

SYMPLECTIC SYMMETRIES OF 4-MANIFOLDS

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ABSTRACT. A study of symplectic actions of a finite group G on smooth 4-manifolds is initiated. The central new idea is the use of G -equivariant Seiberg-Witten-Taubes theory in studying the structure of the fixed-point set of these symmetries. The main result in this paper is a complete description of the fixed-point set structure (and the action around it) of a symplectic cyclic action of prime order on a minimal symplectic 4-manifold with $c_1^2 = 0$. Comparison of this result with the case of locally linear topological actions is made. As an application of these considerations, the triviality of many such actions on a large class of 4-manifolds is established. In particular, we show the triviality of homologically trivial symplectic symmetries of a $K3$ surface (in analogy with holomorphic automorphisms). Various examples and comments illustrating our considerations are also included.

1. INTRODUCTION

When studying smooth or locally linear, topological actions of a finite group G on 4-manifolds the central problem is to describe the structure of the fixed-point set and the action around it. This together with the G -signature theorem of Atiyah-Singer (cf. [1, 21]) leads to a wealth of information about the action of G . For smooth actions, there is additional information provided through the use of gauge theory (cf. e.g. [7, 9, 17, 3]). Indeed, the G -signature theorem (and gauge theory in the case of smooth actions) imposes restrictions on the symmetries of 4-manifolds through the fixed-point set and the action around it. However, in general no substantial insight can be obtained without more specific knowledge of the latter. In fact, Freedman's 4-dimensional topological surgery theory (cf. [14, 15]) allows various constructions of periodic homeomorphisms on simply-connected 4-manifolds. In particular, in the case of topological, locally linear actions results of Edmonds (cf. [10]) demonstrate great flexibility of such actions, and the work of Edmonds and Ewing (cf. [11]) shows that for locally linear, pseudofree (i.e. free in the complement of a finite subset) cyclic topological actions of prime order, the G -signature theorem holds most of the key to the existence. As for smooth finite group actions, a basic question is what additional restrictions the smooth structures of the 4-manifolds may impose on the fixed-point set and the action around it (e.g. nonsmoothability of topological actions).

In this paper, we initiate a study on a class of symmetries of smooth 4-manifolds, which we call *symplectic symmetries*. These are smooth finite group actions which preserve *some* symplectic structure on the 4-manifold. We recall that a symplectic structure is a closed, non-degenerate 2-form; in particular, the manifold is symplectic.

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The study of symplectic structures on smooth 4-manifolds is one of the most rapidly growing research area in manifold topology in recent years (cf. e.g. [20, 27, 18, 8, 40]). One of the fundamental problems in this study is to understand what restrictions a symplectic structure may impose on the underlying smooth structure of the 4-manifold. Thus the following seems to be a very natural question: how restrictive the structure of the fixed-point set and the action around it could be for symplectic symmetries of a smooth 4-manifold, and how this may depend on the underlying smooth structure of the 4-manifold ?

Our approach to this question is based on an equivariant version of the work of C.H. Taubes in [38, 39] (i.e. the G -equivariant Seiberg-Witten-Taubes theory). The basic observation is that when the equivariant versions of Taubes' theorems are applied, the canonical class of the symplectic 4-manifold is represented by a set of 2-dimensional symplectic subvarieties which is invariant under the group action. In principle, the structure of the fixed-point set and the action around it can be recovered from the induced action on the 2-dimensional symplectic subset. This is most effective when the symplectic 4-manifold is minimal with $c_1^2 = 0$; indeed in this case, the set of the 2-dimensional symplectic subvarieties representing the canonical class is relatively simple. While this paper focuses mainly on the structure of the fixed-point set, the question as how the symplectic symmetries may depend on the underlying smooth structure is investigated in [5].

When studying finite group actions on manifolds it is often important and beneficial to consider an induced action on some algebraic invariants associated with the manifold. There is an abundance of such situations, for example, the notion of a Reidemeister torsion and classification of lens spaces (cf. [30]), the classical Hurwitz theorem about rigidity of group actions on surfaces of genus ≥ 2 , rigidity of holomorphic actions on $K3$ surfaces (cf. [2]). The mentioned rigidity of group actions for the real and complex surfaces asserts that these actions are trivial if they are homologically trivial, i.e., inducing a trivial action on homology.

In this paper, we shall mainly consider actions which are either homologically trivial (over \mathbb{Q} coefficients), or slightly more generally, induce a trivial action on the second rational homology. (We remark, however, that our method is applicable to much more general situations.) Our first result is the following rigidity theorem.

Theorem A: *Let M be a symplectic 4-manifold which has trivial canonical class (over \mathbb{Q} coefficients) and nonzero signature, and obeys $b_2^+ \geq 2$. Then any homologically trivial (over \mathbb{Q} coefficients), symplectic action of a finite group G on M must be trivial.*

It should be pointed out that if the action is not required to be homologically trivial, then there are nontrivial holomorphic (in particular symplectic) automorphisms of Kählerian $K3$ surfaces (cf. [32]). Moreover, the condition $b_2^+ \geq 2$ in the theorem is necessary because there exist nontrivial holomorphic involutions on Enriques surfaces (where $b_2^+ = 1$) which are homologically trivial (cf. [31]). Finally, note that the nonvanishing of signature is also necessary as there is an abundance of homologically trivial, symplectic finite group actions on the standard symplectic 4-torus.

Note that the 4-manifold M in Theorem A has a unique Seiberg-Witten basic class by work of Taubes [38, 39]. Hence for any other symplectic structure on M which defines the same orientation, the canonical class is also trivial. In particular, for the case of $K3$ surfaces (given with the canonical orientation), we arrived at the following

Corollary A: *A homologically trivial symplectic symmetry of a $K3$ surface is trivial.*

Remarks: There is an open question (due to A. Edmonds, cf. [22], Problem 4.124 (B)) as whether any homologically trivial, smooth actions of a finite group on a $K3$ surface must be trivial. The question was motivated by the corresponding rigidity for holomorphic actions (cf. [2]), which was known to be false for topological ones (cf. [10]). Corollary A answers the question affirmatively for symplectic symmetries. For smooth actions in general, the homological rigidity was only known to be true for involutions (cf. [25, 36]).

Our main result in this paper is a complete description of the structure of the fixed-point set for a symplectic cyclic action of prime order on a minimal symplectic 4-manifold M with $c_1^2 = 0$ and $b_2^+ \geq 2$, which induces a trivial action on $H^2(M; \mathbb{Q})$. For simplicity, we shall only state the result for the case of pseudofree actions (i.e., actions with only isolated fixed points). Note that in this case, the induced action on the tangent space at a fixed point is called the local representation. (The general case where the fixed point set may contain 2-dimensional components is addressed in Theorem 3.2, with some refined statements contained in Proposition 3.7.)

Theorem B: *Let M be a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, which admits a nontrivial, pseudofree action of $G \cong \mathbb{Z}_p$, where p is prime, such that the symplectic structure is preserved under the action and the induced action on $H^2(M; \mathbb{Q})$ is trivial. Then the set of fixed points of G can be divided into groups each of which belongs to one of the following five possible types. (We set $\mu_p \equiv \exp(\frac{2\pi i}{p})$.)*

- (1) *One fixed point with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some $k \not\equiv 0 \pmod{p}$, i.e., with local representation contained in $SL_2(\mathbb{C})$.*
- (2) *Two fixed points with local representation $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2)$, $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{6k} z_2)$ for some $k \not\equiv 0 \pmod{p}$ respectively. This type of fixed points occurs only when $p > 5$.*
- (3) *Three fixed points, one with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2)$ and the other two with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$ for some $k \not\equiv 0 \pmod{p}$. This type of fixed points occurs only when $p > 3$.*
- (4) *Four fixed points, one with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ and the other three with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2)$ for some $k \not\equiv 0 \pmod{p}$. This type of fixed points occurs only when $p > 3$.*
- (5) *Three fixed points, each with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ for some $k \not\equiv 0 \pmod{p}$. This type of fixed points occurs only when $p = 3$.*

Combined with the G -signature theorem, Theorem B implies the following rigidity for the corresponding homologically trivial actions.

Corollary B: *Let M be a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, which admits a homologically trivial (over \mathbb{Q} coefficients), pseudofree, symplectic \mathbb{Z}_p -action for a prime $p > 1$. Then the following conclusions hold.*

- (a) *The action is trivial if $p \not\equiv 1 \pmod{4}$, $p \not\equiv 1 \pmod{6}$, and the signature of M is nonzero. In particular, if the signature of M is nonzero, then for infinitely many primes p the manifold M does not admit any such nontrivial \mathbb{Z}_p -actions.*
- (b) *The action is trivial as long as there is a fixed point of type (1) in Theorem B.*

Remarks: (1) Corollary B shows that symplectic symmetries are far more restrictive than topological ones. Indeed, there is the following theorem of Edmonds (cf. [10]).

Let M be a closed simply-connected 4-manifold. Then there is a locally linear, pseudofree, homologically trivial action of \mathbb{Z}_p on M for every prime $p > 3$.

On the other hand, one should compare the nonexistence results in Corollary B with a theorem of Edmonds (cf. [26]) which is of a purely topological nature.

Let M be a closed simply-connected 4-manifold with $b_2 \geq 3$. If a finite group G acts on M locally linearly, pseudofreely, and homologically trivially, then G must be a cyclic group.

(2) Corollary B should also be compared with the (stronger) results of Peters (cf. [35]) on homological rigidity of holomorphic actions on Kählerian elliptic surfaces. However, we would like to point out that Peters' result relies essentially on the fact that a homologically trivial holomorphic automorphism on an elliptic surface preserves the holomorphic elliptic fibration, which does not have an analog in the symplectic category. Moreover, symplectic 4-manifolds are a much larger class of manifolds than Kähler surfaces (cf. [18]) (note that this is true even for the case of $c_1^2 = 0$, cf. [13]), and symplectic automorphisms are much more flexible than holomorphic automorphisms — the former form an infinite dimensional Lie Group while the latter only a finite dimensional one.

(3) It is essential that the symplectic automorphisms in Theorem A and Corollary B are of finite order, i.e. our rigidity for symplectic automorphisms is a phenomenon of finite group actions. The necessity of being finite order can be easily seen from the following example: Let $\phi : M \rightarrow M$ be any Hamiltonian symplectomorphism. Then the group $G \equiv \{\phi^n | n \in \mathbb{Z}\}$ acts on M homologically trivially because ϕ is homotopic to identity. Of course, G will be of infinite order in general.

In comparison, the group of holomorphic automorphisms of (compact) Kähler manifolds preserving the Kähler form is compact (cf. [24, 16]); in particular, the group is automatically finite if the Kähler manifold admits no nonzero holomorphic vectorfields. This is certainly not true in the symplectic category.

(4) Theorem A may be regarded as a special case of Corollary B(b). In fact, when applying the equivariant versions of Taubes' theorems in [38, 39] in the context of Theorem A, the 2-dimensional symplectic subvarieties representing the canonical class must be empty because the canonical class is trivial in this case. This has the consequence that the action is pseudofree, and that the fixed-point set consists of only

type (1) fixed points in Theorem B, which is nonempty because the 4-manifold has nonzero signature.

The rest of the paper is contained in two sections. The first one, which consists of two parts, gives a proof of Theorem A. We present the proof of Theorem A separately so as to illustrate the general philosophy of our paper while keeping the necessary technicalities at bay. The second section contains discussion of symplectic \mathbb{Z}_p -actions (not necessarily pseudofree) on a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, and gives a proof for Theorem B and Corollary B. It also contains some examples which illustrate our considerations.

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2. THE PROOF OF THEOREM A

This section is divided into two parts. In part 1 we first briefly discuss the G -equivariant Seiberg-Witten-Taubes theory, then we use it to show that the fixed points of the action are isolated, with local representations all contained in $SL_2(\mathbb{C})$. In part 2 we combine the fixed-point data in part 1 with the G -signature theorem to show that the action is trivial.

Part 1: Let M be a smooth, oriented 4-manifold with an orientation-preserving action of a finite group G . We shall begin by considering the G -equivariant Seiberg-Witten theory on M (which is equivalent to the Seiberg-Witten theory on the orbifold M/G , cf. [4]).

More concretely, we fix a Riemannian metric g on M such that G acts via isometries. Suppose there exists a G - $Spin^{\mathbb{C}}$ structure on M , i.e. a lifting of the principal $SO(4)$ -bundle of orthonormal frames on M to a principal $Spin^{\mathbb{C}}(4)$ -bundle as G -bundles. Then there are associated $U(2)$ vector G -bundles (of rank 2) S_+ , S_- with $\det(S_+) = \det(S_-)$, and a G -equivariant Clifford multiplication which maps T^*M into the skew adjoint endomorphisms of $S_+ \oplus S_-$.

With the preceding understood, we consider the G -equivariant Seiberg-Witten equations associated to the G - $Spin^{\mathbb{C}}$ structure

$$D_A\psi = 0 \text{ and } P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + \mu$$

which are equations for pairs (A, ψ) , where A is a G -equivariant $U(1)$ -connection on $\det(S_+)$ and $\psi \in \Gamma(S_+)$ is a G -equivariant smooth section of S_+ . As for the notations involved, here $D_A : \Gamma(S_+) \rightarrow \Gamma(S_-)$ is the Dirac operator canonically defined from the Levi-Civita connection on M and the $U(1)$ -connection A on $\det(S_+)$, P_+ is the orthogonal projection onto the subspace of self-dual G -equivariant 2-forms, τ (which is canonically defined from the Clifford multiplication) is a map from the space of endomorphisms of S_+ into the space of imaginary valued self-dual G -equivariant 2-forms, and μ is a fixed, imaginary valued self-dual G -equivariant 2-form which is added in as a perturbation term. Note that the G -equivariant Seiberg-Witten equations are

invariant under the gauge transformations $(A, \psi) \mapsto (A - 2\varphi^{-1}d\varphi, \varphi\psi)$, where φ are circle valued G -invariant smooth functions on M .

Let $b_G^{2,+}$ be the dimension of the maximal subspace of $H^2(M; \mathbb{R})$ over which the cup-product is positive and the induced action of G is trivial. Then as in the non-equivariant case, the space \mathcal{M}^G of gauge equivalence classes of the solutions to the G -equivariant Seiberg-Witten equations is compact, and when $b_G^{2,+} \geq 2$, it is an orientable smooth manifold (if nonempty) for a generic choice of (g, μ) , whose cobordism class is independent of the data (g, μ) . An invariant, denoted by $SW_G(S_+)$, can be similarly defined, which, for instance when $\dim \mathcal{M}^G = 0$, is a signed sum of the points in \mathcal{M}^G . Moreover, there is an involution I on the set of G - $Spin^{\mathbb{C}}$ structures which obeys $\det(I(S_+)) = -\det(S_+)$ and $SW_G(I(S_+)) = \pm SW_G(S_+)$.

Given the above terminology and notation, we claim the following

Lemma 2.1. *Let (M, ω) be a symplectic 4-manifold with trivial canonical class $c_1(K)$. Suppose a finite group G acts on M via symplectic automorphisms such that $b_G^{2,+} \geq 2$. Then the canonical bundle K is isomorphic to the trivial bundle $M \times \mathbb{C}$ as G -bundles, where G acts on the second factor of $M \times \mathbb{C}$ trivially.*

Proof. Fix a G -equivariant ω -compatible almost complex structure J , and let $g = \omega(\cdot, J(\cdot))$ be the associated G -equivariant Riemannian metric. Then there is a canonical $Spin^{\mathbb{C}}$ -structure on M such that the associated $U(2)$ bundles are given by $S_+^0 = \mathbb{I} \oplus K^{-1}$ and $S_-^0 = T^{0,1}M$, where \mathbb{I} is the trivial complex line bundle. Clearly, this canonical $Spin^{\mathbb{C}}$ -structure is a G - $Spin^{\mathbb{C}}$ structure on M , with \mathbb{I} being understood as the G -bundle $M \times \mathbb{C}$ where G acts on the second factor trivially.

According to Taubes [38], there is a canonical (up to gauge equivalence) connection A_0 on $K^{-1} = \det(S_+^0)$, such that if we set $u_0 = (1, 0) \in \Gamma(\mathbb{I} \oplus K^{-1})$, then for sufficiently large $r > 0$, $(A_0, \sqrt{r}u_0)$ is the only solution (up to gauge equivalence) to the Seiberg-Witten equations

$$D_A \psi = 0 \text{ and } P_+ F_A = \frac{1}{4} \tau(\psi \otimes \psi^*) + \mu,$$

where $\mu = -\frac{i}{4}r\omega + P_+ F_{A_0}$, and furthermore, $(A_0, \sqrt{r}u_0)$ is a non-degenerate solution. In the present situation, observe that A_0 is G -equivariant, so is the perturbation $\mu = -\frac{i}{4}r\omega + P_+ F_{A_0}$. Hence $(A_0, \sqrt{r}u_0)$, which is G -equivariant, is also a solution to the G -equivariant Seiberg-Witten equations. We claim that $(A_0, \sqrt{r}u_0)$ is the only solution up to G -invariant gauge equivalence. To see this, suppose $(A, \sqrt{r}u)$ is another solution. Then $(A, \sqrt{r}u) = (A_0 - 2\varphi^{-1}d\varphi, \varphi \cdot \sqrt{r}u_0)$ for some circle valued smooth function φ on M , from which we see that φ is G -invariant. Hence the claim. Now note that $(A_0, \sqrt{r}u_0)$ is also non-degenerate as a solution to the G -equivariant Seiberg-Witten equations. This implies that $SW_G(S_+^0) = \pm 1$.

By the assumption $b_G^{2,+} \geq 2$, we have $SW_G(I(S_+^0)) = \pm 1$ as well, where I is the involution on the set of G - $Spin^{\mathbb{C}}$ structures, and $I(S_+^0) = K \oplus \mathbb{I}$. Thus for any $r > 0$, the G -equivariant Seiberg-Witten equations associated to the G - $Spin^{\mathbb{C}}$ structure $I(S_+^0)$, with perturbation $\mu = -\frac{i}{4}r\omega + P_+ F_{A_0}$, has a solution (A, ψ) . If we write $\psi = \sqrt{r}(\alpha, \beta) \in \Gamma(K \oplus \mathbb{I})$, then according to Taubes [39], the zero locus $\alpha^{-1}(0)$, if nonempty, will pointwise converge, as $r \rightarrow +\infty$, to a set of finitely many J -holomorphic

curves with multiplicity, which represents the Poincaré dual of $c_1(K)$. Since $c_1(K)$ is trivial, $\alpha^{-1}(0)$ must be empty when r is sufficiently large. The section α of K is G -equivariant, and is nowhere vanishing for large r , hence it defines isomorphism of G -bundles between K and $M \times \mathbb{C}$, where G acts trivially on the second factor of $M \times \mathbb{C}$. \square

Lemma 2.1 provides an important information about the structure of the fixed-point set M^G of an action of G on M . This information is summarized in the following

Corollary 2.2. *Let (M, ω) be a symplectic 4-manifold with trivial canonical class $c_1(K)$. Suppose a finite group $G \equiv \mathbb{Z}_p$, $p > 1$ prime, acts on M via symplectic automorphisms such that $b_G^{2,+} \geq 2$. Then the fixed-point set $M^G \equiv \{m \in M : gm = m, \forall g \in G\}$, if nonempty, consists of finitely many isolated points, such that with respect to an ω -compatible almost complex structure on M , the complex representation of $G \equiv \mathbb{Z}_p$ at each of these fixed points is given by $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some $k \neq 0 \pmod{p}$. (Here $\mu_p \equiv \exp(\frac{2\pi i}{p})$.)*

Proof. Let $m \in M^G$ be any fixed point. By the equivariant Darboux' theorem, one can choose an ω -compatible almost complex structure on M which is integrable near m , such that there are holomorphic coordinates z_1, z_2 near m within which $\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ and the action of G is given by $(z_1, z_2) \mapsto (\mu_p^{m_1} z_1, \mu_p^{m_2} z_2)$ for some m_1, m_2 , where $m_1 \neq 0 \pmod{p}$. Moreover, $dz_1 \wedge dz_2$ defines a local section of the canonical bundle K , with the induced action of G given by

$$dz_1 \wedge dz_2 \mapsto \mu_p^{-(m_1+m_2)} dz_1 \wedge dz_2.$$

By Lemma 2.1, K and $M \times \mathbb{C}$ are isomorphic as G -bundles, where G acts trivially on the second factor of $M \times \mathbb{C}$. This implies that $m_1 + m_2 = 0 \pmod{p}$, from which the corollary follows easily. \square

Part 2: For readers' convenience, we first recall a version of the G -signature theorem which will be used in this paper.

Let M be a closed, oriented smooth 4-manifold, and let $G \equiv \mathbb{Z}_p$ be a cyclic group of prime order p which acts on M effectively via orientation-preserving diffeomorphisms. Then the fixed point set M^G , if nonempty, will be in general a disjoint union of finitely many isolated points and 2-dimensional orientable submanifolds. Moreover, at each isolated fixed point, G defines a local complex representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$ for some $k, q \neq 0 \pmod{p}$, where q is uniquely determined, and k is determined up to a sign. (Here $\mu_p \equiv \exp(\frac{2\pi i}{p})$).

With the preceding understood, we state the G -signature theorem (cf. [21]).

Theorem 2.3. (G -signature theorem for prime order cyclic actions)

$$|G| \cdot \text{sign}(M/G) = \text{sign}(M) + \sum_m \text{def}_m + \sum_Y \text{def}_Y$$

where m stands for an isolated fixed point, and Y for a 2-dimensional component of M^G . The terms, def_m and def_Y (which are called signature defect), are given by the

following formulae:

$$\text{def}_m = \sum_{k=1}^{p-1} \frac{(1 + \mu_p^k)(1 + \mu_p^{kq})}{(1 - \mu_p^k)(1 - \mu_p^{kq})}$$

if the local representation at m is given by $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$, and

$$\text{def}_Y = \frac{p^2 - 1}{3} \cdot (Y \cdot Y)$$

where $Y \cdot Y$ is the self-intersection of Y .

Now back to the proof of Theorem A. Without loss of generality we may assume that $G \cong \mathbb{Z}_p$ where $p > 1$ is prime. Suppose now that G acts trivially on $H_*(M; \mathbb{Q})$. Then $b_G^{2,+} = b_2^+ \geq 2$ so that Corollary 2.2 is true. The Lefschetz fixed point theorem together with Corollary 2.2 implies that the fixed-point set M^G is nonempty and consists of finitely many isolated points. Indeed, the classical formula relating the Euler characteristic $\chi(M)$ and the signature $\text{sign}(M)$, i.e.,

$$2\chi(M) + 3 \text{sign}(M) = c_1^2(K)[M],$$

gives $\chi(M) = -\frac{3}{2} \text{sign}(M) \neq 0$. Moreover, we would like to point out that the number of fixed points $|M^G|$ equals the Euler characteristic $\chi(M)$ (cf. [23]).

On the other hand, by Corollary 2.2 we observe that, in the G -signature theorem

$$|G| \cdot \text{sign}(M/G) = \text{sign}(M) + \sum_m \text{def}_m + \sum_Y \text{def}_Y,$$

def_m is independent of m and is given by

$$\text{def}_m = \sum_{k=1}^{p-1} \frac{(1 + \mu_p^k)(1 + \mu_p^{-k})}{(1 - \mu_p^k)(1 - \mu_p^{-k})}$$

and there are no terms def_Y . With $\text{sign}(M/G) = \text{sign}(M)$ and $|M^G| = \chi(M) = -\frac{3}{2} \text{sign}(M)$, the G -signature theorem gives rise to

$$\text{def}_m = \frac{1}{|M^G|} \cdot (p-1) \cdot \text{sign}(M) = \frac{2}{3}(1-p).$$

Theorem A follows easily from the following explicit calculation of def_m , which contradicts the above equation.

Lemma 2.4. $\text{def}_m = \frac{1}{3}(p-1)(p-2)$.

Proof. It turns out that def_m can be computed in terms of Dedekind sum $s(q, p)$ (cf. [21], page 92), where

$$s(q, p) = \sum_{k=1}^p \left(\left(\frac{k}{p} \right) \right) \left(\left(\frac{kq}{p} \right) \right)$$

with

$$\left((x) \right) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

(Here $[x]$ stands for the greatest integer less than or equal to x .)

In fact, equation (24) in [21], page 180, gives

$$\text{def}_m = -4p \cdot s(q, p), \text{ with } q = -1.$$

One then computes $6p \cdot s(q, p)$ by (cf. [21], equations (10) and (9) on page 94)

$$6p \cdot s(q, p) = (p - 1)(2pq - q - \frac{3p}{2}) - 6f_p(q),$$

where $f_p(q) = \sum_{k=1}^{p-1} k[\frac{kq}{p}]$. Since $f_p(-1) = \sum_{k=1}^{p-1} k \cdot (-1) = \frac{1}{2}(1 - p)p$, one obtains

$$\text{def}_m = \frac{1}{3}(p - 1)(p - 2)$$

as claimed. □

Remark 2.5. (1) The calculation of def_m remains valid if $q = -1$ is replaced by the equivalent (i.e. congruent mod p) value of $q = p - 1$.

(2) It is quite instructive to compare the proof of homological rigidity of holomorphic actions on Kählerian $K3$ surfaces in [2, 34] with our proof of rigidity of symplectic actions on smooth 4-manifolds. Both consists of the same two basic steps. First, one shows that the fixed-point set of the action consists of isolated fixed points with specific representations around them. In the holomorphic case this is a rather obvious observation, whereas in the symplectic case one needs to use deep results of Taubes on the Seiberg-Witten theory. Next, the holomorphic Lefschetz fixed point theorem is used to reach a contradiction for the $K3$ case. Here the Kähler condition as well as the holomorphicity of the action are crucial; in particular the argument uses information about the action that is encoded in the quadratic form of the $K3$ surfaces (via the Hodge decomposition and the holomorphic Lefschetz fixed point theorem). In our proof the contradiction is reached by computing with the G -signature theorem. This has the advantage of being applicable to any smooth (or even locally linear topological) actions on 4-manifolds.

3. SYMPLECTIC \mathbb{Z}_p -ACTIONS ON 4-MANIFOLDS WITH $c_1^2 = 0$

In this section we push further the techniques of G -equivariant Seiberg-Witten-Taubes theory to the case where $c_1^2 = 0$. The bulk of this section constitutes an analysis of the structure of the fixed-point set of a symplectic \mathbb{Z}_p -action on a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, which induces a trivial action on the second rational cohomology. The results are summarized in Theorem 3.2 below (with some refined statements contained in Proposition 3.7). Theorem B is a special case of Theorem 3.2 where the action is pseudofree. We set $\mu_p \equiv \exp(\frac{2\pi i}{p})$ throughout.

Let (M, ω) be a symplectic 4-manifold and let $G \equiv \mathbb{Z}_p$, where $p > 1$ and is prime, act on M via symplectomorphisms. Then the fixed-point set M^G , if nonempty, consists of two types of connected components in general: an isolated point or an embedded symplectic surface. Moreover, the local representation at each isolated fixed point may be written as $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$, where the local complex coordinates (z_1, z_2) are compatible with the symplectic structure ω ; in particular, the integers k, q are uniquely determined in their congruence classes (mod p).

Now suppose $b_G^{2,+} \geq 2$ (recall that $b_G^{2,+}$ is the dimension of the maximal subspace of $H^2(M; \mathbb{R})$ over which the cup-product is positive and the induced action of G is trivial). Then as we argued in the proof of Lemma 2.1, given any G -equivariant ω -compatible almost complex structure J , there is a solution (A, ψ) , with $\psi \in \Gamma(K \oplus \mathbb{I})$, to the G -equivariant Seiberg-Witten equations with perturbation $\mu = -\frac{i}{4}r\omega + P_+F_{A_0}$ for any $r > 0$. Moreover, if we write $\psi = \sqrt{r}(\alpha, \beta)$, then as $r \rightarrow +\infty$ the zero set $\alpha^{-1}(0)$ will pointwise converge to a union of finitely many J -holomorphic curves $\cup_i C_i$ such that $c_1(K)$ is Poincaré dual to the fundamental class of $\sum_i n_i C_i$ for some $n_i > 0$. Since α is a G -equivariant section of K , we arrived at the following observations.

- The set $\cup_i C_i$ is invariant under the action of G .
- For any fixed point $m \in M^G$, if $m \in M \setminus \cup_i C_i$, then $\alpha(m) \neq 0$ for sufficiently large $r > 0$. Lemma 2.1 and hence Corollary 2.2 hold true locally near m . As a consequence, we see that m is an isolated fixed point with local representation contained in $SL_2(\mathbb{C})$.

We summarize the preceding discussion in the following

Lemma 3.1. *Suppose $b_G^{2,+} \geq 2$. Then given any G -equivariant ω -compatible almost complex structure J , the canonical class $c_1(K)$ is represented by the fundamental class of $\sum_i n_i C_i$ for some $n_i > 0$, where $\{C_i\}$ is a finite set of J -holomorphic curves which has the following significance.*

- *The set $\cup_i C_i$ is invariant under the action of G .*
- *Let m be a fixed point not contained in $\cup_i C_i$. Then m must be an isolated fixed point with local representation contained in $SL_2(\mathbb{C})$. In particular, any 2-dimensional component in M^G is contained in $\cup_i C_i$.*

Lemma 3.1 allows us to extract information about the fixed-point set M^G and the action around it by analyzing the action of G in a neighborhood of $\cup_i C_i$. While this requires a priori knowledge about the structure of $\cup_i C_i$, it can be determined with various additional assumptions on the canonical class of the manifold. In fact, we will show that when (M, ω) is minimal with $c_1^2 = 0$, each connected component of $\cup_i C_i$ is either a nonsingular elliptic curve, or may be identified with a singular fiber of an elliptic fibration. With these understood, we now state

Theorem 3.2. *Let (M, ω) be a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, which admits a nontrivial action of $G \cong \mathbb{Z}_p$, where p is prime, such that the symplectic structure is preserved under the action and the induced action on $H^2(M; \mathbb{Q})$ is trivial. Then for any G -equivariant ω -compatible almost complex structure J , there are J -holomorphic curves $\{C_i\}$ and positive integers $\{n_i\}$, such that $\cup_i C_i$ is G -invariant and the Poincaré dual of the canonical class $c_1(K)$ is represented by the fundamental class of $\sum_i n_i C_i$. Furthermore, the following statements describe the structure of $\cup_i C_i$ as well as that of the fixed-point set M^G .*

- (1) *If a fixed point is not contained in $\cup_i C_i$, it must be isolated with local representation contained in $SL_2(\mathbb{C})$.*
- (2) *The following is a list of all possibilities for a connected component of $\cup_i C_i$:*
 - (I) *An embedded torus with self-intersection 0.*

- (II) *A cusp sphere with self-intersection 0.*
 - (III) *A nodal sphere with self-intersection 0.*
 - (IV) *A union of two embedded (-2) -spheres intersecting at a single point with tangency of order 2.*
 - (V) *A union of embedded (-2) -spheres intersecting transversely.*
- (3) *Accordingly, the possibilities for the associated fixed-point data are listed below, if the connected component of $\cup_i C_i$ contains at least one fixed point:*
- (i) *For a type I component, there are three possibilities: (a) it is entirely fixed by G , (b) it contains four isolated fixed points each of which having a local representation contained in $SL_2(\mathbb{C})$, and (c) it contains three isolated fixed points, all having the same local representation which is either $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ or $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2)$ for some $k \neq 0 \pmod{p}$. Moreover, case (b) occurs only when $p = 2$ and case (c) occurs only when $p = 3$.*
 - (ii) *For a type II component, there are two isolated fixed points contained in it. One of them is the cusp-singularity, which has local representation $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2)$ while the other fixed point has local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{6k} z_2)$, for some $k \neq 0 \pmod{p}$. This case occurs only when $p \geq 5$.*
 - (iii) *A type III component contains only one isolated fixed point, which is the nodal point, with local representation contained in $SL_2(\mathbb{C})$.*
 - (iv) *A type IV component contains three isolated fixed points, one of which is the intersection of the two (-2) -spheres. As for local representations, the intersection point always has $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2)$, and for each of the other two fixed points, there are two possibilities: $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$ which occurs only when $p > 3$, or $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{-2k} z_2)$ which occurs only when $p = 3$, for some $k \neq 0 \pmod{p}$.*
 - (v) *For a type V component, there are three possibilities: (a) it contains at least one 2-dimensional component of M^G , (b) it contains only a number of isolated fixed points whose local representations are all contained in $SL_2(\mathbb{C})$, and (c) it contains exactly four isolated fixed points, one of which has local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ and each of the other three has $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2)$ for some $k \neq 0 \pmod{p}$. The last possibility occurs only when $p \neq 3$.*

(We would like to point out that the fixed points in Theorem B are slightly reorganized. For instance, for type (2) fixed points in Theorem B the range of possible primes p is $p > 5$, while in Theorem 3.2 (3) (ii), the range is $p \geq 5$. This is because when $p = 5$, the two fixed points contained in the type II component all have local representation in $SL_2(\mathbb{C})$, therefore are classified in Theorem B as type (1) fixed points. Similar remarks apply to other cases as well.)

The proof of Theorem 3.2 may be divided into two stages. In the first stage, we determine the structure of $\cup_i C_i$, while in the second stage, we analyze the action of G in a neighborhood of $\cup_i C_i$. The following lemma is the starting point for stage 1.

Lemma 3.3. *Assume (M, ω) is minimal with $c_1^2 = 0$. Then $c_1(K) \cdot C_i = 0$ for all C_i , and $C_i^2 \geq 0$ unless C_i is an embedded (-2) -sphere.*

Proof. Note that for any C_i ,

$$c_1(K) \cdot C_i = \sum_j n_j C_j \cdot C_i \geq n_i C_i^2.$$

Thus if $c_1(K) \cdot C_i < 0$, $C_i^2 < 0$ also, so that

$$C_i^2 + c_1(K) \cdot C_i \leq (-1) + (-1) = -2.$$

By the adjunction formula (cf. [28])

$$C_i^2 + c_1(K) \cdot C_i + 2 \geq 2 \cdot \text{genus}(C_i),$$

C_i is an embedded sphere with $C_i^2 = -1$, which contradicts the minimality of (M, ω) . Hence $c_1(K) \cdot C_i \geq 0$ for all C_i , which implies $c_1(K) \cdot C_i = 0$ for all C_i because $0 = c_1^2 = \sum_i n_i c_1(K) \cdot C_i$.

To see the last statement, we note that the adjunction formula with $c_1(K) \cdot C_i = 0$ gives rise to $C_i^2 \geq 2 \cdot \text{genus}(C_i) - 2$, which implies that $C_i^2 \geq 0$ unless C_i is an embedded (-2) -sphere. □

Let $\{\Lambda_\alpha\}$ be the set of connected components of $\cup_i C_i$, where we write $\Lambda_\alpha = \cup_{i \in I_\alpha} C_i$ for some index set I_α . Denote by $|I_\alpha|$ the cardinality of I_α .

First of all, note that for any α and any $i \in I_\alpha$,

$$0 = c_1(K) \cdot C_i = \sum_{j \in I_\alpha} n_j C_j \cdot C_i = \sum_{j \neq i} n_j C_j \cdot C_i + n_i C_i^2.$$

It follows from Lemma 3.3 that if $|I_\alpha| \geq 2$, then Λ_α is a union of embedded spheres with self-intersection -2 . To analyze the case of $|I_\alpha| = 1$, we need the following refined version of the adjunction formula we used earlier: Let C be a J -holomorphic curve, with δ double points and a number of branch points indexed by j , then

$$C^2 + c_1(K) \cdot C + 2 = 2 \cdot \text{genus}(C) + 2\delta + \sum_j 2\kappa_j$$

where κ_j denotes the Milnor number of the branch point indexed by j (cf. [29], Theorem 7.3). Furthermore, it is well-known that if the Milnor number equals 1, then the branch point must be the cusp-singularity defined by the equation $z^2 + w^3 = 0$. Now suppose $|I_\alpha| = 1$ and C is the J -holomorphic curve contained in Λ_α . Then we have $C^2 = c_1(K) \cdot C = 0$, which implies that C is either an embedded torus, or an immersed sphere with one double point (i.e. a nodal sphere), or a sphere with one cusp-singularity (i.e. a cusp sphere), all with self-intersection 0.

To further analyze the components Λ_α with $|I_\alpha| \geq 2$, we observe the following

Lemma 3.4. *Let Λ_α be any connected component such that $|I_\alpha| \geq 2$ and there exist $i, j \in I_\alpha$ with $C_i \cdot C_j \geq 2$. Then $|I_\alpha| = 2$, and if we denote by C_1, C_2 the two J -holomorphic curves contained in Λ_α , then one of the following is true.*

- (1) C_1, C_2 intersect at a single point with tangency of order 2.

(2) C_1, C_2 intersect at two distinct points transversely.

Proof. Note that $0 = c_1(K) \cdot C_i = \sum_{k \in I_\alpha} n_k C_k \cdot C_i \geq n_j C_j \cdot C_i + n_i C_i^2$ implies

$$2n_j \leq n_j C_i \cdot C_j \leq -n_i C_i^2 = 2n_i.$$

Similarly, one has $2n_i \leq 2n_j$, hence $n_i = n_j$. Moreover, one must also have $C_i \cdot C_j = 2$ and $|I_\alpha| = 2$. The lemma follows easily. \square

Next we analyze the components Λ_α with $|I_\alpha| \geq 2$ and $C_i \cdot C_j = 1$ for any distinct $i, j \in I_\alpha$. In this case, Λ_α is a union of embedded (-2) -spheres with transverse intersections. Such a configuration can be conveniently represented by a graph Γ_α , where each $C_i \subset \Lambda_\alpha$ corresponds to a vertex $v_i \in \Gamma_\alpha$, and each intersection point in $C_i \cap C_j$ corresponds to an edge connecting the vertices v_i, v_j . Moreover, one can associate a matrix $Q_\alpha = (q_{ij})$, where $q_{ii} = 1$ for any $i \in I_\alpha$, $q_{ij} = -\frac{1}{2}$ for any distinct $i, j \in I_\alpha$ with $C_i \cdot C_j \neq 0$, and $q_{ij} = 0$ otherwise. Now observe that $(\sum_{k \in I_\alpha} n_k C_k) \cdot C_i = c_1(K) \cdot C_i = 0$ for all $i \in I_\alpha$, so that the matrix Q_α satisfies the conditions (ii), (iii) and (i)' in Lemma 2.10 of [2], which implies that Q_α is positive semi-definite. By Lemma 2.12 (ii) in [2], the graph Γ_α must be one from the list in Figure 1 below (cf. [2], page 20), with $n \geq 2$ if Γ_α is of type \tilde{A}_n .

We shall summarize the analysis on $\cup_i C_i$ by categorizing the components $\{\Lambda_\alpha\}$ of $\cup_i C_i$ into the following three types.

- (A) $|I_\alpha| = 1$ and Λ_α is either an embedded torus, or a nodal sphere, or a cusp sphere, all with self-intersection 0.
- (B) $|I_\alpha| = 2$ and Λ_α is a union of two embedded (-2) -spheres intersecting at a single point with tangency of order 2.
- (C) $|I_\alpha| \geq 2$ and Λ_α is a union of embedded (-2) -spheres intersecting transversely. The corresponding graph Γ_α is one from the list in Figure 1. Note that we allow $n = 1$ in type A_n graphs, which represents case (2) of Lemma 3.4.

We end the discussion in stage 1 of the proof by identifying the G -invariant components of $\cup_i C_i$ under the assumption that G induces a trivial action on $H^2(M; \mathbb{Q})$.

Lemma 3.5. *Assume G induces a trivial action on $H^2(M; \mathbb{Q})$. Then the following hold true.*

- (1) *If a component Λ_α is not G -invariant, then Λ_α must be of type (A).*
- (2) *If a component Λ_α is G -invariant, then each J -holomorphic curve $C_i \subset \Lambda_\alpha$ must also be G -invariant.*

Proof. First, if a component Λ_α is not G -invariant, then for any $C_i \subset \Lambda_\alpha$, C_i is disjoint from $g \cdot C_i$ for any $1 \neq g \in G$. Particularly, $C_i^2 = (g \cdot C_i) \cdot C_i = 0$ because G induces a trivial action on $H^2(M; \mathbb{Q})$. Clearly such a component Λ_α is of type (A).

Secondly, suppose Λ_α is G -invariant. Then if for some $C_i \subset \Lambda_\alpha$, $g \cdot C_i \neq C_i$ for some $g \in G$, then $|I_\alpha| \geq 2$, and hence Λ_α is either of type (B) or type (C). In any case, $-2 = C_i^2 = (g \cdot C_i) \cdot C_i \geq 0$, which is a contradiction. \square

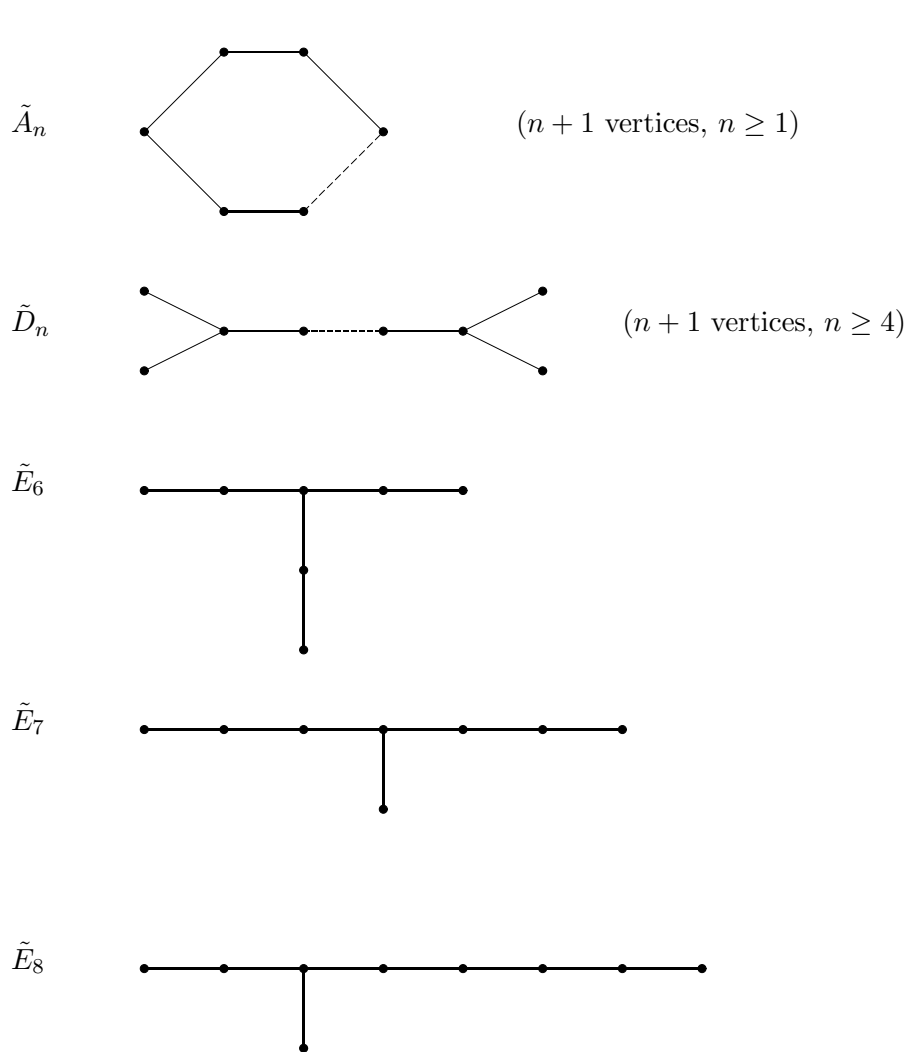


FIGURE 1.

Next we enter stage 2 of the proof where we analyze the action of G in a neighborhood of $\cup_i C_i$. For this purpose, we need to introduce the following terminology. Let C be any G -invariant J -holomorphic curve which is not fixed by G . Suppose C is parametrized by an equivariant J -holomorphic map $f : \Sigma \rightarrow M$, where Σ is a Riemann surface with a $G \cong \mathbb{Z}_p$ holomorphic action. Let $z_i \in \Sigma$, $i = 1, 2, \dots, k$, be the fixed points of G , and for each i , let $g_i \in G$ be the unique element whose action near z_i is given by a counterclockwise rotation of angle $\frac{2\pi}{p}$. Set $p_i = f(z_i)$, which is a fixed point of G , and let $(m_{i,1}, m_{i,2})$, where $0 \leq m_{i,1}, m_{i,2} < p$, be a pair of integers such that the action of g_i on the tangent space at p_i is given by $(w_1, w_2) \mapsto (\mu_p^{m_{i,1}} w_1, \mu_p^{m_{i,2}} w_2)$.

We would like to point out that $(m_{i,1}, m_{i,2})$ is uniquely determined up to order by the J -holomorphic curve C , and f is embedded near z_i iff one of $m_{i,1}, m_{i,2}$ equals 1. We shall call $(m_{i,1}, m_{i,2})$ a pair of *rotation numbers* at p_i associated to C . Note that when C has no double points, each fixed point p_i has a unique pair of rotation numbers associated to C .

We shall consider specially the case where Σ is a Riemann sphere and the J -holomorphic curve C satisfies $c_1(K) \cdot C = 0$. (Note that every C_i in $\cup_i C_i$ satisfies this condition, cf. Lemma 3.3.) It is clear that there are exactly two fixed points z_1, z_2 of the $G \equiv \mathbb{Z}_p$ action on Σ . The following congruence relation for the rotation numbers will be frequently used later in the proof.

Lemma 3.6. *Suppose C is a sphere and $c_1(K) \cdot C = 0$. Then*

$$\sum_{i=1}^2 (m_{i,1} + m_{i,2}) = 0 \pmod{p}$$

where $(m_{i,1}, m_{i,2})$, $i = 1, 2$, are the rotation numbers associated to C .

Proof. To see this, let $f : \mathbb{S}^2 \rightarrow M$ be a G -equivariant J -holomorphic parametrization of C . Then the virtual dimension of the moduli space of the corresponding equivariant J -holomorphic maps at f , which is the index of a first order elliptic differential operator of Cauchy-Riemann type over the orbifold \mathbb{S}^2/G , is of even dimension $2d_f$, where

$$d_f = -\frac{1}{p}c_1(K) \cdot C + 2 - \sum_{i=1}^2 \frac{m_{i,1} + m_{i,2}}{p}.$$

(See the Riemann-Roch theorem for orbit spaces in [1], or Lemma 3.2.4 in [6] for the case of general orbifold Riemann surfaces.) The said congruence relation on the rotation numbers follows easily from $d_f \in \mathbb{Z}$ and the assumption $c_1(K) \cdot C = 0$. \square

With the preceding understood, the next proposition finishes the proof of Theorem 3.2, and moreover, it provides a refinement for some of the statements in Theorem 3.2.

Proposition 3.7. *Under the assumptions in Theorem 3.2, the following hold true for any G -equivariant J , where Λ_α is a connected component of $\cup_i C_i$ which contains at least one fixed point of G .*

- (1) Let $\Lambda_\alpha = C$ be a G -invariant, type (A) component. Then there are three possibilities:
 - (i) C is an embedded torus. In this case, C is either fixed entirely by G , or it contains four isolated fixed points if $p = 2$, or it contains three isolated fixed points if $p = 3$. The rotation numbers at each fixed point are all the same, which is $(1, 1)$ if $p = 2$, and either $(1, 1)$ or $(1, 2)$ if $p = 3$.
 - (ii) C is a cusp sphere. This case occurs only if $p \geq 5$. There are two isolated fixed points, one is the cusp-singularity and the other is a smooth point, with the rotation numbers being $(2, 3)$ and $(1, p - 6)$ respectively if $p > 5$, and $(2, 3)$ and $(1, 4)$ if $p = 5$.

- (iii) C is a nodal sphere. It contains one isolated fixed point, the nodal point, with two pairs of rotation numbers both equaling $(1, p - 1)$.
- (2) Let Λ_α be a type (B) component. This case occurs only when $p \geq 3$. In this case each of the (-2) -spheres in Λ_α contains two isolated fixed points, where one of them is the intersection of the two spheres, with the rotation numbers associated to either sphere being $(1, 2)$, while the other fixed point has rotation numbers $(1, p - 4)$ if $p > 3$, and $(1, 2)$ if $p = 3$.
- (3) Let Λ_α be a type (C) component. Then there are four possibilities:
 - (i) Λ_α contains a 2-dimensional component of M^G , with $n = 4 \bmod p$ if Λ_α is represented by a type \tilde{D}_n graph and $n = -1 \bmod p$ if Λ_α is represented by a type \tilde{A}_n graph.
 - (ii) Λ_α is of type A_n and the intersection of each pair of spheres is an isolated fixed point, with rotation numbers $(1, p - 1)$ associated to either sphere.
 - (iii) Λ_α is of type \tilde{A}_2 where the three spheres intersect at a single point; there are four isolated fixed points, one occurs at the intersection point and each of the other three is contained in each one of the three spheres, with the rotation numbers associated to each sphere being $(1, 1)$ at the intersection point and $(1, |p - 3|)$ at each of the other three fixed points. This case occurs only if $p \neq 3$.
 - (iv) Λ_α is of type \tilde{A}_1 which contains four isolated fixed points. The rotation numbers at each fixed point is $(1, 1)$, and this case occurs only if $p = 2$.

Proof. (1) (i) Suppose C is an embedded torus. Since $C^2 = 0$, a regular neighborhood of C has boundary T^3 . If C is not fixed by G , then the induced G -action on T^3 must be free. Such actions are classified (e.g. see Theorem 4.3 and Table 4.4 in [37]). In particular, since C contains a fixed point by assumption, and p is prime, it follows that this happens only if $p = 2$ or $p = 3$. Moreover, the quotient space T^3/G is naturally a Seifert manifold with base S^2 , and with normalized Seifert invariant $(-2, (2, 1), (2, 1), (2, 1), (2, 1))$ if $p = 2$, and $(-1, (3, 1), (3, 1), (3, 1))$ or $(-2, (3, 2), (3, 2), (3, 2))$ if $p = 3$. The statement about rotation numbers follows easily from the description of the normalized Seifert invariant above.

(ii) Suppose C is a cusp sphere. Then C is not fixed by G because it is not a smooth surface. Moreover, C contains two isolated fixed points, one of which is the cusp-singularity. Because near the cusp-singularity, C can be parametrized by a J -holomorphic map $z \mapsto (z^2, z^3 + \dots)$, it is easily seen that the rotation numbers are $(2, 3)$. Let $(1, m)$, where $1 < m < p$, be the rotation numbers at the other fixed point, near which C is embedded. Then by Lemma 3.6, the congruence relation $(2 + 3) + (1 + m) = 0 \bmod p$ must hold. It follows easily that $p \geq 5$, and moreover, $m = p - 6$ when $p > 5$ and $m = 4$ when $p = 5$.

(iii) Suppose C is a nodal sphere. Again C can not be fixed by G because it is immersed, and furthermore, the double point must be a fixed point which is easily seen the only fixed point contained in C . There are two pairs of rotation numbers. A simple inspection shows that they are both $(1, p - 1)$.

(2) Suppose Λ_α is a type (B) component. Then by Lemma 3.5, Λ_α is G -invariant and so is each of the embedded (-2) -spheres in Λ_α . Note that neither of the two

spheres is fixed by G . This is because if one of them is fixed by G , so is the other as they intersect at a point with tangency of order 2. This, however, is impossible.

It is clear that each of the embedded (-2) -spheres contains two isolated fixed points where one of the fixed points is the intersection of the two spheres. It remains only to check the rotation numbers at each fixed point. For the intersection point, if we choose a local complex coordinate system (w_1, w_2) such that one of the embedded (-2) -spheres is defined by $w_2 = 0$, then the other embedded (-2) -sphere is locally parametrized by $z \mapsto (z, z^2 + \dots)$. It follows easily that the rotation numbers associated to either embedded (-2) -sphere is $(1, 2)$ at the intersection point. On the other hand, if we let $(1, m)$ be the rotation numbers at the other fixed point on the sphere, then the congruence relation in Lemma 3.6 gives $(1 + 2) + (1 + m) = 0 \pmod{p}$, which implies that $p \geq 3$, and moreover, $m = p - 4$ if $p > 3$ and $m = 2$ if $p = 3$.

(3) Suppose Λ_α is a type (C) component. Then by Lemma 3.5, each embedded (-2) -spheres in Λ_α is G -invariant. There are two possible scenarios: (a) a regular neighborhood of Λ_α in M is a plumbing of embedded (-2) -spheres, (b) Λ_α consists of three embedded (-2) -spheres which intersect at a single point.

Let's first consider scenario (a). It is clear that if there are two distinct spheres in Λ_α with one intersection point not fixed by G , then we must have $p = 2$ and Λ_α is of type \tilde{A}_1 , with each sphere containing two isolated fixed points of rotation numbers $(1, 1)$. (This case is listed as (3) (iv) in the proposition.) Now suppose the intersection of any two distinct spheres in Λ_α is fixed by G . Then a sphere in Λ_α which intersects with more than two other spheres must be entirely fixed by G , because a \mathbb{Z}_p -action on \mathbb{S}^2 can not have more than two fixed points unless it is trivial. In particular, Λ_α contains a 2-dimensional component of M^G if the graph representing Λ_α is of type $\tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ or \tilde{E}_8 .

We need to further discuss the cases where Λ_α is of type \tilde{A}_n or \tilde{D}_n .

First we assume Λ_α is of type \tilde{D}_n , and show that $n = 4 \pmod{p}$. The case where $n = 4$ is trivial, so we assume $n > 4$. Then there are two vertices in the representing graph Γ_α , each of which connects to three other vertices, and moreover, there is a linear sub-graph consisting of $(n - 3)$ vertices, with these two vertices at each end. We denote by v_0, v_1, \dots, v_k the vertices along the linear sub-graph, where $k = n - 4$.

According to [33], a regular neighborhood of the configuration of embedded (-2) -spheres represented by the linear sub-graph may be obtained by an \mathbb{S}^1 -equivariant plumbing where the \mathbb{S}^1 -action is linear on each sphere. On the other hand, the action of G on each embedded (-2) -sphere is conjugate to a linear action. It follows that there is an \mathbb{S}^1 -equivariant plumbing such that the $G \equiv \mathbb{Z}_p$ action on the regular neighborhood is induced by the inclusion $\mathbb{Z}_p \subset \mathbb{S}^1$. The congruence relation $n = 4 \pmod{p}$ follows from this consideration.

More concretely, the \mathbb{S}^1 -equivariant plumbing is done as follows. For each $i, 0 \leq i \leq k$, regard the normal bundle of the embedded (-2) -sphere represented by the vertex v_i as the result of \mathbb{S}^1 -equivariantly sewing $D^2 \times D^2$ to $D^2 \times D^2$ (here the second D^2 in each $D^2 \times D^2$ represents the fiber of the normal bundle) by the matrix

$$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix},$$

where in polar coordinates on each factor D^2 , the \mathbb{S}^1 -action on the first $D^2 \times D^2$ is given by $(r, \gamma, s, \delta) \mapsto (r, \gamma + u_{i,1}\theta, s, \delta + v_{i,1}\theta)$, $0 \leq \theta \leq 2\pi$, and on the second $D^2 \times D^2$ it is given by $(r, \gamma, s, \delta) \mapsto (r, \gamma + u_{i,2}\theta, s, \delta + v_{i,2}\theta)$, $0 \leq \theta \leq 2\pi$, for some $u_{i,1}, v_{i,1}, u_{i,2}, v_{i,2} \in \mathbb{Z}$ with $u_{0,1} = 0, v_{0,1} = 1$. Then the plumbing identifies the second $D^2 \times D^2$ associated to the vertex v_i to the first $D^2 \times D^2$ associated to the next vertex v_{i+1} with the two factors of D^2 switched. Moreover, the plumbing is equivariant so that $u_{i,2} = v_{i+1,1}, v_{i,2} = u_{i+1,1}$ must hold.

With the preceding understood, the weights of the \mathbb{S}^1 -action on each $D^2 \times D^2$ can be determined from the following equation where $i \geq 0$ (cf. [33], page 27)

$$\begin{pmatrix} u_{i,2} \\ v_{i,2} \end{pmatrix} = \begin{pmatrix} -(i+1) & 1-(i+1) \\ i+2 & i+1 \end{pmatrix} \begin{pmatrix} u_{0,1} \\ v_{0,1} \end{pmatrix}.$$

On the other hand, the $G \equiv \mathbb{Z}_p$ action on Λ_α is induced by $\mathbb{Z}_p \subset \mathbb{S}^1$, and the embedded (-2) -sphere represented by v_k is fixed by $G \equiv \mathbb{Z}_p$, so that $u_{k,2} = 0 \pmod p$ must hold. With $u_{0,1} = 0$ and $v_{0,1} = 1$, one has $k = -u_{k,2} = 0 \pmod p$, or equivalently, $n = 4 \pmod p$. This finishes the proof for the case when Λ_α is of type \tilde{D}_n .

Next we consider the case where Λ_α is of type \tilde{A}_n . We will first show that n satisfies the congruence relation $n = -1 \pmod p$ (i.e. the number of vertices in Γ_α is divisible by p) if one of the following conditions are satisfied: (1) one of the embedded (-2) -spheres in Λ_α is fixed by G , or (2) one of the isolated fixed points in Λ_α has a pair of rotation numbers (m_1, m_2) such that $m_1 + m_2 \neq 0 \pmod p$.

Consider the linear graph Γ'_α obtained from Γ_α by removing any one of its edges. Denote by v_1, v_2, \dots, v_k the vertices along Γ'_α , where $k = n+1$. Then $n = -1 \pmod p$ is equivalent to $k = 0 \pmod p$, which is what we will show next. To see this, note that as we argued earlier, there exists an \mathbb{S}^1 -equivariant plumbing associated to Γ'_α , such that the $G \equiv \mathbb{Z}_p$ action is induced by the inclusion $\mathbb{Z}_p \subset \mathbb{S}^1$ from the \mathbb{S}^1 -action associated to the equivariant plumbing. Now suppose at the vertex v_i the \mathbb{S}^1 -actions on the two copies of $D^2 \times D^2$ are given (in polar coordinates on each D^2 factor) by $(r, \gamma, s, \delta) \mapsto (r, \gamma + u_{i,1}\theta, s, \delta + v_{i,1}\theta)$, $0 \leq \theta \leq 2\pi$, and $(r, \gamma, s, \delta) \mapsto (r, \gamma + u_{i,2}\theta, s, \delta + v_{i,2}\theta)$, $0 \leq \theta \leq 2\pi$, for some integers $u_{i,1}, v_{i,1}, u_{i,2}, v_{i,2}$. Then

$$\begin{pmatrix} u_{k,2} \\ v_{k,2} \end{pmatrix} = \begin{pmatrix} -k & 1-k \\ k+1 & k \end{pmatrix} \begin{pmatrix} u_{1,1} \\ v_{1,1} \end{pmatrix}$$

must be satisfied (cf. [33], page 27). On the other hand, because the $G \equiv \mathbb{Z}_p$ action on Λ_α is induced by the inclusion $\mathbb{Z}_p \subset \mathbb{S}^1$ from the \mathbb{S}^1 -action associated to the equivariant plumbing, it follows that the following equation must hold in congruence mod p :

$$\begin{pmatrix} u_{1,1} \\ v_{1,1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_{k,2} \\ v_{k,2} \end{pmatrix} = \begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix} \begin{pmatrix} u_{1,1} \\ v_{1,1} \end{pmatrix}$$

which gives rise to the congruence relation $k(u_{1,1} + v_{1,1}) = 0 \pmod p$. Hence $k = 0 \pmod p$ as long as $u_{1,1} + v_{1,1} \neq 0 \pmod p$.

To see that $u_{1,1} + v_{1,1} \neq 0 \pmod p$, note that if one of the embedded (-2) -spheres in Λ_α is fixed by G , which without loss of generality may be assumed to be the one represented by the vertex v_1 , then $u_{1,1} = 0$ and $v_{1,1} = 1$, so that $u_{1,1} + v_{1,1} \neq 0 \pmod p$. If one of the isolated fixed points in Λ_α has a pair of rotation numbers (m_1, m_2) with

$m_1 + m_2 \not\equiv 0 \pmod{p}$, then one may similarly assume that $u_{1,1} = m_1$, $v_{1,1} = m_2$, which also implies $u_{1,1} + v_{1,1} \not\equiv 0 \pmod{p}$.

Therefore for scenario (a) it remains to show that if none of the embedded (-2) -spheres in Λ_α is fixed by G , then the rotation numbers (associated to either sphere) are $(1, p-1)$ at each fixed point. In particular, we will rule out the second possibility in the preceding discussion, i.e., one of the isolated fixed points in Λ_α has a pair of rotation numbers (m_1, m_2) such that $m_1 + m_2 \not\equiv 0 \pmod{p}$.

We begin by introducing the following notations. Let p_i , $1 \leq i \leq k-1$, be the intersection of the i -th sphere (i.e. the one represented by the vertex v_i) with the $(i+1)$ -th sphere, and p_k be the intersection of the k -th sphere with the first sphere. Let $(1, m_i)$, $(1, m'_i)$ be the rotation numbers associated to the i -th sphere at p_i , p_{i-1} respectively (here $p_0 = p_k$).

First of all, some basic properties of the integers m_i , m'_i , $1 \leq i \leq k$. Note first that m'_i and m_{i-1} are mutually determined by each other in the congruence equation $m'_i m_{i-1} = 1 \pmod{p}$. Secondly, for each i , m'_i and m_i satisfy the congruence relation $(1 + m'_i) + (1 + m_i) = 0 \pmod{p}$ (cf. Lemma 3.6), which implies that either $m_i = m'_i = p-1$ or $2 + m'_i + m_i = p$. Finally, with $m'_i m_{i-1} = 1 \pmod{p}$, we see easily that exactly one of the following is true:

- $m_i = m'_i = p-1$ for all $1 \leq i \leq k$, or
- $2 + m'_i + m_i = p$ for all $1 \leq i \leq k$.

Next, we show that the second case, i.e., $2 + m'_i + m_i = p$ for all $1 \leq i \leq k$, can not occur. To see this, we first observe that the sequence m_1, m_2, \dots, m_k is periodic with a period $l \leq p-3$. This is because (1) $1 + m_i \not\equiv 0 \pmod{p}$ implies that $k \equiv 0 \pmod{p}$, so that $p \leq k$, (2) each m_i satisfies $1 \leq m_i \leq p-3$, hence there exist i, l with $1 \leq i \leq k$, $1 \leq l \leq p-3$ such that $m_i = m_{i+l}$, (3) if $m_i = m_{i+l}$ for some i and l , then it holds for all i with that same l . Secondly, a contradiction is reached by showing that the period $l = 1$. Indeed, $l = 1$ means that $m_1 = \dots = m_k = m$ for some m with $1 \leq m \leq p-3$. If we let m' be the unique integer such that $m' m = 1 \pmod{p}$ and $1 < m' < p$. Then $m' + m + 2 = p$ holds, which implies $m = p-1$, a contradiction.

It thus remains to show that $l = 1$. To see this, recall that if we do \mathbb{S}^1 -equivariant plumbing on the linear graph Γ'_α of vertices v_1, v_2, \dots, v_k with $u_{1,1} = 1$ and $v_{1,1} = m'_1$, then for each $i \leq k$, $u_{i,2}$, $v_{i,2}$ are related to $u_{1,1}$, $v_{1,1}$ by (cf. [33], page 27)

$$\begin{pmatrix} u_{i,2} \\ v_{i,2} \end{pmatrix} = \begin{pmatrix} -i & 1-i \\ i+1 & i \end{pmatrix} \begin{pmatrix} u_{1,1} \\ v_{1,1} \end{pmatrix}.$$

On the other hand, we have $v_{i,2} = u_{i,2} m_i \pmod{p}$, because the $G \equiv \mathbb{Z}_p$ action on Λ_α is the restriction of the \mathbb{S}^1 -action associated to the equivariant plumbing to the subgroup $\mathbb{Z}_p \subset \mathbb{S}^1$. By taking $i = p$ (recall that $p \leq k$), we obtain $u_{p,2} = v_{1,1} \pmod{p}$ and $u_{p,2} m_p = v_{p,2} = u_{1,1} \pmod{p}$. With $u_{1,1} = 1$, $v_{1,1} = m'_1$, we see that $m'_1 m_p = 1 \pmod{p}$, which implies that $m'_{p+1} = m'_1$, and hence $m_{p+1} = m_1$. This last equality shows that the period l is a divisor of p . Hence $l = 1$ because $l \leq p-3$ and p is prime.

This shows that either one of the embedded (-2) -spheres in Λ_α is fixed by G , or the rotation numbers are $(1, p-1)$ at each of the isolated fixed points in Λ_α .

To complete the proof of Proposition 3.7, it remains to consider scenario (b) where Λ_α consists of three spheres intersecting at a single point. In this case, it is clear that

the intersection point must be fixed by G . Moreover, since the induced action of G on the tangent space at this point has three distinct eigenspaces, it must be an isolated fixed point and has rotation numbers $(1, 1)$ associated to each sphere. There are three other isolated fixed points in Λ_α , with each sphere containing one of them. The rotation numbers are $(1, 1)$ if $p = 2$, and $(1, p - 3)$ if $p \neq 2$ by the congruence relation in Lemma 3.6. Note that scenario (b) does not occur if $p = 3$, because otherwise the rotation numbers $(1, p - 3)$ would become $(1, 0)$, implying that the normal direction is fixed by G at the isolated fixed point. \square

We have thus proved Theorem 3.2, and particularly Theorem B in the introduction, which is a special case where the action is pseudofree.

With the fixed-point set data in hand, we next give a proof of Corollary B by appealing to the G -signature theorem (cf. Theorem 2.3).

Proof of Corollary B

First of all, because $c_1^2 = 0$ and G acts trivially on $H_*(M; \mathbb{Q})$, one has

$$\text{sign}(M) = -\frac{2}{3}\chi(M) = -\frac{2}{3}|M^G|,$$

where $|M^G|$ is the number of fixed points. Consequently, the G -signature theorem may be rewritten as

$$-\frac{2}{3}(p-1) \cdot |M^G| = -\frac{2}{3}(p-1) \cdot \chi(M) = \sum_{m \in M^G} \text{def}_m.$$

We shall discuss separately two cases: (1) $p = 2$ or $p = 3$, and (2) $p > 3$.

Consider first the case $p = 2$ or $p = 3$. We remark that in this case we do not need to use Theorem B, and the corresponding rigidity for the \mathbb{Z}_p -action is even true for locally linear topological actions, cf. e.g. [10]. For the sake of completeness, we give a proof here.

If $p = 2$, the local representation at each $m \in M^G$ is of the same type, which is $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ with $k \not\equiv 0 \pmod{p}$. Hence $\text{def}_m = \frac{1}{3}(p-1)(p-2) = 0$ by Lemma 2.4. This contradicts the G -signature theorem when $\text{sign}(M) \neq 0$. The case where $p = 3$ is similar. There is another type of local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ with $k \not\equiv 0 \pmod{p}$, for which a similar calculation as in Lemma 2.4 shows that $\text{def}_m = -\frac{1}{3}(p-1)(p-2)$. For both types of local representations, $\text{def}_m > -\frac{2}{3}(p-1)$, which contradicts the G -signature theorem when $|M^G| = -\frac{2}{3}\text{sign}(M) \neq 0$.

For the rest of the proof, we assume $p > 3$. Then the set of fixed points is divided into groups of the first four types in Theorem B. We introduce the following notation. For $1 \leq k \leq 4$, let δ_k be the number of groups of type (k) in Theorem B, and let $\text{def}_{(k)}$ be the total signature defect contributed by one group of type (k) (i.e., the sum of def_m with m running over one group of type (k) fixed points). With this notation, the

G -signature theorem maybe be written as

$$-\frac{2}{3}(p-1)(\delta_1 + 2\delta_2 + 3\delta_3 + 4\delta_4) = \delta_1 \cdot \text{def}_{(1)} + \delta_2 \cdot \text{def}_{(2)} + \delta_3 \cdot \text{def}_{(3)} + \delta_4 \cdot \text{def}_{(4)}.$$

A contradiction will be reached if for all $k = 1, 2, 3$ and 4 ,

$$\text{def}_{(k)} \geq -\frac{2k}{3}(p-1),$$

with the strict inequality holding for some k with $\delta_k > 0$.

The following lemma gives an explicit formula for $\text{def}_{(k)}$, $k \leq 4$.

- Lemma 3.8.** (1) $\text{def}_{(1)} = \frac{1}{3}(p-1)(p-2)$ for all $p > 1$.
 (2) $\text{def}_{(2)} = -8r$ if $p = 6r + 1$, $\text{def}_{(2)} = 8r + 8$ if $p = 6r + 5$.
 (3) $\text{def}_{(3)} = -8r$ if $p = 4r + 1$, $\text{def}_{(3)} = 2$ if $p = 4r + 3$.
 (4) $\text{def}_{(4)} = -8r$ if $p = 3r + 1$, $\text{def}_{(4)} = -4r$ if $p = 3r + 2$.

The proof of Lemma 3.8 is given in the appendix, which is a result of direct calculation. Accepting Lemma 3.8 momentarily, we shall complete the proof of Corollary B for the case $p > 3$. Observe that $\text{def}_{(1)} > -\frac{2}{3}(p-1)$ for all $p > 1$, and $\text{def}_{(2)} \geq -\frac{4}{3}(p-1)$ with equality only if $p = 1 \pmod{6}$, $\text{def}_{(3)} \geq -\frac{6}{3}(p-1)$ with equality only if $p = 1 \pmod{4}$, and $\text{def}_{(4)} \geq -\frac{8}{3}(p-1)$ with equality only if $p = 1 \pmod{6}$ (note that if $p = 3r + 1$ and p is prime, then r must be even and $p = 1 \pmod{6}$). Part (b) of Corollary B follows from this immediately. As for part (a), observe that $\text{def}_{(k)} > -\frac{2k}{3}(p-1)$ for all k if $p \not\equiv 1 \pmod{4}$, $p \not\equiv 1 \pmod{6}$. Hence when $\text{sign}(M) \neq 0$, one of δ_k is nonzero, from which part (a) follows. \square

We close this section with two examples. The first one shows that the ‘‘more exotic’’ types of local representations in Theorem B indeed can occur, at least topologically.

Example 3.9. Let M be a homotopy $K3$ surface (i.e., a manifold homotopy equivalent to a $K3$ surface), which is given with the canonical orientation such that $\text{sign}(M) = -16$. In this example, we will show that there are locally linear, homologically trivial topological actions of \mathbb{Z}_5 and \mathbb{Z}_7 on M with the fixed-point set consisting entirely of type (3) and type (2) or (4) fixed points in Theorem B respectively. (Compare Corollary B(a).) Our construction is based on the work of Edmonds and Ewing [11] concerning realization of certain fixed-point data by a locally linear, topological \mathbb{Z}_p -action of prime order on a simply-connected 4-manifold.

Note that a homologically trivial pseudofree action of \mathbb{Z}_p on M must have 24 fixed points. Consider first the following fixed-point data where $p = 5$: Pick 24 points of M , divide them evenly into two groups, and assign the points in each group with local representations

$$(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2), \quad (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2), \quad \text{and} \quad (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$$

evaluated at $k = 1, 2, 3, 4$.

In order to realize the above fixed-point data, we recall the GSF condition from [11], which in the present case becomes

$$2 \text{def}_{(3)} = \text{sign}(g, M), \quad \forall g \in \mathbb{Z}_5.$$

To verify the GSF condition, we note that the action is assumed to be homologically trivial, so that $\text{sign}(g, M) = \text{sign}(M) = -16$ for any $g \in \mathbb{Z}_5$. On the other hand, we have $\text{def}_{(3)} = -8$ by Lemma 3.8. Hence the GSF condition is satisfied. Now for $p = 5$, GSF is the only condition needed for the realization of the fixed-point data by a homologically trivial action, cf. [11], Corollary 3.2. Consequently, there is a locally linear topological action of \mathbb{Z}_5 on M with the fixed-point set consisting entirely of type (3) fixed points in Theorem B.

Similar arguments lead to a locally linear, homologically trivial topological action of \mathbb{Z}_7 on M with the fixed-point set consisting entirely of type (2) or type (4) fixed points.

The purpose of the second example is to illustrate that when a certain additional information about the canonical class and the symplectic structure is available, Theorem 3.2 (together with Proposition 3.7) may yield rigidity for actions which are not necessarily pseudofree.

Example 3.10. Consider a symplectic 4-manifold (M, ω) with the following properties: M is a homotopy $K3$ surface, ω defines an integral class $[\omega] \in H^2(M; \mathbb{R})$ (or more generally, any sufficiently small perturbation of an integral class), $c_1^2 = 0$, and $c_1(K) \cdot [\omega] < 7$. Note that (M, ω) must be minimal because M has an even intersection form. Moreover, $c_1^2 = 0$ implies $\text{sign}(M) = -16$, so that $b_2^+ = 3$.

We shall next prove that:

There are no nontrivial homologically trivial actions (not necessarily pseudofree) of a finite group on M which preserve the symplectic structure ω .

To see this, suppose there is such an action. Without loss of generality, we may assume that the action is cyclic of prime order p . By Theorem 3.2 and Proposition 3.7, there is a set of J -holomorphic curves $\{C_i\}$ as described therein, such that $c_1(K)$ is represented by the fundamental class of $\sum_i n_i C_i$ for some integers $n_i \geq 1$. Since $[\omega]$ is integral, $[\omega] \cdot C_i \geq 1$ for all C_i . Now the property $c_1(K) \cdot [\omega] < 7$ implies that in $\cup_i C_i$ there is no component Λ_α which is a union of embedded (-2) -spheres represented by a graph of type other than \tilde{A}_n or \tilde{D}_4 . With this understood, it follows easily that there are only two types of 2-dimensional components in the fixed-point set, (1) an embedded torus, (2) an embedded (-2) -sphere contained in a component Λ_α which is represented either by a type \tilde{A}_n graph or by a type \tilde{D}_4 graph. Notice further that a toroidal fixed component does not make any contribution to the signature defect because it has self-intersection 0 (cf. Theorem 2.3).

With the preceding understood, we discuss separately the cases (i) the action is pseudofree, (ii) the action has a (-2) -sphere fixed component.

For case (i), first note that we may assume that there are no fixed points whose local representation is contained in $SL_2(\mathbb{C})$, because otherwise the action is trivial by Corollary B(b). Then we see from Theorem 3.2 and Proposition 3.7 that the number of fixed points, which equals $\chi(M)$ since the action is homologically trivial, equals $2\delta_2 + 3\delta_3 + 4\delta_4$, where δ_2 is the number of cusp-sphere components Λ_α , δ_3 is the number of Λ_α which is a union of two (-2) -spheres, and δ_4 is the number of Λ_α

which is a union of three spheres. A contradiction is reached easily by observing that $\chi(M) = 24$, and that $\delta_2 + 2\delta_3 + 3\delta_4 \leq c_1(K) \cdot [\omega] < 7$.

For case (ii), we first assume that there is a component Λ_α which contains a fixed (-2) -sphere and is represented by a type \tilde{A}_n graph. Then there will be no component represented by a type \tilde{D}_4 graph because of the constraint $c_1(K) \cdot [\omega] < 7$, and moreover, by Proposition 3.7 (3), the number of (-2) -spheres in such a component Λ_α is divisible by the order p of the action. It follows from $c_1(K) \cdot [\omega] < 7$ that $p = 2, 3$ or 5 .

Note that in the context of this example, the G -signature theorem is equivalent to the following equation

$$-16(p-1) = \sum_m \text{def}_m + \sum_Y \text{def}_Y,$$

where m, Y represent an isolated fixed point and a fixed (-2) -sphere respectively.

If $p = 2$, then $\text{def}_Y = \frac{p^2-1}{3} \cdot (Y \cdot Y) = -2$ for any Y and the definition of def_m implies that $\text{def}_m = 0$ for all m . We reach a contradiction to the G -signature theorem because there are at most 3 fixed (-2) -spheres.

Suppose $p = 3$ next. Let δ_m, δ_Y be the number of isolated fixed points m and fixed (-2) -spheres Y respectively. Then δ_m, δ_Y obey $\delta_m + 2\delta_Y = \chi(M) = 24$ (cf. [23]), and moreover, it is clear that $\delta_Y \leq 2$. As for the signature defects def_m and def_Y , we note that for $p = 3$ there are two types of local representations: $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ or $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$. A similar calculation as in Lemma 2.4 shows that in both cases, one has $\text{def}_m \geq -\frac{1}{3}(p-1)(p-2) = -\frac{2}{3}$. On the other hand, $\text{def}_Y = \frac{p^2-1}{3} \cdot (Y \cdot Y) = -\frac{16}{3}$, which gives a contradiction to the G -signature theorem

$$-32 = -16(p-1) \geq \delta_m \cdot \left(-\frac{2}{3}\right) + \delta_Y \cdot \left(-\frac{16}{3}\right) \geq 24 \cdot \left(-\frac{2}{3}\right) + 2 \cdot \left(-\frac{16}{3}\right) = -\frac{80}{3}.$$

Finally, let $p = 5$. First, we observe that there is exactly one component Λ_α of type \tilde{A}_4 which contains a fixed (-2) -sphere. Next we recall that if a (-2) -sphere is not fixed, then it must contain exactly two fixed points, and if we let $(1, m_1), (1, m_2)$ be the corresponding rotation numbers, where $0 \leq m_1, m_2 < p = 5$, the congruence relation $(1 + m_1) + (1 + m_2) = 0 \pmod{5}$ must be satisfied (cf. Lemma 3.6). A simple inspection shows that Λ_α contains, besides the fixed (-2) -sphere, three isolated fixed points of local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$ for some $k \neq 0 \pmod{p}$, with $q = 1, 2, 3$ respectively. On the other hand, for all other fixed points, where there is a total of $\chi(M) - (3 + 2 \cdot 1) = 19$ (cf. [23]), the local representation is $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some $k \neq 0 \pmod{p}$. To see this, note that the type \tilde{A}_4 component Λ_α contributes at least 5 to $c_1(K) \cdot [\omega] < 7$, so that there is at most one more component which must be of type (A), i.e., consisting of a single J -holomorphic curve. The claim follows from Proposition 3.7 (1) immediately.

To continue we recall that the signature defect def_m for an isolated fixed point m with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$ for some $k \neq 0 \pmod{p}$ is

$$I_{p,q} \equiv \sum_{k=1}^{p-1} \frac{(1 + \mu_p^k)(1 + \mu_p^{kq})}{(1 - \mu_p^k)(1 - \mu_p^{kq})}.$$

A direct calculation (cf. Appendix: Proof of Lemma 3.8) gives

$$I_{5,-1} = 4, I_{5,1} = -4, I_{5,2} = 0, I_{5,3} = 0.$$

(Observe the relation $I_{p,q} = -I_{p,-q}$.) We reach a contradiction for $p = 5$ through the G -signature theorem:

$$-16(5-1) = 19I_{5,-1} + I_{5,1} + I_{5,2} + I_{5,3} + \frac{5^2-1}{3} \cdot (-2)$$

or equivalently $-64 = 56$.

It remains to consider for case (ii) the possibility of having a component Λ_α which is represented by a type \tilde{D}_4 graph. In this situation, note that Λ_α must be the only component (notice that the center vertex in the corresponding graph has weight 2, cf. [2], Lemma 2.12 (ii) on page 20, so that Λ_α contributes at least 6 to $c_1(K) \cdot [\omega] < 7$), and besides the fixed (-2) -sphere, it also contains four isolated fixed points of rotation numbers $(1, p-2)$. The rest of the fixed points are isolated, all of local representation contained in $SL_2(\mathbb{C})$. By Lemma 2.4, the G -signature theorem implies

$$-16(p-1) \geq 4 \cdot I_{p,-2} - \frac{2}{3}(p^2-1),$$

which is a contradiction because $I_{p,-2} = -I_{p,2} = \frac{1}{6}(p-1)(p-5)$ (cf. Appendix: Proof of Lemma 3.8).

We have thus shown that there are no nontrivial homologically trivial finite group actions on M which preserve the symplectic structure ω .

Finally, we remark that there are indeed examples of such 4-manifolds (M, ω) , which actually have nontrivial canonical class $c_1(K)$. (This is the case which is not covered in Theorem A.) In fact, such an M may be obtained by the knot surgery construction of Fintushel and Stern [13]. More precisely, consider the $K3$ surface S_4 which is the hypersurface in \mathbb{P}^3 :

$$S_4 \equiv \{[z_0, z_1, z_2, z_3] \in \mathbb{P}^3 \mid z_0^4 - z_1^4 + z_2^4 - z_3^4 = 0\}$$

There is a holomorphic elliptic fibration $\pi : S_4 \rightarrow \mathbb{P}^1$ such that a generic fiber F of π is a cubic curve in a hyperplane in \mathbb{P}^3 (cf. §3.2 of [19]). Let ω_0 be the Kähler form on S_4 which is obtained by restricting the Fubini-Study form to S_4 . Then $[\omega_0] \in H^2(S_4; \mathbb{R})$ is an integral class, and $[\omega_0] \cdot F = 3$ since F is a cubic curve.

Now applying the knot surgery construction of Fintushel and Stern [13] to S_4 at a regular fiber of the holomorphic elliptic fibration with the Kähler form ω_0 and using the trefoil knot, one obtains a symplectic 4-manifold (M, ω) , such that (1) M is homeomorphic to S_4 , (2) the canonical class $c_1(K)$ is the Poincaré dual of $2F$ where F is the fiber class of the holomorphic elliptic fibration. It follows easily that (M, ω) satisfies the required conditions.

We wish to point out that the 4-manifold M constructed above is not a complex surface (with either orientation). To see this, first note that M is not diffeomorphic to any 4-manifold obtained by performing a log transform to a $K3$ surface (cf. [12, 13]). In particular, M can not be a complex surface with Kodaira dimension ≤ 1 (cf. e.g. [19], Lemma 3.3.4 and Theorem 3.4.12). On the other hand, M can not be a complex surface of general type (necessarily with the opposite orientation) because it violates

the Miyaoka-Yau inequality $c_1^2 \leq 3c_2$. Hence M is not a complex surface. Note that this in particular shows that the homological rigidity obtained in this example goes beyond that of holomorphic actions on Kähler surfaces in Peters [35].

Appendix: Proof of Lemma 3.8

Recall that (cf. Theorem 2.3) for an isolated fixed point $m \in M^G$, the signature defect def_m is given by the following expression if the local representation at m is $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$ for some $k \not\equiv 0 \pmod p$ and $q \not\equiv 0 \pmod p$:

$$I_{p,q} \equiv \sum_{k=1}^{p-1} \frac{(1 + \mu_p^k)(1 + \mu_p^{kq})}{(1 - \mu_p^k)(1 - \mu_p^{kq})}$$

With this notation, it is easily seen that

$$\text{def}_{(1)} = I_{p,-1}, \text{def}_{(2)} = I_{p,-6} + I_{p,(p+3)/2}, \text{def}_{(3)} = I_{p,2} + 2I_{p,-4}, \text{def}_{(4)} = I_{p,1} + 3I_{p,-3}.$$

By Lemma 2.4, $\text{def}_{(1)} = \frac{1}{3}(p-1)(p-2)$ and $I_{p,1} = -I_{p,-1} = -\frac{1}{3}(p-1)(p-2)$.

To calculate $\text{def}_{(k)}$ for $k = 2, 3$ and 4, we go back to the proof of Lemma 2.4. Recall that $I_{p,q} = -4p \cdot s(q, p)$, and the Dedekind sum $s(q, p)$ can be computed from

$$6p \cdot s(q, p) = (p-1)(2pq - q - \frac{3p}{2}) - 6f_p(q)$$

where $f_p(q) = \sum_{k=1}^{p-1} k[\frac{kq}{p}]$. Note that $I_{p,2} = -\frac{1}{6}(p-1)(p-5)$ follows directly from

$$f_p(2) = \sum_{k=1}^{p-1} k[\frac{2k}{p}] = \sum_{k=(p+1)/2}^{p-1} k = \frac{1}{8}(3p-1)(p-1).$$

To calculate $I_{p,q}$ for the other values of q , we recall the following equation from [21] (equation (8) on page 94)

$$\sum_{k=1}^{p-1} [\frac{kq}{p}]^2 - \frac{2q}{p} \cdot \sum_{k=1}^{p-1} k[\frac{kq}{p}] = \frac{1}{6p}(1-q^2)(p-1)(2p-1)$$

and eliminate $f_p(q)$ from the expression for $6p \cdot s(q, p)$. Consequently, we have

$$I_{p,q} = (-\frac{2pq}{3} + \frac{q}{3} + \frac{1}{3q} + p - \frac{2p}{3q})(p-1) + \frac{2p}{q} \cdot \sum_{k=1}^{p-1} [\frac{kq}{p}]^2.$$

Next we evaluate $\sum_{k=1}^{p-1} [\frac{kq}{p}]^2$ for $q = -4, -6, \frac{p+3}{2}$ and $q = -3$.

(1) $q = -4$. We shall consider separately the cases $p = 4r + 1$ and $p = 4r + 3$. For $p = 4r + 1$, we have

$$\sum_{k=1}^{p-1} [\frac{-4k}{p}]^2 = \sum_{k=1}^r (-1)^2 + \sum_{k=r+1}^{2r} (-2)^2 + \sum_{k=2r+1}^{3r} (-3)^2 + \sum_{k=3r+1}^{4r} (-4)^2 = 30r.$$

For $p = 4r + 3$, we have

$$\sum_{k=1}^{p-1} \left[\frac{-4k}{p} \right]^2 = \sum_{k=1}^r (-1)^2 + \sum_{k=r+1}^{2r+1} (-2)^2 + \sum_{k=2r+2}^{3r+2} (-3)^2 + \sum_{k=3r+3}^{4r+2} (-4)^2 = 30r + 13.$$

(2) $q = -6$. We shall consider separately the cases $p = 6r + 1$ and $p = 6r + 5$. For $p = 6r + 1$, we have

$$\sum_{k=1}^{p-1} \left[\frac{-6k}{p} \right]^2 = \sum_{j=1}^6 \sum_{k=(j-1)r}^{jr} (-j)^2 = 91r.$$

For $p = 6r + 5$, we have

$$\sum_{k=1}^{p-1} \left[\frac{-6k}{p} \right]^2 = \sum_{k=1}^r (-1)^2 + \sum_{j=1}^4 \sum_{k=jr+j}^{(j+1)r+j} (-j-1)^2 + \sum_{k=5r+5}^{6r+4} (-6)^2 = 91r + 54.$$

(3) $q = \frac{p+3}{2}$. Observe that when $k = (2l - 1)$ is odd, we have

$$\left[\frac{kq}{p} \right] = \begin{cases} l-1 & \text{if } l = 1, \dots, \left[\frac{p+3}{6} \right] \\ l & \text{if } l = \left[\frac{p+3}{6} \right] + 1, \dots, \frac{p-1}{2}, \end{cases}$$

and when $k = 2l$ is even, we have

$$\left[\frac{kq}{p} \right] = \begin{cases} l & \text{if } l = 1, \dots, \left[\frac{p}{3} \right] \\ l+1 & \text{if } l = \left[\frac{p}{3} \right] + 1, \dots, \frac{p-1}{2}. \end{cases}$$

A direct calculation gives

$$\begin{aligned} \sum_{k=1}^{p-1} \left[\frac{kq}{p} \right]^2 &= \sum_{l=1}^{\left[\frac{p+3}{6} \right]} (l-1)^2 + \sum_{l=\left[\frac{p+3}{6} \right] + 1}^{(p-1)/2} l^2 + \sum_{l=1}^{\left[\frac{p}{3} \right]} l^2 + \sum_{l=\left[\frac{p}{3} \right] + 1}^{(p-1)/2} (l+1)^2 \\ &= \begin{cases} r(18r^2 + 13r + 3) & \text{if } p = 6r + 1 \\ (r+1)(18r^2 + 31r + 14) & \text{if } p = 6r + 5. \end{cases} \end{aligned}$$

(4) $q = -3$. We shall consider separately the cases $p = 3r + 1$ and $p = 3r + 2$. For $p = 3r + 1$, we have

$$\sum_{k=1}^{p-1} \left[\frac{-3k}{p} \right]^2 = \sum_{k=1}^r (-1)^2 + \sum_{k=r+1}^{2r} (-2)^2 + \sum_{k=2r+1}^{3r} (-3)^2 = 14r.$$

For $p = 3r + 2$, we have

$$\sum_{k=1}^{p-1} \left[\frac{-3k}{p} \right]^2 = \sum_{k=1}^r (-1)^2 + \sum_{k=r+1}^{2r+1} (-2)^2 + \sum_{k=2r+2}^{3r+1} (-3)^2 = 14r + 4.$$

Now plug these formulas back into the expression for $I_{p,q}$. We obtain

$$\begin{aligned}
 I_{p,-4} &= \begin{cases} \frac{4}{3}r(r-4) & \text{if } p = 4r + 1 \\ \frac{2}{3}(2r^2 + 1) & \text{if } p = 4r + 3, \end{cases} \\
 I_{p,-6} &= \begin{cases} 2r(r-6) & \text{if } p = 6r + 1 \\ 2r^2 + 4r + 4 & \text{if } p = 6r + 5, \end{cases} \\
 I_{p,(p+3)/2} &= \begin{cases} 2r(2-r) & \text{if } p = 6r + 1 \\ -2r^2 + 4r + 4 & \text{if } p = 6r + 5, \end{cases} \\
 I_{p,-3} &= \begin{cases} r(r-3) & \text{if } p = 3r + 1 \\ r(r-1) & \text{if } p = 3r + 2. \end{cases}
 \end{aligned}$$

This gives the formulas of $\text{def}_{(k)}$ for $k = 2, 3$ and 4 , and Lemma 3.8 follows.

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