SYMMETRIES AND EXOTIC SMOOTH STRUCTURES ON A $K3$ SURFACE

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Abstract. Smooth and symplectic symmetries of an infinite family of distinct exotic $K3$ surfaces are studied, and comparison with the corresponding symmetries of the standard $K3$ is made. The action on the $K3$ lattice induced by a smooth finite group action is shown to be strongly restricted, and as a result, nonsmoothability of actions induced by a holomorphic automorphism of a prime order ≥ 7 is proved and nonexistence of smooth actions by several $K3$ groups is established (included among which is the binary tetrahedral group T_{24} which has the smallest order). Concerning symplectic symmetries, the fixed-point set structure of a symplectic cyclic action of a prime order ≥ 5 is explicitly determined, provided that the action is homologically nontrivial.

1. Introduction

The main purpose of this paper is to investigate the effect of a change of a smooth structure on the smooth symmetries of a closed, oriented 4-dimensional smoothable manifold. The influence of symmetries on smooth structures on a manifold is one of the basic questions in the theory of differentiable transformation groups. The following classical theorem of differential geometry gives a beautiful characterization of the standard sphere $\mathbb{S}^n$ among all simply connected manifolds. It led to an extensive study of various degrees of symmetry for the (higher dimensional) exotic spheres in the 1960s and 70s (cf. [30]). Lawson and Yau even found that there exist exotic spheres which support no actions of small groups such as $\mathbb{S}^3$ or $SO(3)$ (cf. [34]). See [47] for a survey.

Theorem (A Characterization of $\mathbb{S}^n$). *Let M^n be a closed, simply connected manifold of dimension n, and let G be a compact Lie group which acts smoothly and effectively on M^n. Then $\dim G \leq n(n+1)/2$, with equality if and only if M^n is diffeomorphic to $\mathbb{S}^n$.*

The subject of symmetries of exotic smooth 4-manifolds, on the other hand, has been so far rather an untested territory. Our investigations of smooth symmetries of 4-manifolds have been focused on the case of $K3$ surfaces. These manifolds exhibit surprisingly rich geometric structures and have been playing one of the central roles in both the theory of complex surfaces and topology of smooth 4-manifolds.

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To be more specific, we will study symmetries of an infinite family of distinct, closed, oriented smooth 4-manifolds, each of which is orientation-preservingly homeomorphic, but not diffeomorphic, to a $K3$ surface (canonically oriented as a complex surface). These exotic $K3$ surfaces, originally due to Fintushel and Stern, are obtained by performing the knot surgery construction simultaneously on three disjoint embedded tori representing distinct homology classes in a Kummer surface (cf. [18], compare also [24]). It is known that none of these 4-manifolds support a complex structure (cf. [18, 25]), however, one may arrange the knot surgeries so that each of these manifolds supports a symplectic structure compatible with the given orientation (cf. [18]).

A $K3$ surface is a simply-connected complex surface with trivial canonical bundle. It is known that all $K3$ surfaces are deformation equivalent as complex surfaces (therefore diffeomorphic as oriented smooth 4-manifolds), and that all $K3$ surfaces are Kähler surfaces (cf. [3]). There is an extensive study on finite subgroups of the automorphism group of a $K3$ surface, beginning with the fundamental work of Nikulin [44]. Special attention has been given to those subgroups of automorphisms which induce a trivial action on the canonical bundle of the $K3$ surface. (Such automorphisms are called symplectic; in Nikulin [44] they were called algebraic.) A finite group G is called a $K3$ group (resp. symplectic $K3$ group) if G can be realized as a subgroup of the automorphism group (resp. symplectic automorphism group) of a $K3$ surface. Finite abelian groups of symplectic automorphisms of a $K3$ surface were first classified by Nikulin in [44]; in particular it was shown that a finite symplectic automorphism must have order ≤ 8. Subsequently, Mukai [42] determined all the symplectic $K3$ groups (see also [31, 51]). There are 11 maximal ones, all of which are characterized as certain subgroups of the Mathieu group M_{23}. Finally, a cyclic group of prime order $p > 7$ is a $K3$ group (necessarily non-symplectic) if and only if $p \leq 19$ (cf. [44, 38]).

We recall three relevant properties of automorphism groups of $K3$ surfaces. First, a finite-order automorphism of a $K3$ surface preserves a Kähler structure, hence by the Hodge theory, it is symplectic if and only if the second cohomology contains a 3-dimensional subspace consisting of invariant elements of positive square. Secondly, since a symplectic automorphism acts trivially on the canonical bundle, it follows that the induced representation at a fixed point (called a local representation) lies in $SL_2(\mathbb{C})$; in particular, the fixed point is isolated. (Such actions are called pseudofree.) Finally, a nontrivial automorphism of a $K3$ surface must act nontrivially on the homology (cf. [3]).

Finite groups of automorphisms of a $K3$ surface are primary sources of smooth and symplectic symmetries of the standard $K3$. (In fact, no examples of smooth symmetries of the standard $K3$ are known to exist that are not automorphisms of a $K3$ surface.) Thus in analyzing symmetry properties of an exotic $K3$ surface, we will use these automorphisms as the base of our comparison.

We shall now state our main theorems. In what follows, we will denote by X_α a member of the infinite family of exotic $K3$ surfaces of Fintushel and Stern we alluded to earlier. (A detailed review of their construction along with some relevant properties will be given in Section 2; we point out here that the index α stands for a triple (d_1, d_2, d_3) of integers which obey $1 < d_1 < d_2 < d_3$ and are pairwise relatively prime.)

The induced action on the quadratic form and the fixed-point set structure are two fundamental pieces of information associated with a finite group action on a simply-connected 4-manifold. In this regard, we have

Theorem 1.1. *Let $G \equiv \mathbb{Z}_p$ where p is an odd prime. The following statements are true for a smooth G-action on X_α.*

1. *The induced action is trivial on a 3-dimensional subspace of $H^2(X_\alpha;\mathbb{R})$ over which the cup-product is positive definite.*
2. *For $p \geq 7$, there is a G-invariant, orthogonal decomposition of the intersection form on $H_2(X_\alpha;\mathbb{Z})$ as*

$$3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8)$$

where the induced G-action on each hyperbolic summand is trivial.

Remark 1.2. (1) For a smooth $\mathbb{Z}_p$-action on a homotopy $K3$ surface, Theorem 1.1 holds true automatically when $p \geq 23$ because in this case the action is necessarily homologically trivial. However, when $p < 23$, nothing is known in general about the induced action on the $K3$ lattice. For a prime order symplectic automorphism of a $K3$ surface, Nikulin showed in [44] that the action on the $K3$ lattice is unique up to conjugacy, which was explicitly determined in [41, 23] by examing some concrete examples of the action. In particular, Theorem 1.1 (2) is false for symplectic automorphisms of a $K3$ surface (cf. the proof of Corollary 1.3).

(2) The G-invariant, orthogonal decomposition in Theorem 1.1 (2) gives severe restrictions on the induced integral G-representation on $H_2(X_\alpha;\mathbb{Z})$; in particular, when $p > 7$ the action must be homologically trivial because Aut (E_8) contains no elements of order > 7. Note that one does not expect such a result in general, as for each prime p with $p < 23$, there exists an automorphism of a $K3$ surface of order p, which is necessarily homologically nontrivial.

(3) Let F be the fixed-point set. A general result of Edmonds (cf. Proposition 3.1) constrains the number of 2-dimensional components of nonzero genus in F by equating the first Betti number of F (if nonempty) and the number of summands of cyclotomic type in the induced integral G-representation on the second homology. For a smooth $\mathbb{Z}_p$-action on a homotopy $K3$ in general, there are no summands of cyclotomic type when $p \geq 13$, and consequently F does not contain any 2-dimensional non-spherical components in these cases. However, when $p = 7$ or 11, such a summand does occur. In fact, for both $p = 7$ and $p = 11$, there exists an automorphism of a $K3$ surface of order p which fixes a regular fiber of an elliptic fibration on the $K3$ surface (cf. [38]). With the above observations, note that Theorem 1.1 (2) implies that for a smooth $\mathbb{Z}_p$-action on X_α of order $p = 7$ or 11, F contains at most 2-dimensional spherical components (cf. Lemma 4.5), which is in contrast with the case of the standard $K3$ we mentioned earlier. Finally, a calculation with the Lefschetz fixed point theorem indicates that for $p \geq 7$, F also has a fairly large size, e.g., $\chi(F) \geq 10$. (In contrast a symplectic automorphism of a $K3$ surface of order 7 has only three isolated fixed points, hence $\chi(F) = 3$.)

We have seen from the discussions in Remark 1.2 that for any prime $p \geq 7$, a smooth $\mathbb{Z}_p$-action on X_α differs in many aspects from an automorphism of a $K3$ surface. In particular, we note the following relative nonsmoothability result as a corollary of Theorem 1.1.

Recall that each X_α is homeomorphic to a $K3$ surface. Thus any finite-order automorphism of a $K3$ surface induces a locally linear topological action on X_α after we fix a homeomorphism between X_α and the standard $K3$.

Corollary 1.3. *Any locally linear topological action induced by an automorphism of a $K3$ surface of a prime order ≥ 7 is nonsmoothable on X_α.*

Proof. Let g be an automorphism of a $K3$ surface of a prime order $p \geq 7$. If g is non-symplectic, then g is not smoothable on X_α by Theorem 1.1 (1). Suppose g is a symplectic automorphism. Then by Nikulin [44], we have $p = 7$, and moreover, the action of g is pseudofree with 3 isolated fixed points. Suppose g is smoothable on X_α. Then by Theorem 1.1 (2) and Lemma 4.5, the trace of the action of g on $H_2(X_\alpha;\mathbb{Z})$ is at least 8, so that by the Lefschetz fixed point theorem (cf. Theorem 3.4), the Euler number of the fixed point set of g is at least 10. A contradiction.

□

Next we turn our attention to smooth involutions, i.e., smooth $\mathbb{Z}_2$-actions on X_α. Let $g : X_\alpha \to X_\alpha$ be any smooth involution. Since X_α is simply-connected, g can be lifted to the spin bundle over X_α, where there are two cases: (1) g is of even type, meaning that the order of lifting to the spin bundle is 2, or (2) g is of odd type, meaning that the order of lifting to the spin bundle is 4. Moreover, g has 8 isolated fixed points in the case of an even type, and g is free or has only 2-dimensional fixed components in the case of an odd type (cf. [1, 6]).

Theorem 1.4. *Suppose $g : X_\alpha \to X_\alpha$ is an odd type smooth involution. Then the fixed-point set of g belongs to one of the following three possibilities:*

1. *An empty set.*
2. *A disjoint union of two tori.*
3. *A disjoint union of spheres or tori where the number of tori is at most one.*

Let τ be an anti-holomorphic involution on a $K3$ surface. (Note that τ is holomorphic with respect to some other complex structure on the smooth 4-manifold, cf. [13].) Then τ falls into one of the following three types according to the fixed point set $\mathrm{Fix}(\tau)$ of τ (cf. [45]); in particular, τ is of odd type:

- $\mathrm{Fix}(\tau) = \emptyset$,
- $\mathrm{Fix}(\tau)$ is a union of two tori,
- $\mathrm{Fix}(\tau)$ is a union of orientable surfaces of genus ≤ 10, such that the number of non-spherical components in $\mathrm{Fix}(\tau)$ is at most one.

Corollary 1.5. *A locally linear topological $\mathbb{Z}_2$-action induced by an anti-holomorphic involution is nonsmoothable on X_α if it has a fixed component of genus ≥ 2.* (*Note that such anti-holomorphic involutions do exist, cf.* [45].)

Remark 1.6. There are previously known examples of locally linear topological actions on closed 4-manifolds which are not smoothable. For example, there is a locally linear, pseudofree, homologically trivial topological action of order 5 on $\mathbb{CP}^2\#\mathbb{CP}^2$ which can not be realized as an equivariant connected sum of two copies of $\mathbb{CP}^2$ (cf. [16]). By the main result of [27], the action is not smoothable with respect to any smooth structure on $\mathbb{CP}^2\#\mathbb{CP}^2$. (For more recent examples of nonsmoothable actions on closed 4-manifolds, see e.g. [36, 37], and for nonsmoothable actions on non-closed 4-manifolds, see [32].) However, the nonsmoothability in Corollary 1.3 and Corollary 1.5 is of a different nature; the action is smooth (even holomorphic) for one smooth structure but not smoothable with respect to some (in fact infinitely many) other smooth structures.

Our investigation of the possible effect of a change of smooth structures on the smooth symmetries of a closed, oriented smoothable 4-manifold is based on the following simple fact. Suppose M^4 is a simply-connected, oriented smooth 4-manifold with an orientation-preserving smooth action of a finite group G. Let L be the primitive sublattice of $H^2(M^4;\mathbb{Z})$ spanned by the Seiberg-Witten basic classes of M^4 (we assume $b_2^+(M^4) > 1$). Then the induced G-action on $H^2(M^4;\mathbb{Z})$ preserves L as it preserves the set of Seiberg-Witten basic classes. One can try to analyze the G-action on $H^2(M^4;\mathbb{Z})$ through the actions on L and $L^\perp$, the orthogonal complement of L in $H^2(M^4;\mathbb{Z})$. With this understood, a crucial ingredient in our investigation is the following property of X_α: L is isotropic and of rank 3, such that

$$L^\perp/L = 2(-E_8).$$

(See Lemma 4.2 for more details.) Furthermore, one can arrange X_α such that each Seiberg-Witten basic class is fixed under the action up to a change of signs; in particular, an odd order G must act trivially on L. Given this, Theorem 1.1 follows readily by analyzing the corresponding action on $L^\perp/L = 2(-E_8)$ where G is cyclic of a prime order ≥ 7.

The above mentioned property of X_α can be further exploited to prove non-existence of effective smooth G-actions on X_α for a certain kind of finite groups G. For instance, suppose G is of odd order and there are no nontrivial G-actions on the E_8 lattice (e.g. G is a p-group with $p > 7$), then any smooth G-actions on X_α must be homologically trivial, and therefore, by a theorem of McCooey [39] G must be abelian of rank at most 2 (cf. Corollary 4.4). In particular, a nonabelian p-group with $p > 7$ can not act smoothly and effectively on X_α. We remark that while for a given finite group G, we do not know a priori any obstructions to the existence of a smooth G-action on a homotopy $K3$ surface, a non-existence result of this sort for X_α may be in fact purely topological in nature.

The following non-existence theorem of smooth actions on X_α covers the cases of several $K3$ groups, hence it must not be purely topological in nature. Its proof requires a deeper analysis of the induced actions on the E_8 lattice.

Theorem 1.7. *Let G be a finite group whose commutator $[G,G]$ contains a subgroup isomorphic to $(\mathbb{Z}_2)^4$ or Q_8, where in the case of Q_8 the elements of order 4 in the subgroup are conjugate in G. Then there are no effective smooth G-actions on X_α.*

A complete list of symplectic $K3$ groups along with their commutators can be found in Xiao [51], Table 2. By examing the list we note that among the 11 maximal $K3$

groups the following can not act smoothly and effectively on X_α (cf. Corollary 5.4):

$$M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, T_{48}.$$

We also note that the binary tetrahedral group T_{24} of order 24 is the $K3$ group of the smallest order which can not act smoothly and effectively on X_α by Theorem 1.7.

As we mentioned earlier, each exotic $K3$ X_α supports an orientation-compatible symplectic structure. In order to investigate how symplectic symmetries may depend on the underlying smooth structure of a 4-manifold, we also analyzed finite group actions on X_α which preserve an orientation-compatible symplectic structure.

Recall that $\text{Aut}(E_8 \oplus E_8)$ is a semi-direct product of $\text{Aut}(E_8) \times \text{Aut}(E_8)$ by $\mathbb{Z}_2$ (cf. [48]). Thus for any smooth G-action on X_α, where $G \equiv \mathbb{Z}_p$ for an odd prime p, there is an associated homomorphism

$$\Theta = (\Theta_1, \Theta_2) : G = \mathbb{Z}_p \to \text{Aut}(E_8) \times \text{Aut}(E_8).$$

The following theorem gives a complete description of the fixed-point set structure of a symplectic $\mathbb{Z}_p$-action on X_α for $p \geq 5$, provided that both homomorphisms $\Theta_1, \Theta_2 :$ $\mathbb{Z}_p \to \text{Aut}(E_8)$ are nontrivial. Note that this implies that $p = 5$ or $p = 7$.

Theorem 1.8. *Let $F \subset X_\alpha$ be the fixed-point set of a symplectic $\mathbb{Z}_p$-action of a prime order $p \geq 5$ such that both Θ_1, Θ_2 are nontrivial. Set $\mu_p \equiv \exp(\frac{2\pi i}{p})$. Then*

(1) if $p = 5$, there are two possibilities:

(i) *F consists of 14 isolated fixed points, two with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$, six with local representation $(z_1, z_2) \mapsto (\mu_p^{4k} z_1, \mu_p^{3k} z_2)$, two with local representation $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{4k} z_2)$, and four with local representation $(z_1, z_2) \mapsto (\mu_p^{3k} z_1, \mu_p^{3k} z_2)$, for some $k \neq 0 \pmod p$.*

(ii) *$F = F_1 \sqcup F_2$, where F_1 consists of 4 isolated fixed points with local representations*

$$(z_1, z_2) \mapsto (\mu_p^q z_1, \mu_p^{-q} z_2), \text{ evaluated at } q = 1, 2, 3, 4,$$

and F_2 is divided into groups of fixed points of the following two types where the number of groups is less than or equal to 2 (in particular, F_2 may be empty):

- *3 isolated points, two with local representation $(z_1, z_2) \mapsto (\mu_p^{-3k} z_1, \mu_p^{-k} z_2)$ and one with local representation $(z_1, z_2) \mapsto (\mu_p^{3k} z_1, \mu_p^{3k} z_2)$, and one fixed (-2)-sphere with local representation $z \mapsto \mu_p^k z$, for some $k \neq 0 \pmod p$,*
- *a fixed torus of self-intersection 0.*

(2) if $p = 7$, F consists of 10 isolated fixed points, two with local representation $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2)$, two with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{-k} z_2)$, two with local representation $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{4k} z_2)$, and four with local representation $(z_1, z_2) \mapsto (\mu_p^{-2k} z_1, \mu_p^k z_2)$, for some $k \neq 0 \pmod p$.

Remark 1.9. (1) We remark that by the work of Edmonds and Ewing [16], the fixed-point set structure of a pseudofree action in Theorem 1.8 can be actually realized by a locally linear, topological action on X_α. On the other hand, none of the known obstructions to smoothability of topological actions (see Section 3) could rule out the possibility that the fixed-point set structure may be realized by a smooth or even symplectic action.

(2) Since the case of small primes p is missing in Theorem 1.1, Theorem 1.8 can be viewed as a complement to Theorem 1.1. We remark that the homological triviality of actions in Theorem 1.1 for the case of $p > 7$ plus the detailed information about the (homologically nontrivial) $\mathbb{Z}_5$ and $\mathbb{Z}_7$ actions in Theorem 1.8 put considerable limitations on the symplectic actions of an arbitrary finite group on X_α.

The proof of Theorem 1.8 is based on a combination of the pseudoholomorphic curve techniques developed in our previous work [10] and a delicate exploitation of the induced actions on the E_8 lattice. Note that the latter is possible only because of the property $L^\perp/L = 2(-E_8)$ of X_α. For a general homotopy $K3$ surface, a symplectic $\mathbb{Z}_p$-action could have a much more complicated fixed-point set structure. However, if the finite group which acts symplectically on a homotopy $K3$ has a relatively complicated group structure (e.g., a maximal symplectic $K3$ group), then the fixed-point set structure can also be explicitly determined. This observation was systematically exploited in our subsequent paper [11] where the following problem was investigated.

Problem *Let X be a homotopy $K3$ surface supporting an effective action of a "large" $K3$ group via symplectic symmetries. What can be said about the smooth structure on X?*

In particular, a characterization of the "standard" smooth structure of $K3$ in terms of symplectic symmetry groups was obtained (compare with the corresponding characterization of $\mathbb{S}^n$ at the beginning of the introduction). See [11] for more details.

The current paper is organized as follows.

In Section 2 we give a detailed description of the Fintushel-Stern exotic $K3$'s that are to be considered in this paper, along with their relevant properties.

In Section 3 we collect various known results concerning topological and smooth actions of finite groups on 4-manifolds. These results are used in our paper (sometimes successfully and sometimes not) to measure the difference between the symmetries of the standard and exotic $K3$ surfaces. In particular, these results are the criteria used in the proof of Theorem 1.8, with which the fixed-point set structure of the group action is analyzed.

Sections 4, 5 and 6 contain proofs of Theorem 1.1, Theorem 1.4, Theorem 1.7 and Theorem 1.8.

2. The Fintushel-Stern exotic $K3$'s

The construction of this type of exotic $K3$'s was briefly mentioned in the paper of Fintushel and Stern [18]. In this section we give a detailed account of one particular family of such exotic $K3$'s that are used in this paper, along with proofs of some relevant properties that will be used in later sections.

The exotic $K3$ surfaces are the 4-manifolds that result from performing the knot surgery construction in [18] simultaneously on three disjoint embedded tori in a Kummer surface. We begin with a topological description of a Kummer surface (following [24]) and establish some relevant properties of the three disjoint tori in it.

Let $\mathbb{S}^1$ be the unit circle in $\mathbb{C}$. Let T^4 denote the 4-torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ and $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ denote the complex conjugation respectively. Moreover, we shall let $\hat{\rho}$ denote the corresponding diagonal involution on T^4 or $T^2 \equiv \mathbb{S}^1 \times \mathbb{S}^1$. Then the underlying 4-manifold X of a Kummer surface is obtained by replacing each of the 16 singularities $(\pm 1, \pm 1, \pm 1, \pm 1)$ in $T^4/\langle \hat{\rho} \rangle$ by a (-2)-sphere. More precisely, for each of the 16 singularities we shall remove a regular neighborhood of it and then glue back a regular neighborhood of an embedded (-2)-sphere (which abstractly is a D^2-bundle over $\mathbb{S}^2$ with Euler number -2). Since the gluing is along $\mathbb{RP}^3$ which has the property that a self-diffeomorphism is isotopic to identity if and only if it is orientation-preserving (cf. [4]), the resulting 4-manifolds for different choices of the gluing map are diffeomorphic to each other. In fact, they can be identified by a diffeomorphism which is identity on $T^4/\langle \hat{\rho} \rangle$ with a regular neighborhood of the 16 singularities removed and sends the corresponding embedded (-2)-spheres diffeomorphically onto each other. Our 4-manifold X is simply a fixed choice of one of these 4-manifolds. As for the orientation of X, we shall orient T^4 by $d\theta_0 \wedge d\theta_1 \wedge d\theta_2 \wedge d\theta_3$, where θ_j, $j = 0, 1, 2, 3$, is the angular coordinate (i.e. $z = \exp(i\theta)$, $z \in \mathbb{S}^1$) on the $(j+1)$-th copy of $\mathbb{S}^1$ in T^4, and the manifold X is oriented by the orientation on $T^4/\langle \hat{\rho} \rangle$, whose smooth part is contained in X.

For $j = 1, 2, 3$, let $\pi_j : T^4/\langle \hat{\rho} \rangle \to T^2/\langle \hat{\rho} \rangle$ be the map induced by the projection

$$(z_0, z_1, z_2, z_3) \mapsto (z_0, z_j).$$

There is a complex structure J_j on T^4, which is compatible with the given orientation on T^4, such that $\pi_j : T^4/\langle \hat{\rho} \rangle \to T^2/\langle \hat{\rho} \rangle$ is holomorphic. Let $X(j)$ be the minimal complex surface obtained by resolving the singularities of $T^4/\langle \hat{\rho} \rangle$. Then π_j induces an elliptic fibration $X(j) \to T^2/\langle \hat{\rho} \rangle \equiv \mathbb{S}^2$. After fixing an identification between $X(j)$ and X in the manner described in the preceding paragraph, we obtain three C^∞-elliptic fibrations (cf. [20]) $\pi_j : X \to \mathbb{S}^2$.

Given this, the three disjoint tori in X which will be used in the knot surgery are some fixed regular fibers $T_j = \pi_j^{-1}(\delta_j, i)$ of $\pi_j : X \to \mathbb{S}^2$, where $\delta_j \in \mathbb{S}^1$, $j = 1, 2, 3$, are not ± 1 and are chosen so that their images are distinct in $\mathbb{S}^1/\langle \rho \rangle$. (Note that T_1, T_2, T_3 are disjoint because the z_0-coordinates $\delta_1, \delta_2, \delta_3$ have distinct images in $\mathbb{S}^1/\langle \rho \rangle$.)

Concerning the relevant properties of the tori T_1, T_2 and T_3, we first observe

Lemma 2.1. *The three disjoint tori T_1, T_2 and T_3 have the following properties.*

1. *There are homology classes $v_1, v_2, v_3 \in H_2(X; \mathbb{Z})$ such that $v_i \cdot [T_j] = 1$ for $i = j$ and $v_i \cdot [T_j] = 0$ otherwise. In particular, $[T_1]$, $[T_2]$, $[T_3]$ are all primitive classes and span a primitive sublattice of rank 3 in $H_2(X; \mathbb{Z})$.*
2. *There are orientation compatible symplectic structures on X with respect to which T_1, T_2 and T_3 are symplectic submanifolds.*

Proof. Observe that for each torus T_j, there is a sphere S_j in the complement of the other two tori in X which intersects T_j transversely at a single point. For instance, for the torus T_1, S_1 may be taken to be the proper transform of the section

$$T^2 \times \{1\} \times \{1\}/\langle\hat{\rho}\rangle$$

of the fibration $\pi_1 : T^4/\langle\hat{\rho}\rangle \to T^2/\langle\hat{\rho}\rangle$ in the complex surface $X(1)$. Here S_1 is regarded as a sphere in X under the fixed identification between $X(1)$ and X. Part (1) of the lemma follows immediately.

Next we show that there are orientation compatible symplectic structures on X with respect to which all three tori T_1, T_2 and T_3 are symplectic. To see this, let θ_j, $j = 0, 1, 2, 3$, be the angular coordinate (i.e. $z = \exp(i\theta)$, $z \in \mathbb{S}^1$) on the $(j+1)$-th copy of $\mathbb{S}^1$ in T^4. Then the following is a symplectic 2-form on T^4 which is equivariant with respect to the diagonal involution $\hat{\rho}$:

$$\sum_{(i,j,k)} (d\theta_0 \wedge d\theta_i + d\theta_j \wedge d\theta_k)$$

where the sum is over $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. This gives rise to a symplectic structure on the orbifold $T^4/\langle\hat{\rho}\rangle$. One can further symplectically resolve the orbifold singularities to obtain a symplectic structure on X as follows. By the equivariant Darboux' theorem, the symplectic structure is standard near each orbifold singularity. In particular, it is modeled on a neighborhood of the origin in $\mathbb{C}^2/\{\pm 1\}$ and admits a Hamiltonian $\mathbb{S}^1$-action with moment map $\mu : (w_1, w_2) \mapsto \frac{1}{4}(|w_1|^2 + |w_2|)$, where w_1, w_2 are the standard coordinates on $\mathbb{C}^2$. Fix a sufficiently small $r > 0$ and remove $\mu^{-1}([0, r))$ from $T^4/\langle\hat{\rho}\rangle$ at each of its singular point. Then X is diffeomorphic to the 4-manifold obtained by collapsing each orbit of the Hamiltonian $\mathbb{S}^1$-action on the boundaries $\mu^{-1}(r)$, which is naturally a symplectic 4-manifold (cf. [35]). It is clear from the construction that all three tori T_1, T_2 and T_3 are symplectic, and moreover, the symplectic structures are orientation compatible.

□

Following [18], we call any Laurent polynomial

$$P(t) = a_0 + \sum_{j=1}^{n} a_j (t^j + t^{-j})$$

in one variable with coefficient sum

$$a_0 + 2\sum_{j=1}^{n} a_j = \pm 1, a_j \in \mathbb{Z}$$

an A-polynomial. According to [18], given any three A-polynomials $P_1(t)$, $P_2(t)$, $P_3(t)$, one can perform the so-called knot surgeries simultaneously along the tori T_1, T_2, T_3 to obtain an oriented 4-manifold $X_{P_1P_2P_3}$, which is orientation-preservingly homeomorphic to X and has Seiberg-Witten invariant

$$SW_{X_{P_1P_2P_3}} = P_1(t_1)P_2(t_2)P_3(t_3),$$

where $t_j = \exp(2[T_j])$, $j = 1, 2, 3$. We remark that the homology classes of $X_{P_1P_2P_3}$ are naturally identified with those of X, and here $[T_j]$ in $t_j = \exp(2[T_j])$ denotes the Poincaré dual of the class in $H_2(X_{P_1P_2P_3}; \mathbb{Z})$ which corresponds to the class of the torus T_j in X under the identification. (In this paper, we shall use $[T_j]$ to denote either the homology class of the torus T_j or the cohomology class that is Poincaré dual to T_j. The actual meaning is always clear from the context.) Moreover, when $P_1(t)$, $P_2(t)$, $P_3(t)$ are monic (i.e., the coefficient $a_n = \pm 1$), the 4-manifold $X_{P_1P_2P_3}$ admits orientation compatible symplectic structures because of Lemma 2.1 (2).

We shall consider one particular infinite family of $(P_1(t), P_2(t), P_3(t))$ where each A-polynomial is monic and has the form

$$P_j(t) = 1 - (t^{d_j} + t^{-d_j}), j = 1, 2, 3.$$

Here d_1, d_2, d_3 are integers which obey $1 < d_1 < d_2 < d_3$ and are pairwise relatively prime. We denote the corresponding 4-manifold $X_{P_1P_2P_3}$ by $X(d_1, d_2, d_3)$.

Lemma 2.2. *For any orientation compatible symplectic structure ω on $X(d_1, d_2, d_3)$, one has $[T_j] \cdot [\omega] \neq 0$ for all j. If we assume (without loss of generality) that $[T_j] \cdot [\omega] > 0$ for all j, then the canonical class is given by*

$$c_1(K) = 2 \sum_{j=1}^{3} d_j [T_j].$$

Proof. Recall that $\beta \in H^2$ is called a Seiberg-Witten basic class if $\exp(\beta)$ appears in the Seiberg-Witten invariant with nonzero coefficient. Given this, the Seiberg-Witten basic classes of $X(d_1, d_2, d_3)$ are the classes $2 \sum_{j=1}^{3} b_j d_j [T_j]$ where $b_j = -1, 0$, or 1.

We first observe that for any orientation compatible symplectic structure ω on $X(d_1, d_2, d_3)$, the canonical class $c_1(K)$ must equal $2 \sum_{j=1}^{3} b_j d_j [T_j]$ where each of b_1, b_2, b_3 is nonzero. The reason is as follows. According to Taubes [50], for any complex line bundle E, if $2c_1(E) - c_1(K)$ is a Seiberg-Witten basic class, then the Poincaré dual of $c_1(E)$ is represented by the fundamental class of a symplectic submanifold; in particular, $c_1(E) \cdot [\omega] > 0$ if $c_1(E) \neq 0$. Now observe that if say $c_1(K) = 2(d_2[T_2] + d_3[T_3])$ (i.e. $b_1 = 0$), then since both $2(\pm d_1[T_1] - d_2[T_2] - d_3[T_3])$ are Seiberg-Witten basic classes, both $\pm d_1[T_1]$ have a positive cup product with $[\omega]$, which is a contradiction.

By replacing $[T_j]$ with $-[T_j]$ if necessary, we may assume without loss of generality that $c_1(K) = 2 \sum_{j=1}^{3} d_j [T_j]$. With this understood, note that for each $j = 1, 2, 3$, $2d_j[T_j] - 2 \sum_{k=1}^{3} d_k [T_k]$ is a Seiberg-Witten basic class, hence by Taubes' theorem in [50], $d_j[T_j]$ is Poincaré dual to the fundamental class of a symplectic submanifold, which implies that $[T_j] \cdot [\omega] > 0$. The lemma follows easily.

□

Lemma 2.3. (1) *Let $f : X(d_1, d_2, d_3) \rightarrow X(d_1', d_2', d_3')$ be any diffeomorphism. Then for $j = 1, 2, 3$, one has $d_j = d_j'$ and $f^*([T_j']) = \pm [T_j]$. (Here $[T_j']$ denotes the corresponding class of $X(d_1', d_2', d_3')$.) In particular, $X(d_1, d_2, d_3)$ are distinct smooth 4-manifolds for distinct triples (d_1, d_2, d_3).*

(2) *Let ω be any orientation compatible symplectic structure on $X(d_1, d_2, d_3)$ and f be any self-diffeomorphism such that $f^*[\omega] = [\omega]$. Then $f^*[T_j] = [T_j]$ for $j = 1, 2, 3$.*

Proof. Observe that f must be orientation-preserving, because under an orientation-reversing diffeomorphism the signature changes by a sign of -1. Consequently, f^*

sends the Seiberg-Witten basic classes of $X(d_1', d_2', d_3')$ to those of $X(d_1, d_2, d_3)$. In particular, there are $b_j \in \mathbb{Z}$, $j = 1, 2, 3$, with each $b_j = -1, 0$ or 1 such that

$$f^*(2d_1'[T_1']) = 2\sum_{j=1}^{3} b_j d_j [T_j].$$

By Lemma 2.1 (1), there are homology classes v_1, v_2, v_3 such that $v_i \cdot [T_j] = 1$ for $i = j$ and $v_i \cdot [T_j] = 0$ otherwise. Taking cup product of each side of the above equation with v_1, v_2, v_3, we see that d_1' is a divisor of d_j if $b_j \neq 0$. Since by assumption $d_1' > 1$ and d_1, d_2, d_3 are pairwise relative prime, it follows that there exists exactly one b_j which is nonzero. Applying the same argument to f^{-1}, we see that one actually has $d_1' = d_j$ and $f^*([T_1']) = \pm[T_j]$. Since each of the triples (d_1, d_2, d_3) and (d_1', d_2', d_3') is assumed to be in the ascending order, we must have $d_j' = d_j$ and $f^*([T_j']) = \pm[T_j]$ for $j = 1, 2, 3$, as claimed in (1).

If ω is an orientation compatible symplectic structure on $X(d_1, d_2, d_3)$ and f is a self-diffeomorphism such that $f^*[\omega] = [\omega]$, then $f^*[T_j] = [T_j]$, $j = 1, 2, 3$, must be true because $[T_j] \cdot [\omega] \neq 0$ by Lemma 2.2.

□

In the remaining sections, we will abbreviate the notation and denote the exotic $K3$ $X(d_1, d_2, d_3)$ by X_α.

3. Recollection of various known results

In this section we collect some theorems (known to date) and some observations that are scattered in the literature, which may be used to provide obstructions to the existence of certain smooth finite group actions on 4-manifolds. (In fact, many of these obstructions also apply to locally linear topological actions.) For symplectic actions of a finite group on a minimal symplectic 4-manifold with $c_1^2 = 0$, there are further results in terms of the fixed-point set structure of the action. These will be briefly reviewed at the beginning of Section 6, and details may be found in [10].

Borel spectral sequence. We review here some relevant results about locally linear topological actions of a finite group on a closed simply-connected 4-manifold. The main technique for deriving these results is the Borel spectral sequence, cf. e.g. [5].

Let $G \equiv \mathbb{Z}_p$, where p is prime, act locally linearly on a closed simply-connected 4-manifold M via orientation-preserving homeomorphisms, and let F be the fixed-point set of the action. We first review a result due to Edmonds which describes a relationship between the fixed-point set F and the existence of certain types of representations of G on $H^2(M)$ induced by the action of G on M.

Recall that by a result of Kwasik and Schultz (cf. [33]), each integral representation of $\mathbb{Z}_p$ on $H^2(M)$ can be expressed as a sum of copies of the group ring $\mathbb{Z}[\mathbb{Z}_p]$ of $\mathbb{Z}$-rank p, the trivial representation $\mathbb{Z}$ of $\mathbb{Z}$-rank 1, and the representation $\mathbb{Z}[\mu_p]$ of cyclotomic type of $\mathbb{Z}$-rank $p-1$, which is the kernel of the augmentation homomorphism $\mathbb{Z}[\mathbb{Z}_p] \to \mathbb{Z}$. Here $\mu_p \equiv \exp(\frac{2\pi i}{p})$, which will be used throughout.

Proposition 3.1. (*cf.* [14], *Prop. 2.4*) *Assume that F is nonempty. Let $b_1(F)$ be the first Betti number of F in $\mathbb{Z}_p$-coefficients and let c be the number of copies of G-representations of cyclotomic type in $H^2(M)$. Then $b_1(F) = c$. In particular, $c = 0$ if the G-action is pseudofree, and $b_1(F) = 0$ if the G-action is homologically trivial.*

Another result of Edmonds gives some homological restrictions on the 2-dimensional components of the fixed-point set F.

Proposition 3.2. (*cf.* [14], *Cor. 2.6*) *If F is not purely 2-dimensional, then the 2-dimensional components of F represent independent elements of $H_2(M;\mathbb{Z}_p)$. If F is purely 2-dimensional, and has k 2-dimensional components, then the 2-dimensional components span a subspace of $H_2(M;\mathbb{Z}_p)$ of dimension at least $k-1$, with any $k-1$ components representing independent elements.*

The next theorem, due to McCooey [39], is concerned with locally linear, homologically trivial topological actions by a compact Lie group (e.g. a finite group) on a closed 4-manifold.

Theorem 3.3. *Let G be a (possibly finite) compact Lie group, and suppose M is a closed 4-manifold with $H_1(M;\mathbb{Z}) = 0$ and $b_2(M) \geq 2$, equipped with an effective, locally linear, homologically trivial G-action. Denote by F the fixed-point set of G.*

1. *If $b_2(M) = 2$ and $F \neq \emptyset$, then G is isomorphic to a subgroup of $\mathbb{S}^1 \times \mathbb{S}^1$.*
2. *If $b_2(M) \geq 3$, then G is isomorphic to a subgroup of $\mathbb{S}^1 \times \mathbb{S}^1$, and $F \neq \emptyset$.*

G-index theorems. Here we collect some formulas which fall into the realm of G-index theorems of Atiyah and Singer (cf. [2]). In particular, these formulas allow us to relate the fixed-point set structure of the group action with the induced representation on the rational cohomology of the manifold.

Let M be a closed, oriented smooth 4-manifold, and let $G \equiv \mathbb{Z}_p$ be a cyclic group of prime order p acting on M effectively via orientation-preserving diffeomorphisms. Then the fixed-point set F, if nonempty, will be in general a disjoint union of finitely many isolated points and orientable surfaces. Fix a generator $g \in G$. Then each isolated fixed point $m \in F$ is associated with a pair of integers (a_m, b_m), where $0 < a_m, b_m < p$, such that the action of g on the tangent space at m is given by the complex linear transformation $(z_1, z_2) \mapsto (\mu_p^{a_m} z_1, \mu_p^{b_m} z_2)$. (Note that a_m, b_m are uniquely determined up to a change of order or a simultaneous change of sign modulo p.) Likewise, at each connected surface $Y \subset F$, there is an integer c_Y with $0 < c_Y < p$, which is uniquely determined up to a sign modulo p, such that the action of g on the normal bundle of Y in M is given by $z \mapsto \mu_p^{c_Y} z$.

Theorem 3.4. (*Lefschetz Fixed Point Theorem*). *$L(g, M) = \chi(F)$, where $\chi(F)$ is the Euler characteristic of the fixed-point set F and $L(g, M)$ is the Lefschetz number of the map $g : M \to M$, which is defined by*

$$L(g, M) = \sum_{k=0}^{4} (-1)^k tr(g)|_{H^k(M;\mathbb{R})}.$$

Note that the above theorem holds true for topological actions, cf. [33].

Theorem 3.5. (*G-signature Theorem*). *Set*

$$Sign(g,M) = tr(g)|_{H^{2,+}(M;\mathbb{R})} - tr(g)|_{H^{2,-}(M;\mathbb{R})}.$$

Then

$$Sign(g,M) = \sum_{m\in F} -\cot(\frac{a_m\pi}{p})\cdot\cot(\frac{b_m\pi}{p}) + \sum_{Y\subset F}\csc^2(\frac{c_Y\pi}{p})\cdot(Y\cdot Y),$$

where $Y\cdot Y$ denotes the self-intersection number of Y.

Note that the G-signature Theorem is also valid for locally linear, topological actions in dimension 4, cf. e.g. [26].

One can average the formula for $\mathrm{Sign}(g,M)$ over $g\in G$ to obtain the following version of the G-signature Theorem.

Theorem 3.6. (*G-signature Theorem – the weaker version*).

$$|G|\cdot Sign(M/G) = Sign(M) + \sum_{m\in F} def_m + \sum_{Y\subset F} def_Y.$$

where the terms def_m and def_Y (called signature defects*) are given by the following formulae:*

$$def_m = \sum_{k=1}^{p-1}\frac{(1+\mu_p^k)(1+\mu_p^{kq})}{(1-\mu_p^k)(1-\mu_p^{kq})}$$

if the local representation of G at m is given by $(z_1,z_2)\mapsto(\mu_p^k z_1,\mu_p^{kq}z_2)$, and

$$def_Y = \frac{p^2-1}{3}\cdot(Y\cdot Y).$$

The above version of the G-signature Theorem is more often used because the signature defect def_m can be computed in terms of Dedekind sum, cf. [29].

Now suppose that the 4-manifold M is spin, and that the G-action on M lifts to the spin structures on M. Then the index of Dirac operator $\mathbb{D}$ gives rise to a character of G. More precisely, for each $g\in G$, one can define the "Spin-number" of g by

$$\mathrm{Spin}(g,M) = tr(g)|_{\mathrm{Ker}\mathbb{D}} - tr(g)|_{\mathrm{Coker}\mathbb{D}}.$$

If we write $\mathrm{Ker}\mathbb{D} = \oplus_{k=0}^{p-1}V_k^+$, $\mathrm{Coker}\mathbb{D} = \oplus_{k=0}^{p-1}V_k^-$, where V_k^+, V_k^- are the eigenspaces of g with eigenvalue μ_p^k, then

$$\mathrm{Spin}(g,M) = \sum_{k=0}^{p-1} d_k\mu_p^k,$$

where $d_k\equiv\dim_{\mathbb{C}}V_k^+ - \dim_{\mathbb{C}}V_k^-$. Since both $\mathrm{Ker}\mathbb{D}$ and $\mathrm{Coker}\mathbb{D}$ are quaternion vector spaces, and the quaternions i and j are anti-commutative, it follows that $V_0^\pm$ are quaternion vector spaces, and when $p=|G|$ is odd, j maps $V_k^\pm$ isomorphically to $V_{p-k}^\pm$ for $1\le k\le p-1$. This particularly implies that d_0 is even, and when p is odd, $d_k = d_{p-k}$ for $1\le k\le p-1$.

Theorem 3.7. (*G-index Theorem for Dirac Operators, cf.* [1]). *Assume further that the action of G on M is spin and that there are only isolated fixed points. Then the "Spin-number" $Spin(g,M)=\sum_{k=0}^{p-1} d_k\mu_p^k$ is given in terms of the fixed-point set structure by the following formula*

$$Spin(g,M)=-\sum_{m\in F}\epsilon(g,m)\cdot\frac{1}{4}\csc(\frac{a_m\pi}{p})\cdot\csc(\frac{b_m\pi}{p}),$$

where $\epsilon(g,m)=\pm 1$ depends on the fixed point m and the lifting of the action of g to the spin structure.

We give a formula below for the sign $\epsilon(g,m)$ with the assumption that the action of G preserves an almost complex structure on M (e.g. the action of G is via symplectic symmetries) and that the order of G is odd.

Lemma 3.8. *Assume further that M is simply-connected, spin, and $|G|=p$ is an odd prime. Then the action of G on M is spin. Moreover, if M is almost complex and the action of G preserves the almost complex structure on M, then the "Spin-number" can be computed by*

$$\begin{aligned} Spin(g,M) &= -\sum_{m\in F}\epsilon(g,m)\cdot\frac{1}{4}\csc(\frac{a_m\pi}{p})\cdot\csc(\frac{b_m\pi}{p}) \\ &\quad +\sum_{Y\subset F}\epsilon(g,Y)\cdot\frac{(Y\cdot Y)}{4}\csc(\frac{c_Y\pi}{p})\cdot\cot(\frac{c_Y\pi}{p}), \end{aligned}$$

where the signs $\epsilon(g,m)$ and $\epsilon(g,Y)$ are determined as follows. First, in the above formula, a_m, b_m and c_Y, which satisfy $0<a_m,b_m<p$ and $0<c_Y<p$, are uniquely determined because the corresponding complex representations $(z_1,z_2)\mapsto(\mu_p^{a_m}z_1,\mu_p^{b_m}z_2)$ and $z\mapsto\mu_p^{c_Y}z$ are chosen to be compatible with the almost complex structure which G preserves. With this convention, $\epsilon(g,m)$ and $\epsilon(g,Y)$ are given by

$$\epsilon(g,m)=(-1)^{k(g,m)},\quad \epsilon(g,Y)=(-1)^{k(g,Y)}$$

where $k(g,m)$ and $k(g,Y)$ are defined by equations

$$k(g,m)\cdot p=2r_m+a_m+b_m,\quad k(g,Y)\cdot p=2r_Y+c_Y$$

for some r_m, r_Y satisfying $0\le r_m<p$ and $0<r_Y<p$.

Proof. We first show that the action of G is spin. Let $E_G\rightarrow B_G$ be the universal principal G-bundle. Then observe that a bundle E over M as a G-bundle corresponds to a bundle E' over $E_G\times_G M$ whose restriction to the fiber M of the fiber bundle $E_G\times_G M\rightarrow B_G$ is E. With this understood, a G-spin structure on M corresponds to a principal $Spin(4)$-bundle over $E_G\times_G M$ whose restriction to the fiber M is a spin structure on M. The obstruction to the existence of such a bundle is a class in $H^2(E_G\times_G M;\mathbb{Z}_2)$ which maps to the second Stiefel-Whitney class $w_2(M)\in H^2(M;\mathbb{Z}_2)$ under the homomorphism $i^*: H^2(E_G\times_G M;\mathbb{Z}_2)\rightarrow H^2(M;\mathbb{Z}_2)$ induced by the inclusion $i: M\rightarrow E_G\times_G M$ as a fiber. The obstruction vanishes because (1) $w_2(M)=0$ since M is spin, (2) the homomorphism $i^*: H^*(E_G\times_G M;\mathbb{Z}_2)\rightarrow H^*(M;\mathbb{Z}_2)$ is a monomorphism, which can be seen easily by the transfer argument (cf. [5]) given that the order $|G| = p$ is odd. This proves that the action of G is spin. Note that there is a unique G-spin structure because the G-spin structures are classified by $H^1(E_G \times_G M; \mathbb{Z}_2) = 0$. Here $H^1(E_G \times_G M; \mathbb{Z}_2) = 0$ because $H^1(M; \mathbb{Z}_2) = 0$ (since M is simply-connected) and $i^* : H^1(E_G \times_G M; \mathbb{Z}_2) \to H^1(M; \mathbb{Z}_2)$ is injective.

Now suppose M is almost complex and G preserves the almost complex structure on M. Then the G-spin structure as the unique G-$Spin^{\mathbb{C}}$ structure with trivial determinant line bundle is given by a (unique) G-complex line bundle L over M such that $L^2 \otimes K^{-1} = M \times \mathbb{C}$ as a G-bundle where the action of G on $\mathbb{C}$ is trivial. Here K is the canonical bundle of the almost complex structure. Moreover, the Dirac operator $\mathbb{D}$ is simply given by the $\bar{\partial}$-complex twisted with the complex line bundle L. The "Spin-number" $\text{Spin}(g, M)$ may be computed using the G-index Theorem for the $\bar{\partial}$-complex (i.e. the holomorphic Lefschetz fixed point theorem), cf. [2].

More concretely, let the action of g at a fixed point $m \in F$ and a fixed component $Y \subset F$ be denoted by $z \mapsto \mu_p^{r_m} z$ and $z \mapsto \mu_p^{r_Y} z$ respectively. Then $L^2 \otimes K^{-1} = M \times \mathbb{C}$ as a trivial G-bundle implies that

$$2r_m + a_m + b_m = 0 \pmod p, \quad 2r_Y + c_Y = 0 \pmod p.$$

We shall impose further conditions that $0 \le r_m < p$ and $0 < r_Y < p$, and define integers $k(g, m)$, $k(g, Y)$ as in the lemma by

$$k(g, m) \cdot p = 2r_m + a_m + b_m, \quad k(g, Y) \cdot p = 2r_Y + c_Y.$$

With these understood, the contribution to $\text{Spin}(g, M)$ from $m \in F$ is

$$I_m = \frac{\mu_p^{r_m}}{(1 - \mu_p^{-a_m})(1 - \mu_p^{-b_m})} = (-1)^{k(g,m)+1} \cdot \frac{1}{4} \csc(\frac{a_m \pi}{p}) \cdot \csc(\frac{b_m \pi}{p}),$$

and the contribution from $Y \subset F$ is

$$I_Y = \frac{\mu_p^{r_Y}(1 + l)(1 + t/2)}{1 - \mu_p^{-c_Y}(1 - n)}[Y] = (-1)^{k(g,Y)} \cdot \frac{(Y \cdot Y)}{4} \csc(\frac{c_Y \pi}{p}) \cdot \cot(\frac{c_Y \pi}{p}),$$

where l, t, n are the first Chern classes of L, TY and the normal bundle of Y in M. The formula for $\text{Spin}(g, M)$ follows immediately.

□

Seiberg-Witten equations. There are obstructions to the existence of smooth finite group actions on 4-manifolds that come from Seiberg-Witten theory, based on the ideas in Furuta [22]. See [6, 17, 21, 43].

Theorem 3.9. (*cf.* [17, 43]) *Let M be a closed, oriented smooth 4-manifold with $b_1 = 0$ and $b_2^+ \ge 2$, which admits a smooth $G \equiv \mathbb{Z}_p$ action of prime order such that $H^2(M; \mathbb{R})$ contains a b_2^+-dimensional subspace consisting of invariant elements of positive square. Let c be a G-$Spin^{\mathbb{C}}$ structure on M such that the G-index of the Dirac operator $ind_G \mathbb{D} = \sum_{k=0}^{p-1} d_k \mathbb{C}_k$ satisfies $2d_k \le b_2^+ - 1$ for all $0 \le k < p$. Then the corresponding Seiberg-Witten invariant obeys*

$$SW_M(c) = 0 \pmod p.$$

Here $\mathbb{C}_k$ denotes the complex 1-dimensional weight k representation of $G \equiv \mathbb{Z}_p$.

Theorem 3.10. (*cf.* [22, 21]) *Suppose a smooth action of a finite group G on a closed, spin 4-manifold M is spin. Let $\mathbb{D}$ be the Dirac operator on the spin 4-orbifold M/G. Then either $ind\,\mathbb{D} = 0$ or $-b_2^-(M/G) < ind\,\mathbb{D} < b_2^+(M/G)$.*

We remark that when the action of G preserves an almost complex structure on M, the index of the Dirac operator, ind $\mathbb{D}$, for the 4-orbifold M/G can be calculated using Lemma 3.8, or using the formula for the dimension of the corresponding Seiberg-Witten moduli space in [8]. (See also [9].)

The Kirby-Siebenmann and the Rochlin invariants. Suppose a locally linear, topological action of a finite group G on a 4-manifold M is spin and pseudofree. Then the quotient space M/G is a spin 4-orbifold with only isolated singular points. Let N be the spin 4-manifold with boundary obtained from M/G by removing a regular neighborhood of the singular set, and denote by $\partial\eta$ the spin structure on ∂N induced from that of N. Then the Kirby-Siebenmann invariant of N, denoted by $\mathrm{ks}(N)$, and the Rochlin invariant of $(\partial N, \partial\eta)$, denoted by $\mathrm{roc}(\partial N, \partial\eta)$, are constrained by the following

Theorem 3.11. (*cf.* *§10.2B in* [19])

$$8 \cdot ks(N) \equiv Sign(N) + roc(\partial N, \partial\eta) \pmod{16}.$$

Note that a necessary condition for the G-action to be smoothable is $\mathrm{ks}(N) = 0$.

4. Smooth cyclic actions

In this section, we give proofs of Theorem 1.1 and Theorem 1.4.

The following lemma, together with Lemma 2.3 (1), settles Theorem 1.1 (1) because Lemma 2.3 (1) implies that the classes $[T_j]$, $j = 1, 2, 3$, are fixed under the action of $\mathbb{Z}_p$ whenever p is odd.

Lemma 4.1. *For any smooth action of a finite group G on X_α which fixes the classes $[T_j]$, $j = 1, 2, 3$, there is a 3-dimensional subspace of $H^2(X_\alpha; \mathbb{R})$ which is fixed under G and over which the cup-product is positive definite.*

Proof. By Lemma 2.1 (1), and since $H^2(X_\alpha; \mathbb{Z})$ and $H^2(X; \mathbb{Z})$ are naturally identified, there are classes v_i, $i = 1, 2, 3$, in $H^2(X_\alpha; \mathbb{Z})$ such that $v_i \cdot [T_j] = 1$ if $i = j$ and $v_i \cdot [T_j] = 0$ otherwise.

For any given $g \in G$, we set $v_j' \equiv \sum_{k=0}^{|g|-1} (g^k)^* v_j$. Then $g^* v_j' = v_j'$. Now for sufficiently small $\epsilon > 0$, we obtain three linearly independent classes

$$u_j \equiv [T_j] + \epsilon v_j', \; j = 1, 2, 3,$$

which are all fixed by g, i.e., $g^* u_j = u_j$ for $j = 1, 2, 3$. On the other hand, set $c \equiv \max\{|(v_1')^2| + 1, |(v_2')^2| + 1, |(v_3')^2| + 1\} > 0$. Then for any i, j,

$$\begin{aligned} u_i \cdot u_j &= [T_i] \cdot [T_j] + \epsilon([T_i] \cdot v_j' + [T_j] \cdot v_i') + \epsilon^2 v_i' \cdot v_j' \\ &= \epsilon|g|([T_i] \cdot v_j + [T_j] \cdot v_i) + \epsilon^2 v_i' \cdot v_j' \\ &= \begin{cases} 2\epsilon|g| + \epsilon^2 v_i' \cdot v_j' & \text{if } i = j \\ \epsilon^2 v_i' \cdot v_j' & \text{if } i \neq j \end{cases} \end{aligned}$$

and for any $a_1, a_2, a_3 \in \mathbb{R}$ and any $0 < \epsilon < 10^{-1}c^{-1}$,

$$\begin{aligned}(\sum_{i=1}^{3} a_i u_i)^2 &\geq 2\epsilon|g|(\sum_{i=1}^{3} a_i^2) - 3c \cdot \epsilon^2(\sum_{i=1}^{3} a_i^2) \\ &\geq \epsilon(\sum_{i=1}^{3} a_i^2).\end{aligned}$$

Consequently, for sufficiently small $\epsilon > 0$, u_1, u_2, u_3 span a 3-dimensional subspace of $H^2(X_\alpha; \mathbb{R})$ which is fixed under the action of G and over which the cup-product is positive definite.

□

Recall that the Kummer surface X (as well as the exotic X_α) has intersection form $3H \oplus 2(-E_8)$, where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$-E_8 = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & 1 \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \\ & & & & 1 & & & -2 \end{pmatrix}.$$

The $-E_8$ form is the intersection matrix of a standard basis $\{f_j | 1 \leq j \leq 8\}$, which may be conveniently described by the graph in Figure 1, where two nodes are connected by an edge if and only if the corresponding basis vectors have intersection product 1.

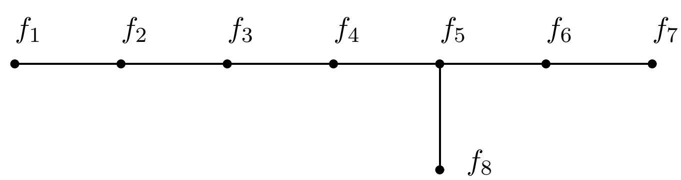

FIGURE 1.

We shall next exhibit some geometric representative of a standard basis for each of the two $(-E_8)$-summands in the intersection form of the Kummer surface X, which plays a crucial role in analyzing the induced action on $H_2(X_\alpha; \mathbb{Z})$ of a smooth finite group action on the exotic $K3$ surface X_α.

Lemma 4.2. *There exist two disjoint geometric representatives of a standard basis of the $-E_8$ form in the Kummer surface X, which both lie in the complement of the tori T_1, T_2, T_3.*

Proof. Recall that X is the 4-manifold obtained by replacing each of the 16 singularities $(\pm 1, \pm 1, \pm 1, \pm 1)$ in $T^4/\langle \hat{\rho} \rangle$ by a (-2)-sphere. We denote the (-2)-spheres by $\Sigma(\pm 1, \pm 1, \pm 1, \pm 1)$ accordingly and call them the exceptional (-2)-spheres in X.

On the other hand, recall from Section 2 that for $j = 1, 2, 3$, there is a minimal complex surface $X(j)$ and an elliptic fibration $X(j) \to \mathbb{S}^2$, where $X(j)$ is obtained by resolving the singularities of $T^4/\langle \hat{\rho} \rangle$ and the elliptic fibration comes from the fibration $\pi_j : T^4/\langle \hat{\rho} \rangle \to T^2/\langle \hat{\rho} \rangle$ induced by the projection

$$(z_0, z_1, z_2, z_3) \mapsto (z_0, z_j).$$

Note that $\pi_j : T^4/\langle \hat{\rho} \rangle \to T^2/\langle \hat{\rho} \rangle$ has 4 singular fibers, which are over $(\pm 1, \pm 1) \in T^2/\langle \hat{\rho} \rangle$. We denote the proper transform of $\pi_j^{-1}(\pm 1, \pm 1)$ in $X(j)$ by $\Sigma_j(\pm 1, \pm 1)$, which is also a (-2)-sphere.

Recall also that for each j we have fixed an identification between X and the complex surface $X(j)$. Note that under such an identification each exceptional (-2)-sphere in X inherits an orientation from the corresponding complex curve in $X(j)$. For the purpose here we shall arrange the identifications between X and the complex surfaces $X(j)$ such that each of the exceptional (-2)-spheres in X inherits a consistant orientation, and as a result, each of them is oriented and defines a homology class in $H_2(X; \mathbb{Z})$. With such identifications between X and the complex surfaces $X(j)$ fixed, we shall regard the (-2)-spheres $\Sigma_j(\pm 1, \pm 1)$ in $X(j)$ as smooth surfaces in X, and call these (-2)-spheres the proper transform (-2)-spheres in X. We orient each $\Sigma_j(\pm 1, \pm 1)$ in X by the canonical orientation of the corresponding complex curve in $X(j)$.

With the choice of orientations on each (-2)-sphere (exceptional or proper transform) understood, we observe that (1) any two distinct exceptional (-2)-spheres have intersection product 0 because they are disjoint, and (2) a proper transform (-2)-sphere and an exceptional (-2)-sphere have intersection product either 0 or 1, depending on whether they are disjoint or not. The intersection product of two distinct proper transform (-2)-spheres are described below.

Claim: *Let $\kappa, \tau, \kappa', \tau'$ take values in $\{1, -1\}$. Then the following hold true:* (1) *If $(\kappa, \tau) \neq (\kappa', \tau')$, then $\Sigma_j(\kappa, \tau)$ and $\Sigma_j(\kappa', \tau')$ are disjoint so that their intersection product is* 0, (2) *If $j \neq j'$, then the intersection product of $\Sigma_j(\kappa, \tau)$ and $\Sigma_{j'}(\kappa', \tau')$ is* 0 *when $\kappa \neq \kappa'$ (in fact the two (-2)-spheres are disjoint), and is -1 when $\kappa = \kappa'$.*

Accepting the above claim momentarily, one can easily verify that the following are two disjoint geometric representatives of a standard basis of the $-E_8$ form and that both lie in the complement of the three tori T_1, T_2 and T_3:

(1) $f_1 = -\Sigma_3(1, -1) - \Sigma(1, -1, -1, -1) - \Sigma(1, 1, -1, -1)$, $f_2 = \Sigma(1, 1, -1, -1)$, $f_3 = \Sigma_2(1, -1) + \Sigma(1, -1, -1, -1)$, $f_4 = \Sigma(1, 1, -1, 1)$, $f_5 = \Sigma_3(1, 1) + \Sigma(1, -1, -1, 1)$, $f_6 = \Sigma(1, 1, 1, 1)$, $f_7 = -\Sigma_2(1, 1) - \Sigma(1, 1, 1, -1) - \Sigma(1, 1, 1, 1)$, $f_8 = \Sigma_1(1, -1) + \Sigma(1, -1, 1, 1)$

(2) $f_1 = -\Sigma_3(-1,-1) - \Sigma(-1,-1,-1,-1) - \Sigma(-1,1,-1,-1)$, $f_2 = \Sigma(-1,1,-1,-1)$, $f_3 = \Sigma_2(-1,-1) + \Sigma(-1,-1,-1,-1)$, $f_4 = \Sigma(-1,1,-1,1)$, $f_5 = \Sigma_3(-1,1) + \Sigma(-1,-1,-1,1)$, $f_6 = \Sigma(-1,1,1,1)$, $f_7 = -\Sigma_2(-1,1) - \Sigma(-1,1,1,-1) - \Sigma(-1,1,1,1)$, $f_8 = \Sigma_1(-1,-1) + \Sigma(-1,-1,1,1)$

It remains to verify the claim. Note that part (1) of the claim follows from the fact that the two proper transform (-2)-spheres lie in two distinct fibers of the C^∞-elliptic fibration $\pi_j : X \to \mathbb{S}^2$. To see part (2), we suppose $j \neq j'$. Then $\Sigma_j(\kappa,\tau)$ and $\Sigma_{j'}(\kappa',\tau')$ are disjoint if $\kappa \neq \kappa'$ (because κ, κ' are the z_0-coordinates), and part (2) of the claim holds true in this case. Therefore we shall assume $\kappa = \kappa'$. Without loss of generality, we may assume that $\kappa = \kappa' = 1$, and for simplicity we shall only check the case where $\tau = \tau' = 1$ and $j = 2$, $j' = 3$. With this understood, note that the fiber class of $\pi_2 : X \to \mathbb{S}^2$, which is the class of the torus T_2, equals

$$2 \cdot \Sigma_2(1,1) + \Sigma(1,1,1,1) + \Sigma(1,1,1,-1) + \Sigma(1,-1,1,1) + \Sigma(1,-1,1,-1)$$

and the fiber class of $\pi_3 : X \to \mathbb{S}^2$, which is the class of the torus T_3, equals

$$2 \cdot \Sigma_3(1,1) + \Sigma(1,1,1,1) + \Sigma(1,1,-1,1) + \Sigma(1,-1,1,1) + \Sigma(1,-1,-1,1).$$

(Note that each C^∞-elliptic fibration $\pi_j : X \to \mathbb{S}^2$ has 4 singular fibers, all of type I_0^*, cf. [3], page 201.) The assertion that the intersection product of $\Sigma_2(1,1)$ and $\Sigma_3(1,1)$ equals -1 follows immediately from the fact that $[T_2] \cdot [T_3] = 0$. This finishes the verification of the claim above, and the proof of Lemma 4.2 is completed. □

Lemma 4.3. *Let G be a finite group acting on $H_2(X_\alpha;\mathbb{Z})$ preserving the intersection form and fixing each $[T_j]$, $j = 1,2,3$. Then there is an induced homomorphism $\Theta : G \to Aut\ (E_8 \oplus E_8)$ such that the action of G on $H_2(X_\alpha;\mathbb{Z})$ is trivial if and only if the induced homomorphism Θ has trivial image.*

Proof. Let ξ_k, η_k, $1 \leq k \leq 8$, be the classes in $H_2(X_\alpha;\mathbb{Z})$ corresponding to the two standard bases of the $-E_8$ form defined in the previous lemma. Then the intersection form of X_α is isomorphic to $3H$ when restricted to the orthogonal complement of Span $(\xi_k, \eta_k | 1 \leq k \leq 8)$. By Lemma 4.2 and Lemma 2.1 (1), there are classes $w_i \in H_2(X_\alpha;\mathbb{Z})$, $i = 1,2,3$, such that

$$w_i \cdot [T_j] = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j, \end{cases} \quad w_i \cdot \xi_k = w_i \cdot \eta_k = 0 \text{ and } w_i \cdot w_j = 0 \ \ \forall i,j,k.$$

Let Ω be the orthogonal complement of Span$([T_1],[T_2],[T_3])$. We shall prove that $\Omega =$ Span $([T_j], \xi_k, \eta_k | j = 1,2,3, 1 \leq k \leq 8)$. To see this, observe that w_1, w_2, w_3, $[T_1]$, $[T_2]$, $[T_3]$, and ξ_k, η_k, $1 \leq k \leq 8$, form a basis of $H_2(X_\alpha;\mathbb{Z})$. For any class $x \in H_2(X_\alpha;\mathbb{Z})$, expand x in the above basis. Then by Lemma 4.2, there are no terms of w_1, w_2, w_3 in the expansion of x if and only if its intersection product with each of $[T_1]$, $[T_2]$, $[T_3]$ is zero. This proves our claim about Ω.

Since G fixes each $[T_j]$, $j = 1,2,3$, the orthogonal complement of Span$([T_1],[T_2],[T_3])$, which is Span $([T_j], \xi_k, \eta_k | j = 1,2,3, 1 \leq k \leq 8)$, is invariant under the action of G. The induced action of G on

$$\text{Span } ([T_j], \xi_k, \eta_k | j = 1,2,3, 1 \leq k \leq 8) / \text{Span } ([T_1],[T_2],[T_3])$$

gives rise to a homomorphism $\Theta : G \to \text{Aut }(E_8 \oplus E_8)$ by Lemma 4.2.

It remains to show that if Θ has trivial image, then the action of G on $H_2(X_\alpha; \mathbb{Z})$ is also trivial. To see this, let $g \in G$ be any given element. Then for each $\alpha \in \text{Span }(\xi_k, \eta_k | 1 \leq k \leq 8)$, there exists a $u_\alpha \in \text{Span }([T_1], [T_2], [T_3])$ such that $g \cdot \alpha = \alpha + u_\alpha$. This gives, for each $n \in \mathbb{Z}^+$,

$$g^n \cdot \alpha = \alpha + n u_\alpha \text{ because } g \cdot u_\alpha = u_\alpha.$$

It follows easily that $u_\alpha = 0$ since g is of finite order, and consequently, $g \cdot \alpha = \alpha$.

For each w_j, $j = 1, 2, 3$, there are $\hat{w}_j \in \text{Span }(w_1, w_2, w_3)$, $u_j \in \text{Span }([T_1], [T_2], [T_3])$ and $\alpha_j \in \text{Span }(\xi_k, \eta_k | 1 \leq k \leq 8)$, such that

$$g \cdot w_j = \hat{w}_j + u_j + \alpha_j.$$

Taking intersection product with $[T_k]$, $k = 1, 2, 3$, we see that $\hat{w}_j = w_j$, and taking intersection product with $\alpha_j = g \cdot \alpha_j$, we see that $\alpha_j^2 = 0$, hence $\alpha_j = 0$ (because $-E_8$ is negative definite). Consequently, we have $g \cdot w_j = w_j + u_j$. Since g fixes u_j and is of finite order, we see as in the earlier argument that $u_j = 0$, and therefore $g \cdot w_j = w_j$. Thus G acts trivially on $H_2(X_\alpha; \mathbb{Z})$ if Θ has trivial image.

□

The following corollary gives Theorem 1.1 (2) for the case of $p > 7$.

Corollary 4.4. *Let G be a p-group with $p > 7$. Then any smooth G-action on X_α must act trivially on homology; in particular, G must be abelian of rank at most* 2.

Proof. Since $|G|$ is odd, the classes $[T_j]$, $j = 1, 2, 3$, are fixed under the G-action by Lemma 2.3 (1). On the other hand, $\Theta : G \to \text{Aut }(E_8 \oplus E_8)$ must have trivial image because the order of Aut $(E_8 \oplus E_8)$ is $2^{29} \cdot 3^{10} \cdot 5^4 \cdot 7^2$ (cf. [48]), which is not divisible by any prime $p > 7$. By Lemma 4.3, the induced G-action on $H_2(X_\alpha; \mathbb{Z})$ must be trivial. It follows that the G-action is homologically trivial because $H_1(X_\alpha; \mathbb{Z}) = H_3(X_\alpha; \mathbb{Z}) = 0$ (X_α is simply-connected). The last assertion follows from McCooey's theorem (cf. Theorem 3.3).

□

The following lemma can be found in [15], however, for completeness we sketch its proof here.

Lemma 4.5. *The following are the only possibilities for integral representations of $\mathbb{Z}_p$ for $p = 3, 5, 7$ induced by $\mathbb{Z}_p \subset$ Aut (E_8):*

$\mathbb{Z}_3$: $\mathbb{Z}[\mathbb{Z}_3] \oplus \mathbb{Z}^5$, $\mathbb{Z}[\mathbb{Z}_3]^2 \oplus \mathbb{Z}^2$, $\mathbb{Z}[\mathbb{Z}_3] \oplus \mathbb{Z}[\mu_3]^2 \oplus \mathbb{Z}$, *and* $\mathbb{Z}[\mu_3]^4$
$\mathbb{Z}_5$: $\mathbb{Z}[\mathbb{Z}_5] \oplus \mathbb{Z}^3$ *and* $\mathbb{Z}[\mu_5]^2$
$\mathbb{Z}_7$: $\mathbb{Z}[\mathbb{Z}_7] \oplus \mathbb{Z}$.

Proof. Since $p < 23$, by a result of Reiner (cf. [12]) such a representation is of the form $\mathbb{Z}[\mathbb{Z}_p]^r \oplus \mathbb{Z}[\mu_p]^s \oplus \mathbb{Z}^t$, where $pr + (p-1)s + t = 8$. By Hambleton and Riehm [28], s must be even. Moreover, observe that $\mathbb{Z}[\mu_p]^s$ and $\mathbb{Z}^t$ are always orthogonal to each other, so that if $r = 0$ one of s or t must be 0 as well because the form E_8 is not splittable. The lemma follows.

□

We remark that the integral representations of $\mathbb{Z}_p$ in the above lemma are all realized by a subgroup of Aut (E_8) of order p, cf. [15].

The following proposition settles the case of $p = 7$ in Theorem 1.1 (2).

Proposition 4.6. *Suppose $G \equiv \mathbb{Z}_p$, where $p = 3, 5, 7$, acts smoothly on X_α such that the integral G-representation given by $\Theta : G \to Aut(E_8 \oplus E_8)$ contains no summands of cyclotomic type. Then the intersection form on $H_2(X_\alpha; \mathbb{Z})$ may be decomposed as $3H \oplus 2(-E_8)$ such that each summand H or $-E_8$ is invariant under the G-action. Moreover, the action of G on each H-summand is trivial.*

Proof. Recall that Aut $(E_8 \oplus E_8)$ is a semi-direct product of Aut $(E_8) \times$ Aut (E_8) by $\mathbb{Z}_2$ (cf. [48]). Since the order $|G| = p$ is odd, G maps trivially to $\mathbb{Z}_2$ under $\Theta : G \to$ Aut $(E_8 \oplus E_8)$ and hence it can not exchange the two E_8-summands. It follows that each of

$$\text{Span } ([T_j], \xi_k | j = 1, 2, 3, 1 \leq k \leq 8), \text{ Span } ([T_j], \eta_k | j = 1, 2, 3, 1 \leq k \leq 8)$$

is invariant under the action of G, and there are two induced integral representations of G on E_8 given by the action on

$$\text{Span } ([T_j], \xi_k | j = 1, 2, 3, 1 \leq k \leq 8)/\text{Span } ([T_1], [T_2], [T_3])$$

and

$$\text{Span } ([T_j], \eta_k | j = 1, 2, 3, 1 \leq k \leq 8)/\text{Span } ([T_1], [T_2], [T_3])$$

respectively.

We claim that there are classes $\xi'_k, \eta'_k \in H_2(X_\alpha; \mathbb{Z})$ such that

(i) $\xi'_k = \xi_k, \eta'_k = \eta_k \pmod{\text{Span } ([T_1], [T_2], [T_3])}$,
(ii) Span $(\xi'_k | 1 \leq k \leq 8)$, Span $(\eta'_k | 1 \leq k \leq 8)$ are invariant under G.

Note that Span $(\xi'_k | 1 \leq k \leq 8)$ and Span $(\eta'_k | 1 \leq k \leq 8)$ split off two G-invariant copies of $-E_8$ from $H_2(X_\alpha; \mathbb{Z})$. The orthogonal complement, which is isomorphic to $3H$ and is also G-invariant, contains Span $([T_1], [T_2], [T_3])$. A similar argument as in the proof of Lemma 4.3 shows that the action of G is trivial on each copy of H.

It remains to verify the above claim. For simplicity, we shall only consider the case of ξ_k's, the other case is completely parallel. Let $g \in G$ be a fixed generator.

The key point of the proof is that a summand of type $\mathbb{Z}$ or $\mathbb{Z}[\mathbb{Z}_p]$ in

$$\text{Span } ([T_j], \xi_k | j = 1, 2, 3, 1 \leq k \leq 8)/\text{Span } ([T_1], [T_2], [T_3])$$

can be lifted to a $\mathbb{Z}[\mathbb{Z}_p]$-submodule of the same type in Span $([T_j], \xi_k | j = 1, 2, 3, 1 \leq k \leq 8)$. By Lemma 4.5 these are the only types of summands if there are no summands of cyclotomic type (which is always the case when $p = 7$).

More concretely, let x be a generator of a $\mathbb{Z}$-summand in

$$\text{Span } ([T_j], \xi_k | j = 1, 2, 3, 1 \leq k \leq 8)/\text{Span } ([T_1], [T_2], [T_3])$$

and let x' be any lift of x in Span $([T_j], \xi_k | j = 1, 2, 3, 1 \leq k \leq 8)$. Then $g \cdot x' = x' + u$ for some $u \in$ Span $([T_1], [T_2], [T_3])$. As we argued in the proof of Lemma 4.3, this implies that $u = 0$ and $g \cdot x' = x'$. Hence x' generates a $\mathbb{Z}[\mathbb{Z}_p]$-submodule of the same type which is a lift of the original $\mathbb{Z}$-summand.

Let y be a generator of a $\mathbb{Z}[\mathbb{Z}_p]$-summand (as a $\mathbb{Z}[\mathbb{Z}_p]$-submodule). Pick any lift y' of y in Span $([T_j], \xi_k | j = 1,2,3, 1 \leq k \leq 8)$, then y' generates a free $\mathbb{Z}[\mathbb{Z}_p]$-submodule in Span $([T_j], \xi_k | j = 1,2,3, 1 \leq k \leq 8)$ which is a lift of the original $\mathbb{Z}[\mathbb{Z}_p]$-summand of the same type.

Now suppose the integral G-representation

$$\text{Span } ([T_j], \xi_k | j = 1,2,3, 1 \leq k \leq 8)/\text{Span } ([T_1], [T_2], [T_3])$$

is decomposed as $\mathbb{Z}^t \oplus \mathbb{Z}[\mathbb{Z}_p]^r$, and let $\{x_i, y_j | 1 \leq i \leq t, 1 \leq j \leq r\}$ be a set of generators of the summands as $\mathbb{Z}[\mathbb{Z}_p]$-submodules. Then the set

$$\{x_i, y_j, g \cdot y_j, \cdots, g^{p-1} \cdot y_j\}$$

forms a $\mathbb{Z}$-basis of

$$\text{Span } ([T_j], \xi_k | j = 1,2,3, 1 \leq k \leq 8)/\text{Span } ([T_1], [T_2], [T_3]).$$

Note that the intersection form on

$$\text{Span } (x_i', y_j', g \cdot y_j', \cdots, g^{p-1} \cdot y_j'),$$

i.e., the span of the lifts, is isomorphic to that on

$$\text{Span } ([T_j], \xi_k | j = 1,2,3, 1 \leq k \leq 8)/\text{Span } ([T_1], [T_2], [T_3]).$$

The existence of ξ_k''s follows immediately.

This completes the proof of the proposition.

□

Remark 4.7. In general, a summand of cyclotomic type may not be lifted to a summand of the same type under a quotient homomorphism. For a simple example, let us consider the integral $\mathbb{Z}_2$-representation on $\mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle y\rangle$ which is defined by

$$g \cdot x = x, \quad g \cdot y = -y + x.$$

One can check easily that the integral $\mathbb{Z}_2$-representation on the quotient modulo $\mathbb{Z}\langle x\rangle$, which is of cyclotomic type, does not lift to a summand of the same type in $\mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle y\rangle$ because $\mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle y\rangle = \mathbb{Z}[\mathbb{Z}_2]\langle (x-y)\rangle$ is of regular type.

Likewise, a summand of cyclotomic type in a $\mathbb{Z}[\mathbb{Z}_p]$-submodule of a $\mathbb{Z}[\mathbb{Z}_p]$-module may not be a summand of the same type in the $\mathbb{Z}[\mathbb{Z}_p]$-module.

We end this section with a proof of Theorem 1.4. Recall that by Bryan [6], a smooth involution $g : X_\alpha \to X_\alpha$ is of odd type if and only if $b_2^+(X_\alpha/\langle g\rangle) = 1$. On the other hand, one can easily check that this condition implies that one of the classes $[T_1]$, $[T_2]$ and $[T_3]$ must be fixed by g. We assume without loss of generality that $g^*[T_1] = [T_1]$.

Lemma 4.8. *Let Σ be a non-spherical fixed component of g. Then (1) $\chi(\Sigma) + \Sigma^2 = 0$, and (2) $\Sigma \cdot [T_1] = 0$.*

Proof. First of all, let $\{\Sigma_j\}$ be the set of fixed components of g. Then the Lefschetz fixed point theorem and the G-signature theorem (cf. Theorem 3.4 and Theorem 3.6) imply that

$$\begin{cases} 2 + t - (22 - t) & = & \sum_j \chi(\Sigma_j) \\ 2(2-t) & = & -16 + \sum_j \frac{2^2-1}{3} \cdot \Sigma_j^2, \end{cases}$$

where t denotes the dimension of the 1-eigenspace of g in $H^2(X_\alpha;\mathbb{R})$. It follows easily from the above equations that $\sum_j(\chi(\Sigma_j)+\Sigma_j^2)=0$.

Now let $\{\Sigma_i\}$ be the set of components in $\{\Sigma_j\}$ such that $\Sigma_i^2<0$, and let $\{\Sigma_k\}$ be the set of components with $\Sigma_k^2\geq 0$. Then since $2d_1[T_1]$ is a Seiberg-Witten basic class, by the generalized adjunction inequality,

$$\text{genus}(\Sigma_k)\geq 1+\frac{1}{2}(|2d_1[T_1]\cdot\Sigma_k|+\Sigma_k^2)$$

for each k. On the other hand, since X_α is even, $\Sigma_i^2\leq -2$ for each i, so that

$$\text{genus}(\Sigma_i)\geq 1+\frac{1}{2}\Sigma_i^2.$$

Putting these two inequalities together, and with $\sum_j(\chi(\Sigma_j)+\Sigma_j^2)=0$, we obtain

$$\sum_k|2d_1[T_1]\cdot\Sigma_k|\leq 0,$$

which implies that $[T_1]\cdot\Sigma_k=0$ for each k and $\chi(\Sigma_j)+\Sigma_j^2=0$ for each j. The lemma follows immediately.

□

Proof of Theorem 1.4

Let Σ be a fixed component of g with genus ≥ 1. We shall prove that Σ must be a torus of self-intersection 0 and that the class $[\Sigma]$ is a multiple of $[T_1]$ over $\mathbb{Q}$. Theorem 1.4 follows easily from this and the result of Edmonds stated in Proposition 3.2.

We fix a g-equivariant decomposition $H^2(X_\alpha;\mathbb{R})=H^+\oplus H^-$ where H^+, H^- are positive definite and negative definite respectively. Since $b_2^+(X_\alpha/\langle g\rangle)=1$, there is a 1-dimensional subspace of H^+ which is fixed under g. We fix a vector $u\in H^+$ in this subspace such that $u^2=1$. Now because both $[T_1]$ and $[\Sigma]$ are fixed under g, we may write

$$[T_1]=a_1u+\beta_1 \text{ and } [\Sigma]=a_2u+\beta_2$$

for some $a_1,a_2\in\mathbb{R}$ and $\beta_1,\beta_2\in H^-$. Note that our claim is trivially true if $[\Sigma]=0$. Assuming $[\Sigma]\neq 0$, and note that $[T_1]\neq 0$, $T_1^2=0$, and $\Sigma^2=-\chi(\Sigma)\geq 0$, we must have $a_1,a_2\neq 0$. We may assume without loss of generality that $a_1,a_2>0$. With this understood, $T_1^2=0$, $\Sigma^2=-\chi(\Sigma)\geq 0$ and $[T_1]\cdot\Sigma=0$ give rise to

$$a_1^2+\beta_1^2=0,\;\; a_2^2+\beta_2^2\geq 0,\text{ and } a_1a_2+\beta_1\cdot\beta_2=0.$$

It follows easily that

$$|\beta_1\cdot\beta_2|=a_1a_2\geq(|\beta_1^2|\cdot|\beta_2^2|)^{1/2},$$

which implies by the triangle inequality that β_1,β_2 must be linearly dependent and that the above must hold with equality. It follows easily that $[\Sigma]$ is a multiple of $[T_1]$, and that Σ is a torus of self-intersection 0.

□

5. Proof of Theorem 1.7 — Involutions in Aut(E_8)

The proof of Theorem 1.7 requires a digression on the conjugacy classes of elements of order 2 in Aut(E_8). We shall give a brief review of this material next, which is taken from Carter [7].

Let $e_1, e_2, \cdots, e_8$ be a standard basis of $\mathbb{R}^8$, i.e., $(e_i, e_j) = \delta_{ij}$. Then the E_8 lattice is the lattice generated by the set of vectors

$$\Phi \equiv \{\pm e_i \pm e_j, \frac{1}{2}\sum_{i=1}^{8} \epsilon_i e_i | \epsilon_i = \pm 1, \prod_{i=1}^{8} \epsilon_i = 1\}.$$

Furthermore, Φ forms the root system of E_8.

For any root $r \in \Phi$, there is an associated reflection $w_r \in \mathrm{Aut}(E_8)$ defined by

$$w_r(x) = x - (r, x)r.$$

It is known that Aut(E_8) is generated by $\{w_r | r \in \Phi\}$.

According to Lemma 5 of [7], every involution $v \in \mathrm{Aut}(E_8)$ can be written as a product $v = w_{r_1} \cdot w_{r_2} \cdot \cdots \cdot w_{r_k}$, where $k = l(v)$ equals the number of -1-eigenvectors of v in $\mathbb{R}^8$, and $r_1, \cdots, r_k$ are mutually orthogonal roots. In particular, by changing v to $(-1) \cdot v$ if necessary, we may assume that $k = l(v) \leq 4$.

Let $f_1 = e_1 - e_2$, $f_2 = e_2 - e_3$, $\cdots$, $f_6 = e_6 - e_7$, $f_7 = e_7 + e_8$, and

$$f_8 = \frac{1}{2}(-e_1 - e_2 - e_3 - e_4 - e_5 + e_6 + e_7 - e_8).$$

Then one can easily check that $f_1, f_2, \cdots, f_8$ form a standard basis for the E_8 lattice. In particular, f_1, f_3, f_5, f_7 are mutually orthogonal roots.

Now according to [7] (see Lemma 11, Lemma 27, and Corollary (iv) following Proposition 38 in [7]), an involution $v \in \mathrm{Aut}(E_8)$ is conjugate to one of

$$w_{f_1}, \quad w_{f_1} \cdot w_{f_3}, \quad w_{f_1} \cdot w_{f_3} \cdot w_{f_5}$$

if $l(v) \leq 3$, and when $l(v) = 4$, v has two different conjugacy classes represented by

$$w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7} \text{ and } w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7'},$$

where $f_7' = e_7 - e_8$. End of digression.

The following lemma is the starting point of our analysis.

Lemma 5.1. *Suppose $\tau : X_\alpha \to X_\alpha$ is a smooth involution which fixes the classes $[T_1]$, $[T_2]$ and $[T_3]$ and such that $\Theta(\tau) = (v_1, v_2) \in Aut(E_8) \times Aut(E_8)$. (Recall that Θ is the homomorphism in Lemma 4.3.) Then both v_1, v_2 are conjugate to the involution $w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7'}$ in $Aut(E_8)$.*

Proof. Since τ fixes the classes $[T_1]$, $[T_2]$ and $[T_3]$, $b_2^+(X_\alpha/\langle\tau\rangle) = 3$ by Lemma 4.1. Consequently τ is an even type involution with 8 isolated fixed points by Bryan [6].

The key property of τ we need here is that for any $x \in H_2(X_\alpha; \mathbb{Z})$, the intersection product of x with $\tau \cdot x$ is even (cf. [15]). To see this, represent x by a smooth surface Γ in X_α which is away from the fixed-point set of τ, then perturb Γ slightly so that Γ and $\tau(\Gamma)$ intersect transversely. It is easily seen that the intersection points of Γ and $\tau(\Gamma)$ are paired up by τ, and hence the claim.

With this understood, we observe that if $v \in \mathrm{Aut}(E_8)$ is an involution which is conjugate to any of the following 4 involutions

$$w_{f_1}, \quad w_{f_1} \cdot w_{f_3}, \quad w_{f_1} \cdot w_{f_3} \cdot w_{f_5}, \quad w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7},$$

then there exists a root $x \in \Phi$ such that $(v(x), x) = 1$. It suffices to check this for the above standard representatives of the conjugacy classes, which is done below.

If $v = w_{f_1}$, we take $x = f_2$. Then $(v(x), x) = (f_2 + f_1, f_2) = 2 - 1 = 1$. If $v = w_{f_1} \cdot w_{f_3}$, we take $x = f_4$. Then $(v(x), x) = (f_4 + f_3, f_4) = 2 - 1 = 1$. If $v = w_{f_1} \cdot w_{f_3} \cdot w_{f_5}$, we take $x = f_6$, and $(v(x), x) = (f_6 + f_5, f_6) = 1$. If $v = w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7}$, we take $x = f_8$, and $(v(x), x) = (f_8 + f_5, f_8) = 1$. (On the other hand, if $v = w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7'}$, then a direct check shows that $(v(f_i), f_i) = 0 \pmod 2$ for any $1 \leq i \leq 8$.)

Consequently, by the classification of conjugacy classes of involutions in $\mathrm{Aut}(E_8)$, we conclude that the involutions v_1, v_2 in $\Theta(\tau) = (v_1, v_2) \in \mathrm{Aut}(E_8) \times \mathrm{Aut}(E_8)$ have the following possibilities: either conjugate to $w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7'}$ or equal to 1, -1.

It remains to show that neither of v_1, v_2 can be 1 or -1. To this end, recall that there are classes $w_i \in H_2(X_\alpha; \mathbb{Z})$, $i = 1, 2, 3$, which are dual to $[T_i]$. Moreover, since τ fixes $[T_1]$, $[T_2]$ and $[T_3]$, for each $i = 1, 2, 3$, there are $u_i \in \mathrm{Span}([T_1], [T_2], [T_3])$ and $\alpha_i \in \mathrm{Span}(\xi_k, \eta_k | 1 \leq k \leq 8)$ (the classes ξ_k, η_k are defined in Lemma 4.3) such that

$$\tau \cdot w_i = w_i + u_i + \alpha_i.$$

(See the proof of Lemma 4.3 for details.) It follows easily from this that

$$tr(\tau)|_{H_2(X_\alpha; \mathbb{Z})} = 6 + tr(v_1) + tr(v_2).$$

On the other hand, τ has 8 isolated fixed points, so by the Lefschetz fixed point theorem (cf. Theorem 3.4), $tr(\tau)|_{H_2(X_\alpha; \mathbb{Z})} = 8 - 2 = 6$, which implies that

$$tr(v_1) + tr(v_2) = 0.$$

Consequently, if $v_1 = 1$ or -1, v_2 must be -1 or 1 respectively. Without loss of generality, we assume that $v_1 = 1$ and $v_2 = -1$.

Now $v_1 = 1$ means that τ acts trivially on

$$\mathrm{Span}(\xi_k, [T_j] | 1 \leq k \leq 8, 1 \leq j \leq 3) / \mathrm{Span}([T_1], [T_2], [T_3]).$$

As we argued in the proof of Lemma 4.3, this implies that each ξ_k is fixed under τ. We thus obtain a τ-invariant decomposition

$$H_2(X_\alpha; \mathbb{Z}) = \mathrm{Span}(\xi_k | 1 \leq k \leq 8) \oplus \mathrm{Span}(\xi_k | 1 \leq k \leq 8)^\perp.$$

(Here we use the fact that the intersection form on $\mathrm{Span}(\xi_k | 1 \leq k \leq 8)$ is $-E_8$ which is unimodular.) Suppose $\mathrm{Span}(\xi_k | 1 \leq k \leq 8)^\perp = \mathbb{Z}[\mathbb{Z}_2]^r \oplus \mathbb{Z}^t \oplus \mathbb{Z}[\mu_2]^s$ is a decomposition of the $\mathbb{Z}_2$-integral representation into a block sum of summands of regular, trivial and cyclotomic types. Then correspondingly we have a decomposition

$$H_2(X_\alpha; \mathbb{Z}) = \mathbb{Z}[\mathbb{Z}_2]^r \oplus \mathbb{Z}^{t+8} \oplus \mathbb{Z}[\mu_2]^s.$$

Now $tr(\tau)|_{H_2(X_\alpha; \mathbb{Z})} = 6$ implies $t + 8 - s = 6$, which implies that $s = t + 2 > 0$. However, τ is pseudofree and has a nonempty fixed-point set, so that s must be 0 by Edmonds' result (cf. Prop. 3.1). This is a contradiction, and the lemma follows.

∎

Lemma 5.1 suggests that one should study 2-subgroups of $\mathrm{Aut}(E_8)$ whose elements of order 2 are conjugate to $w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7'}$. To this end we need to recall a natual subgroup of $\mathrm{Aut}(E_8)$ which contains all the 2-subgroups up to conjugacy.

Consider the following two subgroups H_0 and H_1 of $\mathrm{Aut}(E_8)$,

$$H_0 = \{(\epsilon_i) | 1 \leq i \leq 8, \epsilon_i = \pm 1, \prod_{i=1}^{8} \epsilon_i = 1\} \cong (\mathbb{Z}_2)^7,$$

where H_0 acts by coordinate-wise multiplications on $\mathbb{R}^8$ with respect to the standard basis $e_1, \cdots, e_8$, and

$$H_1 = \{\sigma | \sigma \text{ is a permutation of } e_1, \cdots, e_8\} \cong S_8.$$

Let $H \subset \mathrm{Aut}(E_8)$ be the subgroup generated by H_0 and H_1. Then H is a semi-direct product of H_0 by H_1 with relations

$$(\epsilon_i)\sigma = \sigma(\epsilon_i'), \text{ where } \epsilon_i' = \epsilon_{\sigma(i)}, \ \forall\, (\epsilon_i) \in H_0, \ \ \sigma \in H_1.$$

In particular, the Sylow 2-subgroups of H has order 2^{14} which is the same as the order of Sylow 2-subgroups of $\mathrm{Aut}(E_8)$. Thus by Sylow's theorem, up to conjugacy in $\mathrm{Aut}(E_8)$ any 2-subgroup of $\mathrm{Aut}(E_8)$ is contained in H.

Lemma 5.2. *(1) Let $v = (\epsilon_i)\hat{v} \in H$ where $(\epsilon_i) \in H_0$ and $\hat{v} \in H_1$. Then v is conjugate to $w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7'}$ in Aut(E_8) if and only if the following conditions are satisfied:*

- *either $\hat{v} = 1$ or $\hat{v} = \sigma_1\sigma_2\sigma_3\sigma_4$ where σ_i are disjoint transpositions,*
- *$\epsilon_i = \epsilon_{\hat{v}(i)}$ for any i and $\#\{i | \epsilon_i = -1\} = 0 \pmod 4$,*
- *when $\hat{v} = 1$, $\#\{i | \epsilon_i = -1\} = 4$.*

(2) Let $v = (\epsilon_i)\hat{v} \in H$ be of order 4 such that v^2 is conjugate to $w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7'}$ in Aut(E_8). Then there are the following two possibilities for v:

- *Case (i) $\hat{v}^2 = 1$, where up to conjugacy in H, either*
  - *$\hat{v} = (12)(34)$ with $\epsilon_1\epsilon_2 = \epsilon_3\epsilon_4 = -1$, or*
  - *$\hat{v} = (12)(34)(56)$ with $\epsilon_1\epsilon_2 = \epsilon_3\epsilon_4 = -1$, and $\epsilon_5\epsilon_6 = 1$, or*
  - *$\hat{v} = (12)(34)(56)(78)$ with $\epsilon_1\epsilon_2 = \epsilon_3\epsilon_4 = -1$, and $\epsilon_5\epsilon_6 = \epsilon_7\epsilon_8 = 1$.*
- *Case (ii) $\hat{v}^2 \neq 1$, where up to conjugacy in H, $\hat{v} = \sigma_1\sigma_2$ for two disjoint 4-cycles $\sigma_1, \sigma_2 \in H_1$, and (ϵ_i) satisfies $\epsilon_i = \epsilon_{\hat{v}(i)}$ for any i.*

Proof. (1) Note that up to conjugacy $w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7'}$ may be characterized as the only involution $v \in \mathrm{Aut}(E_8)$ such that $v \neq -1$ and $(v(r), r) = 0 \pmod 2$, $\forall r \in \Phi$.

Now suppose $v = (\epsilon_i)\hat{v} \in H$ is an involution. Then

$$1 = v^2 = (\epsilon_i)\hat{v}(\epsilon_i)\hat{v} = (\epsilon_i)(\epsilon_{\hat{v}^{-1}(i)})\hat{v}\hat{v},$$

which implies $\hat{v}^2 = 1$ and $\epsilon_i = \epsilon_{\hat{v}(i)}$ for any i.

Next we show that if $\hat{v} \neq 1$, then $\hat{v}$ must be a product of 4 disjoint transpositions. To see this, suppose there exist $i \neq j$ such that $\hat{v}(i) = j$ and there exists a $k (\neq i, j)$ such that $\hat{v}(k) = k$, then for the root $e_i + e_k \in \Phi$,

$$(v(e_i + e_k), (e_i + e_k)) = (\epsilon_j e_j + \epsilon_k e_k, e_i + e_k) = \epsilon_k.$$

Hence if v is conjugate to $w_{f_1} \cdot w_{f_3} \cdot w_{f_5} \cdot w_{f_7'}$, then either $\hat{v} \neq 1$ or $\hat{v}$ is a product of 4 disjoint transpositions.

To see that $\#\{i|\epsilon_i=-1\}=0 \pmod 4$, note that

$$\begin{aligned}(v(f_8),f_8) &= \frac{1}{4}(\#\{i|\epsilon_i=1\}-\#\{i|\epsilon_i=-1\})\\ &= \frac{1}{4}(8-2\#\{i|\epsilon_i=-1\})\\ &= 2-\frac{1}{2}\#\{i|\epsilon_i=-1\}.\end{aligned}$$

Thus $(v(f_8),f_8)=0 \pmod 2$ if and only if $\#\{i|\epsilon_i=-1\}=0 \pmod 4$. When $\hat{v}=1$, $v=(\epsilon_i)$. Since $v\neq 1$ or -1, we must have $\#\{i|\epsilon_i=-1\}=4$.

Now suppose $v=(\epsilon_i)\hat{v}$ which satisfies the conditions in (1) of the lemma. If $\hat{v}=1$, then $v=(\epsilon_i)$ where $\#\{i|\epsilon_i=-1\}=4$. In particular, $v\neq 1,-1$. Moreover, for any root $r=\pm e_i\pm e_j$,

$$(v(r),r)=\epsilon_i+\epsilon_j=0 \pmod 2,$$

and for any root $r=\frac{1}{2}(\sum_i \pm e_i)$,

$$(v(r),r)=\frac{1}{4}(\#\{i|\epsilon_i=1\}-\#\{i|\epsilon_i=-1\})=0.$$

Hence v is conjugate to $w_{f_1}\cdot w_{f_3}\cdot w_{f_5}\cdot w_{f_7'}$ by the characterization of $w_{f_1}\cdot w_{f_3}\cdot w_{f_5}\cdot w_{f_7'}$ we mentioned at the beginning of the proof. When $\hat{v}=\sigma_1\sigma_2\sigma_3\sigma_4$ where σ_i are disjoint transpositions, the conditions $\epsilon_i=\epsilon_{\hat{v}(i)}$ for any i and $\#\{i|\epsilon_i=-1\}=0 \pmod 4$ imply that v is conjugate to $\hat{v}=\sigma_1\sigma_2\sigma_3\sigma_4$ by an element of H_0. On the other hand, $\hat{v}=\sigma_1\sigma_2\sigma_3\sigma_4$ is clearly conjugate to $(12)(34)(56)(78)$ in H_1, which is exactly $w_{f_1}\cdot w_{f_3}\cdot w_{f_5}\cdot w_{f_7'}$. This proves part (1) of the lemma.

(2) Let $v=(\epsilon_i)\hat{v}$ be of order 4 such that v^2 is conjugate to $w_{f_1}\cdot w_{f_3}\cdot w_{f_5}\cdot w_{f_7'}$. We have $v^2=(\epsilon_i)(\epsilon_{\hat{v}^{-1}(i)})\hat{v}^2$.

Case (i) where $\hat{v}^2=1$. Then by part (1) and up to conjugacy by an element of H_1,

$$v^2=(\epsilon_i)(\epsilon_{\hat{v}^{-1}(i)})\hat{v}^2=(-1,-1,-1,-1,1,1,1,1).$$

This implies that for $i=1,2,3,4$, $\epsilon_i\epsilon_{\hat{v}(i)}=-1$, and in particular, $\hat{v}(i)\neq i$. Up to further conjugation by an element of H_1, $\hat{v}$ has the following three possibilities:

$$(12)(34),(12)(34)(56),(12)(34)(56)(78).$$

The corresponding conditions that (ϵ_i) must satisfy follow directly from

$$v^2=(\epsilon_i)(\epsilon_{\hat{v}^{-1}(i)})=(-1,-1,-1,-1,1,1,1,1).$$

Case (ii) where $\hat{v}^2\neq 1$. Then $v^2=(\epsilon_i)(\epsilon_{\hat{v}^{-1}(i)})\hat{v}^2$, which, as we have seen in part (1), is conjugate to a product of 4 disjoint transpositions. This implies that $\hat{v}$ is a product of 2 disjoint 4-cycles and $(\epsilon_i)(\epsilon_{\hat{v}^{-1}(i)})=(1)$ implies that $\epsilon_i=\epsilon_{\hat{v}(i)}$ for any i.

□

We remark that with Lemma 5.2 one can easily show that any 2-group of order ≤ 8 (including Q_8 in particular) as well as some other groups of small order (e.g. S_3, A_4, or even S_4) can be realized as a subgroup of H whose order 2 elements are all conjugate to $w_{f_1}\cdot w_{f_3}\cdot w_{f_5}\cdot w_{f_7'}$. (One may even attempt to classify these subgroups of H.) With this understood, the following lemma provides the additional constraints needed for the case of Q_8 in Theorem 1.7.

Lemma 5.3. *Suppose Q_8 acts on X_α smoothly, such that (1) the classes $[T_1]$, $[T_2]$ and $[T_3]$ are fixed under the action, (2) the actions by the elements of order 4 of Q_8 are mutually conjugate. Then each element of order 4 of Q_8 has exactly 4 isolated fixed points in X_α.*

Proof. Let $g \in Q_8$ be an order 4 element. Since $[T_1]$, $[T_2]$ and $[T_3]$ are fixed under the action, $b_2^+(X_\alpha/\langle g\rangle) = b_2^+(X_\alpha/\langle g^2\rangle) = 3$, and in particular, g^2 is an even type involution with 8 isolated fixed points. Since $\mathrm{Fix}(g) \subset \mathrm{Fix}(g^2)$, we see that g has at most 8 isolated fixed points.

We shall prove next that the number of fixed points of g is either 4, 6 or 8. To see this, note that the fixed points of g fall into two different classes according to their local representations. Denote by s_+ the number of fixed points where the weights of the local representation are $(1,3)$ and denote by s_- the number of fixed points where the weights are $(1,1)$ or $(3,3)$. Note that if t is the dimension of the 1-eigenspace of g in $H^2(X_\alpha;\mathbb{R})$, the dimension of the (-1)-eigenspace must be $14-t$, because g^2 has 8 isolated fixed points so that the dimension of the 1-eigenspace of g^2 is 14. Now by the Lefschetz fixed point theorem (cf. Theorem 3.4) and the G-signature theorem (cf. [29]), we have

$$\begin{cases} 2+t-(14-t) & = & s_+ + s_- \\ 4(6-t) & = & -16+2s_+ + (-2)s_-. \end{cases}$$

Here we use the fact that $b_2^+(X_\alpha/g) = 3$, and the fact that the signature defect at a fixed point of g of type $(1,3)$ and type $(1,1)$ or $(3,3)$ is $2,-2$ respectively, and the signature defect at a fixed point of g^2 is 0. The solutions for s_+, s_- (note that $s_+ + s_- \leq 8$) are $s_+ = 4$ and $s_- = 0, 2$ or 4. Our claim about the number of fixed points of g follows immediately.

Now $Q_8 = \{i,j,k | i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j\}$ where by the assumption the actions by i,j,k are all conjugate. In particular, they have the same number of fixed points, which is either 4, 6 or 8. Suppose i,j,k all have 8 fixed points. Then each of the 8 fixed points of -1 is also fixed by the entire group Q_8. But one of them must be a fixed point of i of type $(1,1)$ or $(3,3)$, which, however, contradicts the relation $j^{-1}ij = i^{-1}$. Hence i,j,k can not have 8 fixed points each. Suppose i,j,k each has 6 fixed points. Then i,j each fixes 6 of the 8 fixed points of -1, so that they must have 4 common fixed points, which should also be fixed by $k = ij$. Let $x_1,\cdots,x_4$ denote these 4 common fixed points, and let x_5,x_6 denote the other 2 fixed points of i which is not fixed by j, and let x_7, x_8 be the remaining 2 points which are not fixed by i. Since $j^{-1}ij = i^{-1}$, j has to switch x_5, x_6, so that j must fix both x_7, x_8 because j has 6 fixed points. It follows easily that $k = ij$ does not fix any of the points x_5, x_6, x_7, x_8, which contradicts the assumption that k also has 6 fixed points. Hence i,j,k each has 4 fixed points, and the lemma follows.

□

Proof of Theorem 1.7

Let G be a finite group acting smoothly and effectively on X_α. Then the classes $[T_1]$, $[T_2]$, $[T_3]$ are fixed by the commutator subgroup $[G,G]$. Moreover, under the homomorphism $\Theta: G \to \text{Aut}(E_8 \oplus E_8)$, $[G,G]$ is mapped into the subgroup $\text{Aut}(E_8)\times \text{Aut}(E_8)$ of index 2. We denote by Θ_i, $i=1,2$, the homomorphisms into $\text{Aut}(E_8)$ such that $\Theta=(\Theta_1,\Theta_2)$ on $[G,G]$.

Now suppose K is a subgroup of $[G,G]$ which is isomorphic to $(\mathbb{Z}_2)^4$. Then by Lemma 5.1, for every element $g\in K$ such that $g\neq 1$, $\Theta_1(g)\in \text{Aut}(E_8)$ is conjugate to $w_{f_1}\cdot w_{f_3}\cdot w_{f_5}\cdot w_{f_7'}$. In particular, the trace of $\Theta_1(g)$ equals 0 (cf. Lemma 5.2 (1)). Now consider the 8-dimensional representation V of K induced by Θ_1. We have

$$\dim V^K = \frac{1}{|K|}\sum_{g\in K} tr(g) = \frac{1}{|K|}(8+\sum_{1\neq g\in K} tr(g)) = \frac{8}{16},$$

which is a contradiction. This proves that if $[G,G]$ contains $(\mathbb{Z}_2)^4$ as a subgroup, then G can not act smoothly and effectively on X_α.

Next we assume $K\subset [G,G]$ is a subgroup isomorphic to Q_8, where

$$Q_8=\{i,j,k|i^2=j^2=k^2=-1, ij=k,\ jk=i,\ ki=j\}.$$

By the assumption in Theorem 1.7, the actions of order 4 elements are all conjugate, so that by Lemma 5.3, each of i,j,k has 4 isolated fixed points. It then follows easily from the Lefschetz fixed point theorem (cf. Theorem 3.4) that

$$tr(\Theta_1(g))+tr(\Theta_2(g))=-4, \quad \text{where } g=i,j,k\in Q_8.$$

On the other hand, each of $\Theta_1(g)$, $\Theta_2(g)$, $g=i,j,k$, is an order 4 element of $\text{Aut}(E_8)$ whose square is conjugate to $w_{f_1}\cdot w_{f_3}\cdot w_{f_5}\cdot w_{f_7'}$. From the description in Lemma 5.2 (2), the trace of each of $\Theta_1(g),\Theta_2(g)$, $g=i,j,k$, is even and is bounded between -4 and 4. It follows that for $l=1,2$, $tr(\Theta_l(g))=0,-2,-4$ where $g=i,j,k\in Q_8$.

There are two possibilities which we will discuss separately. First, suppose for g equaling one of $i,j,k\in Q_8$, $tr(\Theta_1(g))=0$ or -4. Note that correspondingly $tr(\Theta_2(g))=-4$ or 0. So without loss of generality we may assume that $tr(\Theta_1(i))=-4$ (i.e. $g=i$). Then from the description in Lemma 5.2 (2), we must have (up to conjugacy) $\Theta_1(i)=(\epsilon_i)\hat{v}$ with $\hat{v}=(12)(34)$, $\epsilon_1\epsilon_2=\epsilon_3\epsilon_4=-1$, $\epsilon_l=-1$ for $l=5,6,7,8$. Note also in this case we have $\Theta_1(-1)=(-1,-1,-1,-1,1,1,1,1)$.

We discuss the possibilities for $tr(\Theta_1(j))$. Suppose $tr(\Theta_1(j))=-4$. Then $\Theta_1(j)=(\epsilon_i)\hat{v}$ where $\hat{v}$ is a product of 2 disjoint transpositions with each of $5,6,7,8$ being fixed, and where $\epsilon_l=-1$ for $l=5,6,7,8$. It follows easily that $\Theta_1(k)=(\epsilon_i)\hat{v}$ where $\hat{v}$ fixes $5,6,7,8$ and $\epsilon_l=1$ for $l=5,6,7,8$. But this implies that $tr(\Theta_1(k))=4$ which is a contradiction. Suppose $tr(\Theta_1(j))=-2$. Then up to conjugacy without effecting $\Theta_1(i)$, $\Theta_1(j)=(\epsilon_i)\hat{v}$ where $\hat{v}=\sigma_1\sigma_2(56)(7)(8)$ and $\epsilon_7=\epsilon_8=-1$. It follows that $\Theta_1(k)=(\epsilon_i)\hat{v}$ with $\hat{v}=\sigma_1'\sigma_2'(56)(7)(8)$ but $\epsilon_7=\epsilon_8=1$. Consequently $tr(\Theta_1(k))=2$ which is also a contradiction. Finally, suppose $tr(\Theta_1(j))=0$. Then either $\Theta_1(j)=(\epsilon_i)\hat{v}$ with $\hat{v}$ a product of 4 disjoint transpositions or with $\hat{v}$ fixing each of $5,6,7,8$ and exactly two of $\epsilon_5,\epsilon_6,\epsilon_7,\epsilon_8$ equal to -1. In any event, it follows that $tr(\Theta_1(k))=0$ also. Now $tr(\Theta_1(j))=tr(\Theta_1(k))=0$ implies that $tr(\Theta_2(j))=tr(\Theta_2(k))=-4$, which is a case that has been already shown impossible. Hence we have eliminated the first possibility that for g equaling one of $i,j,k\in Q_8$, $tr(\Theta_1(g))=0$ or -4.

Next we consider the second possibility that for any $g = i, j, k \in Q_8$, $tr(\Theta_1(g)) = -2$. Then from the description in Lemma 5.2 (2), we see that each $\Theta_1(g) = (\epsilon_i)\hat{v}$, where $\hat{v}$ is a product of 3 disjoint transpositions. This turns out to be impossible, because if we assume without loss of generality that $\Theta_1(i) = (\epsilon_i)\hat{v}$ where $\hat{v} = (12)(34)(56)$ (and $\Theta_1(-1) = (-1, -1, -1, -1, 1, 1, 1, 1))$, then $\Theta_1(j) = (\epsilon_i)\hat{v}$ with $\hat{v} = \sigma_1\sigma_2(56)(7)(8)$ or $\hat{v} = \sigma_1\sigma_2(5)(6)(78)$, which implies that $\Theta_1(k) = (\epsilon_i)\hat{v}$ where $\hat{v} = \sigma_1'\sigma_2'(5)(6)(7)(8)$ or $\hat{v} = \sigma_1'\sigma_2'(56)(78)$ respectively. In any event, it contradicts the assumption that $tr(\Theta_1(k)) = -2$, and hence our claim. This proves that if $[G, G]$ contains Q_8 as a subgroup, then G can not act smoothly and effectively on X_α

The proof of Theorem 1.7 is thus completed.

□

Corollary 5.4. *The following maximal symplectic $K3$ groups*

$$G = M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, T_{48}$$

can not act smoothly and effectively on X_α.

Proof. The structures of these groups and their commutator subgroups are listed below (cf. [42, 51]):

- $G = M_{20} = 2^4 A_5$, $[G, G] = G = 2^4 A_5$.
- $G = F_{384} = 4^2 S_4$, $[G, G] = 4^2 A_4$.
- $G = A_{4,4} = 2^4 A_{3,3}$, $[G, G] = A_4^2$.
- $G = T_{192} = (Q_8 * Q_8) \times_\phi S_3$, $[G, G] = (Q_8 * Q_8) \times_\phi \mathbb{Z}_3$.
- $G = H_{192} = 2^4 D_{12}$, $[G, G] = 2^4 \mathbb{Z}_3$.
- $G = T_{48} = Q_8 \times_\phi S_3$, $[G, G] = T_{24} = Q_8 \times_\phi \mathbb{Z}_3$.

The corollary is evident for all cases except for the case where $G = F_{384}$. We shall prove that in this case $[G, G] = 4^2 A_4$ contains a subgroup isomorphic to $(\mathbb{Z}_2)^4$. To this end, we recall the structure of F_{384} (cf. [42], pages 190-191). $F_{384} = 4^2 S_4$ is a semi-direct product of $(\mathbb{Z}_4)^2$ by S_4, where $(\mathbb{Z}_4)^2 = \{(a, b, c, d) | a+b+c+d = 0\} \subset (\mathbb{Z}_4)^4$ modulo the diagonal subgroup. The action of S_4 is given by permutations of the 4 coordinates. One can check directly that the $(\mathbb{Z}_2)^2$-subgroup of $(\mathbb{Z}_4)^2$ generated by $(2, 2, 0, 0)$ and $(2, 0, 2, 0)$ are fixed under the action of $(12)(34), (13)(24) \in A_4 \subset S_4$, hence the commutator $[G, G] = 4^2 A_4$ contains a subgroup isomorphic to $(\mathbb{Z}_2)^4$.

□

6. Symplectic cyclic actions

In this section we prove Theorem 1.8. The proof draws heavily on our previous work [10] concerning the fixed-point set structure of a symplectic $\mathbb{Z}_p$-action on a minimal symplectic 4-manifold with $c_1^2 = 0$, which we shall recall first.

Let ω be an orientation compatible symplectic structure on X_α, and let G be a finite group acting on X_α which preserves ω. Then by Lemma 2.3 (2), G fixes the classes $[T_1], [T_2], [T_3]$, and therefore by Lemma 4.1, G acts trivially on a 3-dimensional subspace of $H^2(X_\alpha; \mathbb{R})$ which consists of elements of positive square. As we argued in [10], a G-equivariant version of Taubes' work in [49, 50] applies here, so that for any G-equivariant ω-compatible almost complex structure J, the canonical class $c_1(K)$ is represented by a finite set of J-holomorphic curves $\{C_i\}$ with positive weights $\{n_i\}$, i.e., $c_1(K) = \sum_i n_i C_i$, which has the following properties:

- The set $\cup_i C_i$ is G-invariant.
- Any fixed point of G in the complement of $\cup_i C_i$ is isolated with local representation contained in $SL_2(\mathbb{C})$.

One may further analyze the rest of the fixed points through the induced action in a neighborhood of $\cup_i C_i$. To this end, it is useful to take note that the connected components of $\cup_i C_i$ may be divided into the following three types:

(A) A single J-holomorphic curve of self-intersection 0 which is either an embedded torus, or a cusp sphere, or a nodal sphere.
(B) A union of two embedded (-2)-spheres intersecting at a single point with tangency of order 2.
(C) A union of embedded (-2)-spheres intersecting transversely.

A type (C) component may be conveniently represented by one of the graphs of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ listed in Figure 2, where a vertex in a graph represents a (-2)-sphere and an edge connecting two vertices represents a transverse, positive intersection point of the two (-2)-spheres represented by the vertices.

With the preceding understood, the following lemma is specially tailored for the present situation in order to control the number of type (B) or type (C) components.

Lemma 6.1. *Let J be any ω-compatible almost complex structure on X_α, and let $c_1(K) = \sum n_i C_i$ where $\{C_i\}$ is a finite set of J-holomorphic curves and $n_i \geq 1$. Then each C_i lies in the orthogonal complement of Span $([T_1], [T_2], [T_3])$.*

Proof. The key point here is that some multiple of each $[T_j]$ can be represented by J-holomorphic curves. The details of the proof go as follows.

First of all, by Lemma 2.2, we may assume without loss of generality that $c_1(K) = 2(d_1[T_1] + d_2[T_2] + d_3[T_3])$. Because the classes $-2(d_2[T_2] + d_3[T_3])$, $-2(d_1[T_1] + d_3[T_3])$ and $-2(d_1[T_1] + d_2[T_2])$ are Seiberg-Witten basic classes, by the main theorem of Taubes in [50], for a generic ω-compatible almost complex structure J', each $d_j[T_j]$ for $j = 1, 2, 3$ is Poincaré dual to $\sum_{k=1}^{N_j} m_{j,k}\Gamma_{j,k}$, where $m_{j,k} \geq 1$ are integers and $\Gamma_{j,k}$ are (connected) embedded J'-holomorphic curves which are disjoint for each fixed j. Moreover, since X_α is minimal and J' is generic, all $\Gamma_{j,k}$ have nonzero genus. We further notice that for each fixed j the numbers N_j and $m_{j,k}$ and the genus of each $\Gamma_{j,k}$ are bounded by a constant independent of the almost complex structure J'. We take a sequense of generic J' converging in C^∞ to the given J, and by passing to a subsequence we may assume that N_j, $m_{j,k}$ and the genus of $\Gamma_{j,k}$ are independent of J' throughout.

By Gromov compactness theorem (cf. e.g. [40]), each $\Gamma_{j,k}$ converges to a limit $\sum_{l=1}^{M_{j,k}} n_{j,k,l} C_{j,k,l}$ where each $C_{j,k,l}$ is a (nonconstant) J-holomorphic curve, $n_{j,k,l} \geq 1$ and $\cup_{l=1}^{M_{j,k}} C_{j,k,l}$ is connected. Note that

$$c_1(K) = 2(d_1[T_1] + d_2[T_2] + d_3[T_3]) = 2\sum_{j=1}^{3}\sum_{k=1}^{N_j}\sum_{l=1}^{M_{j,k}} m_{j,k} n_{j,k,l} C_{j,k,l}.$$

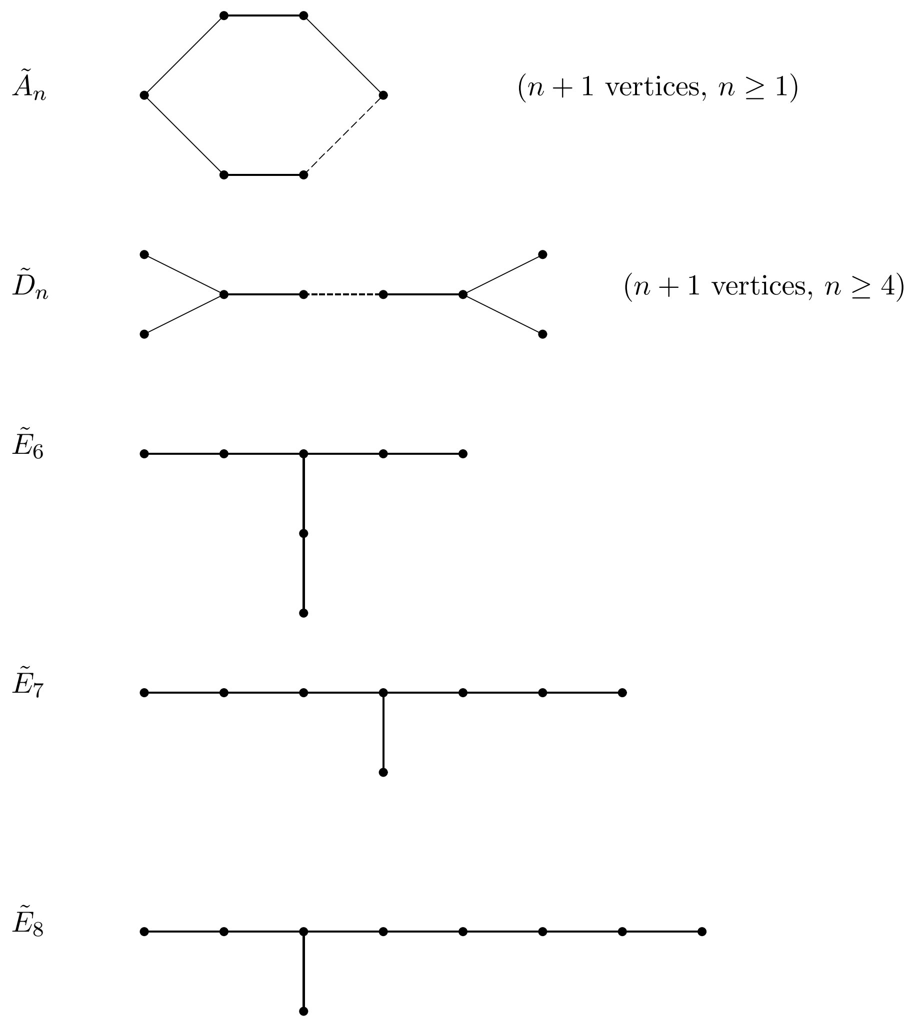

FIGURE 2.

Furthermore, the fact that $c_1(K)^2 = 0$ and X_α is minimal allows us to analyze the structure of $\cup_{j=1}^3 \cup_{k=1}^{N_j} \cup_{l=1}^{M_{j,k}} C_{j,k,l}$, as shown in [10]. In particular, the connected components of the union $\cup_{j=1}^3 \cup_{k=1}^{N_j} \cup_{l=1}^{M_{j,k}} C_{j,k,l}$ may be divided into the following three types (the classification differs slightly from the one we mentioned earlier):

(a) A single J-holomorphic curve of self-intersection 0.
(b) A union of two embedded (-2)-spheres.
(c) A union of at least three embedded (-2)-spheres intersecting transversely.

With the above preparation, we shall prove next that each C_i lies in the orthogonal complement of Span $([T_1],[T_2],[T_3])$. It suffices to show that for each (j,k), $\Gamma_{j,k}\cdot C_i=0$, or equivalently $(\sum_{l=1}^{M_{j,k}} n_{j,k,l}C_{j,k,l})\cdot C_i=0$. We begin by recalling that $c_1(K)\cdot C_i=0$ (cf. [10], Lemma 3.3), so that C_i is either disjoint from $\cup_{j=1}^3\cup_{k=1}^{N_j}\cup_{l=1}^{M_{j,k}}C_{j,k,l}$, in which case $(\sum_{l=1}^{M_{j,k}} n_{j,k,l}C_{j,k,l})\cdot C_i=0$ holds true automatically, or C_i is contained as one of the J-holomorphic curves $C_{j,k,l}$.

At this point, we need to make use of the fact that $\Gamma_{j,k}^2=0$, whose proof is postponed to the end of the proof of this lemma. Accepting it momentarily, we shall continue with the proof of the lemma. It is clear that we only need to verify the case where $\cup_{l=1}^{M_{j,k}}C_{j,k,l}$ and C_i lie in the same component of $\cup_{j=1}^3\cup_{k=1}^{N_j}\cup_{l=1}^{M_{j,k}}C_{j,k,l}$. Since each $\cup_{l=1}^{M_{j,k}}C_{j,k,l}$ is connected and $(\sum_{l=1}^{M_{j,k}} n_{j,k,l}C_{j,k,l})^2=\Gamma_{j,k}^2=0$, it follows easily that if C_i lies in a type (a) or (b) component of $\cup_{j=1}^3\cup_{k=1}^{N_j}\cup_{l=1}^{M_{j,k}}C_{j,k,l}$, then $(\sum_{l=1}^{M_{j,k}} n_{j,k,l}C_{j,k,l})\cdot C_i=0$ holds true. It remains to check the case where C_i lies in a type (c) component. To this end we recall that a type (c) component corresponds to a graph of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ as discussed in [3], Lemma 2.12 (ii). Each graph defines a positive semi-definite quadratic form which is canonically associated with the intersection form of the J-holomorphic curves $C_{j,k,l}$ in the type (c) component. The key property we will use here is that the positive semi-definite quadratic form has a 1-dimensional annihilator. Now it is clear that $(\sum_{l=1}^{M_{j,k}} n_{j,k,l}C_{j,k,l})^2=0$ implies that $\sum_{l=1}^{M_{j,k}} n_{j,k,l}C_{j,k,l}$ must be an annihilator for the positive semi-definite quadratic form, which implies that $(\sum_{l=1}^{M_{j,k}} n_{j,k,l}C_{j,k,l})\cdot C_i=0$.

We end the proof by showing that $\Gamma_{j,k}^2=0$. This follows from the fact that $[T_j]^2=0$ by a standard argument involving the Sard-Smale theorem and the adjunction formula for pseudoholomorphic curves (cf. [40]). The details are sketched below. The dimension of the moduli space of J'-holomorphic curves which contains $\Gamma_{j,k}$ equals $d=2(-c_1(K)\cdot\Gamma_{j,k}+\text{genus}(\Gamma_{j,k})-1)$. (Here we use the fact that $\Gamma_{j,k}$ has nonzero genus.) Since $\Gamma_{j,k}$ is embedded, the adjunction formula

$$2\cdot\text{genus}(\Gamma_{j,k})-2=\Gamma_{j,k}^2+c_1(K)\cdot\Gamma_{j,k}$$

gives rise to $d=\Gamma_{j,k}^2-c_1(K)\cdot\Gamma_{j,k}$. Now J' is chosen generic so that $d\geq 0$ must hold, which implies that $\Gamma_{j,k}^2\geq c_1(K)\cdot\Gamma_{j,k}$. Again by the adjunction formula, we have

$$\Gamma_{j,k}^2\geq\frac{1}{2}(\Gamma_{j,k}^2+c_1(K)\cdot\Gamma_{j,k})=\text{genus}(\Gamma_{j,k})-1\geq 0.$$

With this, $\Gamma_{j,k}^2=0$ follows easily from $(\sum_{k=1}^{N_j} m_{j,k}\Gamma_{j,k})^2=(d_j[T_j])^2=0$.

□

The preceding lemma has the following useful corollary. Let Λ be a component of $\cup_i C_i$ of either type (B) or type (C), and let C be a (-2)-sphere in Λ. Recall that the orthogonal complement of Span $([T_1],[T_2],[T_3])$ is

$$\text{Span }([T_j],\xi_k,\eta_k|j=1,2,3,1\leq k\leq 8)$$

where ξ_k, η_k are the classes in $H_2(X_\alpha;\mathbb{Z})$ which correspond to the two standard bases of the $-E_8$ form defined in Lemma 4.2. We denote by $\underline{C}$ the projection into

$$\text{Span}\ ([T_j], \xi_k, \eta_k | j = 1,2,3, 1 \le k \le 8)/\text{Span}\ ([T_1],[T_2],[T_3]).$$

Since C has nontrivial self-intersection, its projection $\underline{C}$ must be nonzero. We denote by L_Λ the sublattice spanned by the projections of (-2)-spheres in Λ.

Lemma 6.2. *For any component Λ, L_Λ is contained in either*

$$\textit{Span}\ ([T_j], \xi_k | j = 1,2,3, 1 \le k \le 8)/\textit{Span}\ ([T_1],[T_2],[T_3])$$

or

$$\textit{Span}\ ([T_j], \eta_k | j = 1,2,3, 1 \le k \le 8)/\textit{Span}\ ([T_1],[T_2],[T_3]),$$

and for any two distinct components Λ, Λ', the corresponding sublattices L_Λ, $L_{\Lambda'}$ are orthogonal to each other. Moreover, if Λ is of type (B), then $L_\Lambda = \langle -2 \rangle$ (i.e. L_Λ is a A_1-root lattice), and if Λ is of type (C), represented by a graph of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ listed in Figure 2, then L_Λ is a root lattice of the corresponding type (i.e. of type A_n, D_n, E_6, E_7 or E_8).

Proof. Let C be a (-2)-sphere in Λ. Write $\underline{C} = \xi + \eta$, where

$$\xi \in \text{Span}\ ([T_j], \xi_k | j = 1,2,3, 1 \le k \le 8)/\text{Span}\ ([T_1],[T_2],[T_3])$$

and

$$\eta \in \text{Span}\ ([T_j], \eta_k | j = 1,2,3, 1 \le k \le 8)/\text{Span}\ ([T_1],[T_2],[T_3]).$$

We claim that either ξ or η is zero. Suppose to the contrary that neither of them is zero. Then since $-E_8$ is negative definite and even, $\xi^2, \eta^2 \le -2$, which implies that $\underline{C}^2 = \xi^2 + \eta^2 \le -4$. But this contradicts $\underline{C}^2 = C^2 = -2$, and the claim follows.

Now for each (-2)-sphere C in Λ, its projection $\underline{C}$ lies in either

$$\text{Span}\ ([T_j], \xi_k | j = 1,2,3, 1 \le k \le 8)/\text{Span}\ ([T_1],[T_2],[T_3])$$

or

$$\text{Span}\ ([T_j], \eta_k | j = 1,2,3, 1 \le k \le 8)/\text{Span}\ ([T_1],[T_2],[T_3]).$$

Since Λ is connected, the projections of its (-2)-spheres must lie in the same lattice. This proves that for any component Λ, L_Λ is contained in either

$$\text{Span}\ ([T_j], \xi_k | j = 1,2,3, 1 \le k \le 8)/\text{Span}\ ([T_1],[T_2],[T_3])$$

or

$$\text{Span}\ ([T_j], \eta_k | j = 1,2,3, 1 \le k \le 8)/\text{Span}\ ([T_1],[T_2],[T_3]).$$

Similarly, one can show that for any two distinct components Λ, Λ', the corresponding sublattices L_Λ, $L_{\Lambda'}$ are orthogonal to each other.

Now let Λ be a type (B) component, which consists of two (-2)-spheres C_1, C_2 intersecting at a single point with tangency of order 2. Because

$$(C_1 + C_2)^2 = (-2) + 2 \cdot 2 + (-2) = 0,$$

$\underline{C}_1 + \underline{C}_2$ must be 0 and hence $L_\Lambda = \langle -2 \rangle$ in this case. Suppose Λ is a type (C) component represented by a graph Γ of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ listed in Figure 2, and let $\{C_i\}$ be the (-2)-spheres corresponding to the vertices in Γ. Then there are weights $\{m_i\}$, $m_i > 0$, such that (1) $(\sum_i m_i C_i)^2 = 0$, (2) there exists a weight

$1 = m_0 \in \{m_i\}$ which has the property that if the corresponding vertex (and the edge connecting to it) in Γ is removed, the resulting graph is the Dynkin diagram for the root lattice corresponding to Γ (cf. [3], Lemma 2.12 (ii)). It follows easily that L_Λ is isomorphic to the corresponding root lattice.

□

With the preceding preparation, we give a proof of Theorem 1.8 next.

We assume $G \equiv \mathbb{Z}_p$ where $p = 5$ or 7. First, we observe that the main results in [10] (Theorem B, Theorem 3.1 and Prop. 3.7) concerning the fixed-point set structure of a symplectic $\mathbb{Z}_p$-action apply to the current situation, even though there was an additional assumption in [10] that the symplectic $\mathbb{Z}_p$-action acts trivially on the second homology. This is because the said additional assumption was mainly used to ensure that the induced action of G on each component of the union of J-holomorphic curves $\cup_i C_i$ leaves each (-2)-sphere in the component invariant if the component contains at least one fixed point, which is automatically true in the current case. (Note that for $G \equiv \mathbb{Z}_5$ or $\mathbb{Z}_7$, the graphs of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ listed in Figure 2 do not have any nontrivial G-symmetries except for the case of $\tilde{A}_n$, in which G acts freely so that the corresponding component of $\cup_i C_i$ can not contain any fixed points of the G-action on X_α.) With this understood, according to [10] the fixed points of G can be divided into groups of the following types:

(1) One fixed point with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some $k \neq 0 \bmod p$, i.e., with local representation contained in $SL_2(\mathbb{C})$.

(2) Two fixed points with local representation $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2)$, $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{6k} z_2)$ for some $k \neq 0 \bmod p$ respectively. (This type of fixed points occurs only when $p > 5$.)

(3) Three fixed points, one with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2)$ and the other two with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$ for some $k \neq 0 \bmod p$.

(4) Four fixed points, one with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ and the other three with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2)$ for some $k \neq 0 \bmod p$.

(Γ) The subset of fixed points which are contained in a component Λ of $\cup_i C_i$, where Λ is of type (C) and is represented by graph Γ of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, such that at least one of the (-2)-spheres in Λ is fixed under the action. Note that according to [10], $n = -1 \pmod p$ if Γ is of type $\tilde{A}_n$, and $n = 4 \pmod p$ if Γ is of type $\tilde{D}_n$.

(T^2) An embedded torus of self-intersection 0.

We shall consider the cases $p = 5$ and $p = 7$ separately.

Case(a) $p = 5$. In this case, there are no type (2) fixed points. Moreover, by the assumption that both Θ_1, Θ_2 are nontrivial, and by Lemma 4.5 and Lemma 6.2, there are no type (Γ) fixed points unless Γ is of type $\tilde{A}_4$ or $\tilde{D}_4$. The next lemma further eliminates type (Γ) fixed points where Γ is of type $\tilde{D}_4$.

Lemma 6.3. *Let $G \subset Aut(E_8)$ be a subgroup of order 5. There are no sublattices of E_8 fixed under G, which are isomorphic either to a D_4-root lattice or to a direct sum of two copies of a A_2-root lattice.*

Proof. By Lemma 4.5, there are two different integral G-representations associated to the subgroup $G \subset \mathrm{Aut}(E_8)$: $\mathbb{Z}[\mathbb{Z}_5] \oplus \mathbb{Z}^3$ and $\mathbb{Z}[\mu_5]^2$. The latter does not fix any vector in the lattice, so we only need to consider the case of $\mathbb{Z}[\mathbb{Z}_5] \oplus \mathbb{Z}^3$.

By Carter [7] (Table 3, page 23), an element of order 5 in $\mathrm{Aut}(E_8)$ is uniquely determined up to conjugacy by the characteristic polynomial. Hence a subgroup G of order 5 whose corresponding integral G-representation is $\mathbb{Z}[\mathbb{Z}_5] \oplus \mathbb{Z}^3$ must be conjugate to the subgroup generated by the permutation

$$e_1 \mapsto e_2, e_2 \mapsto e_3, e_3 \mapsto e_4, e_4 \mapsto e_5, e_5 \mapsto e_1, e_l \mapsto e_l, l = 6, 7, 8,$$

where $\{e_1, \cdots, e_8\}$ is a standard basis of $\mathbb{R}^8$. Clearly, the roots of E_8 which are fixed under the permutation can be put in two groups

$$\Omega_1 = \{\pm e_i \pm e_j | i \neq j, i, j = 6, 7, 8\}$$

and

$$\Omega_2 = \{\frac{1}{2}\sum_{i=1}^{8} \epsilon_i e_i | \epsilon_1 = \cdots = \epsilon_5, \prod_{i=1}^{8} \epsilon_i = 1\}.$$

Note that for any roots $r_1, r_2 \in \Omega_1$, $(r_1, r_2) = 0$ if and only if $r_1 = \pm(e_i + e_j)$ and $r_2 = \pm(e_i - e_j)$ (or vice versa), and $(r_1, r_2) \neq 0$ for any $r_1, r_2 \in \Omega_2$.

With these preparations, we shall prove next that there are no sublattices of E_8 isomorphic to a D_4-root lattice that are fixed under G. To see this, note that amongst the three roots represented by the vertices other than the central one in a D_4-Dynkin diagram, exactly two of them must belong to Ω_1, which are of the form $\pm(e_i + e_j)$, $\pm(e_i - e_j)$ for some $i \neq j$, $i, j = 6, 7, 8$. On the other hand, a root $r = \frac{1}{2}\sum_{i=1}^{8} \epsilon_i e_i \in \Omega_2$ is orthogonal to $\pm(e_i + e_j)$ if and only if $\epsilon_i = -\epsilon_j$. But such a root certainly is not orthogonal to $\pm(e_i - e_j)$. Our claim follows easily.

It remains to show that G can not fix a direct sum of two copies of a A_2-root lattice. To see this, let r_1, r_2 be the two roots generating the first copy, and let r_3, r_4 generate the second copy. Then note first that one of the r_i's must belong to Ω_2. Assume it is r_1 without loss of generality. Then r_3, r_4, both being orthogonal to r_1, must belong to Ω_1. Without loss of generality we may only consider the case $r_3 = e_6 - e_7$ and $r_4 = e_7 - e_8$. The root r_1, being orthogonal to both r_3, r_4, must be $\pm\frac{1}{2}(e_1 + \cdots + e_8)$. But then the root r_2, which has the property that $(r_1, r_2) = -1$, can not be possibly orthogonal to both r_3, r_4. A contradiction. The other cases are analogous, and this finishes the proof of the lemma.

□

We remark that there are sublattices of E_8 isomorphic to a A_4-root lattice which are fixed under G. For example, the following 4 roots

$$e_6 - e_7, -\frac{1}{2}(e_1 + \cdots + e_5 + e_6 - e_7 - e_8), \frac{1}{2}(e_1 + \cdots + e_5 - e_6 - e_7 + e_8), e_6 + e_7$$

generate a A_4-root lattice which is fixed under G.

With the preceding understood, let u, v, w and A be the number of groups of fixed points of G of type (1), (3), (4) and $(\tilde{A}_4)$ respectively. We will next determine the possibilities of u, v, w and A using the Lefschetz fixed point theorem and the G-signature theorem (i.e. Theorem 3.6). Note that since a fixed torus of self-intersection 0 makes no contribution in the calculation with the Lefschetz fixed point theorem and the G-signature theorem, we will ignore it in the consideration. The number of such components in the fixed-point set will be determined later by the number of cyclotomic summands in the integral representation on the middle homology.

To this end, recall that the total signature defect of a group of fixed points of type (1), (3), (4) is 4, -8, and -4 respectively (cf. Lemma 3.8 of [10]). For the total signature defect of a group of fixed points of type $(\tilde{A}_4)$, we note that such a component Λ of $\cup_i C_i$ contains exactly 1 fixed (-2)-sphere plus 3 isolated fixed points of local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$ for some $k \neq 0 \pmod p$ (where $p = 5$), with $q = 1, 2, 3$ respectively (cf. [10]). The signature defect for each of the isolated fixed points is correspondingly given by

$$I_{p,q} = \sum_{k=1}^{p-1} \frac{(1+\mu_p^k)(1+\mu_p^{kq})}{(1-\mu_p^k)(1-\mu_p^{kq})}.$$

(Observe the relation $I_{p,q} = -I_{p,-q}$.) It follows easily that $I_{5,1} = -4$ and $I_{5,2} = I_{5,3} = 0$ (cf. [10], Appendix). Hence the total signature defect of a group of fixed points of type $(\tilde{A}_4)$ is

$$-4 + 0 + 0 + \frac{5^2-1}{3} \cdot (-2) = -20.$$

For $i = 1, 2$, let $\mathbb{Z}[\mathbb{Z}_5]^{r_i} \oplus \mathbb{Z}^{t_i} \oplus \mathbb{Z}[\mu_5]^{s_i}$ be the integral G-representation associated to $\Theta_i : G \to \mathrm{Aut}(E_8)$. Then by Lemma 4.5, there are the following three possibilities if we assume both Θ_1, Θ_2 are nontrivial: $(r_1, t_1, s_1) = (r_2, t_2, s_2) = (1, 3, 0)$, $(r_1, t_1, s_1) = (r_2, t_2, s_2) = (0, 0, 2)$ and $(r_1, t_1, s_1) = (1, 3, 0)$, $(r_2, t_2, s_2) = (0, 0, 2)$. The G-signature theorem as stated in Theorem 3.6 and the Lefschetz fixed point theorem (cf. Theorem 3.4) give rise to the following equations:

$$\begin{cases} 1 + 3\cdot 2 + t_1 - s_1 + t_2 - s_2 + 1 = & u + 3v + 4w + 5A \\ p\cdot(-r_1 - t_1 - r_2 - t_2) = & -16 + 4u - 8v - 4w - 20A \text{ (with } p = 5). \end{cases}$$

The solution to the above system of equations is

$$(u, v) = \begin{cases} (2 - w + A, 4 - w - 2A) & \text{if } (r_1, t_1, s_1) = (r_2, t_2, s_2) = (1, 3, 0) \\ (3 - w + A, 2 - w - 2A) & \text{if } (r_1, t_1, s_1) = (1, 3, 0), (r_2, t_2, s_2) = (0, 0, 2) \end{cases}$$

and $(u, v, w, A) = (4, 0, 0, 0)$ if $(r_1, t_1, s_1) = (r_2, t_2, s_2) = (0, 0, 2)$.

We shall further analyze the fixed-point set with help of the G-signature theorem as stated in Theorem 3.5 and with help of the G-index theorem for Dirac operators as stated in Lemma 3.8.

We consider first the cases where $A = 0$, i.e., there are no type $(\tilde{A}_4)$ fixed points. To apply the G-signature theorem in Theorem 3.5, we fix a $g \in G$ and recall, with $p = 5$ below, that each isolated fixed point m of g is associated with a pair of integers (a_m, b_m), where $0 < a_m, b_m < p$, such that the action of g on the tangent space at m is given by the complex linear transformation $(z_1, z_2) \mapsto (\mu_p^{a_m} z_1, \mu_p^{b_m} z_2)$, and moreover, the contribution to Sign (g, X_α) from m is given by

$$\delta_m = -\cot(\frac{a_m\pi}{p}) \cdot \cot(\frac{b_m\pi}{p}).$$

Now divide the fixed points of g into three groups I, II, III according to their local representations: group I consists of fixed points with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ for some $k \neq 0 \pmod p$, group II consists of fixed points with $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2)$ for some $k \neq 0 \pmod p$, and group III consists of fixed points with $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{4k} z_2)$ for some $k \neq 0 \pmod p$. Then one observes that δ_m has only two possible values for the fixed points in each of the groups I, II, III. For group I, the values are $-\cot^2(\frac{\pi}{5})$, $-\cot^2(\frac{2\pi}{5})$, for group II, the values are $-\cot(\frac{\pi}{5})\cot(\frac{2\pi}{5})$, $\cot(\frac{\pi}{5})\cot(\frac{2\pi}{5})$, and for group III, the values are $\cot^2(\frac{\pi}{5})$, $\cot^2(\frac{2\pi}{5})$. We let x_1, x_2, y_1, y_2 and z_1, z_2 be the number of fixed points at which δ_m takes these values respectively.

By the G-signature theorem as stated in Theorem 3.5, we have

$$\begin{aligned} s_1 + s_2 - t_1 - t_2 = \text{Sign } (g, X_\alpha) &= -x_1 \cot^2(\frac{\pi}{5}) - x_2 \cot^2(\frac{2\pi}{5}) \\ &\quad -y_1 \cot(\frac{\pi}{5})\cot(\frac{2\pi}{5}) + y_2 \cot(\frac{\pi}{5})\cot(\frac{2\pi}{5}) \\ &\quad +z_1 \cot^2(\frac{\pi}{5}) + z_2 \cot^2(\frac{2\pi}{5}). \end{aligned}$$

On the other hand, if we replace g by g^2, δ_m will correspondingly be switched between the two values it assumes, and consequently, we have

$$\begin{aligned} s_1 + s_2 - t_1 - t_2 = \text{Sign } (g^2, X_\alpha) &= -x_1 \cot^2(\frac{2\pi}{5}) - x_2 \cot^2(\frac{\pi}{5}) \\ &\quad y_1 \cot(\frac{\pi}{5})\cot(\frac{2\pi}{5}) - y_2 \cot(\frac{\pi}{5})\cot(\frac{2\pi}{5}) \\ &\quad +z_1 \cot^2(\frac{2\pi}{5}) + z_2 \cot^2(\frac{\pi}{5}). \end{aligned}$$

Combining these two equations, one obtains

$$\begin{aligned} [(z_1 - z_2) - (x_1 - x_2)] \cot^2(\frac{\pi}{5}) - 2(y_1 - y_2) \cot(\frac{\pi}{5})\cot(\frac{2\pi}{5}) \\ -[(z_1 - z_2) - (x_1 - x_2)] \cot^2(\frac{2\pi}{5}) = 0. \end{aligned}$$

Lemma 6.4. $\cot(\frac{\pi}{5})/\cot(\frac{2\pi}{5})$ *satisfies the algebraic equation* $t^2 - 4t - 1 = 0$*, which is irreducible over* $\mathbb{Q}$*.*

Proof. We start with the equation $1 + \mu_5 + \cdots + \mu_5^4 = 0$, from which one sees that $\cos(\frac{\pi}{5})$ satisfies $4t^2 - 2t - 1 = 0$, and hence $\cos(\frac{\pi}{5}) = (1 + \sqrt{5})/4$.

Now observe that

$$\frac{\cot(\frac{2\pi}{5})}{\cot(\frac{\pi}{5})} = 1 - \frac{1}{2\cos^2(\frac{\pi}{5})}.$$

Using the fact that $\cos(\frac{\pi}{5}) = (1+\sqrt{5})/4$, one can check that $\cot(\frac{\pi}{5})/\cot(\frac{2\pi}{5})$ is a solution of $t^2-4t-1=0$, which is clearly irreducible over $\mathbb{Q}$.

□

The preceding lemma implies the following relations

$$(z_1-z_2)-(x_1-x_2)=c,\ y_1-y_2=2c \text{ for some } c\in\mathbb{Z}.$$

Now observe that type (1) fixed points contribute exclusively to z_1 or z_2, and we have $u=z_1+z_2$, and on the other hand, a group of type (3) or type (4) fixed points contributes nontrivially to x_1 (resp. x_2) if and only if it contributes nontrivially to y_1 (resp. y_2), and we have

$$x_1=2v_1+w_1,\ \ x_2=2v_2+w_2,\ \ y_1=v_1+3w_1,\ \ y_2=v_2+3w_2$$

where $v=v_1+v_2$ and $w=w_1+w_2$. From these equations we obtain

$$2(z_1-z_2)=2(x_1-x_2)+(y_1-y_2)=5(v_1+w_1-v_2-w_2).$$

Since $|z_1-z_2|\leq z_1+z_2=u<5$ in all the cases, we must have

$$z_1-z_2=v_1+w_1-v_2-w_2=0.$$

In particular, note that $u=z_1+z_2=2z_1$ is an even number.

The solutions which satisfy the above constraints are given below (up to changing from g to g^2)

(a) $x_1=x_2=4, y_1=y_2=2, z_1=z_2=1$, and $(u,v,w)=(2,4,0)$,
(b) $x_1=x_2=3, y_1=y_2=4, z_1=z_2=0$, and $(u,v,w)=(0,2,2)$,
(c) $x_1=4$, $x_2=2$, $y_1=2$, $y_2=6$, $z_1=z_2=0$, and $(u,v,w)=(0,2,2)$,
(d) $x_1=2$, $x_2=1$, $y_1=1$, $y_2=3$, $z_1=z_2=1$, and $(u,v,w)=(2,1,1)$.

We next use Lemma 3.8 and Theorem 3.9 to rule out the cases (a), (b) where

$$x_1-x_2=y_1-y_2=z_1-z_2=0 \text{ and } x_1-z_1=3.$$

Observe that by the formula for the "Spin-number" in Lemma 3.8, the contribution to Spin (g,X_α) from a fixed point m is

$$\nu_m=-(-1)^{k(g,m)}\cdot\frac{1}{4}\cdot\csc(\frac{a_m\pi}{5})\csc(\frac{b_m\pi}{5}),$$

where $0<a_m,b_m<5$ and $k(g,m)\cdot 5=2r_m+a_m+b_m$ for some $0\leq r_m<5$. One can check that ν_m takes values $-\frac{1}{4}\csc^2(\frac{\pi}{5})$, $-\frac{1}{4}\csc^2(\frac{2\pi}{5})$ if m belongs to group I; for group II, the values of ν_m are $\frac{1}{4}\csc(\frac{\pi}{5})\csc(\frac{2\pi}{5})$, $-\frac{1}{4}\csc(\frac{\pi}{5})\csc(\frac{2\pi}{5})$, and for group III, the values are $\frac{1}{4}\csc^2(\frac{\pi}{5})$, $\frac{1}{4}\csc^2(\frac{2\pi}{5})$. The number of fixed points at which ν_m takes these values is x_1,x_2,y_1,y_2,z_1,z_2 respectively.

With the above understood, for the cases (a), (b) we obtain from Lemma 3.8

$$\begin{aligned}
\text{Spin } (g, X_\alpha) &= -\frac{x_1}{4}\csc^2(\frac{\pi}{5}) - \frac{x_2}{4}\csc^2(\frac{2\pi}{5}) \\
&\quad +\frac{y_1}{4}\csc(\frac{\pi}{5})\csc(\frac{2\pi}{5}) - \frac{y_2}{4}\csc(\frac{\pi}{5})\csc(\frac{2\pi}{5}) \\
&\quad +\frac{z_1}{4}\csc^2(\frac{\pi}{5}) + \frac{z_2}{4}\csc^2(\frac{2\pi}{5}) \\
&= \frac{z_1 - x_1}{4}\csc^2(\frac{\pi}{5}) + \frac{z_2 - x_2}{4}\csc^2(\frac{2\pi}{5}) \\
&= z_1 - x_1 \\
&= -3
\end{aligned}$$

because $z_1 - x_1 = z_2 - x_2$ and $\csc^2(\frac{\pi}{5}) + \csc^2(\frac{2\pi}{5}) = 4$.

On the other hand, there are integers $d_0, \cdots, d_4$ such that

$$\text{Spin } (g, X_\alpha) = d_0 + d_1\mu_5 + \cdots + d_4\mu_5^4.$$

Since $1 + t + \cdots + t^4 = 0$ is irreducible over $\mathbb{Q}$, one must have

$$d_0 + 3 = d_1 = \cdots = d_4.$$

With the fact that the index of the Dirac operator on X_α, which is given by the sum $d_0 + \cdots + d_4$, equals $-\text{Sign}(X_\alpha)/8 = 2$, we obtain $d_0 = -2$ and $d_1 = \cdots = d_4 = 1$. By Fang's theorem (cf. Theorem 3.9), the Seiberg-Witten invariant

$$SW_{X_\alpha}(0) = 0 \pmod 5$$

for the trivial $Spin^{\mathbb{C}}$-structure on X_α. (Note that the trivial $Spin^{\mathbb{C}}$-structure on X_α is a G-$Spin^{\mathbb{C}}$ structure because by Lemma 3.8, the action of G is spin.) However, this is a contradiction, because by construction $SW_{X_\alpha}(0) = 1$, cf. Section 2. This proves our claim regarding the cases (a), (b).

For case (c), a similar calculation shows that

$$\text{Spin } (g, X_\alpha) = -2 + 2\mu_5^2 + 2\mu_5^3$$

(i.e., $d_0 = -2$, $d_2 = d_3 = 2$ and $d_1 = d_4 = 0$), which does not violate Fang's theorem (cf. Theorem 3.9). With $d_0 = -2$, this set of fixed-point data does not violate Theorem 3.10 either.

For case (d), we have by a similar calculation that

$$\text{Spin } (g, X_\alpha) = \mu_5^2 + \mu_5^3,$$

which violates Fang's theorem, and so it is eliminated.

To finish the analysis for the cases where $A = 0$, it remains to check case (c) against Theorem 3.11, a constraint coming from the Kirby-Siebenmann and the Rochlin invariants. One finds easily from the fixed-point set structure that the corresponding 4-manifold with boundary N has 14 boundary components: there are six $L(5,1)$, six $L(5,2)$, and two $L(5,3)$. By Corollary 2.24 in [46], the Rochlin invariant of $L(5,1)$,

$L(5,2)$, and $L(5,3)$ equals 4, 0, and 0 respectively. On the other hand, one finds easily that $\mathrm{Sign}(N)=-8$. The equation in Theorem 3.11 becomes

$$8\cdot \mathrm{ks}(N)\equiv -8+6\times 4+6\times 0+2\times 0 \pmod{16},$$

which implies that $\mathrm{ks}(N)=0$. Hence case (c) can not be ruled out by Theorem 3.11.

Now we consider the cases where there are type $(\tilde{A}_4)$ fixed points, i.e., $A\neq 0$. Then by the analysis in [10] (cf. Lemma 3.6 of [10]), we see that there exists a $k\neq 0 \pmod p$ such that the local representations of g at the 3 isolated points in a group of type $(\tilde{A}_4)$ fixed points are

$$(z_1,z_2)\mapsto(\mu_p^{-3k}z_1,\mu_p^{-k}z_2),(z_1,z_2)\mapsto(\mu_p^{3k}z_1,\mu_p^{3k}z_2),(z_1,z_2)\mapsto(\mu_p^{-3k}z_1,\mu_p^{-k}z_2),$$

and the local representation at the fixed (-2)-sphere is $z\mapsto\mu_p^k z$ (here $p=5$). The key observation is that the total contribution from a group of type $(\tilde{A}_4)$ fixed points to Sign (g,X_α) equals -5, which is independent of g. More concretely, by a direct calculation the total contribution is

$$-2\cot(\frac{-3k\pi}{5})\cdot\cot(\frac{-k\pi}{5})-\cot^2(\frac{3k\pi}{5})-2\csc^2(\frac{k\pi}{5})=-5,\ \forall k.$$

Consequently, one can similarly introduce the numbers x_1,x_2,y_1,y_2,z_1,z_2 and v_1,v_2, w_1,w_2 for the fixed points of type (1), type (3), or type (4) as in the case of $A=0$, and the same argument implies that

$$z_1-z_2=v_1+w_1-v_2-w_2=0.$$

In particular, $u=z_1+z_2=2z_1$ is an even number.

With Lemma 6.3, the solutions (up to changing from g to g^2) which satisfy these constraints are

(i) $x_1=x_2=y_1=y_2=0, z_1=z_2=2,$ and $(u,v,w,A)=(4,0,0,2)$,
(ii) $x_1=2,\ x_2=1,\ y_1=1,\ y_2=3,\ z_1=z_2=1$, and $(u,v,w,A)=(2,1,1,1)$.
(iii) $x_1=x_2=y_1=y_2=0, z_1=z_2=2,$ and $(u,v,w,A)=(4,0,0,1)$.

Next we use Lemma 3.8 and Fang's theorem (cf. Theorem 3.9) to examine these fixed-point data. To this end, we need to determine the possible values of the total contribution of a group of type $(\tilde{A}_4)$ fixed points to the "Spin-number" Spin (g,X_α). A direct calculation shows that for $k=1,4$, the total contribution is

$$-\frac{2}{4}\csc(\frac{2\pi}{5})\cdot\csc(\frac{4\pi}{5})-\frac{1}{4}\csc^2(\frac{3\pi}{5})-\frac{-2}{4}\csc(\frac{\pi}{5})\cdot\cot(\frac{\pi}{5})=0,$$

and for $k=2,3$, it is

$$\frac{2}{4}\csc(\frac{4\pi}{5})\cdot\csc(\frac{3\pi}{5})-\frac{1}{4}\csc^2(\frac{\pi}{5})+\frac{-2}{4}\csc(\frac{2\pi}{5})\cdot\cot(\frac{2\pi}{5})=0.$$

As an immediate consequence we obtain that the "Spin-number"

$$\text{Spin } (g,X_\alpha)=\begin{cases} 2 & \text{in case (i)}\\ \mu_5^2+\mu_5^3 & \text{in case (ii)}\\ 2 & \text{in case (iii)}\end{cases}$$

Note that case (ii) violates Fang's theorem, hence is eliminated. However, the remaining cases (i) and (iii) can not be ruled out by Theorem 3.10 (in both cases $d_0=2$).

It remains to show that the number of fixed tori in the fixed-point set is bounded from above by one half of the number of copies of $\mathbb{Z}[\mu_5]$ in the associated integral G-representation of $\Theta = (\Theta_1, \Theta_2) : G \to \mathrm{Aut}(E_8 \oplus E_8)$. To see this, let $H_2(X_\alpha; \mathbb{Z}) = \mathbb{Z}[\mathbb{Z}_5]^r \oplus \mathbb{Z}^t \oplus \mathbb{Z}[\mu_5]^s$ be the decomposition into summands of regular, trivial and cyclotomic types. Since the classes $[T_1]$, $[T_2]$, $[T_3]$ are fixed under the G-action on X_α, we see that $\mathbb{Z}[\mu_5]^s$ is orthogonal to $[T_1]$, $[T_2]$, and $[T_3]$, hence is contained in

$$\text{Span}\ ([T_j], \xi_k, \eta_k | j = 1, 2, 3, 1 \le k \le 8).$$

(Here ξ_k, η_k are the classes in $H_2(X_\alpha; \mathbb{Z})$ which correspond to the two standard bases of the $-E_8$ form defined in Lemma 4.2.) The number s is bounded from above by the number of copies of $\mathbb{Z}[\mu_5]$ in the associated integral G-representation of Θ follows from Proposition 4.6 plus the fact that $\mathbb{Z}[\mu_5]^s$ is mapped injectively into

$$\text{Span}\ ([T_j], \xi_k, \eta_k | j = 1, 2, 3, 1 \le k \le 8)/\text{Span}\ ([T_1], [T_2], [T_3])$$

under the quotient map. Our claim then follows from Edmonds' result (cf. Prop. 3.1). The case of Theorem 1.8 where $p = 5$ follows easily.

Case(b) $p = 7$. In this case, there are no type (Γ) fixed points by Lemma 4.5 and Lemma 6.2, because by the assumption Θ_1, Θ_2 are nontrivial. Moreover, there are no fixed tori of self-intersection 0 either by Proposition 4.6. The next lemma eliminates the possibility of having type (4) fixed points.

Lemma 6.5. *Let $G \subset Aut(E_8)$ be a subgroup of order 7. There are no sublattices of E_8 fixed under G, which are isomorphic to a A_2-root lattice.*

Proof. By Carter [7] (Table 3, page 23), an element of order 7 in $\mathrm{Aut}(E_8)$ is uniquely determined up to conjugacy by the characteristic polynomial. Hence a subgroup G of order 7, with the corresponding integral G-representation being $\mathbb{Z}[\mathbb{Z}_5] \oplus \mathbb{Z}^3$ (cf. Lemma 4.5), must be conjugate to the subgroup generated by the permutation

$$e_1 \mapsto e_2, e_2 \mapsto e_3, \cdots, e_6 \mapsto e_7, e_7 \mapsto e_1, \text{ and } e_8 \mapsto e_8,$$

where $\{e_1, \cdots, e_8\}$ is a standard basis of $\mathbb{R}^8$. The only roots which are fixed under the permutation are

$$r = \pm\frac{1}{2}(e_1 + e_2 + \cdots + e_7 + e_8),$$

which do not generate a A_2-root lattice. The lemma follows.

□

With the preceding understood, we shall next determine the number of groups of type (1), type (2), and type (3) fixed points, which is denoted by u, v, w respectively. By Lemma 3.8 of [10], the total signature defect of each of such groups is 10, -8, and 2 respectively. The Lefschetz fixed point theorem and the G-signature theorem (as in Theorem 3.6) give rise to the following equations

$$\begin{cases} 1 + 3 \cdot 2 + 1 + 1 + 1 = & u + 2v + 3w \\ p \cdot (-1 - 1 - 1 - 1) = & -16 + 10u - 8v + 2w \text{ (with } p = 7). \end{cases}$$

The solutions are $(u, v, w) = (0, 2, 2), (1, 3, 1), (2, 4, 0)$.

We examine these data with the G-signature theorem as in Theorem 3.5 and the G-index theorem for Dirac operators in Lemma 3.8.

Let $\delta_1, \delta_2, \delta_3$ be the total contributions to $\text{Sign}(g, X_\alpha)$ of a group of fixed points of type (1), type (2), type (3) respectively. With a direct calculation we list all the possible values of them (in approximations) below, taken at $k = 1, 2, 3$ respectively.

- $\delta_1 = 4.31194, \quad 0.63596, \quad 0.05210,$
- $\delta_2 = -4.49396, \quad -1.10992, \quad 1.60388,$
- $\delta_3 = -2.60388, \quad 3.49396, \quad 0.10992.$

The cases where $(u, v, w) = (1, 3, 1), (2, 4, 0)$ can be eliminated as follows. Consider the case of $(u, v, w) = (1, 3, 1)$ first. There are 3 groups of type (2) fixed points. If all three values of δ_2 are assumed, then because the sum of these three values of δ_2 equals -4 and $\text{Sign}(g, X_\alpha) = -2$, the sum of the values of δ_1 and δ_3 must equal 2, which is easily seen impossible by examining the possible values of δ_1 and δ_3. If not all three values of δ_2 are assumed, then there must be one value of δ_2 which is assumed by at least 2 groups of type (2) fixed points. By changing $g \in G$ to a suitable power of g, we may assume this value of δ_2 is -4.49396. Then the sum of the rest of the values, one for each of $\delta_1, \delta_2, \delta_3$, must be $2 \times 4.49396 - 2 = 6.98792$. But this is easily seen impossible by examining the possible values. This ruled out the case where $(u, v, w) = (1, 3, 1)$. By a similar argument, the case where $(u, v, w) = (2, 4, 0)$ can be also ruled out.

It remains to examine the case where $(u, v, w) = (0, 2, 2)$. Observe first that for each value of δ_2, there is a unique value of δ_3 such that the sum of the two values equals -1. It follows easily from this that the fixed points of g can be divided into two groups, where each group consists of 5 points with local representations $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2)$, $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{-k} z_2)$, $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{4k} z_2)$, $(z_1, z_2) \mapsto (\mu_p^{-2k} z_1, \mu_p^{k} z_2)$, $(z_1, z_2) \mapsto (\mu_p^{-2k} z_1, \mu_p^{k} z_2)$ respectively, for some $k \neq 0 \pmod p$. The question is whether the number $k \pmod p$ (which is only determined up to a sign) must be the same for the two groups. We will show that it must be the same using Lemma 3.8.

To this end, we let ν_2, ν_3 be the total contributions to the "Spin-number" $\text{Spin}(g, X_\alpha)$ of a group of fixed points of type (2), type (3) respectively. With a direct calculation we list all the possible values of them (in approximations) below, taken at $k = 1, 2, 3$ respectively.

- $\nu_2 = -1, \quad -1, \quad -1,$
- $\nu_3 = -0.44504, \quad -1.80194, \quad 1.24698.$

On the other hand, $\text{Spin}(g, X_\alpha) = \sum_{l=0}^{p-1} d_l \mu_p^l$ where d_0 is even and $d_l = d_{p-l}$ for $l \neq 0$. With the observation that

$$2\cos(\frac{2\pi}{7}) = 1.24698, \quad 2\cos(\frac{4\pi}{7}) = -0.44504, \quad 2\cos(\frac{6\pi}{7}) = -1.80194,$$

we can easily conclude that in $\text{Spin}(g, X_\alpha) = \sum_{l=0}^{p-1} d_l \mu_p^l$, $d_0 = -2$ and $d_1, \cdots, d_6$ contain two 0's and four 1's if ν_3 assumes two distinct values, and $d_1, \cdots, d_6$ contain four 0's and two 2's if ν_3 assumes only one value. The former case violates Fang's theorem (cf. Theorem 3.9), so it can not occur. This proves that the number $k \pmod p$ in the local representations must be the same for the two different groups of fixed points. Finally, we note that since $d_0 = -2$ in the "Spin-number" $\mathrm{Spin}(g, X_\alpha) = \sum_{l=0}^{p-1} d_l \mu_p^l$, the remaining case can not be ruled out by Theorem 3.10. Moreover, by a similar argument as in the case of $p = 5$, one can check easily that Theorem 3.11 is not violated either. The proof for the case of $p = 7$ in Theorem 1.8 is completed.

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