

Fractional dynamical systems and applications in mechanics and economics

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Abstract

Using the fractional integration and differentiation on \mathbb{R} we build the fractional jet fibre bundle on a differentiable manifold and we emphasize some important geometrical objects. Euler-Lagrange fractional equations are described. Some significant examples from mechanics and economics are presented.

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1 Introduction

The operators of fractional differentiation have been introduced by Leibnitz, Liouville, Riemann, Grunwal and Letnikov [6]. The fractional derivatives and integrals are used in the description of some models in mechanics, physics [6], economics [4] and medicine [11]. The fractional variational calculus [1] is an important instrument in the analysis of such models. The Euler-Lagrange equations are non-autonomous fractional differential equations in those models.

In this paper we present the fractional jet fibre bundle of order k on a differentiable manifold as being $J^{\alpha k}(\mathbb{R}, M) = \mathbb{R} \times Osc^{\alpha k}(M)$, $\alpha \in (0, 1)$, $k \in \mathbb{N}^*$.

The fibre bundle $J^{\alpha k}$ is built in a similar way as the fibre bundle E^k by R. Miron [9]. Among the geometrical structures defined on $J^\alpha(\mathbb{R}, M)$ we consider the dynamical fractional connection and the fractional Euler-Lagrange equations associated with a function defined on $J^{\alpha k}(\mathbb{R}, M)$.

In section 2 we describe the fractional operators on \mathbb{R} and some of their properties which are used in the paper. In section 3 we describe the fractional osculator bundle of order k . In section 4 the fractional jet fibre bundle $J^\alpha(\mathbb{R}, M)$ is defined, the fractional dynamical connection is built and the fractional Euler-Lagrange equations are established using the notion of fractional extremal value and classical extremal value on $J^{\alpha k}(\mathbb{R}, M)$. In section 5 we consider some examples and applications.

2 Elements of fractional integration and differentiation on \mathbb{R}

Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function and $\alpha \in (0, 1)$. The left-sided (right-sided) fractional derivative of f is the function

$$\begin{aligned} (-D_t^\alpha f)(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)-f(a)}{(t-s)^\alpha} ds \\ (+D_t^\alpha f)(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(b)-f(s)}{(s-t)^\alpha} ds, \end{aligned} \tag{1}$$

where $t \in [a, b)$ and Γ is Euler's gamma function.

Proposition 1. (see [6]) The operators $-D_t^\alpha$ and $+D_t^\alpha$ have the properties:

1. If f_1 and f_2 are defined on $[a, b]$ and $-D_t^\alpha, +D_t^\alpha$ exists, then

$$-D_t^\alpha(c_1 f_1 + c_2 f_2)(t) = c_1 (-D_t^\alpha f_1)(t) + c_2 (-D_t^\alpha f_2)(t). \tag{2}$$

2. If $\{\alpha_n\}_{n \geq 0}$ is a real number sequence with $\lim_{n \rightarrow \infty} \alpha_n = 1$ then

$$\lim_{n \rightarrow \infty} (-D_t^{\alpha_n} f)(t) = (-D_t^1 f)(t) = \frac{d}{dt} f(t). \tag{3}$$

3. a) If $f(t) = c, t \in [a, b], c \in \mathbb{R}$ then

$$(-D_t^\alpha f)(t) = 0. \tag{4}$$

b) If $f(t) = t^\gamma, t \in (a, b], \gamma \in \mathbb{R}$, then

$$(-D_t^\alpha f)(t) = \frac{t^{\gamma-\alpha} \Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}. \tag{5}$$

c) If $f(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$, then

$$(-D_t^\alpha f)(t) = 1. \quad (6)$$

4. If f_1 and f_2 are analytic functions on $[a, b]$ then

$$(-D_t^\alpha (f_1 f_2))(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-D_t^{\alpha-k} f_1)(t) \frac{d^k}{(dt)^k} f_2(t), \quad (7)$$

where $\frac{d^k}{(dt)^k} = \frac{d}{dt} \circ \frac{d}{dt} \circ \dots \circ \frac{d}{dt}$.

5. It also holds true

$$\int_a^b f_1(t) (-D_t^\alpha f_2)(t) dt = - \int_a^b f_2(t) (+D_t^\alpha f_1)(t) dt. \quad (8)$$

6. a) If $f : [a, b] \rightarrow \mathbb{R}$ admits fractional derivatives of order $a\alpha$, $a \in \mathbb{N}$, then

$$f(t+h) = E_\alpha((ht)^\alpha -D_t^\alpha) f(t), \quad (9)$$

where E_α is the Mittag-Leffler function given by

$$E_\alpha(t) = \sum_{a=0}^{\infty} \frac{t^{\alpha a}}{\Gamma(1 + \alpha a)}. \quad (10)$$

b) If $f : [a, b] \rightarrow \mathbb{R}$ is analytic and $0 \in (a, b)$ then the fractional McLaurin series is

$$f(t) = \sum_{a=0}^{\infty} \frac{t^{\alpha a}}{\Gamma(1 + \alpha a)} (-D_t^{\alpha a} f)(t) |_{t=0}. \quad (11)$$

The physical and geometrical interpretation of the fractional derivative on \mathbb{R} is suggested by the interpretation of the Stieltjes integral, because the integral used in the definition of the fractional derivative is a Riemann-Stieltjes integral [10].

By definition, the left-sided (right-sided) fractional derivative of f , of order α , $m = [\alpha] + 1$, is the function

$$\begin{aligned} D_t^\alpha f(t) &= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_{-\infty}^t \frac{f(s)-f(0)}{(t-s)^\alpha} ds, \quad 0 \in (-\infty, t) \\ {}^*D_t^\alpha f(t) &= \frac{1}{\Gamma(m-\alpha)} \left(-\frac{d}{dt}\right)^m \int_t^\infty \frac{f(s)-f(0)}{(s-t)^\alpha} ds, \quad 0 \in (t, \infty). \end{aligned} \quad (12)$$

If $\overline{\text{supp} f} \subset [a, b]$, then $D_t^\alpha f = -D_t^\alpha f$, ${}^*D_t^\alpha f = +D_t^\alpha f$.

Let us consider the seminorms

$$\begin{aligned} |x|_{J_L^\alpha(\mathbb{R})} &= \|D_t^\alpha x\|_{L^2(\mathbb{R})} \\ |x|_{J_R^\alpha(\mathbb{R})} &= \|{}^*D_t^\alpha x\|_{L^2(\mathbb{R})}, \end{aligned}$$

and the norms

$$\begin{aligned}\|x\|_{J_L^\alpha(\mathbb{R})} &= \left(\|x\|_{L^2(\mathbb{R})}^2 + |x|_{J_L^\alpha(\mathbb{R})}^2 \right)^{1/2} \\ \|x\|_{J_R^\alpha(\mathbb{R})} &= \left(\|x\|_{L^2(\mathbb{R})}^2 + |x|_{J_R^\alpha(\mathbb{R})}^2 \right)^{1/2},\end{aligned}$$

and $J_{0L}^\alpha(\mathbb{R})$, $J_{0R}^\alpha(\mathbb{R})$ the closures of $C_0^\infty(\mathbb{R})$ with respect to the two norms from above, respectively. In [6] it is proved that the operators D_t^α and $*D_t^\alpha$ satisfy the properties:

Proposition 2. *Let $I \subset \mathbb{R}$ and let $J_{0L}^\alpha(I)$ and $J_{0R}^\alpha(I)$ be the closures of $C_0^\infty(I)$ with respect to the norms from above. For any $x \in J_{0L}^\beta(I)$, $0 < \alpha < \beta$, the following relation holds:*

$$D_t^\beta x(t) = D_t^\alpha D_t^{\beta-\alpha} x(t).$$

For any $x \in J_{0R}^\beta(I)$, $0 < \alpha < \beta$, it also holds

$$*D_t^\beta x(t) = *D_t^\alpha *D_t^{\beta-\alpha} x(t).$$

In the following we shall consider the fractional derivatives defined above.

3 The fractional osculator bundle of order k on a differentiable manifold

Let $\alpha \in (0, 1]$ be fixed and M a differentiable manifold of dimension n . Two curves $\rho, \sigma : I \rightarrow \mathbb{R}$, with $\rho(0) = \sigma(0) = x_0 \in M$, $0 \in I$, have a fractional contact α of order $k \in \mathbb{N}^*$ in x_0 , if for any $f \in \mathcal{F}(U)$, $x_0 \in U$, U a chart on M , it holds

$$D_t^{\alpha a}(f \circ \rho)|_{t=0} = D_t^{\alpha a}(f \circ \sigma)|_{t=0} \quad (13)$$

where $a = \overline{1, k}$. The relation (13) is an equivalence relation. The equivalence class $[\rho]_{x_0}^{\alpha k}$ is called the fractional k -osculator space of M in x_0 and it will be denoted by $Osc_{x_0}^{\alpha k}(M)$. If the curve $\rho : I \rightarrow M$ is given by $x^i = x^i(t)$, $t \in I$, $i = \overline{1, n}$, then, considering the formula (11), the class $[\rho]_{x_0}^{\alpha k}$, may be written as

$$x^i(t) = x^i(0) + \frac{t^\alpha}{\Gamma(1+\alpha)} D_t^\alpha x^i(t)|_{t=0} + \dots + \frac{t^{\alpha k}}{\Gamma(1+\alpha k)} D_t^{\alpha k} x^i(t)|_{t=0}, \quad (14)$$

where $t \in (-\varepsilon, \varepsilon)$. We shall use the notation

$$x^i(0) = x^i, \quad y^{i(\alpha a)} = \frac{1}{\Gamma(1+\alpha a)} D_t^{\alpha a} x^i(t)|_{t=0}, \quad (15)$$

for $i = \overline{1, n}$ and $a = \overline{1, k}$.

By definition, the fractional osculator bundle of order r is the fibre bundle $(Osc^{\alpha k}(M), M)$ where $Osc^{\alpha k}(M) = \bigcup_{x_0 \in M} Osc_{x_0}^{\alpha k}(M)$ and $\pi_0^{\alpha k} : Osc^{\alpha k}(M) \rightarrow M$ is defined by $\pi_0^{\alpha k}([\rho]_{x_0}^{\alpha k}) = x_0$, $(\forall) [\rho]_{x_0}^{\alpha k} \in Osc_{x_0}^{\alpha k}(M)$.

For $f \in \mathcal{F}(U)$, the fractional derivative of order α , $\alpha \in (0, 1)$, with respect to the variable x^i , is defined by

$$(D_{x^i}^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x^i} \int_{a^i}^{x^i} \frac{f(x^1, \dots, x^{i-1}, s, x^{i+1}, \dots, x^n) - f(x^1, \dots, x^{i-1}, a^i, x^{i+1}, \dots, x^n)}{(x^i - s)^\alpha} ds, \quad (16)$$

where x^i are the coordinate functions on U , $\frac{\partial}{\partial x^i}$, $i = \overline{1, n}$, is the canonical base of the vector fields on U and $U_{ab} = \{x \in U, a^i \leq x^i \leq b^i, i = \overline{1, n}\} \subset U$. Let $U, U' \subset M$ be two charts on M , $U \cap U' \neq \emptyset$ and consider the change of variable

$$\bar{x}^i = \bar{x}^i(x^1, \dots, x^n) \quad (17)$$

with $\det \left(\frac{\partial \bar{x}^i}{\partial x^j} \right) \neq 0$. Let $\{dx^i\}_{i=\overline{1, n}}$ be the canonical base of 1-forms of $\mathcal{D}^1(U)$ and let us define the 1-forms $d(x^i)^\alpha = \alpha(x^i)^{\alpha-1} dx^i$, $i = \overline{1, n}$. The exterior differential $d^\alpha : \mathcal{F}(U \cap U') \rightarrow \mathcal{D}^1(U \cap U')$ is defined by

$$d^\alpha = d(x^j)^\alpha D_{x^j}^\alpha = d(\bar{x}^j)^\alpha D_{\bar{x}^j}^\alpha. \quad (18)$$

Using (18) and the property $D_{x^i}^\alpha \left(\frac{(x^i)^\alpha}{\Gamma(1+\alpha)} \right) = 1$, it follows that

$$d(x^j)^\alpha = \frac{1}{\Gamma(1+\alpha)} D_{\bar{x}^i}^\alpha (x^j)^\alpha d(\bar{x}^i)^\alpha. \quad (19)$$

Using the notation

$$J_i^\alpha(x, \bar{x}) = \frac{1}{\Gamma(1+\alpha)} D_{\bar{x}^i}^\alpha (x^j)^\alpha, \quad (20)$$

from (19) we get

$$d(x^j)^\alpha = J_i^\alpha(x, \bar{x}) d(\bar{x}^i)^\alpha. \quad (21)$$

From (21) it follows that

$$J_i^\alpha(x, \bar{x}) J_h^\alpha(x, \bar{x}) = \delta_h^i. \quad (22)$$

Consider $x^i = x^i(t)$ and $\bar{x}^i(t) = \bar{x}^i(x(t))$, $i = \overline{1, n}$, $t \in I$. Applying the operator D_t^α we get

$$(D_t^\alpha \bar{x}^i)(t) = D_{x^j}^\alpha \bar{x}^i(x) (D_t^\alpha x^j)(t) = J_j^\alpha(\bar{x}, x) (D_t^\alpha x^j)(t). \quad (23)$$

Considering the notation from (15) we have

$$y^{i(\alpha)} = J_j^i(\bar{x}, x)\bar{y}^{j(\alpha)}. \quad (24)$$

Also, from (15) we deduce

$$D_t^\alpha y^{i(\alpha a)} = \frac{\Gamma(\alpha a)}{\Gamma(\alpha(a-1))} y^{i(\alpha a)}, \quad (25)$$

where $i = \overline{1, n}$. Applying the operator D_t^α in the relation (24) we find

$$\begin{aligned} \frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} \bar{y}^{i(\alpha a)} &= \Gamma(1 + \alpha) J_j^i(\bar{y}^{\alpha(a-1)}, x) y^{j(\alpha)} + \\ \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} J_j^i(y^{\alpha(a-1)}, y^\alpha) y^{j(2\alpha)} &+ \dots + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} J_j^i(\bar{y}^{\alpha(a-1)}, y^{\alpha b}) y^{j((b+1)\alpha)} + \\ \dots + \frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} y^{i(\alpha a)}, \end{aligned} \quad (26)$$

where $a = \overline{1, k}$.

Proposition 3. (see [2], [5])

- a) The coordinate transformation on $Osc^{(\alpha k)}(M)$,
 $(x^i, y^{i(\alpha)}, \dots, y^{i(\alpha k)}) \rightarrow (\bar{x}^i, \bar{y}^{i(\alpha)}, \dots, \bar{y}^{i(\alpha k)})$ are given by the formulas (17) and (26).
b) The operators $D_{x^i}^\alpha$ and the 1-forms $(dx^i)^\alpha$, $i = \overline{1, n}$, transform by the formulas

$$\begin{aligned} D_{\bar{x}^i}^\alpha &= J_j^i(x, \bar{x}) D_{x^j}^\alpha \\ d(\bar{x}^i)^\alpha &= J_j^i(\bar{x}, x) d(x^j)^\alpha. \end{aligned} \quad (27)$$

4 The fractional jet bundle of order k on a differentiable manifold; geometrical objects

By definition, the k -order fractional jet bundle is the space $J^{\alpha k}(\mathbb{R}, M) = \mathbb{R} \times Osc^{k\alpha}(M)$. A system of local coordinates on $J^{\alpha k}(\mathbb{R}, M)$ will be denoted by $(t, x, y^{(\alpha)}, y^{(2\alpha)}, \dots, y^{(k\alpha)})$. Consider the projections $\pi_0^{\alpha k} : J^{\alpha k}(\mathbb{R}, M) \rightarrow M$ defined by

$$\pi_0^{\alpha k}(t, x, y^{(\alpha)}, \dots, y^{(k\alpha)}) = x. \quad (28)$$

Let $U, U' \subset M$ be two charts on M with $U \cap U' \neq \emptyset$, $(\pi_0^\alpha)^{-1}(U)$, $(\pi_0^\alpha)^{-1}(U') \subset J^\alpha(\mathbb{R}, M)$ the corresponding charts on $J^\alpha(\mathbb{R}, M)$ and, respectively, the corresponding coordinates (x^i) , (\bar{x}^i) and $(t, x^i, y^{i(\alpha)})$, $(t, \bar{x}^i, \bar{y}^{i(\alpha)})$. The transformations of coordinates are given by

$$\begin{aligned}\bar{x}^i &= \bar{x}^i(x^1, \dots, x^n) \\ \bar{y}^{i(\alpha)} &= \overset{\alpha}{J}(x, \bar{x})y^{i(\alpha)}.\end{aligned}\tag{29}$$

Consider the functions $(t)^\alpha$, $(x^i)^\alpha$, $(y^{i(\alpha)})^\alpha \in \mathcal{F}((\pi_0^\alpha)^{-1}(U))$, the 1-forms $\frac{1}{\Gamma(1+\alpha)}d(t)^\alpha$, $\frac{1}{\Gamma(1+\alpha)}d(x^i)^\alpha$, $\frac{1}{\Gamma(1+\alpha)}d(y^{i(\alpha)})^\alpha \in \mathcal{D}^1((\pi_0^\alpha)^{-1}(U))$ and the operators D_t^α , $D_{x^i}^\alpha$, $D_{y^{i(\alpha)}}^\alpha$ on $(\pi_0^\alpha)^{-1}(U)$, $i = \overline{1, n}$. The following relations hold:

$$\begin{aligned}D_t^\alpha\left(\frac{1}{\Gamma(1+\alpha)}t^\alpha\right) &= 1, \quad D_{x^i}^\alpha\left(\frac{1}{\Gamma(1+\alpha)}(x^j)^\alpha\right) = \delta_i^j, \\ D_{y^{i(\alpha)}}^\alpha\left(\frac{1}{\Gamma(1+\alpha)}(y^{j(\alpha)})^\alpha\right) &= \delta_i^j, \quad \frac{1}{\Gamma(1+\alpha)}d(t^\alpha)(D_t^\alpha) = 1, \\ \frac{1}{\Gamma(1+\alpha)}d(x^i)^\alpha(D_{x^j}^\alpha) &= \delta_j^i, \quad \frac{1}{\Gamma(1+\alpha)}d(y^{i(\alpha)})^\alpha(D_{y^{j(\alpha)}}^\alpha) = \delta_j^i.\end{aligned}\tag{30}$$

On $J^\alpha(\mathbb{R}, M)$ we may define the canonical structures

$$\begin{aligned}\overset{\alpha}{\theta}_1 &= d(t^\alpha) \otimes (D_t^\alpha + y^{i(\alpha)}D_{x^i}^\alpha) \\ \overset{\alpha}{\theta}_2 &= \overset{\alpha}{\theta}^i \otimes D_{x^i}^\alpha, \quad \overset{\alpha}{\theta}^i = \frac{1}{\Gamma(1+\alpha)}(d(x^i)^\alpha - y^{i(\alpha)}d(t)^\alpha) \\ \overset{\alpha}{S} &= \overset{\alpha}{\theta}^i \otimes D_{y^{i(\alpha)}}^\alpha \\ \overset{\alpha}{V}_i &= D_{y^{i(\alpha)}}^\alpha.\end{aligned}\tag{31}$$

Using (29) it is easy to show that the structures (31) have geometrical character. The space of the operators generated by the operators $\{D_t^\alpha, D_{x^i}^\alpha, D_{y^{i(\alpha)}}^\alpha\}$, $i = \overline{1, n}$, will be denoted by $\chi^\alpha((\pi_0^\alpha)^{-1}(U))$. For $\alpha \rightarrow 1$ the space of these operators represents the space of the vector fields on $\pi_0^{-1}(U)$.

A vector field $\overset{\alpha}{\Gamma} \in \chi^\alpha((\pi_0^\alpha)^{-1}(U))$ is called *FODE* (fractional ordinary differential equation) iff

$$\begin{aligned}d(t)^\alpha(\overset{\alpha}{\Gamma}) &= 1 \\ \overset{\alpha}{\theta}^i(\overset{\alpha}{\Gamma}) &= 0,\end{aligned}\tag{32}$$

for $i = \overline{1, n}$. In local coordinates *FODE* is given by

$$\overset{\alpha}{\Gamma} = D_t^\alpha + y^{i(\alpha)}D_{x^i}^\alpha + F^i D_{y^{i(\alpha)}}^\alpha,\tag{33}$$

where $F^i \in C^\infty((\pi_0^\alpha)^{-1}(U))$, $i = \overline{1, n}$. The integral curves of the field *FODE* are the solutions of the fractional differential equation (*EDF*)

$$D_t^{2\alpha} x^i(t) = F^i(t, x(t), D_t^\alpha x(t)), \quad i = \overline{1, n}. \quad (34)$$

The fractional dynamical connection on $J^\alpha(\mathbb{R}, M)$ is defined by the fractional tensor fields $\overset{\alpha}{H}$ of type (1, 1) which satisfy the conditions

$$\begin{aligned} \theta_1 \circ \overset{\alpha}{H} &= 0 \\ \theta_2 \circ \overset{\alpha}{H} &= \theta_2 \\ \overset{\alpha}{H} \Big|_{\overset{\alpha}{V}} &= -id \Big|_{\overset{\alpha}{V}}, \end{aligned} \quad (35)$$

where $\overset{\alpha}{V}$ is formed by operators generated by $\{D_{y^{i(\alpha)}}^\alpha\}_{i=\overline{1, n}}$. In the chart $(\pi_0^\alpha)^{-1}(U)$ the fractional tensor field $\overset{\alpha}{H}$ has the expression

$$\begin{aligned} \overset{\alpha}{H} &= \overset{1}{H} d(t)^\alpha + \overset{2}{H}_j^i d(x^j)^\alpha + \overset{3}{H}_i d(y^{i(\alpha)})^\alpha \otimes D_t^\alpha + \\ &(\overset{4}{H}_j^i (dt)^\alpha + \overset{5}{H}_j^i d(x^j)^\alpha + \overset{6}{H}_j^i d(y^{i(\alpha)})^\alpha) \otimes D_{x^i}^\alpha + \\ &(\overset{7}{H} d(t)^\alpha + \overset{8}{H}_j^i d(x^j)^\alpha + \overset{9}{H}_j^i d(y^{i(\alpha)})^\alpha) \otimes D_{y^{i(\alpha)}}^\alpha. \end{aligned} \quad (36)$$

The tensor field $\overset{\alpha}{H}$ has a geometrical character, fact which results by using the relations (29), and is called a d^α -tensor field. Using the relations (30) and (31) we get

Proposition 4. *a) The fractional dynamical connection $\overset{\alpha}{H}$, in the chart $(\pi_0^\alpha)^{-1}(U)$, is given by*

$$\begin{aligned} \overset{\alpha}{H} &= \frac{1}{\Gamma(1+\alpha)} [(-y^{i(\alpha)} D_{x^i}^\alpha + H^i D_{y^{i(\alpha)}}^\alpha) \otimes d(t)^\alpha + \\ &(D_{x^i}^\alpha + H_i^j D_{y^j(\alpha)}^\alpha) \otimes d(x^i)^\alpha - D_{y^{i(\alpha)}}^\alpha \otimes d(y^{i(\alpha)})^\alpha]. \end{aligned} \quad (37)$$

b) The fractional dynamical connection $\overset{\alpha}{H}$ defines a $f(3, -1)$ fractional structure on $J^\alpha(\mathbb{R}, M)$, i.e., $\left(\overset{\alpha}{H}\right)^3 = \overset{\alpha}{H}$.

c) The fractional tensor fields $\overset{\alpha}{l}$ and $\overset{\alpha}{m}$ which are defined by

$$\begin{aligned} \overset{\alpha}{l} &= \overset{\alpha}{H} \circ \overset{\alpha}{H} \\ \overset{\alpha}{m} &= -\overset{\alpha}{H} \circ \overset{\alpha}{H} + I, \end{aligned} \quad (38)$$

where I is the identity map, satisfy the relations

$$\begin{aligned}
\overset{\alpha}{l} \circ \overset{\alpha}{l} &= \overset{\alpha}{l}, \quad \overset{\alpha}{m} \circ \overset{\alpha}{m} = \overset{\alpha}{m} \circ \overset{\alpha}{l}, \quad \overset{\alpha}{l} + \overset{\alpha}{m} = I \\
\overset{\alpha}{l}(D_t^\alpha) &= -y^{i(\alpha)} D_{x^i} - (y^{i(\alpha)} \overset{\alpha}{H}_i^j + \overset{\alpha}{H}^j) D_{y^{i(\alpha)}} \\
\overset{\alpha}{l}(D_{x^i}^\alpha) &= D_{x^i}^\alpha, \quad \overset{\alpha}{l}(D_{y^{i(\alpha)}}^\alpha) = D_{y^{i(\alpha)}}^\alpha \\
\overset{\alpha}{m}(D_t^\alpha) &= D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha + (y^{i(\alpha)} \overset{\alpha}{H}_i^j + \overset{\alpha}{H}^j) D_{y^{i(\alpha)}}^\alpha \\
\overset{\alpha}{m}(D_{x^i}^\alpha) &= 0, \quad \overset{\alpha}{m}(D_{y^{i(\alpha)}}^\alpha) = 0.
\end{aligned} \tag{39}$$

d) The fractional vector field $\overset{\alpha}{\Gamma} \in \chi^\alpha(J^\alpha(\mathbb{R}, M))$ given by

$$\overset{\alpha}{\Gamma} = \overset{\alpha}{m}(D_t^\alpha) = D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha + (y^{i(\alpha)} \overset{\alpha}{H}_i^j + \overset{\alpha}{H}^j) D_{y^{j(\alpha)}}^\alpha \tag{40}$$

defines a field FODE associated to the fractional dynamical connection. The integral curves are the solutions of the EDF

$$D_t^{2\alpha} x^i(t) = D_t^\alpha x^i(t) \overset{\alpha}{H}_i^j + \Gamma(1 + \alpha) \overset{\alpha}{H}^j \tag{41}$$

where $\overset{\alpha}{H}_i^j$ and $\overset{\alpha}{H}^j$ are functions of $(t, x(t), y^{(\alpha)}(t))$.

Let $L \in C^\infty(J^\alpha(\mathbb{R}, M))$ be a fractional Lagrange function. By definition, the Cartan fractional 1-form is the 1-form $\overset{\alpha}{\theta}_L$ given by

$$\overset{\alpha}{\theta}_L = L d(t)^\alpha + \overset{\alpha}{S}(L). \tag{42}$$

We call the Cartan fractional 2-form, the 2-form $\overset{\alpha}{\omega}_L$ given by

$$\overset{\alpha}{\omega}_L = d^\alpha \overset{\alpha}{\theta}_L \tag{43}$$

where d^α is the fractional exterior differential:

$$d^\alpha = d(t)^\alpha D_t^\alpha + d(x^i)^\alpha D_{x^i}^\alpha + d(y^{i(\alpha)})^\alpha D_{y^{i(\alpha)}}^\alpha. \tag{44}$$

In the chart $(\pi_0^\alpha)^{-1}(U)$, $\overset{\alpha}{\theta}_L$ and $\overset{\alpha}{\omega}_L$ are given by

$$\begin{aligned}
\overset{\alpha}{\theta}_L &= (L - \frac{1}{\Gamma(1+\alpha)} y^{i(\alpha)} D_{y^{i(\alpha)}}^\alpha(L) d(t)^\alpha + \frac{1}{\Gamma(1+\alpha)} D_{y^{i(\alpha)}}^\alpha(L) d(x^i)^\alpha \\
\overset{\alpha}{\omega}_L &= A_i d(t)^\alpha \wedge d(x^i)^\alpha + B_i d(t)^\alpha \wedge d(y^{i(\alpha)})^\alpha + \\
&\quad A_{ij} d(x^i)^\alpha \wedge d(x^j)^\alpha + B_{ij} d(x^i)^\alpha \wedge d(y^{j(\alpha)})^\alpha,
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
A_i &= \frac{1}{\Gamma(1+\alpha)} D_t^\alpha D_{y^{i(\alpha)}}^\alpha(L) + \frac{1}{\Gamma(1+\alpha)} y^{j(\alpha)} D_{x^i}^\alpha D_{y^{j(\alpha)}}^\alpha(L) - D_{x^i}^\alpha(L) \\
B_i &= \frac{1}{\Gamma(1+\alpha)} D_{y^{i(\alpha)}}^\alpha(y^{j(\alpha)} D_j^\alpha(L)) \\
A_{ij} &= D_{x^i}^\alpha D_{y^{i(\alpha)}}^\alpha(L), \quad B_{ij} = -D_{y^{j(\alpha)}}^\alpha D_{y^{i(\alpha)}}^\alpha(L).
\end{aligned} \tag{46}$$

Proposition 5. *If L is regular (i.e., $\det\left(\frac{\partial^2 L}{\partial y^{i(\alpha)} \partial y^{j(\alpha)}}\right) \neq 0$) then there exists a fractional field FODE Γ_L^α such that $i_{\Gamma_L^\alpha} \omega_L^\alpha = 0$. In the chart $(\pi_0^\alpha)^{-1}(U)$ we have*

$$\Gamma_L^\alpha = D_t^\alpha + y^{i(\alpha)} D_{x^i} + M^i D_{y^{i(\alpha)}}^\alpha, \tag{47}$$

where

$$\begin{aligned}
M^i &= g^{ik} (D_k^\alpha(L) - d_t^\alpha(\frac{\partial^\alpha L}{\partial y^{k(\alpha)}})) \\
d_t^\alpha &= D_t^\alpha + y^{i(\alpha)} D_{x^i} \\
(g^{ik}) &= (D_{y^{i(\alpha)}}^\alpha D_{y^{k(\alpha)}}^\alpha(L))^{-1}.
\end{aligned} \tag{48}$$

An important structure on $J^\alpha(\mathbb{R}, M)$ is described by the fractional Euler-Lagrange equations. Let $c : t \in [0, 1] \rightarrow (x^i(t)) \in M$ be a parameterized curve, such that $Imc \subset U \subset M$. The extension of the curve c to $J^\alpha(\mathbb{R}, M)$ is the curve $c^\alpha : t \in [0, 1] \rightarrow (t, x^i(t), y^{i(\alpha)}(t)) \in J^\alpha(\mathbb{R}, M)$. Consider $L \in C^\infty(J^\alpha(\mathbb{R}, M))$. The action of L along the curve c^α is defined by

$$\mathcal{A}(c^\alpha) = \int_0^1 L(t, x(t), y^\alpha(t)) dt. \tag{49}$$

Let $c_\varepsilon : t \in [0, 1] \rightarrow (x^i(t, \varepsilon)) \in M$ be a family of curves, where ε is sufficiently small so that $Imc_\varepsilon \subset U$, $c_0(t) = c(t)$, $D_\varepsilon^\alpha c_\varepsilon(0) = D_\varepsilon^\alpha c_\varepsilon(1) = 0$. The action of L along the curves c_ε is

$$\mathcal{A}(c_\varepsilon^\alpha) = \int_0^1 L(t, x(t, \varepsilon), y^\alpha(t, \varepsilon)) dt, \tag{50}$$

where $y^{i(\alpha)}(t, \varepsilon) = \frac{1}{\Gamma(1+\alpha)} D_t^\alpha x^i(t, \varepsilon)$. The action (50) has a fractional extremal value if

$$D_\varepsilon^\alpha \mathcal{A}(c_\varepsilon^\alpha) |_{\varepsilon=0} = 0. \tag{51}$$

The action (50) has an extremal value if

$$D_\varepsilon^1 \mathcal{A}(c_\varepsilon^\alpha) |_{\varepsilon=0} = 0. \tag{52}$$

Using the properties of the fractional derivative we obtain

Proposition 6. a) A necessary condition for the action (50) to reach a fractional extremal value is that $c(t)$ satisfies the fractional Euler-Lagrange equations

$$\begin{aligned} D_{x^i}^\alpha L - d_t^{2\alpha}(D_{y^{i(\alpha)}}^\alpha L) &= 0 \\ d_t^\alpha &= D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha + y^{i(2\alpha)} D_{y^{i(\alpha)}}^\alpha, \end{aligned} \quad (53)$$

where $i = \overline{1, n}$.

b) A necessary condition for the action (50) to reach an extremal value is that $c(t)$ satisfies the Euler-Lagrange equations

$$\begin{aligned} D_{x^i}^1 L - d_t^2(D_{y^{i(\alpha)}}^1 L) &= 0 \\ d_t^2 &= D_t^\alpha + y^{i(\alpha)} D_{x^i}^1 + y^{i(2\alpha)} D_{y^{i(\alpha)}}^1, \end{aligned} \quad (54)$$

where $i = \overline{1, n}$.

The equations (53) may be written in the form

$$D_{x^i}^\alpha L - d_t^\alpha(D_{y^{i(\alpha)}}^\alpha L) - y^{j(2\alpha)} D_{y^{j(\alpha)}}^\alpha(D_{y^{i(\alpha)}}^\alpha L) = 0, \quad (55)$$

for $i = \overline{1, n}$. The equations (54) may be written as

$$\frac{\partial L}{\partial x^i} - d_t^\alpha \left(\frac{\partial L}{\partial y^{i(\alpha)}} \right) - y^{j(2\alpha)} \frac{\partial^2 L}{\partial y^{i(\alpha)} \partial y^{j(\alpha)}} = 0, \quad (56)$$

where $i = \overline{1, n}$. Let us denote by

$$g_{ij}^\alpha = D_{y^{i(\alpha)}}^\alpha(D_{y^{j(\alpha)}}^\alpha L), \quad (57)$$

and by $\left(g^{ik}\right)^\alpha = (g_{ij}^\alpha)^{-1}$, if $\det(g_{ij}^\alpha) \neq 0$. From (55) and from Proposition 5,

we get the fractional field $FODE \Gamma_L^\alpha$ associated to L .

Let $c : t \in [0, 1] \rightarrow (x^i(t)) \subset U$ be a parameterized curve. The extension of c to $J^{\alpha k}(\mathbb{R}, M)$ is the curve $c^{\alpha k} : t \in [0, 1] \rightarrow (t, x^i(t), y^{\alpha a}(t)) \in J^{\alpha k}(\mathbb{R}, M)$, $a = \overline{1, k}$. Let $L : J^{\alpha k}(\mathbb{R}, M) \rightarrow \mathbb{R}$ be a Lagrange function. The action of L along the curve $c^{\alpha k}$ is

$$\mathcal{A}(c^{\alpha k}) = \int_0^1 L(t, x(t), y^{\alpha a}(t)) dt. \quad (58)$$

Let $c_\varepsilon : t \in [0, 1] \rightarrow (x^i(t, \varepsilon)) \in M$ be a family of curves, where the absolute value of ε is sufficiently small so that $Imc_\varepsilon \subset U \subset M$, $c_0(t) = c(t)$, $D_\varepsilon^\alpha c(\varepsilon)|_{\varepsilon=0} = D_\varepsilon^\alpha c(\varepsilon)|_{\varepsilon=1} = 0$. The action of L on the curve c_ε is given by

$$\mathcal{A}(c_\varepsilon^{\alpha k}) = \int_0^1 L(t, x(t, \varepsilon), y^{\alpha a}(t, \varepsilon)) dt \quad (59)$$

where $y^{i(\alpha a)}(t, \varepsilon) = \frac{1}{\Gamma(1+\alpha a)} D_t^{\alpha a} x^i(t, \varepsilon)$, $a = \overline{1, k}$. The action (59) has a fractional extremal value if

$$D_\varepsilon^\alpha(\mathcal{A}(c_\varepsilon^{\alpha k}))|_{\varepsilon=0} = 0. \quad (60)$$

The action (59) has an extremal value if

$$D_\varepsilon^1(\mathcal{A}(c_\varepsilon^{\alpha k}))|_{\varepsilon=0} = 0. \quad (61)$$

Proposition 7. *a) A necessary condition for the action (58) to reach a fractional extremal value is that $c(t)$ satisfies the fractional Euler-Lagrange equations*

$$D_{x^i}^\alpha L + \sum_{a=1}^k (-1)^a d_t^{\alpha a} (D_{y^{i(\alpha a)}}^\alpha L) = 0, \quad (62)$$

where

$$d_t^{\alpha a} = D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha + y^{i(2\alpha)} D_{y^{i(\alpha)}}^\alpha + \dots + y^{i(\alpha a)} D_{y^{i(\alpha(a-1))}}^\alpha, \quad (63)$$

and $i = \overline{1, n}$.

b) A necessary condition that the action (58) reaches an extremal value is that $c(t)$ satisfies the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^i} + \sum_{a=1}^k (-1)^a d_t^a (D_{y^{i(\alpha a)}}^\alpha L) = 0, \quad (64)$$

where

$$d_t^a = D_t^1 + y^{i(\alpha)} D_{x^i}^1 + \dots + y^{i(\alpha a)} D_{y^{i(\alpha(a-1))}}^1. \quad (65)$$

Example. Consider the fractional differential equation

$$\begin{aligned} & \frac{c\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}(t) f(t) + a_1 \Gamma(1+2\alpha) y^{(2\alpha)} + \\ & a_2 \Gamma(1+3\alpha) y^{(3\alpha)} = 0. \end{aligned} \quad (66)$$

The equation (66) is the fractional Euler-Lagrange equation (62) for the function

$$\begin{aligned} L = & \frac{c}{1+\gamma-\alpha} x^\gamma - a_1 \Gamma(1+2\alpha) (y^\alpha)^\alpha + \\ & a_2 \Gamma(1+3\alpha) (y^{2\alpha})^\alpha. \end{aligned}$$

The equation (66) is the fractional Euler-Lagrange equation (64) for the function

$$L = \frac{c\Gamma(1+\gamma)x^{\gamma-\alpha+1}}{\Gamma(1+\gamma-\alpha)^{(1+\gamma-\alpha)}} f - \frac{a_1}{2} \Gamma(1+2\alpha) (y^\alpha)^2 + \frac{a_2}{2} \Gamma(1+3\alpha) (y^{2\alpha})^2.$$

5 Examples and applications

1. The nonhomogeneous Bagley-Torvik equation

The dynamics of a flat rigid body embedded in a Newton fluid is described by the equation

$$aD_t^2x(t) + bD_t^{3/2}x(t) + cx(t) - f(t) = 0, \quad (67)$$

where $a, b, c \in \mathbb{R}$ and the initial conditions are $x(0) = 0$, $D_t^1x(0) = 0$. The equation (67) is a fractional differential equation on the bundle $J^\alpha(\mathbb{R}, \mathbb{R})$ for $\alpha = \frac{1}{4}$. Indeed, let's consider the fractional differential equation

$$aD_t^{8\alpha}x(t) + bD_t^{6\alpha}x(t) + cx(t) - f(t) = 0, \quad (68)$$

with $\alpha > 0$. For $\alpha = \frac{1}{4}$ the equation (68) reduces to (67). With the notations (15), the equation (68) becomes

$$a\Gamma(1 + 8\alpha)y^{(8\alpha)}(t) + b\Gamma(1 + 6\alpha)y^{(6\alpha)}(t) + cx(t) - f(t) = 0. \quad (69)$$

On the bundle $J^{4\alpha}(\mathbb{R}, \mathbb{R})$ let us consider the Lagrange function

$$L(t, x, y^{(3\alpha)}, y^{(4\alpha)}) = \frac{1}{2}cx^2 - fx - \frac{b}{2}\Gamma(1 + 6\alpha)(y^{(3\alpha)})^2 + \frac{a}{2}\Gamma(1 + 8\alpha)(y^{(4\alpha)})^2. \quad (70)$$

Using the relation (65), the Euler-Lagrange equation for (70) is

$$\begin{aligned} D_x^1L - D_t^{3\alpha}(D_{y^{(3\alpha)}}^1L) + D_t^{4\alpha}(D_{y^{(4\alpha)}}^1L) = \\ cx - f + b\Gamma(1 + 6\alpha)D_t^{3\alpha}y^{(3\alpha)} + a\Gamma(1 + 8\alpha)D_t^{4\alpha}y^{(4\alpha)} = \\ cx - f + b\Gamma(1 + 6\alpha)y^{(6\alpha)} + a\Gamma(1 + 8\alpha)y^{(8\alpha)} = 0. \end{aligned} \quad (71)$$

Proposition 8. *The equation (67) represents the Euler-Lagrange equation on the bundle $J^{4\alpha}(\mathbb{R}, \mathbb{R})$ for $\alpha = \frac{1}{4}$, with the Lagrange function given by*

$$L(t, x, y^{(3/2)}, y^{(2)}) = \frac{1}{2}cx^2 - fx - \frac{b}{2}\Gamma(5/2)(y^{(3/2)})^2 + \frac{a}{2}\Gamma(3)(y^{(2)})^2. \quad (72)$$

2. Differential equations of order one, two and three which admit fractional Lagrangians

The following differential equations don't have classical Lagrangians such that the Euler-Lagrange equation represents the given equation:

$$\dot{x}(t) + V_1(t, x) = 0, \quad V_1(t, x) = \frac{\partial U_1(t, x)}{\partial x}, \quad (73)$$

$$\ddot{x}(t) + a_1\dot{x}(t) + V_2(t, x) = 0, \quad V_2(t, x) = \frac{\partial U_2(t, x)}{\partial x}, \quad (74)$$

$$\ddot{x}(t) + a_2\ddot{x}(t) + a_1\dot{x}(t) + V_3(t, x) = 0, \quad V_3(t, x) = \frac{\partial U_3(t, x)}{\partial x}. \quad (75)$$

Let us associate the fractional equations from below to the equations (73), (74) and (75), respectively:

$$D_t^{2\alpha}x(t) + V_1(t, x) = 0, \quad (76)$$

$$D_t^{4\alpha}x(t) + a_1D_t^{2\alpha}x(t) + V_2(t, x) = 0, \quad (77)$$

$$D_t^{6\alpha}x(t) + a_2D_t^{4\alpha}x(t) + a_1D_t^{2\alpha}x(t) + V_3(t, x) = 0. \quad (78)$$

Proposition 9. a) Let $J^\alpha(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ be the fractional bundle and consider $L : J^\alpha(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ given by

$$L(t, x, y^{(\alpha)}) = U_1(t, x) - \frac{1}{2}\Gamma(1 + 2\alpha)(y^\alpha)^2. \quad (79)$$

The Euler-Lagrange equation of (79) is

$$\begin{aligned} \frac{\partial L}{\partial x} - D_t^\alpha \left(\frac{\partial L}{\partial y^\alpha} \right) &= \frac{\partial U_1(t, x)}{\partial x} + \Gamma(1 + 2\alpha)y^{(2\alpha)} = \\ V_1(t, x) + D_t^{2\alpha}x(t) &= 0. \end{aligned} \quad (80)$$

b) Let $J^{2\alpha}(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ be the fractional bundle and the Lagrangian $L : J^{2\alpha}(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ given by

$$\begin{aligned} L(t, x, y^{(\alpha)}, y^{(2\alpha)}) &= U_2(t, x) - \frac{1}{2}a_1\Gamma(1 + 2\alpha)(y^\alpha)^2 + \\ \frac{1}{2}\Gamma(1 + 4\alpha)(y^{(2\alpha)})^2. \end{aligned} \quad (81)$$

The Euler-Lagrange equation of (81) is

$$\begin{aligned} \frac{\partial L}{\partial x} - D_t^\alpha \left(\frac{\partial L}{\partial y^\alpha} \right) + D_t^{2\alpha} \left(\frac{\partial L}{\partial y^{2\alpha}} \right) &= \\ V_2(t, x) + a_1\Gamma(1 + 2\alpha)y^{(2\alpha)} + \\ a_2\Gamma(1 + 4\alpha)y^{(4\alpha)} &= \\ V_2(t, x) + a_1D_t^{2\alpha}x(t) + D_t^{4\alpha}x(t) &= 0. \end{aligned} \quad (82)$$

c) Let $J^{3\alpha}(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ be the fractional bundle and $L : J^{3\alpha}(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ given by

$$\begin{aligned} L(t, x, y^{(\alpha)}, y^{(2\alpha)}, y^{(3\alpha)}) &= V_3(t, x) - \frac{a_1}{2}\Gamma(1 + 2\alpha)(y^{(\alpha)})^2 + \\ \frac{a_2}{2}\Gamma(1 + 4\alpha)(y^{(2\alpha)})^2 - \frac{1}{2}\Gamma(1 + 6\alpha)(y^{(3\alpha)})^2. \end{aligned} \quad (83)$$

The Euler-Lagrange equation of (83) is

$$\begin{aligned} \frac{\partial L}{\partial x} - D_t^\alpha \left(\frac{\partial L}{\partial y^\alpha} \right) + D_t^{2\alpha} \left(\frac{\partial L}{\partial y^{(2\alpha)}} \right) - D_t^{3\alpha} \left(\frac{\partial L}{\partial y^{(3\alpha)}} \right) &= V_3(t, x) + \\ a_1 \Gamma(1 + 2\alpha) y^{(2\alpha)} + a_2 \Gamma(1 + 4\alpha) y^{(4\alpha)} + \Gamma(1 + 6\alpha) y^{(6\alpha)} &= \\ V_3(t, x) + a_1 D_t^{2\alpha} x(t) + a_2 D_t^{4\alpha} x(t) + D_t^{6\alpha} x(t) &= 0. \end{aligned} \quad (84)$$

d) For $\alpha = \frac{1}{2}$ we obtain the fractional Lagrangians that describe the equations (73), (74), (75), respectively

$$\begin{aligned} L(t, x, y^{(1/2)}) &= U_1(t, x) - \frac{1}{2} \Gamma(2) (y^{(1/2)})^2 \\ L(t, x, y^{(1/2)}, y^{(1)}) &= U_2(t, x) - \frac{1}{2} a_1 \Gamma(2) (y^{(1/2)})^2 + \frac{1}{2} \Gamma(3) (y^{(1)})^2 \\ L(t, x, y^{(1/2)}, y^{(1)}, y^{(3/2)}) &= U_3(t, x) - \frac{a_1}{2} \Gamma(2) (y^{(1/2)})^2 + \\ \frac{a_2}{2} \Gamma(2) (y^{(1)})^2 - \frac{1}{2} \Gamma(4) (y^{(3/2)})^2. & \end{aligned} \quad (85)$$

In the category of the equations (74) and (75) there are:

a) the nonhomogeneous classical friction equation

$$m\ddot{x}(t) + \gamma\dot{x}(t) - \frac{\partial U(t, x)}{\partial x} = 0, \quad (86)$$

b) the nonhomogeneous model of Phillips [8]

$$\ddot{x}(t) + a_1 \dot{x}(t) + b_1 x(t) + f(t) = 0, \quad (87)$$

c) the nonhomogeneous business cycle with innovation [8]

$$\ddot{y}(t) + a_2 \dot{y}(t) + a_1 y(t) + b_1 x(t) + f(t) = 0. \quad (88)$$

Conclusions

The paper presents the main differentiable structures on $J^\alpha(\mathbb{R}, M)$, in order to describe fractional differential equations and ordinary differential equations, using Lagrange functions defined on $J^\alpha(\mathbb{R}, M)$.

With the help of the methods shown, there may be analyzed other models, such as those found in [4] and [11].

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