t,\xi) = h_j(t;\xi/|\xi|),$$

and denote

$$v_j = |\xi|^{m-j-1} D_t^j v, \quad j = 0,1,\cdots,m-1.$$

It is easy to see that

$$D_t v_j = |\xi| v_{j+1}, \quad j = 0,1,\cdots,m-2,$$

holds. Then (2.1) can be written as

$$D_t v_{m-1} + \sum_{j=0}^{m-1} H_{m-j}(t,\xi)|\xi| v_j = 0.$$

Hence, if we put

$$\mathcal{H}(t,\xi)=\begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ -H_m(t,\xi) & -H_{m-1}(t,\xi) & \dots & -H_1(t,\xi) \end{pmatrix},$$

then (2.1) can be written by

$$D_t\boldsymbol{v}=\mathcal{H}(t,\xi)|\xi|\boldsymbol{v}, \tag{2.2}$$

where $\boldsymbol{v}={}^T(v_0,v_1,\dots,v_{m-1})$.

We will use the following lemma.

Lemma 2.1 ([15] Proposition 6.4)**.** *Assume* (1.3)–(1.5)*. Then there exists a matrix* $\mathcal{N}=\mathcal{N}(t;\xi)$ *of homogeneous degree* 0 *satisfying the following properties*:
(i) $\mathcal{N}\mathcal{H}(t,\xi)=\mathcal{D}\mathcal{N}$*, where*

$$\mathcal{D}=\mathcal{D}(t;\xi)=\operatorname{diag}\left\{\varphi_1(t;\xi/|\xi|),\dots,\varphi_m(t;\xi/|\xi|)\right\},$$

(ii) $\inf_{\xi\in\mathbb{R}^n\setminus 0,t\in\mathbb{R}}|\det\mathcal{N}(t;\xi)|>0$,
(iii) $\mathcal{N}(t;\xi)$ *is* C^∞ *in* $\xi\neq 0$*,* C^1 *in* $t\in\mathbb{R}$ *and* $\partial_t\mathcal{N}(t;\xi)$ *belongs to* $L^1(\mathbb{R})$ *for each* $\xi\neq 0$.

We will first derive the energy estimates.

Lemma 2.2. *Assume* (1.3)–(1.5)*. Let* $\boldsymbol{v}=\boldsymbol{v}(t;\xi)$ *be a general solution of* (2.2)*. Then, for all* $t\in\mathbb{R}$*, we have*

$$|\boldsymbol{v}(t;\xi)|^2\le C|\boldsymbol{v}(0;\xi)|^2\mathrm{e}^{\int_{-\infty}^{+\infty}2\|\partial_t\mathcal{N}(s;\xi)\|\,ds}. \tag{2.3}$$

Proof. Multiplying (2.2) by $\mathcal{N}=\mathcal{N}(t;\xi)$ from Lemma 2.1, we get

$$D_t(\mathcal{N}\boldsymbol{v})-\mathcal{N}\mathcal{H}|\xi|\boldsymbol{v}-(D_t\mathcal{N})\boldsymbol{v}=0.$$

Putting $\mathcal{N}\boldsymbol{v}=\boldsymbol{w}$, we have

$$D_t\boldsymbol{w}-\mathcal{D}|\xi|\boldsymbol{w}-(D_t\mathcal{N})\boldsymbol{v}=0, \tag{2.4}$$

since $\mathcal{N}\mathcal{H}=\mathcal{D}\mathcal{N}$ by Lemma 2.1. This implies that

$$\partial_t|\boldsymbol{w}|^2=2\mathrm{Re}\left(\partial_t\boldsymbol{w}\cdot\overline{\boldsymbol{w}}\right)=2\mathrm{Re}\left(i\mathcal{D}|\xi|\boldsymbol{w}\cdot\overline{\boldsymbol{w}}\right)+2\mathrm{Re}\left(i(D_t\mathcal{N})\boldsymbol{v}\cdot\overline{\boldsymbol{w}}\right).$$

Taking account that $\overline{i\mathcal{D}|\xi|}=-i\mathcal{D}|\xi|$ and $\mathcal{D}$ is real and diagonal, we have

$$\mathrm{Re}\left(i\mathcal{D}|\xi|\boldsymbol{w}\cdot\overline{\boldsymbol{w}}\right)=0,$$

hence,

$$\partial_t|\boldsymbol{w}|^2\le 2\|\partial_t\mathcal{N}\||\boldsymbol{v}||\boldsymbol{w}|. \tag{2.5}$$

Here, $|\boldsymbol{v}|$ and $|\boldsymbol{w}|$ are equivalent to each other. Indeed, there exists $C_1,C_2>0$ such that $C_1|\boldsymbol{v}|\le|\boldsymbol{w}|\le C_2|\boldsymbol{v}|$ on account of Lemma 2.1. Thus, integrating (2.5), we arrive at

$$|\boldsymbol{v}(t;\xi)|^2\le C\left(|\boldsymbol{v}(0;\xi)|^2+\int_0^{|t|}\|\partial_t\mathcal{N}(s;\xi)\||\boldsymbol{v}(s;\xi)|^2\,ds\right).$$

Since $\partial_t\mathcal{N}\in L^1(\mathbb{R})$ by Lemma 2.1 (iii), we conclude from Gronwall's lemma that (2.3) is true. The proof of Lemma 2.2 is complete. □

As a consequence of (2.4) in the proof of Lemma 2.2, we have derived

$$D_t\boldsymbol{w} - \mathcal{D}|\xi|\boldsymbol{w} - (D_t\mathcal{N})\mathcal{N}^{-1}\boldsymbol{w} = 0, \tag{2.6}$$

where we put $\boldsymbol{w} = \mathcal{N}\boldsymbol{v}$. We can expect that the solution of (2.6) is asymptotic to the solution of

$$D_t\boldsymbol{y} = \mathcal{D}|\xi|\boldsymbol{y}. \tag{2.7}$$

Let $\Phi(t;\xi)$ be the fundamental matrix of (2.7), i.e.,

$$\Phi(t;\xi) = \mathrm{diag}\left\{e^{i\vartheta_1(t;\xi)}, \cdots, e^{i\vartheta_m(t;\xi)}\right\},$$

where we put

$$\vartheta_j(t;\xi) = \int_0^t \varphi_j(s;\xi)\,ds, \qquad j = 1, \ldots, m.$$

Let us first analyse certain basic properties of characteristic roots $\varphi_k(t;\xi)$ of (1.4).

Proposition 2.3. *Let the operator $L(t, D_t, D_x)$ satisfy the properties (1.4)–(1.5). Then each $\partial_t\varphi_k(t;\xi)$, $k = 1, \ldots, m$, is homogeneous of order one in ξ, and there exist a constant $C > 0$ such that*

$$|\partial_t\varphi_k(t;\xi)| \le C|\xi| \quad \text{for all } \xi \in \mathbb{R}^n,\ \ t \in \mathbb{R},\ \ k = 1, \ldots, m. \tag{2.8}$$

Moreover, if $a'_{\nu,j}(\cdot) \in L^1(\mathbb{R})$ for all ν, j, then we have also $\partial_t\varphi_k(\cdot;\xi) \in L^1(\mathbb{R})$ for all $\xi \in \mathbb{R}^n$. Furthermore, there exist functions $\varphi_k^{\pm} \in C^\infty(\mathbb{R}^n\backslash 0)$, homogeneous of order one, such that

$$\partial_\xi^\alpha\varphi_k(t;\xi) \to \partial_\xi^\alpha\varphi_k^{\pm}(\xi) \ \text{ as } t \to \pm\infty, \tag{2.9}$$

for all $\xi \in \mathbb{R}^n$, all α, and $k = 1, \ldots, m$. Finally, we have the following formula for the derivatives of characteristic roots:

$$\partial_t\varphi_k(t;\xi) = -\sum_{|\nu|+j=m} a'_{\nu,j}(t)\varphi_k(t;\xi)^j\xi^\nu \prod_{r\ne k}\left(\varphi_k(t;\xi) - \varphi_r(t;\xi)\right)^{-1}. \tag{2.10}$$

Proof. Let us show first that $\varphi_k(t;\xi)$ is bounded with respect to $t \in \mathbb{R}$, i.e.,

$$|\varphi_k(t;\xi)| \le C|\xi|, \quad \text{for all } \xi \in \mathbb{R}^n,\ t \in \mathbb{R},\ k = 1, \ldots, m. \tag{2.11}$$

We will use the fact that $\varphi_k(t;\xi)$ are roots of the polynomial L of the form

$$L(t,\tau,\xi) = \tau^m + c_1(t,\xi)\tau^{m-1} + \cdots + c_m(t,\xi)$$

with $|c_j(t,\xi)| \le M|\xi|^j$, for some $M \ge 1$. Suppose that one of its roots τ satisfies $|\tau(t,\xi)| \ge 2M|\xi|$. Then

$$\begin{aligned}
|L(t,\tau,\xi)| &\ge |\tau|^m\left(1 - \frac{|c_1(t,\xi|)}{|\tau|} - \cdots - \frac{|c_m(t,\xi)|}{|\tau|^m}\right)\\
&\ge 2M|\xi|^m\left(1 - \frac{1}{2} - \frac{1}{4M} - \cdots - \frac{1}{2^m M^{m-1}}\right) > 0,
\end{aligned}$$

hence $|\tau(t,\xi)| \le 2M|\xi|$ for all $\xi \in \mathbb{R}^n$. Thus we establish (2.11).

Differentiating (1.4) with respect to t, we get

$$\frac{\partial L(t,\tau,\xi)}{\partial t} = \sum_{|\nu|+j=m} a'_{\nu,j}(t)\tau^j\xi^\nu = -\sum_{k=1}^m \partial_t\varphi_k(t;\xi)\prod_{r\ne k}(\tau - \varphi_r(t;\xi)).$$

Setting $\tau = \varphi_k(t;\xi)$, we obtain

$$\partial_t \varphi_k(t;\xi) \prod_{r\neq k} (\varphi_k(t;\xi) - \varphi_r(t;\xi)) = - \sum_{|\nu|+j=m} a'_{\nu,j}(t)\varphi_k(t;\xi)^j \xi^\nu, \tag{2.12}$$

implying (2.10). Now, using (1.5), (2.11), and the assumption that $a'_{\nu,j}(\cdot) \in L^1(\mathbb{R})$ for all ν, j, we conclude that (2.8) holds and $\partial_t \varphi_k(\cdot;\xi) \in L^1(\mathbb{R})$ for all $\xi \in \mathbb{R}^n$ and $k = 1, \dots, m$. The homogeneity of order one of $\partial_t\varphi_k(t;\xi)$ is an immediate consequence of (2.12) and its derivatives.

Finally, setting

$$\varphi^\pm_\ell(\xi) = \varphi_k(0;\xi) + \int_0^{\pm\infty} \partial_t\varphi_k(t;\xi)\, dt,$$

we get (2.9) with $\alpha = 0$. Differentiating this equality with respect to ξ, we get (2.9) for all α. The proof is complete. □

We note that under the assumptions of Proposition 2.3 the coefficients $a_{\nu,j}(t)$ of the operator $L(t, D_t, D_x)$ in (1.1) have limits $a^\pm_{\nu,j}$ as $t \to \pm\infty$, namely

$$a^\pm_{\nu,j} = a_{\nu,j}(0) + \int_0^{\pm\infty} a'_{\nu,j}(t)\, dt.$$

Functions $\varphi^\pm_k(\xi)$ are characteristics of the limiting strictly hyperbolic operator

$$L^\pm(D_t, D_x)u \equiv D_t^m u + \sum_{\substack{|\nu|+j=m\\ j\le m-1}} a^\pm_{\nu,j} D_x^\nu D_t^j u, \tag{2.13}$$

and their geometric properties are responsible for the time decay of solutions to the Cauchy problems for both operators $L(t, D_t, D_x)$ and $L^\pm(D_t, D_x)$. This will be analysed in §4. We also note that since operator $L^\pm(D_t, D_x)$ has constant coefficients its solution can be represented as a sum of oscillatory integrals in the standard way. The dependence of coefficients of $L(t, D_t, D_x)$ on time brings corrections to the phases and amplitudes of this representation.

Next we make the representation formulae of solutions for our equation. The following proposition is known as Levinson's lemma (see Coddington and Levinson [6]) in the theory of ordinary differential equations, the new feature here is the additional dependence on ξ. For the convenience of the readers, we shall prove it along the method of Ascoli [2] and Wintner [27] (cf. Hartman [7]).

Proposition 2.4. *Assume* (1.3)–(1.5). *Then, for every nontrivial solution $v(t;\xi)$ of* (2.1), *there exist vectors of C^∞-amplitude functions $\boldsymbol{\alpha}_\pm(\xi)$ and error functions $\boldsymbol{\varepsilon}_\pm(t;\xi)$ such that*

$$\boldsymbol{v}(t;\xi) = \mathcal{N}(t;\xi)^{-1}\Phi(t;\xi)\left(\boldsymbol{\alpha}_\pm(\xi) + \boldsymbol{\varepsilon}_\pm(t;\xi)\right), \quad t \gtrless 0, \tag{2.14}$$

where

$$\boldsymbol{\varepsilon}_\pm(t;\xi) \to \mathbf{0} \quad \text{for any fixed } \xi \neq 0 \text{ as } t \to \pm\infty. \tag{2.15}$$

Furthermore we have

$$D_t\boldsymbol{\varepsilon}_\pm(t;\xi) = C(t;\xi)\left(\boldsymbol{\alpha}_\pm(\xi) + \boldsymbol{\varepsilon}_\pm(t;\xi)\right), \tag{2.16}$$

where $C(t;\xi)$ belongs to $L^1(\mathbb{R})$ in t, and has the following form:

$$C(t;\xi) = \Phi(t;\xi)^{-1}(D_t\mathcal{N}(t;\xi))\mathcal{N}(t;\xi)^{-1}\Phi(t;\xi). \tag{2.17}$$

Proof. We can expect that every solution $\boldsymbol{w} = \boldsymbol{w}(t;\xi)$ of (2.6) is asymptotic to some solution $\boldsymbol{y} = \boldsymbol{y}(t;\xi)$ of (2.7). If we perform the Wronskian transform $\boldsymbol{z} = \Phi(t;\xi)^{-1}\boldsymbol{w}$, then the system (2.6) reduces to a system $D_t\boldsymbol{z} = C(t;\xi)\boldsymbol{z}$, where $C(t;\xi)$ is given by (2.17).

We will now prove that (2.14)–(2.15) hold for every nontrivial solution $v = v(t;\xi)$. It follows from Lemma 2.2 that

$$|\boldsymbol{z}(t;\xi)| \le \|\Phi(t;\xi)^{-1}\||\boldsymbol{w}(t;\xi)| \le c|\boldsymbol{w}(t;\xi)| \le c_1|\boldsymbol{w}(0;\xi)|$$

for all $t \in \mathbb{R}$ and some constant c_1. Using this bound and equation $D_t\boldsymbol{z} = C(t;\xi)\boldsymbol{z}$, we have

$$|D_t\boldsymbol{z}(t;\xi)| \le \|C(t;\xi)\|\,|\boldsymbol{z}(t;\xi)| \le c_1|\boldsymbol{w}(0;\xi)|\|C(t;\xi)\|,$$

hence $D_t\boldsymbol{z}(\cdot;\xi) \in L^1(\mathbb{R})$ on account of $C(\cdot;\xi) \in L^1(\mathbb{R})$. Thus $\{\boldsymbol{z}(t;\xi)\}_{t\in\mathbb{R}}$ is a convergent function, and there exists

$$\lim_{t\to\pm\infty} \boldsymbol{z}(t;\xi) =: \boldsymbol{\alpha}_\pm(\xi).$$

If we set

$$\boldsymbol{\varepsilon}_\pm(t;\xi) = \boldsymbol{z}(\xi,t) - \boldsymbol{\alpha}_\pm(\xi), \tag{2.18}$$

then $\boldsymbol{z}(t;\xi)$ can be written as $\boldsymbol{z}(t;\xi) = \boldsymbol{\alpha}_\pm(\xi) + \boldsymbol{\varepsilon}_\pm(t;\xi)$ for $t \gtrless 0$, and further, $\boldsymbol{\varepsilon}_\pm(t;\xi)$ decays as $t \to \pm\infty$ for any fixed $\xi \neq 0$, which proves (2.15). Since $\boldsymbol{w}(t;\xi) = \Phi(t;\xi)\boldsymbol{z}(t;\xi)$, we get the formula (2.14). Finally, differentiating (2.18) with respect to t and using the equation $D_t\boldsymbol{z} = C(t;\xi)\boldsymbol{z}$, we get (2.16). The proof of Proposition 2.4 is now complete. □

Finally, we will need the estimates for higher order derivatives of $C(t;\xi)$ appearing in Proposition 2.4.

Lemma 2.5. *Assume* (1.3)–(1.5). *Then the μ^{th} derivatives for each entry $c_{jk}(t;\xi)$ of $C(t;\xi)$ satisfy*

$$\left|\partial_\xi^\mu c_{jk}(t;\xi)\right| \le c|\xi|^{-|\mu|}(1+|t|)^{|\mu|}\Psi(t), \quad j,k = 1,\dots,m, \tag{2.19}$$

for $|\mu| \ge 1$ and $|\xi| \ge 1$, where

$$\Psi(t) = \sum_{\substack{|\nu|+j=m \\ j\le m-1}} \left|a'_{\nu,j}(t)\right|. \tag{2.20}$$

For $0 < |\xi| < 1$ and $|\mu| \ge 1$, we have

$$\left|\partial_\xi^\mu c_{jk}(t;\xi)\right| \le c|\xi|^{-|\mu|}(1+|t|)^{|\mu|}\Psi(t), \quad j,k = 1,\dots,m. \tag{2.21}$$

Moreover, assume that $(1+|t|)^{|\mu|}a'_{\nu,j}(t) \in L^1(\mathbb{R})$ for all ν, j, and for some μ with $|\mu| \ge 1$. Then we have $\partial_\xi^\mu c_{jk}(t;\xi) \in L^1(\mathbb{R})$.

Proof. Since $\varphi_j(t;\xi)$ is homogeneous of order one, we have

$$|\partial_\xi^\mu \varphi_j(t;\xi)| \le c|\xi|^{-|\mu|+1} \quad \text{for } \xi \in \mathbb{R}^n \setminus 0,\ j = 1,\dots,m,$$

and hence,

$$|\partial_\xi^\mu \vartheta_j(t;\xi)| \le \int_0^{|t|} |\partial_\xi^\mu \varphi_j(s;\xi)|\,ds \le c|t||\xi|^{-|\mu|+1}, \quad \text{for } \xi \in \mathbb{R}^n \setminus 0,\ j = 1,\dots,m.$$

Thus we get, for every multi-index μ,

$$\left|\partial_\xi^\mu e^{i\vartheta_j(t;\xi)}\right| \le \begin{cases} c(1+|t|)^{|\mu|}, & |\xi| \ge 1, \\ c(1+|t|)^{|\mu|}|\xi|^{-|\mu|+1}, & 0 < |\xi| < 1. \end{cases} \tag{2.22}$$

Now let us go back to (2.17). It follows from (2.10) that $\partial_t \mathcal{N}$ is represented by $a'_{\nu,j}$. Hence, using (2.22) and differentiating (2.17) with respect to ξ, we conclude that μ^{th} derivative of $c_{jk}(t;\xi)$ with respect to ξ is bounded by $|\xi|^{-|\mu|}(1+|t|)^{|\mu|}\Psi(t)$ for $|\xi| \ge 1$, where $\Psi(t)$ is given in (2.20). $\partial_\xi^\mu c_{jk}(t;\xi) \in L^1(\mathbb{R})$ follows from the assumption $(1+|t|)^{|\mu|}a'_{\nu,j}(t) \in L^1(\mathbb{R})$ for all ν, j. The proof of Lemma 2.5 is complete. □

3. Representation of solution

In this section we will establish the representation formulae for solutions of the Cauchy problem (2.1) in the form of the oscillatory integrals. Let

$$V(t;\xi) = {}^T(\boldsymbol{v}_0(t;\xi),\dots,\boldsymbol{v}_{m-1}(t;\xi))$$

be the fundamental matrix of (2.6). This means that $V(0;\xi) = I$. Then it follows from Proposition 2.4 that each $\boldsymbol{v}_j(t;\xi)$ can be represented by

$$\boldsymbol{v}_j(t;\xi) = \mathcal{N}(t,\xi)^{-1}\Phi(t;\xi)\left(\boldsymbol{\alpha}_{j,\pm}(\xi) + \boldsymbol{\varepsilon}_{j,\pm}(t;\xi)\right). \tag{3.1}$$

Let $u(t,x)$ be the solution to (1.1) with the Cauchy data $D_t^k u(0,x) = f_k(x)$. Put

$$\widehat{\boldsymbol{u}}(t,\xi) = \left(|\xi|^{m-1}\widehat{u}(t,\xi),\dots,|\xi|^{m-1-l}D_t^l\widehat{u}(t,\xi),\dots,D_t^{m-1}\widehat{u}(t,\xi)\right)^T.$$

Then we can write $\widehat{\boldsymbol{u}}(t,\xi) = V(t;\xi)\widehat{\boldsymbol{u}}(0,\xi)$; thus we arrive at

$$|\xi|^{m-1-l}D_t^l\widehat{u}(t,\xi) = \sum_{j=1}^{m}\sum_{k=0}^{m-1} e^{i\vartheta_j(t;\xi)}n^{lj}(t;\xi)\left(\alpha_{j,\pm}^k(\xi) + \varepsilon_{j,\pm}^k(t;\xi)\right)|\xi|^{m-1-k}\widehat{f}_k(\xi), \tag{3.2}$$

where $n^{lj}(t;\xi)$ is the entry of $\mathcal{N}(t;\xi)^{-1}$:

$$\mathcal{N}(t;\xi)^{-1} = \left(n^{lj}(t;\xi)\right)_{\substack{l=0,\dots,m-1\\ j=1,\dots,m}}.$$

Summarizing the above argument, we have the representation formulae of (1.1)–(1.2).

Theorem 3.1. *Assume* (1.3)–(1.5). *Then there exists* $\alpha_{k,\pm}^j(\xi)$ *and* $\varepsilon_{k,\pm}^j(t;\xi)$ *such that the solution* $u(t,x)$ *of our problem* (1.1)–(1.2) *is represented by*

$$D_t^l u(t,x) = \sum_{j=1}^{m}\sum_{k=0}^{m-1}\mathcal{F}^{-1}\left[e^{i\vartheta_j(t;\xi)}n^{lj}(t;\xi)\left(\alpha_{j,\pm}^k(\xi) + \varepsilon_{j,\pm}^k(t;\xi)\right)|\xi|^{l-k}\widehat{f}_k(\xi)\right](x), \quad t \gtrless 0,$$

for $l = 0, \dots, m-1$, *where*

$$\left|\alpha^k_{j,\pm}(\xi)\right| \le c, \quad \left|\varepsilon^k_{j,\pm}(t;\xi)\right| \le c\int_{|t|}^{+\infty} \Psi(s)\,ds,$$

and $\Psi(t)$ *is defined by* (2.20) *from Lemma 2.5. For the higher order derivatives of amplitude functions, we have, for* $|\mu| \ge 1$,

$$\left|\partial^\mu_\xi \alpha^k_{j,\pm}(\xi)\right| \le c, \quad \left|\partial^\mu_\xi \varepsilon^k_{j,\pm}(t;\xi)\right| \le c\, e^{\int_0^{|t|}(1+s)^{|\mu|}\Psi(s)\,ds}, \qquad |\xi| \ge 1,$$

$$\left|\partial^\mu_\xi \alpha^k_{j,\pm}(\xi)\right| \le c|\xi|^{-|\mu|}, \quad \left|\partial^\mu_\xi \varepsilon^k_{j,\pm}(t;\xi)\right| \le c\, e^{\int_0^{|t|}(1+s)^{|\mu|}\Psi(s)\,ds}|\xi|^{-|\mu|}, \quad 0 < |\xi| < 1.$$

If in addition to (1.3)–(1.5)*, we further assume that* $(1+|t|)^{|\mu|}a'_{\nu,j}(t) \in L^1(\mathbb{R})$ *for some* μ *with* $|\mu| \ge 1$*, and for all* ν, j*, then the bound for each* $\partial^\mu_\xi \varepsilon^k_{j,\pm}(t;\xi)$ *is uniform in* t.

Proof of Theorem 3.1. We must determine the precise growth order of $\alpha^k_{j,\pm}(\xi)$ and $\varepsilon^k_{j,\pm}(t;\xi)$ with respect to ξ.

Lemma 3.2. *Assume* (1.3)–(1.5)*. Then there exists a constant* $c > 0$ *such that, for* $j = 1, \dots, m$ *and* $k = 0, \dots, m-1$,

$$\left|\alpha^k_{j,\pm}(\xi)\right| \le c, \quad \left|\varepsilon^k_{j,\pm}(t;\xi)\right| \le c\int_{|t|}^{+\infty} \Psi(s)\,ds.$$

Proof. Let us go back to (3.1). Then we have, by using Lemma 2.2,

$$|\boldsymbol{\alpha}_{j,\pm}(\xi) + \boldsymbol{\varepsilon}_{j,\pm}(t;\xi)| \le \left|\Phi(t;\xi)^{-1}\mathcal{N}(t;\xi)\boldsymbol{v}_j(t,\xi)\right| \le c_0 \tag{3.3}$$

for $j = 1, \dots, m$. Since $\boldsymbol{\varepsilon}_{j,\pm}(t;\xi)$ decays to $\mathbf{0}$, it follows that

$$|\boldsymbol{\alpha}_{j,\pm}(\xi)| \le c_0.$$

On the other hand, (2.16) is equivalent to the following integral equation:

$$\boldsymbol{\varepsilon}_{j,\pm}(t;\xi) = i\int_{|t|}^{+\infty} C(s;\xi)\left(\boldsymbol{\varepsilon}_{j,\pm}(s;\xi) + \boldsymbol{\alpha}_{j,\pm}(\xi)\right)\,ds.$$

Thus combining this equation and (3.3), we get

$$|\boldsymbol{\varepsilon}_{j,\pm}(t;\xi)| \le c_0\int_{|t|}^{+\infty} \|C(s;\xi)\|\,ds \le c\int_{|t|}^{+\infty}\Psi(s)\,ds.$$

The proof of Lemma 3.2 is finished. □

We need the estimates of higher order derivatives of amplitude functions.

Lemma 3.3. *Assume* (1.3)–(1.5)*. Then we have, for* $|\mu| \ge 1$,

$$\left|\partial^\mu_\xi \alpha^k_{j,\pm}(\xi)\right| \le c, \quad |\xi| \ge 1,$$

$$\left|\partial^\mu_\xi \alpha^k_{j,\pm}(\xi)\right| \le c|\xi|^{-|\mu|}, \quad 0 < |\xi| < 1,$$

$$\left|\partial^\mu_\xi \varepsilon^k_{j,\pm}(t;\xi)\right| \le c\, e^{\int_0^{|t|}(1+s)^{|\mu|}\Psi(s)\,ds}, \quad |\xi| \ge 1, \tag{3.4}$$

$$\left|\partial^\mu_\xi \varepsilon^k_{j,\pm}(t;\xi)\right| \le c\, e^{\int_0^{|t|}(1+s)^{|\mu|}\Psi(s)\,ds}|\xi|^{-|\mu|}, \quad 0 < |\xi| < 1. \tag{3.5}$$

In addition to (1.3)–(1.5), *if we assume that* $(1+|t|)^{|\mu|}C(t;\xi) \in L^1(\mathbb{R})$ *for some* μ *with* $|\mu| \geq 1$, *then* (3.4)–(3.5) *is uniform in* t.

Proof. Putting

$$Q(t,\xi) = (\boldsymbol{\alpha}_{0,\pm}(\xi) + \boldsymbol{\varepsilon}_{0,\pm}(t;\xi), \cdots, \boldsymbol{\alpha}_{m-1,\pm}(\xi) + \boldsymbol{\varepsilon}_{m-1,\pm}(t;\xi))$$

we see that the matrix $Q(t,\xi)$ satisfies

$$D_t Q(t,\xi) = C(t;\xi) Q(t,\xi)$$

with the initial data

$$Q(0,\xi) = \Phi(0;\xi)^{-1}\mathcal{N}(0;\xi)(\boldsymbol{v}_0(0;\xi), \cdots, \boldsymbol{v}_{m-1}(0;\xi)) = \mathcal{N}(0;\xi).$$

Then it follows from the theory of ordinary differential equations that $Q(t,\xi)$ can be written by Picard series:

$$Q(t,\xi) = \Big(I + i\int_0^t C(\tau_1;\xi)\,d\tau_1 + i^2\int_0^t C(\tau_1;\xi)\,d\tau_1\int_0^{\tau_1} C(\tau_2;\xi)\,d\tau_2 + \cdots\Big)\mathcal{N}(0;\xi). \tag{3.6}$$

We note from Lemma 2.5 that

$$\big\|\partial_\xi^\mu C(t;\xi)\big\| \leq \begin{cases} c(1+|t|)^{|\mu|}\Psi(t), & |\xi| \geq 1, \\ c(1+|t|)^{|\mu|}\Psi(t)|\xi|^{-|\mu|}, & 0 < |\xi| < 1. \end{cases} \tag{3.7}$$

where

$$\Psi(t) = \sum_{\substack{j+|\nu|=m \\ j \leq m-1}} |a'_{\nu,j}(t)| \in L^1(\mathbb{R}).$$

Differentiating (3.6) with respect to ξ, we have, by using (3.7),

$$\big|\partial_\xi^\mu\left(\boldsymbol{\alpha}_{j,\pm}(\xi) + \boldsymbol{\varepsilon}_{j,\pm}(t;\xi)\right)\big| \leq c\, e^{\int_0^{|t|}(1+s)^{|\mu|}\Psi(s)\,ds} \tag{3.8}$$

for all $t \in \mathbb{R}$, $|\xi| \geq 1$, $|\mu| \geq 1$ and $j = 0, \ldots, m-1$, where we have used the following:
Fact. *Let* $f(t) \in C(\mathbb{R})$. *Then*

$$e^{\int_s^t f(\tau)\,d\tau} = 1 + \int_s^t f(\tau_1)\,d\tau_1 + \int_s^t f(\tau_1)\,d\tau_1\int_s^{\tau_1} f(\tau_2)\,d\tau_2 + \cdots.$$

Since $\partial_t\partial_\xi^\mu Q(t,\xi) = \big(\partial_t\partial_\xi^\mu \boldsymbol{\varepsilon}^j_{k,\pm}(t;\xi)\big)$, we combine $D_tQ = CQ$ and (3.8) to deduce that $\partial_t\partial_\xi^\mu\boldsymbol{\varepsilon}^j_\pm(0;\xi)$ exists and is uniformly bounded in $|\xi| \geq 1$, which ensures the existence of $\partial_\xi^\mu\boldsymbol{\varepsilon}^j_\pm(0;\xi)$. Using again (3.8) with $t = 0$, we conclude

$$\big|\partial_\xi^\mu\boldsymbol{\alpha}_{j,\pm}(\xi)\big| \leq c, \quad |\xi| \geq 1.$$

In a similar way, we get the bound for $\partial_\xi^\mu\boldsymbol{\alpha}_{j,\pm}(\xi)$ in low frequency part $0 < |\xi| < 1$. If we combine these estimates with (3.8), we have the bound for $\partial_\xi^\mu\boldsymbol{\varepsilon}_{j,\pm}(t;\xi)$:

$$\big|\partial_\xi^\mu\boldsymbol{\varepsilon}_{j,\pm}(t;\xi)\big| \leq \begin{cases} c\, e^{\int_0^{|t|}(1+s)^{|\mu|}\Psi(s)\,ds}, & |\xi| \geq 1, \\ c\, e^{\int_0^{|t|}(1+s)^{|\mu|}\Psi(s)\,ds}|\xi|^{-|\mu|}, & 0 < |\xi| < 1. \end{cases}$$

The proof of Lemma 3.3 is complete. □

Completion of the proof of Theorem 3.1. The estimates of the amplitude and error functions have been derived in Lemmas 3.2–3.3. The proof of Theorem 3.1 is now finished. □

4. Estimates for oscillatory integrals; Proof of Theorem 1.1

The aim of this section is to establish time decay estimates for L^p–L^q norms of propagators for the Cauchy problem (1.1), which gives the proof of Theorem 1.1. The analysis of high frequencies will give estimates dependent on the geometry of the level sets of characteristic roots of the equation. For small frequencies estimates are independent of the geometry of the level set and are given by Proposition 4.1 below. We recall that Theorem 3.1 assures in particular that the solution to the Cauchy problem (1.1) is of the form

$$u(t,x) = \sum_{k=0}^{m-1}\sum_{j=1}^{m} \mathcal{F}^{-1}\left[\left(\alpha^j_{k,\pm}(\xi) + \varepsilon^j_{k,\pm}(t;\xi)\right) e^{i\vartheta_j(t;\xi)} \widehat{f}_k(\xi)\right](x), \quad t \gtrless 0. \tag{4.1}$$

The following Proposition 4.1 provides the time decay estimate for small frequencies for each of the terms in this sum. To simplify the notation, we formulate it in a more general form for general oscillatory integrals of the form

$$T_t f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \vartheta(t;\xi))} a(t,\xi)\widehat{f}(\xi)\, d\xi.$$

In the analysis of oscillatory integrals in the sum (4.1) we will actually make time-dependent cut-offs and analyse separately different ranges of frequencies. We can obtain the following proposition for small frequencies $|\xi| \le t^{-1}$. Higher frequencies $|\xi| \ge t^{-1}$ will be analysed later. Thus, we introduce a cut-off function of the form $\psi((1+|t|)\xi)$ for some $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi(\xi) \equiv 1$ for $|\xi| \le \frac{1}{2}$, and 0 for $|\xi| \ge 1$. We recall that we use the notation $\dot{L}^p_\kappa(\mathbb{R}^n)$ for the homogeneous Sobolev space $\dot{W}^\kappa_p(\mathbb{R}^n)$.

Proposition 4.1. *Let T_t, $t \in \mathbb{R}$, be an operator defined by*

$$T_t f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \vartheta(t;\xi))} \psi((1+|t|)\xi) a(t,\xi)\widehat{f}(\xi)\, d\xi,$$

where $\vartheta(t;\xi)$ is real valued, positively homogeneous of order one in ξ. Assume that the amplitude $a(t,\xi)$ satisfies

$$|\xi|^\kappa |a(t,\xi)| \le C \tag{4.2}$$

for some $\kappa \in \mathbb{R}$, and for all $t \in \mathbb{R}$ and all $\xi \in \operatorname{supp}\psi((1+|t|)\xi)$. Let $1 \le p \le 2 \le q \le +\infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $t \in \mathbb{R}$ we have the estimate

$$\|T_t f\|_{L^q(\mathbb{R}^n)} \le C(1+|t|)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{\dot{L}^p_{-\kappa}(\mathbb{R}^n)}, \tag{4.3}$$

where constant C depends on n,p,q and the norm $\||\xi|^\kappa a\|_{L^\infty}$.

Proof. We easily obtain

$$\|T_t f\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2_{-\kappa}(\mathbb{R}^n)}$$

by the Plancherel identity. Thus (4.3) would follow by analytic interpolation from an estimate:

$$\|T_t f\|_{L^\infty(\mathbb{R}^n)} \le C(1+|t|)^{-n}\|f\|_{\dot{L}^1_{-\kappa}(\mathbb{R}^n)}. \tag{4.4}$$

In fact, since $|a(t,\xi)||\widehat{f}(\xi)| \le C\||\xi|^\kappa a\|_{L^\infty}\|f\|_{L^1_{-\kappa}(\mathbb{R}^n)}$, we can estimate

$$\begin{aligned}\|T_t f\|_{L^\infty(\mathbb{R}^n)} \le& \int_{|\xi|\le(1+|t|)^{-1}} |a(t,\xi)||\widehat{f}(\xi)|\,d\xi\\ \le& C\left(\int_{|\xi|\le(1+|t|)^{-1}} d\xi\right)\|f\|_{L^1_{-\kappa}(\mathbb{R}^n)} \le C(1+|t|)^{-n}\|f\|_{\dot{L}^1_{-\kappa}(\mathbb{R}^n)},\end{aligned}$$

for all $t \in \mathbb{R}$. This proves (4.4). The proof of Proposition 4.1 is complete. □

We now turn to the analysis of larger frequencies. The following proposition provides the necessary background to obtain the time decay estimate for large frequencies for each of the terms in the sum (4.1). In fact, it will be used for frequencies $|\xi| \ge 1$ but will be formulated here in a slightly more general form. The relation with the sum (4.1) and a refinement for low frequencies will be also made in Proposition 4.6.

Proposition 4.2. *Let T_t, $t > 0$, be an operator defined by*

$$T_t f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\vartheta(t;\xi))} a(t,\xi)\widehat{f}(\xi)\,d\xi,$$

where $\vartheta(t;\xi)$ is real valued, continuous in t, smooth in $\xi \in \mathbb{R}^n\backslash 0$, homogeneous of order one in ξ. Assume that the set

$$\Sigma_\varphi = \{\xi \in \mathbb{R}^n\backslash 0 : \varphi(\xi) = 1\}$$

is strictly convex and let $\gamma = \gamma(\Sigma_\varphi)$ be the convex Sugimoto index of Σ_φ, as defined in (1.10)*. Suppose that*

$$|\vartheta(t;\xi)| \le C(1+t)|\xi| \quad \textit{for all} \quad t > 0,\, \xi \in \mathbb{R}^n,$$

and that there is some $\varphi \in C^\infty(\mathbb{R}^n\backslash 0)$, $\varphi > 0$, such that

$$t^{-1}\partial_\xi^\alpha\vartheta(t;\xi) \to \partial_\xi^\alpha\varphi(\xi) \ \textit{as}\ t\to\infty,\ \textit{for all}\ \ \xi\in\mathbb{R}^n\backslash 0,\ |\alpha|\le\gamma. \tag{4.5}$$

Assume also that the amplitude $a(t,\xi)$ satisfies

$$|\partial_\xi^\alpha a(t,\xi)| \le C_\alpha \quad \textit{for all} \quad |\alpha| \le [(n-1)/\gamma]+1. \tag{4.6}$$

Let $1 < p \le 2 \le q < \infty$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Then for $t > 0$ we have the estimate

$$\|T_t f\|_{L^q(\mathbb{R}^n)} \le Ct^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p_{N_p}(\mathbb{R}^n)}, \tag{4.7}$$

where $N_p = \left(n-\frac{n-1}{\gamma}+\left[\frac{n-1}{\gamma}\right]+1\right)\left(\frac{1}{p}-\frac{1}{q}\right)$.

The number of derivatives $N_p = \left(n-\frac{n-1}{\gamma}+\left[\frac{n-1}{\gamma}\right]+1\right)\left(\frac{1}{p}-\frac{1}{q}\right)$ required for the estimate (4.7) is determined by the fact that the amplitude $a(t,\xi)$ in (4.6) is in the symbol class $S^0_{0,0}$ rather than the usual $S^0_{1,0}$. In fact, if $a(t,\xi)$ satisfies inequalities

$$|\partial_\xi^\alpha a(t,\xi)| \le C_\alpha\langle\xi\rangle^{-|\alpha|} \quad \text{for all} \quad |\alpha| \le \left[\frac{n-1}{\gamma}\right]+1, \tag{4.8}$$

then we can take the Sobolev index $N_p = \left(n - \frac{n-1}{\gamma}\right)\left(\frac{1}{p} - \frac{1}{q}\right)$ for the estimate (4.7) to hold. However, the method of asymptotic integration and the statement of Theorem 3.1 forces us to assume (4.6) rather than (4.8).

Let us now discuss other assumptions we make in this proposition from the point of view of the original Cauchy problem (1.1). We recall from (2.13) that functions $\varphi_k^{\pm}(\xi)$ are characteristics of the limiting strictly hyperbolic operator

$$L^{\pm}(D_t, D_x)u \equiv D_t^m u + \sum_{\substack{|\nu|+j=m \\ j\leq m-1}} a_{\nu,j}^{\pm} D_x^{\nu} D_t^j u, \tag{4.9}$$

and their geometric properties are responsible for the time decay of solutions to the Cauchy problems for both operators $L(t, D_t, D_x)$ and $L^{\pm}(D_t, D_x)$. The fact that $\varphi_k^{\pm}(\xi)$ are characteristics of (4.9), implies that they are real analytic for $\xi \neq 0$ and that we have the following statement, which was established for operators with constant coefficients by Sugimoto [24].

Proposition 4.3. *Let $\varphi_k(\xi)$, $k = 1, \dots, m$, be characteristics of operator* (4.9)*, ordered by $\varphi_1(\xi) > \varphi_2(\xi) > \cdots > \varphi_m(\xi)$ for $\xi \neq 0$. Suppose that all the Hessians $\varphi_k''(\xi)$ are semi-definite for $\xi \neq 0$. Then there exists a polynomial $\alpha(\xi)$ of order one such that $\varphi_{m/2}(\xi) > \alpha(\xi) > \varphi_{m/2+1}$ (if m is even) or $\alpha(\xi) = \varphi_{(m+1)/2}(\xi)$ (if m is odd). Moreover, the hypersurfaces $\Sigma_k = \{\xi \in \mathbb{R}^n;\ \widetilde{\varphi}_k = \pm 1\}$ with $\widetilde{\varphi}_k(\xi) = \varphi_k(\xi) - \alpha(\xi)$ $(k \neq (m+1)/2)$ are convex and $\gamma(\Sigma_k) \leq 2[m/2]$.*

In particular, in our arguments we can replace φ_k by $\widetilde{\varphi}_k$ since the addition of a linear function does not change the decay rate nor the index $\gamma(\Sigma_{\varphi_k})$. This also ensures that the limiting phase φ in Proposition 4.2 may be taken to be strictly positive. Indeed, it can be taken to be nonzero, and if it is strictly negative we simply replace φ by $-\varphi$. Moreover, the assumption that Σ_φ is strictly convex in Proposition 4.2 can be replaced by the assumption that it is only convex. Indeed, since $\varphi(\xi)$ is a characteristic root of (4.9), it is real analytic for $\xi \neq 0$. Then, the convexity, the real analyticity and the compactness imply that it is actually strictly convex. In particular, it also implies that $\gamma(\Sigma_\varphi)$ is *finite and even.*

In the case when the level set Σ_φ in Proposition 4.2 is not convex, we get a weaker result based on the one-dimensional van der Corput lemma. In this case we use the non-convex Sugimoto index of Σ_φ which was defined in (1.11) in the introduction.

Proposition 4.4. *Let T_t, $t > 0$, be an operator defined by*

$$T_t f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \vartheta(t;\xi))} a(t,\xi)\widehat{f}(\xi)\, d\xi,$$

where $\vartheta(t;\xi)$ is real valued, continuous in t, smooth in $\xi \in \mathbb{R}^n\backslash 0$, homogeneous of order one in ξ. Let $\gamma_0 = \gamma_0(\Sigma_\varphi)$ be the non-convex Sugimoto index of the level surface $\Sigma_\varphi = \{\xi \in \mathbb{R}^n\backslash 0 : \varphi(\xi) = 1\}$. Suppose that

$$|\vartheta(t;\xi)| \leq C(1+t)|\xi| \quad \textit{for all} \quad t > 0,\, \xi \in \mathbb{R}^n,$$

and that there is some $\varphi \in C^\infty(\mathbb{R}^n\backslash 0)$, $\varphi > 0$, such that

$$t^{-1}\partial_\xi^\alpha \vartheta(t;\xi) \to \partial_\xi^\alpha \varphi(\xi) \quad \textit{as} \quad t \to \infty, \quad \textit{for all} \quad \xi \in \mathbb{R}^n,\ |\alpha| \leq \gamma_0.$$

Assume also that the amplitude $a(t,\xi)$ *satisfies*

$$|\partial_\xi^\alpha a(t,\xi)| \le C_\alpha \quad \textit{for all} \quad |\alpha| \le 1. \tag{4.10}$$

Let $1 < p \le 2 \le q < +\infty$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then for* $t > 0$ *we have the estimate*

$$\|T_t f\|_{L^q(\mathbb{R}^n)} \le C t^{-\frac{1}{\gamma_0}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p_{N_p}(\mathbb{R}^n)}, \tag{4.11}$$

where $N_p = \left(n - \frac{1}{\gamma_0} + 1\right)\left(\frac{1}{p} - \frac{1}{q}\right)$.

We will first prove Proposition 4.2 and then indicate the changes necessary for the proof of Proposition 4.4.

Proof of Proposition 4.2. First we observe that $\|T_t f\|_{L^2} \le C\|f\|_{L^2}$ by the Plancherel identity. In order to simplify the proof somewhat, we will absorb the Sobolev index N_p into the amplitude $a(t,\xi)$, so that we will estimate $||T_t f||_{L^q}$ in terms of $||f||_{L^p}$, and not in terms of $||f||_{L^p_{N_p}}$. Thus, instead of (4.6), from now on we will assume that the amplitude $a(t,\xi)$ satisfies

$$|\partial_\xi^\alpha a(t,\xi)| \le C_\alpha \langle\xi\rangle^{-k}$$

for all $|\alpha| \le [(n-1)/\gamma] + 1$ and $k = N_p$, so that estimate (4.7) would follow by interpolation from the estimate

$$\|T_t f\|_{L^\infty(\mathbb{R}^n)} \le C t^{-\frac{n-1}{\gamma}} \|f\|_{L^1(\mathbb{R}^n)}, \tag{4.12}$$

and where we take $k = N_1 = n - \frac{n-1}{\gamma} + \left[\frac{n-1}{\gamma}\right] + 1$. Note that since we assume that the amplitude is bounded for small frequencies, we can work with standard Sobolev spaces here. By using Besov spaces, we can microlocalise the desired estimate to discs in the frequency space. Indeed, let $\{\Phi_j\}_{j=0}^\infty$ be the Littlewood-Paley partition of unity, and let

$$\|u\|_{B^s_{p,q}} = \left(\sum_{j=0}^\infty \left(2^{js}\|\mathcal{F}^{-1}\Phi_j(\xi)\mathcal{F}u\|_{L^q(\mathbb{R}^n)}\right)^q\right)^{1/p}$$

be the norm of the Besov space $B^s_{p,q}$. Then, because of the continuous embeddings $L^p \subset B^0_{p,2}$ for $1 < p \le 2$, and $B^0_{q,2} \subset L^q$ for $2 \le q < +\infty$ (see [3]), it is sufficient to prove the uniform estimate for the operators with amplitudes $a(t,\xi)\Phi_j(\xi)$. Let us denote

$$\widetilde{\vartheta}(t,\xi) = t^{-1}\vartheta(t,\xi),$$

so that by the assumption we have $\widetilde{\vartheta}(t;\xi) \to \varphi(\xi)$ as $t \to \infty$. Now, writing

$$\Phi_j(\xi) = \Phi_j(\xi)\Psi\left(\frac{\widetilde{\vartheta}(t;\xi)}{2^j}\right)$$

with some function $\Psi \in C_0^\infty(0,\infty)$, we may prove the uniform estimate for operators with amplitudes $a(t,\xi)\Psi\left(\frac{\widetilde{\vartheta}(t;\xi)}{2^j}\right)$. Such choice of Ψ is possible due to our assumption that $\widetilde{\vartheta}(t;\xi) \to \varphi(\xi)$ as $t \to \infty$, and we restrict the analysis for large enough t. Let

$$I(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\vartheta(t;\xi))} a(t,\xi) \Psi\left(\frac{\widetilde{\vartheta}(t;\xi)}{2^j}\right) d\xi \tag{4.13}$$

be the kernel of the corresponding operator. Since we easily have the L^2–L^2 estimate by the Plancherel identity, by analytic interpolation we only need to prove the L^1–L^∞ case of (4.12). In turn, this follows from the estimate $|I(t,x)| \le Ct^{-\frac{n-1}{\gamma}}$, with constant C independent of j.

Let $\kappa \in C_0^\infty(\mathbb{R}^n)$ be supported in a ball with some radius $r > 0$ centred at the origin. We split the integral in

$$\begin{aligned} I(t,x) &= I_1(t,x) + I_2(t,x) \\ &= \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\vartheta(t;\xi))} a(t,\xi) \kappa\left(t^{-1}x + t^{-1}\nabla_\xi \vartheta(t;\xi)\right) \Psi\left(\frac{\widetilde{\vartheta}(t;\xi)}{2^j}\right) d\xi \\ &\quad + \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\vartheta(t;\xi))} a(t,\xi)(1-\kappa)\left(t^{-1}x + t^{-1}\nabla_\xi \vartheta(t;\xi)\right) \Psi\left(\frac{\widetilde{\vartheta}(t;\xi)}{2^j}\right) d\xi. \end{aligned}$$

We can easily see that $|I_2(t,x)| \le Ct^{-\frac{n-1}{\gamma}}$. In fact, we can show $|I_2(t,x)| \le Ct^{-l}$ for $l = [(n-1)/\gamma] + 1$ and then the required estimate simply follows since $l > (n-1)/\gamma$. Indeed, on the support of $1-\kappa$, we have $|x + \nabla_\xi \vartheta(t;\xi)| \ge rt > 0$. Thus, integrating by parts with operator $P = \frac{x+\nabla_\xi \vartheta(t;\xi)}{i|x+\nabla_\xi \vartheta(t;\xi)|^2} \cdot \nabla_\xi$, we get

$$\begin{aligned} &I_2(t,x) = \\ &\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\vartheta(t;\xi))} (P^*)^l \left[a(t,\xi)(1-\kappa)\left(t^{-1}x + t^{-1}\nabla_\xi \vartheta(t;\xi)\right) \Psi\left(\frac{\widetilde{\vartheta}(t;\xi)}{2^j}\right)\right] d\xi. \end{aligned} \tag{4.14}$$

Using the fact that $|\partial_\xi^\alpha \vartheta(t;\xi)| \le C(1+t)|\xi|^{1-|\alpha|}$, we readily observe from (4.14) that the required estimate $|I_2(t,x)| \le Ct^{-l}$ holds. Here we also used the condition (4.6) which assures that we can perform the integration by parts $[(n-1)/\gamma] + 1$ times.

Now we will turn to estimating $I_1(t,x)$. Recall that $\widetilde{\vartheta}(t;\xi) = t^{-1}\vartheta(t;\xi)$ and $\widetilde{\vartheta}(t;\xi) \to \varphi(\xi)$ as $t \to \infty$. Let us denote

$$\Sigma^t = \{\xi \in \mathbb{R}^n : \widetilde{\vartheta}(t;\xi) = 1\}.$$

It can be readily checked that $\gamma(\Sigma^t) \to \gamma(\Sigma_\varphi) = \gamma$ as $t \to \infty$. So we can restrict our attention to t large enough for which we have $\gamma(\Sigma^t) = \gamma$. By rotation, we can always microlocalise in some narrow cone around $e_n = (0,\dots,0,1)$ and in this cone we can parameterise

$$\Sigma^t = \{(y, h_t(y)) : y \in U\}$$

for some open $U \subset \mathbb{R}^{n-1}$. In other words, we have $\widetilde{\vartheta}(t; y, h_t(y)) = 1$, and it follows that h_t is smooth and $\nabla h_t : U \to \nabla h_t(U) \subset \mathbb{R}^{n-1}$ is a homeomorphism. The function h_t is concave if Σ^t is convex. We claim that

$$|\partial_y^\alpha h_t(y)| \le C_\alpha, \quad \text{for all} \quad y \in U \quad \text{and large enough} \quad t. \tag{4.15}$$

Indeed, let us look at $|\alpha|=1$ first. From $\widetilde{\vartheta}(t;y,h_t(y))=1$ we get that

$$\nabla_y\widetilde{\vartheta}+\partial_{\xi_n}\widetilde{\vartheta}\cdot\nabla h_t(y)=0.$$

From homogeneity we have $|\nabla_\xi\widetilde{\vartheta}|\leq C$, so also $|\nabla_y\widetilde{\vartheta}|\leq C$. By Euler's identity we have

$$\partial_{\xi_n}\widetilde{\vartheta}(t;e_n)=\widetilde{\vartheta}(t;e_n)\to\varphi(e_n)>0 \text{ as } t\to\infty, \tag{4.16}$$

so we have $|\partial_{\xi_n}\widetilde{\vartheta}|\geq c>0$ since we are in a narrow cone around e_n. From this it follows that $|\nabla_y h_t(y)|\leq C$ for all $y\in U$ and t large enough. A similar argument proves the boundedness of higher order derivatives in (4.15).

Now, let us turn to analyse the structure of the sets Σ^t. We have the Gauss map

$$\nu:\Sigma^t\ni\zeta\mapsto\frac{\nabla_\zeta\widetilde{\vartheta}(t;\zeta)}{|\nabla_\zeta\widetilde{\vartheta}(t;\zeta)|}\in\mathbb{S}^{n-1},$$

and for $x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$ near the point $-\nabla_\zeta\widetilde{\vartheta}(t;e_n)$ we define $z_t\in U$ by

$$(z_t,h_t(z_t))=\nu^{-1}(-x/|x|).$$

Then $(-\nabla_y h_t(y),1)$ is normal to Σ^t at $(y,h_t(y))$, so we get

$$-\frac{x}{|x|}=\frac{(-\nabla_y h_t(z_t),1)}{|(-\nabla_y h_t(z_t),1)|} \quad\text{and}\quad \frac{x'}{x_n}=-\nabla_y h_t(z_t).$$

Making change of variables $\xi=(\widetilde{\lambda}y,\widetilde{\lambda}h_t(y))$ and using $\widetilde{\vartheta}(t;\xi)=\widetilde{\lambda}$, we get

$$\begin{aligned}
I_1(t,x) &= \int_0^\infty\int_U e^{i\widetilde{\lambda}(x'\cdot y+x_nh_t(y)+t)}a(t,\widetilde{\lambda}y,\widetilde{\lambda}h_t(y))\Psi\left(\frac{\widetilde{\lambda}}{2^j}\right)\kappa_0(t,x,y)\left|\frac{d\xi}{d(\widetilde{\lambda},y)}\right|dyd\widetilde{\lambda}\\
&= \int_0^\infty\int_U e^{i\widetilde{\lambda}(-x_n\nabla_yh_t(z_t)\cdot y+x_nh_t(y)+t)}\left[\widetilde{\lambda}^l a(t,\widetilde{\lambda}y,\widetilde{\lambda}h_t(y))\right]\widetilde{\lambda}^{n-1-l}\times\\
&\qquad\times\Psi\left(\frac{\widetilde{\lambda}}{2^j}\right)\kappa_0(t,x,y)\chi(t,y)\,dyd\widetilde{\lambda}\\
&= \int_0^\infty\int_U e^{i\lambda(-\nabla_yh_t(z_t)\cdot y+h_t(y)+tx_n^{-1})}\widetilde{a}(t,x_n,\lambda y,\lambda h_t(y))\lambda^{n-1-l}\times\\
&\qquad\times\Psi\left(\frac{\lambda}{2^jx_n}\right)x_n^{-n+1+l-1}\kappa_0(t,x,y)\chi(t,y)\,dyd\lambda,
\end{aligned}\tag{4.17}$$

where $\kappa_0(t,x,y)=\kappa\left(t^{-1}x+\nabla_\xi\widetilde{\vartheta}(t;y,h_t(y))\right)$, and

$$\widetilde{a}(t,x_n,\lambda y,\lambda h_t(y))=(x_n^{-1}\lambda)^l a\left(t,x_n^{-1}\lambda y,x_n^{-1}\lambda h_t(y)\right),$$

and where we made a change $\widetilde{\lambda}=x_n^{-1}\lambda$ in the last equality. Here also we used $\left|\frac{d\xi}{d(\widetilde{\lambda},y)}\right|=\widetilde{\lambda}^{n-1}\chi(t,y)$, where $\chi(t,y)$ and all of its derivatives with respect to y are bounded because of (4.15).

If we choose r in the definition of the cut-off function κ sufficiently small, then on its support we have $|x| \approx |x_n| \approx t$, and we can estimate

$$\begin{aligned} |I_1(t,x)| &\leq Ct^{-n+l} \int_0^\infty \left| J(\lambda, z_t)\Psi\left(\frac{\lambda}{2^j t}\right)\lambda^{n-1-l}\right| d\lambda \\ &= Ct^{-n+l}2^{j(n-l)} \int_0^\infty \left| J(2^j\lambda, z_t)\Psi\left(\frac{\lambda}{t}\right)\lambda^{n-1-l}\right| d\lambda \end{aligned} \tag{4.18}$$

with

$$J(\lambda, z_t) = \int_U e^{i\lambda(-\nabla_y h_t(z_t)\cdot y + h_t(y) + tx_n^{-1})}\widetilde{a}(t, x_n, \lambda y, \lambda h_t(y))\kappa_0(t,x,y)\chi(t,y)\,dy.$$

We will show that

$$|J(\lambda, z_t)| \leq C(1+\lambda)^{-\frac{n-1}{\gamma}}, \quad \lambda > 0. \tag{4.19}$$

Then, if we take $l = n - \frac{n-1}{\gamma}$, and use (4.18) and (4.19), we get

$$\begin{aligned} |I_1(t,x)| &\leq Ct^{-\frac{n-1}{\gamma}}2^{j\frac{n-1}{\gamma}} \int_0^\infty (2^j\lambda)^{-\frac{n-1}{\gamma}}\Psi\left(\frac{\lambda}{t}\right)\lambda^{\frac{n-1}{\gamma}-1}\,d\lambda \\ &= Ct^{-\frac{n-1}{\gamma}} \int_0^\infty \lambda^{-1}\Psi\left(\frac{\lambda}{t}\right)d\lambda = Ct^{-\frac{n-1}{\gamma}} \int_0^\infty \lambda^{-1}\Psi(\lambda)\,d\lambda \\ &\leq Ct^{-\frac{n-1}{\gamma}}, \end{aligned} \tag{4.20}$$

which is the desired estimate for $I_1(t,x)$.

Let us now prove (4.19). It will, in turn, follow from Theorem 4.5 below. First of all we note that since we assumed that $|\partial_\xi^\alpha a(t,\xi)| \leq C_\alpha \langle\xi\rangle^{-N_1}$ for all $|\alpha| \leq [(n-1)/\gamma]+1$ and $N_1 = n - \frac{n-1}{\gamma} + \left[\frac{n-1}{\gamma}\right] + 1$, we get that

$$|\partial_y^\alpha \widetilde{a}| \leq C \quad \text{for all} \quad |\alpha| \leq [(n-1)/\gamma]+1. \tag{4.21}$$

To write $J(\lambda, z_t)$ in a suitable form, we change to polar coordinates (ρ, ω) with $y = \rho\omega + z_t$, so that

$$J(\lambda, z_t) = \int_{\mathbb{S}^{n-2}} \int_0^\infty e^{i\lambda F(\rho, z_t, \omega)}\beta(\rho, z_t, \omega)\rho^{n-2}\,d\rho d\omega, \tag{4.22}$$

with

$$F(\rho, z_t, \omega) = h_t(\rho\omega + z_t) - h_t(z_t) - \rho\nabla_y h_t(z_t)\cdot\omega, \tag{4.23}$$

$$\beta(\rho, z_t, \omega) = \widetilde{a}\left(t, x_n, \lambda(\rho\omega + z_t), \lambda h_t(\rho\omega + z_t)\right)\kappa_0(t, x, \rho\omega + z_t)\chi(t, \rho\omega + z_t), \tag{4.24}$$

where we can assume in addition that $\chi = 0$ unless $\rho\omega + z_t \in U$, so both ρ and ω vary over bounded sets.

Now we can apply the following result, which has appeared in [20] for more general complex valued phases Φ, thus including the real-valued case of the phase function F in (4.23). The estimate (4.19) follows from the following theorem with $N = n-1$.

Theorem 4.5 ([20]). *Consider the oscillatory integral*

$$I(\lambda, \nu) = \int_{\mathbb{R}^N} e^{i\lambda\Phi(x,\nu)}a(x,\nu)\chi(x)\,dx\,,$$

where $N \geq 1$, and ν is a parameter. Let $\gamma \geq 2$ be an integer. Assume that

(A1) *there exists a sufficiently small* $\delta > 0$ *such that* $\chi \in C_0^\infty(B_{\delta/2}(0))$, *where* $B_{\delta/2}(0)$ *is the ball with radius* $\delta/2$ *around* 0;

(A2) $\Phi(x,\nu)$ *is a complex valued function such that* $\mathrm{Im}\Phi(x,\nu) \geq 0$ *for all* $x \in \operatorname{supp}\chi$ *and all parameters* ν;

(A3) *for some fixed* $z \in \operatorname{supp}\chi$, *the function*

$$F(\rho,\omega,\nu) := \Phi(z+\rho\omega,\nu),\ |\omega| = 1,$$

satisfies the following conditions. Assume that for each $\mu = (\omega,\nu)$, *function* $F(\cdot,\mu)$ *is of class* $C^{\gamma+1}$ *on* $\operatorname{supp}\chi$, *and let us write its* γ^{th} *order Taylor expansion in* ρ *at* 0 *as*

$$F(\rho,\mu) = \sum_{j=0}^{\gamma} a_j(\mu)\rho^j + R_{\gamma+1}(\rho,\mu)\,,$$

where $R_{\gamma+1}$ *is the remainder term. Assume that we have*

(F1) $a_0(\mu) = a_1(\mu) = 0$ *for all* μ;

(F2) *there exists a constant* $C > 0$ *such that* $\sum_{j=2}^{\gamma} |a_j(\mu)| \geq C$ *for all* μ;

(F3) *for each* μ, $|\partial_\rho F(\rho,\mu)|$ *is increasing in* ρ *for* $0 < \rho < \delta$;

(F4) *for each* $k \leq \gamma+1$, $\partial_\rho^k F(\rho,\mu)$ *is bounded uniformly in* $0 < \rho < \delta$ *and* μ;

(A4) *for each multi-index* α *of length* $|\alpha| \leq \left[\frac{N}{\gamma}\right] + 1$, *there exists a constant* $C_\alpha > 0$ *such that* $|\partial_x^\alpha a(x,\nu)| \leq C_\alpha$ *for all* $x \in \operatorname{supp}\chi$ *and all parameters* ν.

Then there exists a constant $C = C_{N,\gamma} > 0$ *such that*

$$|I(\lambda,\nu)| \leq C(1+\lambda)^{-\frac{N}{\gamma}} \quad \text{for all } \lambda \in [0,\infty) \text{ and all parameters } \nu. \tag{4.25}$$

We refer to [20] and [22] for details. Now, the function F in (4.23) satisfies condition (A3) of Theorem 4.5 because of the definition of the convex Sugimoto index γ and because h_t is concave. Since $\partial_y^\alpha h_t$, $|\alpha| \leq \gamma$, can be expressed via $\partial_\xi^\alpha \varphi$, $|\alpha| \leq \gamma$, and since we have (4.16), it also follows from (2.9) and (4.5) that function F satisfies property (F2) of Theorem 4.5. The proof of Proposition 4.2 is now complete. □

Let us now show that we can actually also insert the cut-off $1 - \psi((1+|t|)\xi)$ in Proposition 4.2 which is necessary for the analysis of the representation (4.1). Here $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi(\xi) \equiv 1$ for $|\xi| \leq \frac{1}{2}$, and 0 for $|\xi| \geq 1$. The case of high frequencies $|\xi| \geq 1$ (for solutions) is covered by Proposition 4.2, and the proof about the insertion of $1 - \psi((1+|t|)\xi)$ is similar to the proof of the following Proposition 4.6. So we now restrict to $t^{-1} < |\xi| \leq 1$, since the case $|\xi| < t^{-1}$ was covered in Proposition 4.1.

Proposition 4.6. *Let* T_t, $t \neq 0$, *be an operator defined by*

$$T_t f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \vartheta(t;\xi))}[1 - \psi((1+|t|)\xi)]a(t,\xi)\widehat{f}(\xi)\,d\xi, \tag{4.26}$$

where $\vartheta(t;\xi)$, *and* γ *are as in Proposition 4.2. Assume that the amplitude* $a(t,\xi)$ *satisfies* $a(t,\xi) = 0$ *for all* $|\xi| \geq 1$ *and that*

$$|\partial_\xi^\alpha a(t,\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \quad \text{for all} \quad |\alpha| \leq [(n-1)/\gamma] + 1.$$

Let $1 < p \le 2 \le q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $t \ne 0$ we have the estimate

$$\|T_t f\|_{L^q(\mathbb{R}^n)} \le C|t|^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{\dot{L}^p_{N_p}(\mathbb{R}^n)}, \tag{4.27}$$

where $N_p = \left(n - \frac{n-1}{\gamma} + \left[\frac{n-1}{\gamma}\right] + 1\right)\left(\frac{1}{p} - \frac{1}{q}\right)$.

Proof. The proof of this proposition is almost the same as the proof of Proposition 4.2 with several differences that we will point out here. Again, by interpolation, it is sufficient to prove estimate

$$\|T_t f\|_{L^\infty(\mathbb{R}^n)} \le C|t|^{-\frac{n-1}{\gamma}} \|f\|_{L^1(\mathbb{R}^n)},$$

with amplitude $a(t, \xi)$ satisfying

$$|\partial_\xi^\alpha a(t, \xi)| \le C_\alpha |\xi|^{-N_1 - |\alpha|} \quad \text{for all} \quad |\alpha| \le [(n-1)/\gamma] + 1, \tag{4.28}$$

with $N_1 = n - \frac{n-1}{\gamma} + \left[\frac{n-1}{\gamma}\right] + 1$.

Further differences concern estimates for $I_1(t, x)$ and $I_2(t, x)$. In general, since we work with low frequencies $|\xi| < 1$ only, no Besov space decomposition is necessary, so we do not need to introduce function Ψ and Φ_j, so we can take $\Psi = 1$.

Some additional complications are related to the fact that in principle derivatives of the amplitudes of operators T_t from (4.26) may introduce an additional growth with respect to t. In the estimate for $I_2(t, x)$ we performed integration by parts with operator P. Now after integration by parts the amplitude of this integral in (4.14) is

$$(P^*)^l \left[[1 - \psi((1 + |t|)\xi)] a(t, \xi)(1 - \kappa) \left(t^{-1}x + t^{-1}\nabla_\xi \vartheta(t; \xi)\right) \Psi\left(\frac{\widetilde{\vartheta}(t; \xi)}{2^j}\right) \right].$$

Now, if any of the ξ-derivatives falls on $[1 - \psi((1 + |t|)\xi)]$, we get an extra factor t which is cancelled with t^{-1} in the definition of P. However, in this case we can then restrict to the support of $\nabla\psi$ which is contained in the ball with radius $(1 + |t|)^{-1}$, so we are in the situation of low frequencies $|\xi| \le t^{-1}$ again. Consequently, we can apply Proposition 4.1 to this integral to actually get a better decay rate of Proposition 4.1. If none of the derivatives in $(P^*)^l$ fall on $[1 - \psi((1 + |t|)\xi)]$, the argument is the same as in the proof of the estimate for $I_2(t, x)$ in Proposition 4.2.

The other main difference with the proof of Proposition 4.2 is in the estimate for $I_1(t, x)$. Recall now that in formula (4.17) we made a change of variables $\widetilde{\lambda} = x_n^{-1}\lambda$. As it was then pointed out, if r in the definition of the cut-off function κ is chosen sufficiently small, on its support we have $|x_n| \approx |t|$. On the other hand, we have $|\xi| \approx \widetilde{\lambda}$ by the definition of $\widetilde{\lambda}$, since we assume that the limiting phase function φ is strictly positive. It then follows that $(1 + |t|)\xi \approx \widetilde{\lambda}|x_n| \approx \lambda$, and so the change of variables $\widetilde{\lambda} = x_n^{-1}\lambda$ changes $[1 - \psi((1 + |t|)\xi)]$ into $[1 - \psi(\lambda)]$ in the amplitude of $I_1(t, x)$. Justifying this argument, we can then continue as in the proof of Proposition 4.2. The crucial condition for the use of Theorem 4.5 is the boundedness of derivatives of $\widetilde{a}$ in (4.21). Here, every differentiation of a with respect to y introduces a factor $x_n^{-1}\lambda$ which is then cancelled in view of assumption (4.28). It follows that $[(n-1)/\gamma]+1$ y-derivatives of $\widetilde{a}$ are bounded, implying the conclusion of Theorem 4.5. This yields estimate (4.27) in the way that is similar to the proof of Proposition 4.2. □

Let us now turn to prove Proposition 4.4.

Proof of Proposition 4.4. Let us show how the proof of Proposition 4.4 differs from the proof of Proposition 4.2. We need to prove that $|I(t,x)| \le Ct^{-\frac{1}{\gamma_0}}$, $t>0$, for $I(t,x)$ as in (4.13). We note that $\gamma_0 \ge 1$, so to prove the estimate for $I_2(t,x)$ we can show that $|I_2(t,x)| \le Ct^{-1}$. This can be done by integrating by parts with the same operator P and using (4.10) instead of (4.6). As for the proof of the estimate for $I_1(t,x)$, we can reason in the same way as in Proposition 4.2 to arrive at the estimate (4.18), i.e.,

$$|I_1(t,x)| \le Ct^{-n+l}2^{j(n-l)} \int_0^\infty \left| J(2^j\lambda, z_t)\Psi\left(\frac{\lambda}{t}\right)\lambda^{n-1-l}\right| d\lambda$$

with the same operator

$$J(\lambda, z_t) = \int_U e^{i\lambda(-\nabla_y h_t(z_t)\cdot y + h_t(y) + tx_n^{-1})}\widetilde{a}(t, x_n, \lambda y, \lambda h_t(y))\kappa_0(t,x,y)\chi(t,y)\,dy.$$

Now, instead of (4.19) we will show that

$$|J(\lambda, z_t)| \le C\lambda^{-\frac{1}{\gamma_0}}, \quad \lambda > 0. \tag{4.29}$$

Then, taking $l = n - \frac{1}{\gamma_0}$, we get the estimate $|I_1(t,x)| \le Ct^{-\frac{1}{\gamma_0}}$ in the same way as in estimate (4.20). Now, estimate (4.29) follows from Theorem 4.5 with $N=1$. Indeed, let us write $J(\lambda, z_t)$ in the form (4.22)–(4.24) with phase

$$F(\rho, z_t, \omega) = h_t(\rho\omega + z_t) - h_t(z_t) - \rho\nabla_z h_t(z_t)\cdot\omega.$$

Now, by rotation we may assume that in some direction, say $e_1 = (1, 0, \dots, 0)$, we have by definition of the index γ_0 that

$$\gamma_0 = \min\left\{k \in \mathbb{N} : \partial_{\omega_1}^k F(\rho, z_t, \omega)|_{\omega_1=0} \ne 0\right\}.$$

Then by taking $N=1$ and $y=\omega_1$ in Theorem 4.5, we get the required estimate (4.29). ☐

Now we will state the corollary of the proof of Proposition 4.4 which is similar to Proposition 4.6 to ensure its application to our Cauchy problem. The proof is similar to the proof of Proposition 4.6.

Proposition 4.7. *Let T_t, $t \ne 0$, be an operator defined by*

$$T_t f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \vartheta(t;\xi))}[1 - \psi((1+|t|)\xi)]a(t,\xi)\widehat{f}(\xi)\,d\xi,$$

where $\vartheta(t;\xi)$ and γ_0 are as in Proposition 4.4. Assume that the amplitude $a(t,\xi)$ satisfies $a(t,\xi)=0$ for all $|\xi| \ge 1$ and that

$$|\partial_\xi^\alpha a(t,\xi)| \le C_\alpha |\xi|^{-|\alpha|} \quad \text{for all} \quad |\alpha| \le 1.$$

Let $1 < p \le 2 \le q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $t \ne 0$ we have the estimate

$$\|T_t f\|_{L^q(\mathbb{R}^n)} \le C|t|^{-\frac{1}{\gamma_0}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{\dot{L}^p_{N_p}(\mathbb{R}^n)}, \tag{4.30}$$

where $N_p = \left(n - \frac{1}{\gamma_0} + 1\right)\left(\frac{1}{p} - \frac{1}{q}\right)$.

Let us finally estimate our Fourier multiplier for small t. For the analysis of very small t we will use the following Littlewood-Paley type theorem.

Lemma 4.8 ([9](Theorem 1.11)). *Let $h = h(\xi)$ be a tempered distribution on $\mathbb{R}^n$ ($n \ge 1$) such that*

$$\sup_{0<l<+\infty} l^b \mathrm{meas}\,\{\xi : |h(\xi)| \ge l\} < +\infty,$$

for some $1 < b < +\infty$. Then the convolution operator with $\mathcal{F}^{-1}[h]$ is L^p–L^q bounded provided that $1 < p \le 2 \le q < +\infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{b}$, i.e. we have the estimate

$$\left\|\mathcal{F}^{-1}[h] * u\right\|_{L^q} \le C\|u\|_{L^p} \quad \textit{for all} \quad u \in L^p(\mathbb{R}^n).$$

Using this fact, we obtain:

Proposition 4.9. *Let T_t be an operator defined by*

$$T_t f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\vartheta(t;\xi))}[1-\psi((1+|t|)\xi)]a(t,\xi)\widehat{f}(\xi)\,d\xi,$$

where $\vartheta(t;\xi)$ and $a(t,\xi)$ are as in Propositions 4.2 or 4.4. Let $n \ge 1$, $1 < p \le 2 \le q < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for small t we have the estimate

$$\|T_t f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{\dot{L}^p_{\widetilde{N}_p}(\mathbb{R}^n)}, \tag{4.31}$$

where $\widetilde{N}_p = n\left(\frac{1}{p} - \frac{1}{q}\right)$.

In fact, the proof will yield the Besov norm $B^{1,1}_p$ on the right hand side of the estimate, which is a known improvement for this type of estimates.

Proof. In the following argument we need not use the Van der Corpt lemma, and the proof relies only on the Littlewood–Paley type theorem. We put $K(t) = (1+|t|)^{-1}$.

It suffices to prove (4.31) for $p \ne q$, since the case $p = q = 2$ follows from the Plancherel theorem. Noting that $\vartheta(t;\xi)$ and is homogeneous of order one, and making change of variable $\eta = \frac{\xi}{K(t)}$ and $y = K(t)x$, we get

$$\|T_t f\|_{L^q(\mathbb{R}^n)} = K(t)^{n-\frac{n}{q}} \left\|\mathcal{F}^{-1}\left[m_t(\eta)\right] * \mathcal{F}^{-1}\left[|\eta|^{\widetilde{N}_p}\widehat{f}(K(t)\eta)\right]\right\|_{L^q(\mathbb{R}^n)}, \tag{4.32}$$

where we set

$$m_t(\eta) = e^{iK(t)\vartheta(t;\eta)}a(t,K(t)\eta)(1-\psi(\eta))|\eta|^{-\widetilde{N}_p}, \quad \widetilde{N}_p = n\left(\frac{1}{p} - \frac{1}{q}\right).$$

Since $a(t,\xi)$ is bounded, we have

$$\mathrm{meas}\,\{\eta : |m_t(\eta)| \ge l\} \le \mathrm{meas}\left\{\eta : |\eta| \le C^{1/\widetilde{N}_p} l^{-1/\widetilde{N}_p}\right\} = C^{n/\widetilde{N}_p} l^{-n/\widetilde{N}_p}$$

for each $l > 0$. Hence it follows from Lemma 4.8 that the convolution operator with m_t is L^p–L^q bounded, which implies that

$$\begin{aligned}
\left\|\mathcal{F}^{-1}[m_t] * \mathcal{F}^{-1}\left[|\eta|^{\widetilde{N}_p}\widehat{f}(K(t)\eta)\right]\right\|_{L^q(\mathbb{R}^n)} &\le C\left\|\mathcal{F}^{-1}\left[|\eta|^{\widetilde{N}_p}\widehat{f}(K(t)\eta)\right]\right\|_{L^p(\mathbb{R}^n)} \\
&= CK(t)^{-n+\frac{n}{p}-\widetilde{N}_p}\|f\|_{\dot{L}^p_{\widetilde{N}_p}(\mathbb{R}^n)},
\end{aligned}$$

where we performed the transformations $K(t)\eta=\xi$ and $\frac{x}{K(t)}=z$ in the last step. Thus, combining this estimate with (4.32), we obtain the desired estimate (4.31). The proof of Proposition 4.9 is complete. □

Proof of Theorem 1.1. The proof of Theorem 1.1 now follows from Proposition 4.1 for low frequencies $|\xi|<t^{-1}$, from Propositions 4.2 and 4.4 for large frequencies $|\xi|\geq 1$, and from Propositions 4.6 and 4.7 for intermediate frequencies $t^{-1}\leq|\xi|<1$. We also use Proposition 4.9 for small times. We can note that all these propositions give different Sobolev orders on the regularity of the Cauchy data.

Indeed, using representation formula for the solution established in Theorem 3.1, we can write the solution as

$$u(t,x)=\sum_{k=0}^{m-1}u^k(t,x),$$

with

$$u^k(t,x)=\sum_{j=1}^{m}\mathcal{F}^{-1}\left[\left(\alpha^j_{k,\pm}(\xi)+\varepsilon^j_{k,\pm}(t;\xi)\right)e^{i\vartheta_j(t;\xi)}\widehat{f}_k(\xi)\right](x).$$

Now, we decompose

$$\begin{aligned}
u^k(t,x)&=u^k_1(t,x)+u^k_2(t,x)+u^k_3(t,x)\\
&=\sum_{j=1}^{m}\mathcal{F}^{-1}\left[\left(\alpha^j_{k,\pm}(\xi)+\varepsilon^j_{k,\pm}(t;\xi)\right)\psi((1+|t|)\xi)e^{i\vartheta_j(t;\xi)}\widehat{f}_k(\xi)\right](x)\\
&+\sum_{j=1}^{m}\mathcal{F}^{-1}\left[\left(\alpha^j_{k,\pm}(\xi)+\varepsilon^j_{k,\pm}(t;\xi)\right)(1-\psi((1+|t|)\xi))\chi(\xi)e^{i\vartheta_j(t;\xi)}\widehat{f}_k(\xi)\right](x)\\
&+\sum_{j=1}^{m}\mathcal{F}^{-1}\left[\left(\alpha^j_{k,\pm}(\xi)+\varepsilon^j_{k,\pm}(t;\xi)\right)(1-\psi((1+|t|)\xi))(1-\chi(\xi))e^{i\vartheta_j(t;\xi)}\widehat{f}_k(\xi)\right](x),
\end{aligned}$$

with $\psi,\chi\in C_0^\infty(\mathbb{R}^n)$ such that $\psi(\xi)=\chi(\xi)\equiv 1$ for $|\xi|\leq\frac{1}{2}$, and 0 for $|\xi|\geq 1$. Assume conditions of part (i) of Theorem 1.1. Then we have estimates

$$\|u^k_1(t,\cdot)\|_{L^q(\mathbb{R}^n)}\leq C(1+|t|)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|f_k\|_{\dot{L}^p_{-k}(\mathbb{R}^n)}, \tag{4.33}$$

by Proposition 4.1,

$$\|u^k_2(t,\cdot)\|_{L^q(\mathbb{R}^n)}\leq C|t|^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f_k\|_{\dot{L}^p_{M_p-k}(\mathbb{R}^n)}, \tag{4.34}$$

with $M_p=\left(n-\frac{n-1}{\gamma}\right)\left(\frac{1}{p}-\frac{1}{q}\right)$ by Proposition 4.6, and

$$\|u^k_3(t,\cdot)\|_{L^q(\mathbb{R}^n)}\leq C|t|^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f_k\|_{L^p_{N_p-k}(\mathbb{R}^n)}, \tag{4.35}$$

with $N_p=\left(n-\frac{n-1}{\gamma}+\left[\frac{n-1}{\gamma}\right]+1\right)\left(\frac{1}{p}-\frac{1}{q}\right)$, by Proposition 4.2. For small t we have the estimate

$$\|u^k_2(t,\cdot)\|_{L^q(\mathbb{R}^n)}+\|u^k_3(t,\cdot)\|_{L^q(\mathbb{R}^n)}\leq C\|f_k\|_{\dot{L}^p_{\widetilde{N}_p-k}(\mathbb{R}^n)}, \tag{4.36}$$

with $\widetilde{N}_p = n\left(\frac{1}{p}-\frac{1}{q}\right)$, by Proposition 4.9. Putting all these estimates (4.33)–(4.36) together with similar estimates for derivatives, implies the statement of Theorem 1.1. □

As a corollary of this proof and all the propositions above, we have the following refinement of Theorem 1.1, providing quantitatively different estimates for different frequency regions.

Theorem 4.10. *Assume* (1.3)–(1.5). *Let* $\chi \in C_0^\infty(\mathbb{R})$ *be such that* $\chi(\rho) \equiv 1$ *for* $|\rho| \le \frac{1}{2}$, *and* 0 *for* $|\rho| \ge 1$. *Let us denote*

$$u_1 = \chi((1+|t|)|D|)u, \quad u_2 = (1-\chi((1+|t|)|D|)\chi(|D|)u, \quad \text{and}$$

$$u_3 = (1-\chi((1+|t|)|D|)(1-\chi(|D|))u.$$

Then the solution $u(t,x)$ *of* (1.1) *satisfies the following estimates*:

(i) *Suppose that the sets*

$$\Sigma_{\varphi_k^\pm} = \{\xi \in \mathbb{R}^n : \varphi_k^\pm(\xi) = 1\}$$

are convex for all $k = 1,\dots,m$, *and set* $\gamma = \max_{k=1,\dots,m} \gamma(\Sigma_{\varphi_k^\pm})$. *In addition, suppose that* $(1+|t|)^r a'_{\nu,j} \in L^1(\mathbb{R})$ *for* $1 \le r \le [(n-1)/\gamma]+1$, *and for all* ν, j *with* $|\nu|+j=m$. *Let* $1<p\le 2\le q<+\infty$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then we have the estimates*

$$\|D_t^l D_x^\alpha u_1(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C(1+|t|)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{k=0}^{m-1} \|f_k\|_{\dot{L}^p_{l+|\alpha|-k}(\mathbb{R}^n)}, \quad (t\in\mathbb{R}),$$

$$\|D_t^l D_x^\alpha u_2(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C|t|^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{k=0}^{m-1} \|f_k\|_{\dot{L}^p_{l+|\alpha|+M_p-k}(\mathbb{R}^n)}, \quad (|t|\ge 1),$$

$$\|D_t^l D_x^\alpha u_3(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C|t|^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{k=0}^{m-1} \|f_k\|_{L^p_{l+|\alpha|+N_p-k}(\mathbb{R}^n)}, \quad (|t|\ge 1),$$

$$\|D_t^l D_x^\alpha u_2(t,\cdot)\|_{L^q(\mathbb{R}^n)} + \|D_t^l D_x^\alpha u_3(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C \sum_{k=0}^{m-1} \|f_k\|_{\dot{L}^p_{l+|\alpha|+\widetilde{N}_p-k}(\mathbb{R}^n)}, \quad (|t|<1),$$

with $M_p = \left(n-\frac{n-1}{\gamma}\right)\left(\frac{1}{p}-\frac{1}{q}\right)$, $N_p = \left(n-\frac{n-1}{\gamma}+\left[\frac{n-1}{\gamma}\right]+1\right)\left(\frac{1}{p}-\frac{1}{q}\right)$, $\widetilde{N}_p = n\left(\frac{1}{p}-\frac{1}{q}\right)$, $l=0,\dots,m-1$, *and* α *any multi-index.*

(ii) *Suppose that* $\Sigma_{\varphi_k^\pm}$ *is non-convex for some* $k=1,\dots,m$, *and set* $\gamma_0 = \max_{k=1,\dots,m} \gamma_0(\Sigma_{\varphi_k^\pm})$. *In addition, suppose that* $(1+|t|)a'_{\nu,j} \in L^1(\mathbb{R})$ *for all* ν, j *with* $|\nu|+j=m$. *Let*

$1 < p \le 2 \le q < +\infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then we have the estimates*

$$\|D_t^l D_x^\alpha u_1(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C(1+|t|)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{k=0}^{m-1} \|f_k\|_{\dot{L}^p_{l+|\alpha|-k}(\mathbb{R}^n)}, \quad (t \in \mathbb{R}),$$

$$\|D_t^l D_x^\alpha u_2(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C|t|^{-\frac{1}{\gamma_0}\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{k=0}^{m-1} \|f_k\|_{\dot{L}^p_{l+|\alpha|+M_p-k}(\mathbb{R}^n)}, \quad (|t| \ge 1),$$

$$\|D_t^l D_x^\alpha u_3(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C|t|^{-\frac{1}{\gamma_0}\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{k=0}^{m-1} \|f_k\|_{L^p_{l+|\alpha|+N_p-k}(\mathbb{R}^n)}, \quad (|t| \ge 1),$$

$$\|D_t^l D_x^\alpha u_2(t,\cdot)\|_{L^q(\mathbb{R}^n)} + \|D_t^l D_x^\alpha u_3(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C \sum_{k=0}^{m-1} \|f_k\|_{\dot{L}^p_{l+|\alpha|+\widetilde{N}_p-k}(\mathbb{R}^n)}, \quad (|t| < 1),$$

with $M_p = \left(n - \frac{1}{\gamma_0}\right)\left(\frac{1}{p} - \frac{1}{q}\right)$, $N_p = \left(n - \frac{1}{\gamma_0} + 1\right)\left(\frac{1}{p} - \frac{1}{q}\right)$, $\widetilde{N}_p = n\left(\frac{1}{p} - \frac{1}{q}\right)$, $l = 0, \ldots, m-1$, *and* α *any multi-index.*

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