# A HYPERELLIPTIC VIEW ON TEICHMÜLLER SPACE. I

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ABSTRACT. We explicitly describe the Teichmüller space  $\mathcal{T}H_n$  of hyperelliptic surfaces in terms of natural and effective coordinates as the space of certain (2n-6)-tuples of distinct points on the ideal boundary of the Poincaré disc. We essentially use the concept of a simple earthquake which is a particular case of a Fenchel-Nielsen twist deformation. Such earthquakes generate a group that acts transitively on  $\mathcal{T}H_n$ . This fact can be interpreted as a continuous analog of the well-known Dehn theorem saying that the mapping class group is generated by Dehn twists. We find a simple and effective criterion that verifies if a given representation of the surface group  $\pi_1\Sigma$  in the group of isometries of the hyperbolic plane is faithful and discrete. The article also contains simple and elementary proofs of several known results, for instance, of W. M. Goldman's theorem [Gol1] characterizing the faithful discrete representations as having maximal Toledo invariant (which is essentially the area of the representation in the two-dimensional case).

## 1. Introduction

This article is an attempt to an elementary study of Teichmüller spaces and we hope it does not require from the reader any specific knowledge in the field. We try to avoid the analytic methods typical in the classic theory and worry more about the way of the proofs than about the facts *per se*, having no prejudice against proving well-known ones. Such elementary approach is motivated by its possible extension to complex hyperbolic Teichmüller spaces and originates from [Ana1].

Let  $\Sigma = \mathbb{D}/\pi_1\Sigma$  be a hyperelliptic Riemann surface of genus  $g \geq 2$ , where  $\mathbb{D}$  stands for the Poincaré disc. It is well known [Mac] (and proven in Proposition 4.1) that the extension  $H_n$  of the fundamental group  $\pi_1\Sigma$  with an isometry of  $\mathbb{D}$  induced by the hyperelliptic involution of  $\Sigma$  is a group with generators  $r_1, \ldots, r_n$  and defining relations  $r_n \ldots r_1 = 1, r_i^2 = 1$ , where n = 2g+2. Moreover, every  $r_i$  is a reflection in some point  $q_i \in \mathbb{D}$ . In other words, a hyperelliptic surface can be described as a certain geometric configuration of n points.

The following two concepts are crucial in this article. As is easy to see, while moving the points  $q_{i-1}$  and  $q_i$  along the geodesic they generate and preserving the distance between these two points, new configurations provide new hyperelliptic surfaces, i.e., the relation  $r_n \ldots r_1 = 1$  remains valid. We call such a deformation a *simple earthquake* (SE for short). This concept is nothing more than a particular case of a Fenchel-Nielsen twist deformation [ImT]. It appears naturally in the context of [Ana1]. The earthquake group  $\mathcal{E}_n$ , i.e., the formal group generated by the SEs, acts on the Teichmüller space  $\mathcal{T}H_n$  of the group  $H_n$ .

The other concept is the area of a surface. It is better to call this area the Toledo invariant of a representation. The remarkable results of W. M. Goldman [Gol1, Corollary C] and D. Toledo [Tol] say that a representation is faithful and discrete if (and only if, in the case of the classic hyperbolic geometry) the 'area' of the representation is 'maximal.' In literature (see, for instance, [BIW] and [KMa]), there are several proofs of Toledo's theorem and neither of them is simple.

First, we study hyperelliptic surfaces. We prove the analog of W. M. Goldman's theorem for hyperelliptic surfaces (Theorem 3.15). The Teichmüller space  $\mathcal{T}H_n$  turns out to be supplied with natural

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coordinates: The space  $\mathcal{T}H_n$  can be described as the space of all (2n-6)-tuples  $(z_1, z_2, \ldots, z_{2n-6})$  of distinct points on the ideal boundary  $\partial \mathbb{D}$  that appear in the cyclic order  $z_1, z_2, \ldots, z_{2n-6}$  when running once over  $\partial \mathbb{D}$  (Corollary 3.17). These coordinates are natural in the sense that they have a clear geometric nature and are not related to any arbitrary choice. Also, they are effective and easily calculable. Besides, following these ideas, we arrive at a simple and effective criterion allowing to verify that a given representation is faithful and discrete. It is worthwhile mentioning a curious fact (we did not find it in literature) : Every pentagon, i.e., every<sup>1</sup> representation  $\varrho : H_5 \to \mathrm{PU}(1,1)$  such that  $\varrho(r_i) \neq 1$ , is faithful and discrete (Corollary 3.16). (A complex hyperbolic version of this fact is discussed in [Ana1, Conjecture 1.2].)

Next, we show that the earthquake group  $\mathcal{E}_n$  acts transitively on  $\mathcal{T}H_n$  (Theorem 4.5). This fact can be considered as a continuous analog<sup>2</sup> of the well-known Dehn theorem saying that the mapping class group can be generated by the Dehn twists. (The Dehn twists we use are 'integer' SEs.) Then we prove a discrete variant of Theorem 4.5 — a sort of the Dehn theorem: The subgroup of index 2 in Aut  $H_n$  is generated by the 'integer' SEs (Theorem 4.6).

Finally, we prove W. M. Goldman's theorem [Gol1, Corollary C] in general case (Theorem 5.1). The idea of the proof is reflected by the title of this article. We pretend to view a general Riemann surface  $\Sigma$  as if it were a hyperelliptic one and, with a certain precaution, apply to  $\Sigma$  the methods developed in the previous sections. As in the hyperelliptic case, we establish an effective and simple criterion of discreteness of a representation of  $G_n := \pi_1 \Sigma$  that involves the construction of a natural fundamental domain (Remark 5.10). This fundamental domain allows to visualize the universal family  $\mathcal{F} \to \mathcal{T}_n$  of Riemann surfaces, where  $\mathcal{T}_n$  denotes the classic Teichmüller space:  $G_n$  acting fibrewise on the trivial bundle  $\mathbb{D} \times \mathcal{T}_n \to \mathcal{T}_n$  provides  $\mathcal{F} = \mathbb{D} \times \mathcal{T}_n/G_n$ . The union of the natural fundamental domains over all fibres is a fundamental domain for the action of  $G_n$  on  $\mathbb{D} \times \mathcal{T}_n$ . Yet, we cannot describe  $\mathcal{T}_n$  as explicitely as  $\mathcal{T}H_n$ . Nevertheless, it is easy to extend the action of  $\mathcal{E}_n$  to  $\mathcal{T}_n$  (see Remark 5.24).

Our way of proving the discreteness of a representation, where SEs are extensively used, resembles a kind of hidden Maskit combination theorems [Mas]. We think that there is no satisfactory complex hyperbolic analog of these theorems. The reason is that it is quite difficult to deduce the discreteness of a 'cocompact' group from the discreteness of its 'noncocompact' subgroups appearing after cutting the corresponding manifold. In our approach, we escape passing to 'noncocompact' groups.

As expected, the complex hyperbolic Toledo theorem [Tol] can be easily proven (see [Ana2]) by literally repeating the arguments presented in this article. Another (unexpected) consequence of our methods is the fact that  $\mathcal{T}_n$  is fibred twice over  $\mathcal{T}H_n \subset \mathcal{T}_n$ . Moreover, every point in  $\mathcal{T}_n$  is uniquely determined by its projections to  $\mathcal{T}H_n$  [Ana2].

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#### 2. Preliminaries

In our notation, we follow [AGr], except that, for the sake of convenience, we change the hermitian metric in order to have the curvature -1.

Let W be a two-dimensional  $\mathbb{C}$ -vector space equipped with a hermitian form of signature +-. For a nonisotropic  $p \in \mathbb{CPW}$ , define a hermitian form in  $T_p \mathbb{CPW} \simeq \langle -, p \rangle p^{\perp}$  as  $\langle t_1, t_2 \rangle := -4 \langle p, p \rangle \langle v_1, v_2 \rangle$ , where  $t_1, t_2 \in T_p \mathbb{CPW}$ ,  $t_i = \langle -, p \rangle v_i$ , and  $v_i \in p^{\perp}$ . The set BW of negative points in  $\mathbb{CPW}$  is simply the open Poincaré disc. The set  $\overline{BW}$  of nonpositive and the set SW of isotropic points in  $\mathbb{CPW}$  form the closed Poincaré disc and its boundary; all geometrical objects we deal with live in  $\overline{BW}$ . For distinct  $p_1, p_2 \in \overline{BW}$ , denote by  $G[p_1, p_2]$ ,  $G(p_1, p_2)$ ,  $G(p_1, p_2]$ ,  $G \prec p_1, p_2 \succ$ , etc. the geodesic segments oriented from  $p_1$  to  $p_2$ : closed, open, semiopen, full geodesic, etc.

<sup>&</sup>lt;sup>1</sup>We interpret as PU(1,1) the group of all orientation-preserving isometries of  $\mathbb{D}$ .

<sup>&</sup>lt;sup>2</sup>Maxim Kontsevich convinced us that  $\mathcal{E}_n$  is not finite-dimensional modulo the kernel of its action on  $\mathcal{T}H_n$ .

Let  $\mathbb{B}^2$  denote a closed disc and let  $\varphi : \mathbb{B}^2 \to \overline{\mathbb{B}}W$  be a piecewise smooth map such that  $\varphi(\partial \mathbb{B}^2)$  is the union of a finite number of geodesics and such that  $\varphi^{-1}(\mathbb{S}W) \subset \partial \mathbb{B}^2$  is finite. Clearly,  $\int_{\varphi} \omega = \int_{\partial \varphi} P$ , where

 $\omega$  and P stand for the Kähler form and its potential. In particular, for  $p_1, p_2, p_3 \in \overline{B}W$ , the oriented area of the triangle  $\Delta(p_1, p_2, p_3)$  is given by<sup>3</sup>

(2.1) 
$$\operatorname{Area} \Delta(p_1, p_2, p_3) = 2 \arg\left(-\langle p_1, p_2 \rangle \langle p_2, p_3 \rangle \langle p_3, p_1 \rangle\right)$$

(see, for instance, [Gol2] or [AGr, Subsection 5.9]). This formula works for triangles having no coinciding isotropic vertices. Obviously, the area of  $\Delta(p, p, q)$  vanishes for isotropic p. Thus, Area  $\Delta(p_1, p_2, p_3)$  is continuous while  $p_1, p_2, p_3$  run over  $\overline{B}W$ , assuming different isotropic vertices not to coincide during the deformation.

Integrating a Kähler potential over a closed piecewise geodesic path C (not necessarily simple), we obtain the 'area' of the 'polygon limited by C.' In order to express this area in explicit terms, take an arbitrary 'centre'  $c \in \overline{BW}$ . Let  $p_1, p_2, \ldots, p_n$  be successive vertices of C. Define

(2.2) 
$$\operatorname{Area}(c;C) := \operatorname{Area}(c;p_1,p_2,\ldots,p_n) := \sum_{i=1}^n \operatorname{Area}\Delta(c,p_i,p_{i+1})$$

(the indices are modulo n). Intuitively, this area does not depend on the choice of c. We prefer to give a formal proof of this fact since it can be useful when we will deal with other invariants different from the Toledo one.

**2.3. Remark.** For arbitrary  $c, p, q, p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_l \in \overline{B}W$ , we have

$$Area(c; p, p_1, p_2, \dots, p_k, q, q_1, q_2, \dots, q_l) = Area(c; p, p_1, p_2, \dots, p_k, q) + Area(c; q, q_1, q_2, \dots, q_l, p)$$

because Area  $\Delta(c, q, p) + \operatorname{Area} \Delta(c, p, q) = 0.$ 

So, in order to prove that (2.2) is independent of c, we can assume n = 3 and the  $p_i$ 's pairwise distinct. Now, it follows from (2.1) that

Area 
$$\Delta(c; p_1, p_2, p_3) \equiv 2 \arg(\langle p_1, p_2 \rangle \langle p_2, p_3 \rangle \langle p_3, p_1 \rangle) \mod 2\pi$$

for c different from the isotropic  $p_i$ 's. For such c, the independence follows from the continuity of the triangle area. It is immediate that  $\operatorname{Area}(c; p_1, p_2, p_3) = \operatorname{Area} \Delta(p_1, p_2, p_3)$  for  $c = p_i$ . Therefore, it remains to observe that  $\operatorname{Area}(c; p_1, p_2, p_3) = \operatorname{Area} \Delta(p_1, p_2, p_3)$  for c isotropic and the  $p_i$ 's pairwise distinct and isotropic, which is straightforward.

For  $n \geq 5$ , let  $H_n$  denote the group generated by  $r_1, r_2, \ldots, r_n$  with the defining relations  $r_i^2 = 1$ ,  $i = 1, \ldots, n$ , and  $r_n \ldots r_2 r_1 = 1$ . For even n, there is a unique fully characteristic torsion-free subgroup  $G_n$  of index 2 in  $H_n$ . It is constituted by the words of even length in  $r_i$ 's. As is well known (see also Proposition 4.1),  $G_n$  is the fundamental group of a closed orientable Riemann surface of genus  $\frac{n}{2} - 1$ . For odd n, there is a torsion-free subgroup  $T_n$  of index 4 in  $H_n$  which is the fundamental group of a closed orientable Riemann surface of genus n - 3 (see, for instance, [AGG, Subsection 2.1]).

Let  $\mathcal{L} := \text{PU}W$  denote the Lie group of all orientation-preserving isometries of BW. Denote by  $\mathcal{R}H_n$  and  $\mathcal{R}G_n$  the spaces of faithful discrete representations of  $H_n$  and of  $G_n$  into  $\mathcal{L}$ , respectively. The spaces  $\mathcal{H}_n := \mathcal{T}H_n := \mathcal{R}H_n/\mathcal{L}$  and  $\mathcal{T}_n := \mathcal{T}G_n := \mathcal{R}G_n/\mathcal{L}$  are the Teichmüller spaces of the groups  $H_n$  and  $G_n$ , i.e., the spaces of conjugacy classes of the above representations. Each of the two connected

<sup>&</sup>lt;sup>3</sup>The function arg takes values in  $[-\pi,\pi]$ . In the presented formula, the values of arg lie in fact in  $[-\frac{\pi}{2},\frac{\pi}{2}]$ .

components  $\mathcal{T}_n^-$  and  $\mathcal{T}_n^+$  of  $\mathcal{T}_n$  can be interpreted as the classic Teichmüller space. (The latter appears if we take for  $\mathcal{L}$  the Lie group of all isometries of BW.) Similarly, we introduce  $\mathcal{H}_n^{\pm}$ . The part of  $\mathcal{T}_n^{\pm}$ corresponding to hyperelliptic surfaces possesses infinitely many connected components [Mac] which are copies of  $\mathcal{H}_n^{\pm}$  provided by the action of the mapping class group.

It is easy to see that the involutions in  $\mathcal{L}$  are exactly the reflections in points in BW. Explicitly, in terms of SUW, such a reflection R(q) is given by  $R(q) : x \mapsto i\left(x - 2\frac{\langle x, q \rangle}{\langle q, q \rangle}q\right), q \in BW, i^2 = -1$ . Note that R(q)R(q) = -1.

### 3. Hyperelliptic Teichmüller Space

Let  $\rho: H_n \to \mathcal{L}$  be a representation. For an arbitrary  $p \in \overline{B}W$ , define

(3.1) 
$$\operatorname{Area}(p; \varrho) := \operatorname{Area}(p_1, p_2, \dots, p_n),$$

where  $p_0 := p$  and  $p_i := \varrho(r_i)p_{i-1}$  (the indices are modulo n). Clearly, we can also define the  $p_i$ 's starting from  $p = p_j \in \overline{BW}$  for an arbitrary j instead of j = n.

**3.2. Lemma.** Area $(p; \varrho)$  is independent of the choice of p. If  $\varrho(r_i) \neq 1$  for all i, then Area $(p; \varrho) \equiv n\pi \mod 2\pi$ .

**Proof.** Without loss of generality, we can assume that  $\varrho(r_i) \neq 1$  for all *i* and choose a representative  $R(q_i) \in SUW$ ,  $q_i \in BW$ , for every  $\varrho(r_i)$ . Hence,  $p_i \neq p_{i+1}$  if *p* is isotropic. It follows from the defining relations of  $H_n$  that  $R(q_n) \dots R(q_1) = \varepsilon$ , where  $\varepsilon = \pm 1$ . Take representatives  $p_i \in W$  so that  $p_i = R(q_i)p_{i-1}$ . In particular,  $p_{i+n} = \varepsilon p_i$ . It follows from (2.1) that

Area
$$(p; \varrho) \equiv 2 \arg (\langle p_1, p_2 \rangle \langle p_2, p_3 \rangle \dots \langle p_n, p_{n+1} \rangle) \mod 2\pi.$$

Since  $R(q_i) \in SUW$  and  $R(q_i)R(q_i) = -1$ , we obtain

$$0 \neq \langle p_i, p_{i+1} \rangle = \langle p_i, R(q_{i+1})p_i \rangle = \langle R(q_{i+1})p_i, R(q_{i+1})R(q_{i+1})p_i \rangle = -\langle p_{i+1}, p_i \rangle.$$

So,  $\operatorname{Area}(p; \varrho) \equiv 2 \arg i^n \equiv n\pi \mod 2\pi$ , being  $\operatorname{Area}(p; \varrho)$  continuous in  $p \blacksquare$ 

**3.3. Remark.** For a given representation  $\varrho : H_n \to \mathcal{L}$ , define  $\varrho J : H_n \to \mathcal{L}$  by  $\varrho J(r_i) := \varrho(r_{n-i})$ . Obviously, Area  $\varrho J = -$  Area  $\varrho$ . In other words, changing the cyclic order of the generators alters the sign of the area.

In the sequel, we assume without loss of generality that Area  $\rho \geq 0$ .

**3.4. Remark.** Let  $p_{i-2} \in \overline{B}W$  be a fixed point of  $\rho(r_i r_{i-1})$ . Then, by taking  $c = p_{i-2} = p_i$ , we can see that

$$\operatorname{Area} \Delta(c, p_{i-3}, p_{i-2}) = \operatorname{Area} \Delta(c, p_{i-2}, p_{i-1}) =$$
$$= \operatorname{Area} \Delta(c, p_{i-1}, p_i) = \operatorname{Area} \Delta(c, p_i, p_{i+1}) = 0$$



and, hence, Area  $\rho \leq (n-4)\pi$ . When Area  $\rho = (n-4)\pi$ , we say that Area  $\rho$  is maximal.

If  $\varrho(r_i) = 1$ , then Area  $\varrho \leq (n-5)\pi$ : 'excluding' the generator  $r_i$  we deal in fact with a representation of  $H_{n-1}$ .

Analogously, if  $\varrho(r_i r_{i-1}) = 1$ , then 'excluding' the generators  $r_{i-1}$  and  $r_i$ , we arrive at the representation  $\varrho' : H_{n-2} \to \mathcal{L}$ . Note that Area  $\varrho = \operatorname{Area} \varrho'$  since  $p_{i-2} = p_i$  and Area  $\Delta(c, p_{i-2}, p_{i-1}) = -\operatorname{Area} \Delta(c, p_{i-1}, p_i)$ . Therefore, Area  $\varrho \leq (n-6)\pi$  in this case.

**3.5. Definition.** Let  $q_1, q_2 \in BW$  be distinct. Clearly,  $h^2 := R(q_2)R(q_1)$ for some hyperbolic  $h \in \mathcal{L}$ . It is easy to see that  $R(h^t q_k) = h^t R(q_k) h^{-t}$ , k = 1, 2, and that  $R(q_2)R(q_1) = R(h^tq_2)R(h^tq_1)$  for every  $t \in \mathbb{R}$ .

Let  $\varrho: H_n \to \mathcal{L}$  be a representation such that  $h^2 := \varrho(r_i r_{i-1})$  is hyperbolic. For every  $t \in \mathbb{R}$ , define a representation  $\rho E_i(t) : H_n \to \mathcal{L}$  as follows:  $\varrho E_i(t)(r_j) := \varrho(r_j)$  if  $j \notin \{i-1, i\}$  and  $\varrho E_i(t)(r_j) := h^t \varrho(r_j) h^{-t}$ , otherwise.

This defines a partial right action of the group  $(\mathbb{R}, +)$  on representations. We call  $E_i(t)$  a simple earthquake involving  $q_{i-1}, q_i$  (SE for short), where  $\varrho(r_j) = R(q_j), j = 1, 2, \dots n$ . Denote by  $E_i := E_i(1)$  the Dehn twist involving  $q_{i-1}, q_i$  (DT for short).

**3.6 Definition.** If a cycle of isotropic points  $p_1, p_2, \ldots, p_k \in SW, k \geq 3$ , is listed in the counterclockwise (clockwise) sense (in particular, the points have to be pairwise distinct), the cycle is said to be *positive* (*negative*).

**3.7. Remark.** Given  $p_1, p_2, q_1, q_2 \in SW$ , the cycle  $p_1, q_1, p_2, q_2$  is positive or negative if and only if  $G(p_1, p_2)$  and  $G(q_1, q_2)$  intersect in a single point.

If the cycles  $p_1, p_2, \ldots, p_k \in SW$ ,  $k \ge 3$ , and  $p_k, p_{k+1}, p_1 \in SW$  are positive, then the cycle  $p_1, p_2, \ldots, p_k, p_{k+1}$  is positive.

**3.8. Remark.** Suppose that  $\rho(r_i r_{i-1})$  is hyperbolic. Then Area  $\rho$  = Area  $\rho E_i(t)$ . Indeed, taking for  $p_{i-2}$  a fixed point of  $\rho(r_i r_{i-1})$ , we can see that the  $p_i$ 's are independent of t and so is Area  $\rho E_i(t)$ . (See the picture close to Remark 3.4.)

**3.9. Lemma.** Let  $\varrho: H_n \to \mathcal{L}$  be a representation with maximal Area  $\varrho$ . Then, for every *i*, there exists a suitable  $q_i \in BW$  such that  $\varrho(r_i) = R(q_i), q_{i-1} \neq q_i$ , and  $\varrho(r_i r_{i-1})$  is hyperbolic. If we take in (3.1) a fixed point of  $\varrho(r_i r_{i-1})$  for  $p_{i-2} = p_i$ , then the cycle  $p_i, p_{i+1}, \ldots, p_{i+n-3} \in SW$  is positive.

**Proof.** The first three assertions follow from Remark 3.4 in view of the fact that the involutions in  $\mathcal{L}$  are reflections in points. As in Remark 3.4, take  $c = p_{i-2} = p_i$ . The four triangles indicated in Remark 3.4 are degenerated. Hence, each of the remaining n-4 ideal triangles should have area  $+\pi$ . In other words, the triangles  $\Delta(c, p_{j-1}, p_j), j = i+1, \ldots, i+n-3$ , are oriented in the counterclockwise sense. This implies the fourth assertion  $\blacksquare$ 

**3.10. Lemma.** In the situation of Lemma 3.9, there are no three collinear points among the  $q_i$ 's. Moreover,  $q_1, q_2, \ldots, q_n$  are successive vertices of a convex polygon.

**Proof.** Suppose that  $q_j, q_k, q_l$  are collinear. Acting by  $E_j$  or by  $E_{j+1}^{-1}$   $q_{j-1}$ several times, we can reach a position where  $q_{k-1}, q_k, q_l$  are collinear



(we diminish |j - k| > 1). Next, applying  $E_l$  or  $E_{l+1}^{-1}$ several times, we arrive at collinear  $q_{k-1}, q_k, q_{k+1}$ . Finally, by means of some  $E_k(t)$ , we obtain  $q_k = q_{k+1}$ . This contradicts Lemma 3.9.

If  $q_k$  and  $q_l$  are on different sides from  $G \prec q_{j-1}, q_j \succ$ , then  $G \prec q_{j-1}, q_j \succ$  and  $G[q_k, q_l]$ intersect in some  $q \in BW$ . With a suitable  $E_j(t)$ , we obtain  $q_j = q$ , hence,  $q_j, q_k, q_l$ 

become collinear  $\blacksquare$ 

**3.11. Lemma.** In the situation of Lemma 3.9, the points  $q_j$ ,  $j \notin \{i-1, i\}$ , are on the side of the normal vector to  $\mathbf{G} \prec q_{i-1}, q_i \succ$ .







**Proof.** Due to Lemma 3.10, we can assume all the points  $q_j$ ,  $j \notin \{i-1,i\}$ , on the opposite side of the normal vector to  $G \prec q_{i-1}, q_i \succ$ . By Lemma 3.10, this implies that  $q_{i-2}$  is in the region given by the normal vectors to  $G \prec q_i, q_{i-1} \succ$  and to  $G \prec q_{i+1}, q_i \succ$ , i.e., in the grey region on the first picture. On the other hand, by Lemma 3.9, the cycle  $p_i, p_{i+1}, p_{i+n-3} \in SW$  is positive, where  $p_{i-2} = p_i \in SW$  stands for the attractor of  $\varrho(r_i r_{i-1})$ . This implies that the geodesic  $G \prec p_{i+n-3}, p_{i-2} \succ \ni q_{i+n-2} = q_{i-2}$  is entirely on the side of the normal vector to  $G \prec p_i, p_{i+1} \succ$  as illustrated on the second picture. Therefore, the point  $q_{i-2}$  is in the region given by the normal vectors to  $G \prec q_{i+1}, q_i \succ$  and to  $G \prec p_i, p_{i+1} \succ$  and, thus, the geodesics  $G \prec q_{i-2}, q_{i+1} \succ$  and  $G \prec q_{i-1}, q_i \succ$  intersect in some point in BW (see the third picture)

**3.12. Definition.** Let  $\varrho: H_n \to \mathcal{L}$  be a representation such that  $\varrho(r_i r_{i-1})$  is hyperbolic. Denote by  $b_i \in SW$  and by  $e_i \in SW$  the repeller and the attractor of  $\varrho(r_i r_{i-1})$ . Put  $b_i^i := b_i$ ,  $e_i^i := e_i$ ,  $b_i^j := \varrho(r_j)b_i^{j-1}$ , and  $e_i^j := \varrho(r_j)e_i^{j-1}$ . It follows from the defining relations of  $H_n$  that  $b_i^{i+n-2} = b_i$  and  $e_i^{i+n-2} = e_i$ . We call  $b_i^i, e_i^i, b_i^{i+1}, e_i^{i+1}, \dots, b_i^{i+n-3}, e_i^{i+n-3} \in SW$  the *i*-cycle of  $\varrho$ .

**3.13.** Proposition. Let  $\varrho: H_n \to \mathcal{L}$  be a representation with maximal Area  $\varrho$ . Then the *i*-cycle of  $\varrho$  is positive.

**Proof.** By Lemma 3.9, the cycles  $b_i^i, b_i^{i+1}, \ldots, b_i^{i+n-3}$  and  $e_i^i, e_i^{i+1}, \ldots, e_i^{i+n-3}$  are positive. For suitable points  $q_j \in BW$ , we have  $\varrho(r_j) = R(q_j)$ . By Lemma 3.11,  $q_{i+n-2}$  and  $q_{i+1}$  are in the region D given

by the normal vector to  $\mathbf{G} \prec q_{i-1}, q_i \succ = \mathbf{G}[b_i^i, e_i^i]$ . So,  $e_i^{i+n-3} = R(q_{i+n-2})e_i \in D$ . In other words, the cycle  $e_i^{i+n-3}, b_i^i, e_i^i$  is positive. Since the geodesics  $\mathbf{G}[e_i^i, e_i^{i+1}]$ and  $\mathbf{G}[b_i^i, b_i^{i+1}]$  intersect in  $q_{i+1} \in D \cap \mathbf{B}W$ , we have  $b_i^{i+1}, e_i^{i+1} \in D$  and the cycle  $b_i^i, e_i^i, b_i^{i+1}, e_i^{i+1}$  is positive by Remark 3.7. The fact that the cycles  $e_i^i, b_i^{i+1}, e_i^{i+1}$ and  $e_i^{i+1}, e_i^{i+n-3}, e_i^i$  are positive implies that the cycle  $e_i^{i+n-3}, e_i^i, b_i^{i+1}, e_i^{i+1}$  is positive by Remark 3.7. Taking into account that the cycle  $e_i^{i+n-3}, b_i^i, e_i^i$  is positive, by Remark 3.7, we get the positive cycle  $e_i^{i+n-3}, b_i^i, e_i^i$ ,  $b_i^{i+1}, e_i^{i+1}$ .

By induction on j > i, we can assume that the cycle  $e_i^{i+n-3}, b_i^i, e_i^i, \ldots, b_i^j, e_i^j$  is positive. The cycle  $e_i^{i+n-3}, e_i^j, e_i^{j+1}$  is positive. Hence, the cycle  $e_i^{i+n-3}, b_i^i, e_i^i, \ldots, b_i^j, e_i^j, e_i^{j+1}$  is positive by Remark 3.7. In particular,  $b_i^j, e_i^j, e_i^{j+1}$  is positive. The geodesics  $G[e_i^j, e_i^{j+1}]$  and



$$\begin{split} \mathbf{G}[b_i^j, b_i^{j+1}] \text{ intersect (in } q_{j+1} \in \mathbf{B}W). \text{ By Remark 3.7, the cycle } b_i^j, e_i^j, b_i^{j+1}, e_i^{j+1} \text{ is positive or negative.} \\ \text{Knowing that the cycle } b_i^j, e_i^j, e_i^{j+1} \text{ is positive, we infer that } b_i^j, e_i^j, b_i^{j+1}, e_i^{j+1} \text{ is positive and imply that } e_i^j, b_i^{j+1}, e_i^{j+1} \text{ is positive.} \text{ Since } e_i^{i+n-3}, b_i^i, e_i^i, \dots, b_i^j, e_i^j, e_i^{j+1} \text{ is positive, } e_i^{i+n-3}, b_i^i, e_i^j, \dots, b_i^j, e_i^j, e_i^{j+1} \text{ is positive, } e_i^{i+n-3}, b_i^i, e_i^j, \dots, b_i^j, e_i^{j+1}, e_i^{j+1} \text{ is positive by Remark 3.7} \end{split}$$

**3.14. Proposition.** Let  $\varrho: H_n \to \mathcal{L}$  be a representation with hyperbolic  $\varrho(r_i r_{i-1})$ . If the *i*-cycle of  $\varrho$  is positive, then  $\varrho \in \mathcal{R}H_n$ .

**Proof.** Taking  $b_i^i$  for  $p_{i-2}$  in (3.1), we obtain the points  $p_{i-2}, p_{i-1}, \ldots, p_{i+n-3}$  which are in fact the points  $b_i^i, e_i^i, b_i^i, b_i^{i+1}, b_i^{i+2}, \ldots, b_i^{i+n-3}$ . Since the *i*-cycle of  $\rho$  is positive, the cycle  $b_i^i, b_i^{i+1}, b_i^{i+2}, \ldots, b_i^{i+n-3}$  is positive and we conclude that Area  $\rho = (n-4)\pi$ .

Following the natural orientation of SW, we draw an arc  $a_j \,\subset\, SW$  from  $b_i^j$  to  $e_i^j$  for every  $j = i, i + 1, \ldots, i + n - 3$ . The arcs  $a_j$  are pairwise disjoint because the *i*-cycle is positive. We take an arbitrary  $p_{i-1} \in G(q_{i-1}, q_i)$  and generate the points  $p_j := \varrho(r_j)p_{j-1}$  so that  $p_{i+n-2}, p_{i-1}, p_i \in G_i$ , where  $G_j := G[b_i^j, e_j^j]$ . We claim that  $p_{i-1}, p_i, \ldots, p_{i+n-2}$  are the successive vertices of a convex geodesic



*n*-gon  $P_n$ . Indeed,  $p_j \in G_j$  for j = i, i + 1, ..., i + n - 3 because  $G_{j+1} = R(q_{j+1})G_j$ . For such *j*'s, the vertices of the geodesic  $\Gamma_{j+1} := G \prec p_j, p_{j+1} \succ$  belong to  $a_j$  and  $a_{j+1}$  (by convention,  $a_{i+n-2} := a_i$ ). Hence,  $\Gamma_j$  and  $\Gamma_{j+1}$  intersect in  $p_j$  and these are the only intersections between the  $\Gamma_j$ 's. Since Area  $\rho = Area(p_i, p_{i+1}, \ldots, p_{i+n-1}) = Area P_n$ , the sum of the interior angles of  $P_n$  equals  $(n-2)\pi$ -Area  $P_n = 2\pi$ . By Poincaré's Polyhedron Theorem,  $P_n$  is a fundamental polygon for the group generated by  $\rho(r_j)$  (it has

one cycle of vertices) and  $\rho$  is faithful and discrete

**3.15. Theorem.** Let  $\rho: H_n \to \mathcal{L}$  be a representation. Then the following statements are equivalent: •  $\rho \in \mathcal{R}H_n$ , • Area  $\rho = \pm (n-4)\pi$ , • the *i*-cycle of  $\rho$  is positive or negative.

**Proof** explores standard arguments. We will deal with even n (similar arguments work for odd n). Let  $\rho \in \mathcal{R}H_n$ . Clearly,  $\rho|_{G_n} \in \mathcal{R}G_n$ . By definition, Area  $\rho = \operatorname{Area}(p_1, p_2, \ldots, p_n)$ , where  $p_j = \rho(r_j)p_{j-1}$  for suitable  $p_j \in \mathcal{B}W$ .

Let  $P_n$  be a simple geodesic polygon such that the sum of its interior angles equals  $2\pi$  and let  $v_1, v_2, \ldots, v_n$  stand for the successive vertices of  $P_n$  listed in the counterclockwise sense. Let  $q_j$  denote the middle point of  $G[v_{j-1}, v_j]$ . By Poincaré's Polyhedron Theorem,  $P_n$  is a fundamental polygon for the group generated by  $R(q_j)$  and, thus, we arrive at some  $\rho_0 \in \mathcal{R}H_n$ .

Let us define a continuous  $H_n$ -equivariant map  $\varphi : \mathbb{B} W \to \mathbb{B} W$ . Put  $\varphi v_j = p_j$  and define  $\varphi$  linearly on the geodesic  $G[v_{j-1}, v_j]$ ; so,  $\varphi G[v_{j-1}, v_j] = G[p_{j-1}, p_j]$ . Next, extend  $\varphi$  continuously to  $\varphi : P_n \to \mathbb{B} W$ . Finally, put  $\varphi(\varrho_0(h)p) = \varrho(h)\varphi(p)$  for all  $h \in H_n$  and  $p \in P_n$ . The map  $\varphi$  induces a continuous map  $\psi : \Sigma_0 \to \Sigma$ , where  $\Sigma_0 := \mathbb{B} W/\varrho_0 G_n$  and  $\Sigma := \mathbb{B} W/\varrho G_n$  are Riemann surfaces

of genus  $\frac{n}{2} - 1$ . By construction,  $\pi_1 \psi : \pi_1 \Sigma_0 \to \pi_1 \Sigma$  is an isomorphism, hence,  $H_2 \psi : H_2(\Sigma_0, \mathbb{Z}) \to H_2(\Sigma, \mathbb{Z})$  is an isomorphism and  $\int_{\psi} \omega' = \mp \operatorname{Area} \Sigma = \pm 2\pi \chi(\Sigma) =$ 

 $\mp 2(n-4)\pi$ , where  $\omega'$  stands for the Kähler form of  $\Sigma$ . On the other hand,  $P_n \cup \varrho_0(r_i)P_n$  is a fundamental polygon for  $\varrho_0G_n$ , therefore,  $\int_{\psi} \omega' = 2 \int_{\varphi|_{P_n}} \omega = 2 \int_{\varphi|_{\partial P_n}} P = 0$ 

 $-2 \operatorname{Area}(p_1, p_2, \dots, p_n) = -2 \operatorname{Area} \varrho$ , where P stands for a Kähler potential of BW. Consequently,  $\operatorname{Area} \varrho = \pm (n-4)\pi$ 



From Remark 3.3 and Lemma 3.2, we obtain the **3.16. Corollary.** Let  $\rho: H_5 \to \mathcal{L}$  be a representation such that  $\rho(r_i) \neq 1$  for all i. Then  $\rho \in \mathcal{R}H_5$ 

Note that Theorem 3.15 provides an effective criterion of discreteness: In order to verify that some  $q_1, \ldots, q_n \in BW$  subject to the relation  $R(q_n) \ldots R(q_1) = \pm 1$ provide a representation  $\rho \in \mathcal{R}H_n$ , we can explicitly find the  $b_i^j$ 's and  $e_i^j$ 's and check if the *i*-cycle of  $\rho$ 

is positive or negative. Also, Theorem 3.15 yields some explicit description of the two components  $\mathcal{H}_n^+$  and  $\mathcal{H}_n^-$  (related to the sign of Area  $\varrho$ ) of  $\mathcal{H}_n$ . Let  $\mathbb{S}^1_+ := \{z \in \mathbb{C} \mid |z| = 1, \text{Im } z > 0\}$  and let

 $\mathcal{K}_n^+ := \left\{ (z_1, z_2, \dots, z_{2n-7}, z_{2n-6}) \mid z_j \in \mathbb{S}_+^1, \text{ the cycle } z_1, z_2, \dots, z_{2n-7}, z_{2n-6} \text{ is positive} \right\}$ 

(similarly, we define  $\mathbb{S}^1_-$  and  $\mathcal{K}^-_n$ ).



**3.17.** Corollary.  $\mathcal{H}_n^{\pm} \simeq \mathcal{K}_n^{\pm}$ .

**Proof.** Identify  $\overline{B}W$  with the unitary disc  $\{z \in \mathbb{C} \mid |z| \leq 1\}$ . Let  $[\varrho] \in \mathcal{H}_n^+$ . Conjugating  $\varrho$  with an element in  $\mathcal{L}$ , we can assume that  $b_i^i = -1$ ,  $e_i^i = 1$ , and  $q_i = 0$ . This provides

$$(z_1, z_2, \dots, z_{2n-7}, z_{2n-6}) := (b_i^{i+1}, e_i^{i+1}, \dots, b_i^{i+n-3}, e_i^{i+n-3}).$$

In other words, we obtain a map  $\mathcal{H}_n^+ \to \mathcal{K}_n^+$ .

Conversely, for given  $(z_1, z_2, \ldots, z_{2n-7}, z_{2n-6}) \in \mathcal{K}_n^+$ , define  $q_i := 0$ ,  $q_{i+1} := G[-1, z_1] \cap G[1, z_2]$ ,  $q_{i+k} := G[z_{2k-3}, z_{2k-1}] \cap G[z_{2k-2}, z_{2k}]$  for  $k = 2, 3, \ldots, n-3$ , and  $q_{i+n-2} := G[z_{2n-7}, -1] \cap G[z_{2n-6}, 1]$ . It is easy to see that the isometry  $h := R(q_{i+n-2}) \ldots R(q_{i+2})R(q_{i+1}) \in SUW$  fixes the points -1and 1. If  $h = \pm 1$ , we obtain a representation  $\varrho_0 : H_{n-2} \to \mathcal{L}$ . Taking  $p_i = 1$ , we arrive at Area  $\varrho_0 =$ Area $(1, z_2, z_4, \ldots, z_{2n-6}) = (n-4)\pi$ , which contradicts Remark 3.4. Therefore, h is hyperbolic with the axis G[-1, 1] and there exists a unique  $q_{i+n-1} \in G(-1, 1)$  such that  $h = R(q_{i+n-1})R(q_i)$ . In other words,  $R(q_{i+n-1}) \ldots R(q_{i+1})R(q_i) = \pm 1$ , providing a representation  $\varrho$  whose *i*-cycle is positive

Note that the indicated identification is effectively calculable with a simple algorithm. It is easy to show that the points  $q_j$  can be algebraically expressed in terms of the  $z_k$ 's (not involving radicals, when using the Klein model). This is why we can treat  $\mathcal{H}_n^{\pm}$  as a 'rational variety.'

We are going to study the space  $\mathcal{H}_n^{\pm}$  in detail in subsequent articles. In particular, we would like to describe the standard hermitian and complex structures of  $\mathcal{H}_n^{\pm}$  in terms of the  $z_k$ 's. We can introduce a complex structure on  $\mathcal{H}_n^{\pm}$  by taking the  $q_i$ 's,  $i = 2, \ldots, n-2$ , as complex coordinates that vary in the open upper half-disc. Taking  $q_1 = 0$ , we can reconstruct  $q_n \in (-1,0)$  and  $q_{n-1}$  from given  $q_i$ 's,  $i = 2, \ldots, n-2$ . However, it is easy to see that the DT  $E_3$  is not holomorphic with respect to this structure. So, it is not the genuine one. (The DT  $E_3$  belongs to the hyperelliptic mapping class group (see Section 4) which is known to be the group of holomorphic automorphisms of  $\mathcal{H}_n^+$ .)

### 4. Earthquake Group and Hyperlliptic Mapping Class Group

Let  $\tau : \Sigma \to \Sigma/\iota \simeq \mathbb{CP}^1$  be a hyperelliptic Riemann surface of genus g, where  $\iota : \Sigma \to \Sigma$  stands for the hyperelliptic involution of  $\Sigma$ . Put n := 2g + 2 and denote by  $f_1, f_2, \ldots, f_n \in \Sigma$  the fixed points of  $\iota$ . Let  $F \leq \mathcal{L}$  stand for the fundamental group of  $\Sigma = BW/F$  and  $\pi : BW \to \Sigma$ , for the universal covering of  $\Sigma$ .

**4.1. Proposition** [Mac].  $\Sigma \simeq BW/\rho G_n$  for some  $\rho \in \mathcal{R}H_n$ . If  $\rho \in \mathcal{R}H_n$ , then  $\Sigma \simeq BW/\rho G_n$  is hyperelliptic.



For  $q_1, q_2 \in Q$ , the product  $R(q_1)R(q_2)$  induces in  $\Sigma$  the isometry  $\iota^2 = 1$ . This implies  $R(q_1)R(q_2) \in F$ . Choose and fix a point  $p \in BW$  that belongs to no geodesic joining points in Q. Let  $q_i$  denote a point in  $\pi^{-1}(f_i)$  closest to p, i.e., dist $(p, q_i) \leq \text{dist}(p, fq_i)$  for all  $f \in F$ . Note that  $q_i$  is also a point in  $\pi^{-1}(f_i)$  closest to  $R(q_i)p$ , i.e.,  $\operatorname{dist}(R(q_i)p, q_i) \leq \operatorname{dist}(R(q_i)p, fq_i)$  for all  $f \in F$ . This follows from  $f^{R(q_i)} \in F$ .



Clearly,  $\pi$  identifies the  $d_i^1$ 's. Let us show that  $\pi$  identifies in  $\Gamma$  only the points  $d_i^1$ . The point  $\pi p$  is not a fixed point of  $\iota$ , hence,  $\pi$  cannot identify  $d_i^1$  and  $d_j^0 = p$ . Suppose that  $\pi c_i = \pi c_j$  for  $c_i \in \Gamma_i$  and  $c_j \in \Gamma_j$ . This means that  $fc_j = c_i$  for some  $f \in F$ . We assume that  $q_i \neq fq_j$  since i = j and f = 1, otherwise. Let  $d_j^{\delta}$  denote an end of  $\Gamma_j$  closest to  $c_j$  and  $d_i^{\varepsilon}$ , an end of  $\Gamma_i$  closest to  $c_i$ . Since  $p \notin G \prec q_i, fq_j \succ$  by the choice of p, we have  $fq_j \notin G \prec d_i^{\varepsilon}, q_i \succ \ni p$ . Without loss of generality, we can assume that  $\gamma := G[d_i^{\varepsilon}, c_i] \cup G[fc_j, fq_j]$  is less or equal than that of  $G[fd_j^{\delta}, fc_j] \cup G[fc_j, fq_j] = G[fd_j^{\delta}, fq_j]$ . Consequently,  $dis(d_i^{\varepsilon}, fq_j) \leq dist(fd_j^{\delta}, fq_j)$ . Since

$$\operatorname{dist}(fd_j^{\delta}, fq_j) = \operatorname{dist}(d_j^{\delta}, q_j) = \operatorname{dist}(p, q_j) \le \operatorname{dist}(d_i^{\varepsilon}, fq_j),$$

we conclude that  $\operatorname{dist}(d_i^{\varepsilon}, fq_j) = \operatorname{dist}(fd_j^{\delta}, fq_j)$ . Thus,  $\gamma = \operatorname{G}[d_i^{\varepsilon}, fq_j]$  and, in view of  $fq_j \notin \operatorname{G} \prec d_i^{\varepsilon}, q_i \succ$ , the point  $c_i$  has to coincide with  $d_i^{\varepsilon}$ . From  $\operatorname{dist}(fd_j^{\delta}, fq_j) = \operatorname{dist}(d_i^{\varepsilon}, fq_j)$ , we conclude that  $fc_j = c_i$  $= fd_j^{\delta}$ . A contradiction.

The involution  $\iota$  identifies one half of  $\pi\Gamma_i$  with the other since  $R(q_i)$  induces in  $\Sigma$  the isometry  $\iota$ . Those are the only identifications in  $\pi\Gamma$  by  $\iota$ . The curve  $\mu_i := \tau\pi\Gamma_i$  begins with  $\tau\pi p$  and ends with  $\tau f_i$ . The only pairwise intersection between the  $\mu_i$ 's is  $\tau\pi p$ . We can assume that  $\mu_1, \mu_2, \ldots, \mu_n$  are listed in the clockwise sense with respect to the standard orientation of  $\mathbb{CP}^1$ . For every i, choose a small open disc  $D_i \subset$  $\mathbb{CP}^1$  centred at  $\tau f_i$  such that  $D_i$  intersects  $\mu_i$  in some final segment  $s_i \subset \mu_i$  and such that the  $D_i$ 's are pairwise disjoint. Also, choose a simple closed curve  $\omega_i \subset D_i$  that begins with  $p_i \in s_i, p_i \neq \tau f_i$ , and winds once around  $\tau f_i$  in the clockwise sense. Let  $\sigma_i \subset \mu_i$  denote the segment that begins with  $\tau\pi p$  and ends



<sup>4</sup>When a path x ends with the start point of a path y, we denote by  $x \cup y$  their path-product.

with  $p_i$ . The lift  $\eta$  of  $\bigcup (\sigma_i \cup \omega_i \cup \sigma_i^{-1})$  based at  $\pi p$  is contractible in  $\Sigma$  and runs over almost all  $\pi \Gamma$ . Deforming inside the open discs  $\tau^{-1}D_i$  the parts of  $\eta$  that are lifts of the  $\omega_i$ 's, we arrive at a curve  $\gamma \subset \pi\Gamma$ contractible in  $\Sigma$ . Clearly,  $\gamma$  runs over all  $\pi\Gamma$ , once over each  $\pi\Gamma_i$ . For the same reason, every element in  $F = \pi_1(\Sigma, \pi p)$  is represented by a curve included in  $\pi\Gamma$  since the group  $\pi_1(\mathbb{CP}^1 \setminus \{\tau f_1, \tau f_2, \dots, \tau f_n\}, \tau \pi p)$ is generated by the elements  $[\sigma_i \cup \omega_i \cup \sigma_i^{-1}]$ . We assume that  $\gamma_i := \pi \Gamma_i$  begins with  $\pi p$ , passes through  $f_i$ , and ends with  $\iota \pi p$ . The group  $F = \pi_1(\Sigma, \pi p)$  is generated by the elements  $[\gamma_{i-1} \cup \gamma_i^{-1}]$  (the indices are modulo n). Therefore, the elements  $\lambda_{2i} := \gamma_1 \cup \gamma_2^{-1} \cup \cdots \cup \gamma_{2i-1} \cup \gamma_{2i}^{-1}$  and  $\lambda_{2i+1} := \gamma_2 \cup \gamma_3^{-1} \cup \cdots \cup \gamma_{2i} \cup \gamma_{2i+1}^{-1}$ ,  $i = 1, 2, \ldots, \frac{n}{2}$ , also generate F. Note that  $\lambda_n = \gamma$  is contractible in  $\Sigma$  by construction.

The lift  $\Lambda_{2i}$  of  $\lambda_{2i}$  based at p is formed by  $\Gamma_1, \Gamma'_2, \ldots, \Gamma'_{2i}$ , conjugates of  $\Gamma_i$ 's. Let  $q'_i \in Q$  denote the middle point of  $\Gamma'_i$ . Then  $R(q'_{2i}) \dots R(q'_2) R(q_1) p$  is the end of  $\Lambda_{2i}$ . Hence,  $[\lambda_{2i}] = R(q'_{2i}) \dots R(q'_2) R(q_1)$ . The lift  $\Lambda_{2i+1}$  of  $\lambda_{2i+1}$  based at p is formed by  $\Gamma_2, \Gamma''_3, \ldots, \Gamma''_{2i+1}$ , conjugates of  $\Gamma_i$ 's. Let  $q''_i \in Q$  denote the middle point of  $\Gamma''_i$ . Then  $R(q''_{2i+1}) \dots R(q''_3) R(q_2) p$  is the

end of  $\Lambda_{2i+1}$ . Hence,  $[\lambda_{2i+1}] = R(q''_{2i+1}) \dots R(q''_3) R(q_2)$ . We put  $q_2'' := q_2$  and  $q_1' := q_1$ .

Let us show that  $q''_i = R(q_1)q'_i$ . Note that if some  $f \in F$ maps a point in  $\Gamma'_i$  to a point in  $\Gamma''_i$ , then  $\Gamma''_i = f\Gamma'_i$ . Since  $R(q_1)$  maps the beginning of  $\Gamma'_2$  to p, we conclude that 



maps the end of  $\Gamma'_{j}$  (which is the beginning of  $\Gamma'_{j+1}$ ) to the end of  $\Gamma''_{j}$  (which is the beginning of  $\Gamma''_{j+1}$ ), we conclude that  $R(q_1)R(q'_{j+1})$  maps the end of  $\Gamma'_{j+1}$  to the beginning of  $\Gamma''_{j+1}$ . Hence,  $\Gamma''_{j+1} = \Gamma''_{j+1}$  $R(q_1)R(q'_{j+1})\Gamma'_{j+1} = R(q_1)\Gamma'_{j+1}.$ 

Consequently,  $R(q'_n) \dots R(q'_2) R(q'_1) = 1$ , which generates a representation  $\varrho : H_n \to \mathcal{L}$  such that  $\rho G_n = F$ . Being  $G_n$  and F the fundamental groups of Riemann surfaces of the same genus,  $\rho|_{G_n}$  is an isomorphism. So is  $\rho$ .

The converse can be readily shown with the help of the fundamental polygon for  $\rho H_n$  constructed in the proof of Proposition 3.14

The formal multiplicative group generated by n copies  $\{E_i(t) \mid t \in \mathbb{R}\}, i = 1, 2, ..., n$ , of  $(\mathbb{R}, +)$  is denoted by  $\mathcal{E}_n$  and called *earthquake group*. We distinguish the parts  $\mathcal{R}^+H_n$  and  $\mathcal{R}^-H_n$  of  $\mathcal{R}H_n$  related to the sign of the area of a representation. Due to Remark 3.8,  $\mathcal{E}_n$  acts from the right by means of SEs on  $\mathcal{R}^{\pm}H_n$  and, hence, on  $\mathcal{H}_n^{\pm}$ . Later (see Remark 5.24) we will extend

this action to  $\mathcal{R}G_n$  and to  $\mathcal{T}_n$ .





**4.3. Remark.** Let  $p, q \in BW$  be distinct and let G be a full geodesic different from  $G \prec p, q \succ$  and intersecting  $G \prec p, q \succ$  in some point in BW. Then, on any side from  $G \prec p, q \succ$ , there exists some  $d \in$  $G \cap BW$  such that R(d)R(q)R(p) is hyperbolic. Indeed, the points  $d \in$ B W making R(d)R(q)R(p) parabolic form two curves (hypercycles) equidistant from  $G \prec p, q \succ$ . The isometry R(d)R(q)R(p) is hyperbolic

exactly when d is outside the band limited by these curves.

**4.4. Lemma.**  $\mathcal{E}_5$  acts transitively on  $\mathcal{R}^{\pm}H_5$ .

G

 $\langle p, q \rangle$ 

**Proof.** Let  $\varrho, \varrho' \in \mathcal{R}^+ H_5$ , that is, Area  $\varrho$  = Area  $\varrho' = \pi$ . For suitable  $q_i, q'_i \in \mathcal{B}W$ , we have  $\varrho(r_i) = R(q_i)$  and  $\varrho'(r_i) = R(q'_i)$ .

By Lemma 3.11, the points  $q_1, q_2, q_3$  are on the side of the normal vector to  $G \prec q_4, q_5 \succ$ . Let G be a geodesic passing through  $q'_1$  and intersecting  $G \prec q_4, q_5 \succ$  in some point in BW. By Remark 4.3,  $R(q''_1)R(q_5)R(q_4)$  is hyperbolic for some  $q''_1 \in$  $G \cap BW$  on the mentioned side. Hence,  $R(q''_1)R(q_5)R(q_4) =$ 

 $R(q_2'')R(q_3'')$  for some  $q_2'', q_3'' \in BW$ , which provides some  $\varrho'' \in \mathcal{R}H_5$  by Corollary 3.16. Since  $q_1''$  is on the side of the normal vector to  $G \prec q_4, q_5 \succ$ , it follows that Area  $\varrho'' = \pi$  by Lemma 3.11. By Lemma 4.2, after applying to  $\varrho$  a finite number of SEs, we can assume that  $q_i = q_i'', i = 1, 2, 3$ . Some SE involving  $q_4, q_5$  puts  $q_5$  into  $G \prec q_1, q_1' \succ = G$ . Now, some SE involving  $q_5, q_1$  provides  $q_1 = q_1'$ .

 $E_{3}(t) = q_{1}^{\prime} = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq q_{2}^{\prime}, \text{ after applying (if necessary) some SE} \\ E_{3}(t) = q_{1}^{\prime} \neq$ 



**4.5. Theorem.**  $\mathcal{E}_n$  acts transitively on  $\mathcal{R}^{\pm}H_n$ .

**Proof.** Let  $\varrho, \varrho' \in \mathcal{R}^+ H_n$ , i.e., Area  $\varrho = \operatorname{Area} \varrho' = (n-4)\pi$ . For suitable  $q_i, q'_i \in \mathcal{B}W$ , we have  $\varrho(r_i) = R(q_i)$  and  $\varrho'(r_i) = R(q'_i)$ .



The isometry  $R(q_3)R(q_2)R(q_1)$  is hyperbolic because  $\varrho \in \mathcal{R}H_n$ . Indeed, if it is parabolic,  $q_3$  belongs to the hypercycle  $H = \{q \in BW \mid R(q)R(q_2)R(q_1) \text{ is parabolic}\}$ . Applying a 'small' SE involving  $q_4, q_5$  if necessary, we can assume  $G \prec q_3, q_4 \succ$  to be transversal to H at  $q_3$ . Now a suitable SE involving  $q_3, q_4$  provides an elliptic  $R(q_3)R(q_2)R(q_1)$  (see Remark 4.3). A contradiction.

 $q_2''$ 

 $q_3$ 

Hence, there exist  $b, d \in BW$  such that R(d)R(b) $R(q_3)R(q_2)R(q_1) = 1$ , which generates a representation  $\rho_0: H_5 \to \mathcal{L}$ . The relation  $R(q_n) \dots R(q_5)R(q_4)R(b)$ R(d) = 1 generates a representation  $\rho_1: H_{n-1} \to \mathcal{L}$ . Take for  $p_0 \in SW$  a fixed point of  $R(b)R(d) = R(q_3)$  $R(q_2)R(q_1)$ . By Remark 2.3,



Area  $\rho = Area(p_0, p_1, p_2, p_3, p_4, \dots, p_{n-1}) = Area(p_0, p_1, p_2, p_3) + Area(p_3, p_4, \dots, p_{n-1}, p_0) = Area(p_0, p_1, p_2, p_3, p_4, \dots, p_{n-1}, p_0) = Area(p_0, p_1, p_2, p_3, p_4, \dots, p_{n-1}) = Area(p_0, p_1, p_2, p_3) + Area(p_0, p_1, p_2, p_3) + Area(p_0, p_1, p_2, p_3) + Area(p_0, p_1, p_2, p_3) = Area(p_0, p_1, p_2, p_3) + Area(p_0, p_1, p_2, p_3) + Area(p_0, p_1, p_2, p_3) = Area(p_0, p_1, p_2, p_3) + Area(p_0, p_1, p_2, p_3) + Area(p_0, p_1, p_2, p_3) + Area(p_0, p_1, p_2, p_3) = Area(p_0, p_1, p_2, p_3) + Area(p_0, p_1, p_2) + Area(p_0, p_2$ 

$$= \operatorname{Area}(p_0, p_1, p_2) + \operatorname{Area}(p_3, p_4, \dots, p_{n-1}) = \operatorname{Area} \varrho_0 + \operatorname{Area} \varrho_1$$

due to  $p_0 = p_3$ . By Remark 3.4, Area  $\rho_0 = \pi$  and Area  $\rho_1 = (n-5)\pi$ . By Theorem 3.15,  $\rho_0 \in \mathcal{R}^+ H_5$ and  $\rho_1 \in \mathcal{R}^+ H_{n-1}$ .

We are going to express every SE of  $\rho_1$  in terms of suitable SEs of  $\rho$  and an SE involving d, b (the latter is simply a rechoice of b and d). The SEs of  $\rho_1$  involving  $q_{i-1}, q_i, i = 5, 6, \ldots, n$ , are in fact some SEs of  $\rho$ . All we need is to execute the SEs of  $\rho_1$  involving the pairs  $b, q_4$  and  $q_n, d$ . By symmetry, we deal only with the first one.

By Remark 4.3, we can find  $q_3'' \in G \prec b, q_4 \succ \cap BW$  on the side of the normal vector to  $G \prec b, d \succ$  such that  $R(d)R(b)R(q_3'')$  is hyperbolic and, hence,  $R(d)R(b)R(q_3'') = R(q_1'')R(q_2'')$  for some  $q_1'', q_2'' \in BW$ . As in the proof of Lemma 4.4 (that is, by Corollary 3.16, Lemma 3.11, and Lemma 4.2), we obtain



 $q_i = q_i'', i = 1, 2, 3$ , after a few SEs involving  $q_1, q_2$  and  $q_2, q_3$ . Now, the point  $q_3$  is in  $G \prec b, q_4 \succ$ . Thus, in order to execute a given SE of  $\rho_1$  involving  $b, q_4$ , we can simply apply a suitable SE of  $\rho$  involving  $q_3, q_4$ .

In the same manner, we can 'cut'  $\varrho'$  into  $\varrho'_0$  and  $\varrho'_1$  by means of appropriate  $b', d' \in BW$ . By induction on n, we can assume that  $\varrho_1 = \varrho'_1$ . It remains to apply Lemma 4.2

The relation  $R(q_5)R(q_4)R(q_3)R(q_2)R(q_1) = 1$  suits [Ana1, Conjecture 1.1]. We strongly believe that this conjecture is valid for the Poincaré disc.

Let  $\varrho \in \mathcal{R}^{\pm}H_n$ ,  $\varrho(r_i) = R(q_i)$ ,  $q_i \in BW$ . Following the proof of Proposition 3.14, associate to  $\varrho$  a standard fundamental polygon  $P_{\varrho}$  for  $\varrho H_n$  with vertices  $p_1, p_2, \ldots, p_{n-1}$  by taking  $p_0 := q_n$  and  $p_i := R(q_i)p_{i-1}$ ,  $i = 1, 2, \ldots, n-1$ . The polygon  $P_{\varrho}$  is convex and the sum of its interior angles equals  $\pm \pi$ . In order to describe  $\varrho$ , it suffices to mark the vertex  $q_n$  and the middle points  $q_1, q_2, \ldots, q_{n-1}$  of the edges of  $P_{\varrho}$ . Clearly,  $p_{n-1} = p_n = p_0 = q_n$ . We alter our convention concerning the notation of the vertices of  $P_{\varrho}$ : the indices of the vertices  $p_1, p_2, \ldots, p_{n-1}$  are modulo n-1. According to the new convention,  $p_0 = p_{n-1}$  and  $p_n = p_1$ .

We are going to study the group Aut  $H_n$ . Fix some discrete subgroup  $H_n \leq \mathcal{L}$  and consider the representations  $\varrho \in \mathcal{R}^{\pm}H_n$  such that  $\varrho H_n = H_n$ . The group Aut  $H_n$  acts from the right on these representations. In particular, every DT can be regarded as an element in Aut  $H_n$ : the automorphism corresponding to  $E_i$  is given by  $E_i r_{i-1} = r_i$ ,  $E_i r_i = r_i r_{i-1} r_i$ , and  $E_i r_j = r_j$  for  $j \notin \{i-1, i\}$ .

Denote by  $\operatorname{Aut}^+ H_n$  the subgroup in  $\operatorname{Aut} H_n$  generated by all the  $E_i$ 's. In addition, there is an automorphism  $J \in \operatorname{Aut} H_n$  given by  $Jr_i := r_{n-i}$  (cf. Remark 3.3). Obviously,  $J^2 = 1$ .

Define  $S \in \operatorname{Aut} H_n$  as  $Sr_i := r_{i+1}$  for all *i*. It is immediate that  $E_i^S = E_{i+1}$ . Looking at the polygon  $P_{\rho}$ ,



we can see that  $S = E_1 E_2 \dots E_{n-1} \in \operatorname{Aut}^+ H_n$ . Also, the vertices  $p'_i$  of the standard polygon  $P_{\varrho'}$  for the representation  $\varrho' := \varrho S E_n = \varrho E_1 E_2 \dots E_n$  are given by  $p'_i = p_{i+1}$ , where the  $p_i$ 's stand for the vertices of  $P_{\varrho}$ . Therefore, acting by  $\operatorname{Aut}^+ H_n$  on the representations, we can shift the indices both of the vertices and of the marks of the middle points of the edges of  $P_{\varrho}$ .

Denote by  $I_h \in \text{Aut } H_n$  the conjugation by  $h \in H_n$ . Clearly,  $I_h^A = I_{Ah}$  for all  $A \in \text{Aut } H_n$ . Looking at the polygon  $P_{\varrho} \cup R(q_1)P_{\varrho}$ ,





we can see that  $I_{r_1} = E_2^{-1} E_3^{-1} \dots E_{n-1}^{-1} (SE_n)^{-1}$ . Hence,

$$I_{r_1} = E_1 E_2 \dots E_{n-1} E_n E_{n-1} \dots E_3 E_2.$$

It follows from  $I_{r_i}^S = I_{r_{i+1}}$  that  $I_{H_n} \subset \operatorname{Aut}^+ H_n$ .

**4.6. Theorem.** The group  $\operatorname{Aut} H_n$  is generated by J and by the normal subgroup  $\operatorname{Aut}^+ H_n$  of index 2.

**Proof.** Given  $\rho, \rho' \in \mathcal{R}H_n$  such that  $\rho H_n = \rho' H_n = H_n \leq \mathcal{L}$ , we can assume that  $\rho, \rho' \in \mathcal{R}^+ H_n$  acting by J if necessary. Hence, the vertices  $p_1, p_2, \ldots, p_{n-1}$  and  $p'_1, p'_2, \ldots, p'_{n-1}$  of the convex polygons  $P := P_{\rho}$  and  $P' := P_{\rho'}$  are listed in  $\partial P$  and in  $\partial P'$  in the counterclockwise sense. It suffices to show that, acting by Aut<sup>+</sup>  $H_n$  on both  $\rho$  and  $\rho'$ , we can make them coincide.

Note that the 'DT E involving  $q_{n-1}, q_1$ ' is expressible in terms of  $E_i$ 's :  $E = E_1 E_n E_1^{-1}$ .



Dealing with the representations  $\rho$  and  $\rho'$  modulo the action by  $\operatorname{Aut}^+ H_n$  and taking into account that the automorphism  $SE_n \in \operatorname{Aut}^+ H_n$  shifts the indices of the vertices and of the marks of the middle points of the edges, we can actually think of the representations as their standard counterclockwise-oriented polygons P and P', but with unmarked vertices and middle points. As shown above, we are able to execute any DT that involves the middle points of adjacent edges of the unmarked polygons, acting by  $\operatorname{Aut}^+ H_n$ . Also, the inclusion  $I_{H_n} \subset \operatorname{Aut}^+ H_n$  allows us to change P and P' by their conjugates.

Let  $\varrho(r_i) = R(q_i)$  and  $\varrho'(r_i) = R(q'_i)$ ,  $q_i, q'_i \in BW$ . Every involution  $r \in H_n$  is determined by its fixed point q and induces in  $\Sigma$  the hyperelliptic involution  $\iota$ . In particular,  $\pi q = f_i$  for a suitable  $i = 1, \ldots, n$  (see the proof of Proposition 4.1). Two involutions r, r' are conjugated in  $H_n$  (equivalently, by an element in  $G_n$ ) if and only if their fixed points q, q' satisfy the relation  $\pi q = \pi q'$ . Hence, the  $R(q_i)$ 's list all conjugate classes of the involutions in  $H_n$ . Obviously, the  $R(q'_i)$ 's represent different conjugate classes. Therefore, every  $q_i$  is a conjugate of some  $q'_i$ 

and vice versa. The edge  $e_i$  of P has the ends  $p_{i-1}, p_i$  and the middle point  $q_i, i = 1, 2, ..., n-1$ . Similarly, we introduce the edges  $e'_i$  of P'. If  $e_i$  is a conjugate of some  $e'_j$ , we say that  $e_i$  and  $e'_j$  are good. Note that  $e_i$  cannot be a conjugate of two  $e'_j$ 's at the same

time. Let k denote the number of good  $e_i$ 's. We proceed by induction on  $k \ge 0$ .





Suppose that  $e_{i-1}$  is a good edge and that  $e_i$  is not (the indices are modulo n-1). Apply to  $\rho$  the DT that involves  $q_{i-1}, q_i$ . This does not alter the edges  $e_j$ ,  $j \neq i-1, i$ . The new  $e_i$  is a conjugate of the old  $e_{i-1}$  and, hence, is good. We can assume that the new  $e_{i-1}$  is bad since, otherwise, we are done by induction on k. So, we are able to permute the types of any two adjacent edges, one good and the other bad, finally reaching the situation where the good edges of P (and of P') form a sequence in  $\partial P$  (and in  $\partial P'$ ). Moreover, we can assume that the first edge e in the sequence in  $\partial P$  and the first edge e' in the sequence in  $\partial P'$  are conjugated (both sequences are read in the counterclockwise sense). By means of DT's, we can change P' by any of its conjugates. Also, by means of DT's, we can shift

the marks of the vertices and of the middle points in P and in P'. So, we assume that P and P' are on the same side from  $e_1 = e = e' = e'_1$ . (If k = 0, we assume only that  $p_{n-1} = p'_{n-1}$ .)

The fact that conjugated points in P are necessarily in  $\partial P$  and the same fact concerning P' imply that  $e_2 = e'_2$ . In this way, we can show that  $e_i = e'_i$ ,  $i = 1, 2, \ldots, k$ . Denote by  $s \subset \partial P$  the segment formed by all the good edges of P. Clearly,  $s \subset \partial P'$  is the segment formed by all the good edges of P'. (If k = 0, we have  $s = p_{n-1} = p'_{n-1}$ .)

Suppose that  $k \neq n$ . We will study how the conjugates of bad edges of P' intersect the polygon P. Let  $b' \neq b''$  be such edges. Looking at the tessellation of BW related to P', we see that

(4.6.1) The edges b' and b'' can intersect only in points that are conjugates of  $p'_i$ 's, i.e., conjugates of  $p_{n-1} = p'_{n-1} = q_n$ . Therefore, b' and b'' do not intersect in the interior of P.

(4.6.2) If b' intersects the interior of P, it does not intersect the interior of s. Otherwise, b' enters the interior of P right after its intersection  $s \cap b'$  since P is convex. Hence, it enters the interior of P'. A contradiction.

(4.6.3) The edge b' cannot pass through two middle points of edges of P because the conjugates of middle points of edges of P coincide with those for P'.

(4.6.4) For every middle point  $q_i$  of a bad edge of P, there exists a unique conjugate b' of an edge of P', necessarily bad, that passes through  $q_i$  and, therefore, through the interior of P.

We say that the intersection of  $\partial P$  with some conjugate of a bad edge of P' is proper if this intersection is different from the vertices of P and from the middle points of the edges of P. It is immediate that the number of proper intersections is the same in each half of a bad edge of P. Let l denote the total number of proper intersections in  $\partial P$ . We proceed by induction on l.

Let  $q_i$  be the middle point of a bad edge of P and let b' be a conjugate of a bad edge of P' that passes through  $q_i$  and through the interior of P according to (4.6.4). By (4.6.2), b' cuts P into two closed parts and s is entirely included in one of them. If the other part contains a single middle point of an edge of P, namely  $q_i$ , we arrive at the desired situation to be studied later. Otherwise, by (4.6.4), we take a conjugate b'' of a bad edge of P' passing through the extra middle point  $q_i$  and through the interior of P. Note that  $q_i \notin b'$  by (4.6.3).



By (4.6.1), b' and b'' do not intersect in the interior of P. Now we take b'' in place of b' and so on ... Finally, we arrive at the situation (or at the one symmetric to it) where  $b' \cap \partial P = \{q_i, q\}$  and  $q \in G(p_i, q_{i+1})$ .

In this situation, we execute the DT  $E_{i+1}$ . By induction on k, we can assume that the new  $e_i$  is bad. We will show that the new l is strictly less than the old one.

Note that  $E_{i+1}$  removes from P the triangle  $\Delta(p_i, q_{i+1}, p_{i-1})$  and glue to P the triangle  $\Delta(p_{i+1}, q_{i+1}, R(q_{i+1})p_{i-1})$ . Since these triangles are conjugated, it suffices to show that the number of proper intersections included in  $G(q_{i+1}, p_{i-1})$  is strictly less than that in  $G(p_i, q_{i+1})$ . So, we consider only those parts of conjugates of bad edges of P' that pass via the interior of  $\Delta(p_{i-1}, p_i, q_{i+1})$ .

The following types and quantities of such parts are possible:



•  $l_1$  parts whose ends are a point in  $G(p_i, q_i)$ and a point in  $G(p_i, q)$ ,

• 1 part with ends  $q_i$  and  $q_j$ ,

•  $l_2$  parts whose ends are a point in  $G(q_i, p_{i-1})$ and a point in  $G(q, q_{i+1})$ ,

•  $l_3 = 0, 1$  parts whose ends are a point in  $G(q_i, p_{i-1})$  and  $q_{i+1}$ ,

•  $l_4$  parts whose ends are a point in  $G(q_i, p_{i-1})$ and a point in  $G(q_{i+1}, p_{i-1})$ ,

•  $l_5$  parts whose ends are  $p_{i-1}$  and a point in  $G(q, q_{i+1})$ ,

•  $l_6$  parts whose ends are a point in  $G(q, q_{i+1})$ and a point in  $G(q_{i+1}, p_{i-1})$ .

Since the number of proper intersections is the same in each half of  $e_i = G(p_i, p_{i-1})$ , we obtain  $l_1 = l_2 + l_3 + l_4$ . The number of proper intersections included in  $G(p_i, q_{i+1})$  equals  $l_1 + 1 + l_2 + l_5 + l_6$ . The number of such intersections related to  $G(q_{i+1}, p_{i-1})$  is equal to  $l_6 + l_4 \equiv$ 

A straightforward verification shows that  $E_i^J = E_{n+1-i}^{-1}$ . Denote

$$S := E_1 E_2 \dots E_{n-1}, \qquad \widehat{S} := E_{n-1} \dots E_2 E_1, \qquad I := I_{r_n}$$

It follows from  $I_{r_i}^S = I_{r_{i+1}}$ ,  $E_i^S = E_{i+1}$ , and  $I_{r_1} = E_1 E_2 \dots E_{n-1} E_n E_{n-1} \dots E_3 E_2$  that  $I_{r_1} = S \widehat{S}^S$  and  $I = I_{r_n} = I_{r_1}^{S^{-1}} = S \widehat{S}$ . Hence,  $S \widehat{S} S \widehat{S} = 1$ . The relations  $r_n \dots r_2 r_1 = 1$ ,  $I_{r_i}^S = I_{r_{i+1}}$ , and  $S^n = 1$  imply the relation  $I^{S^n} \dots I^{S^2} I^S = 1$  which can be rewritten as  $(S^{-1}I)^n = 1$ , i.e., as  $\widehat{S}^n = 1$ . It is immediate that  $E_i E_j = E_j E_i$  if  $|i - j| \ge 2$ . As is easy to see, the relation



 $E_i E_{i+1} E_i = E_{i+1} E_i E_{i+1}$  is valid for all *i*. It is possible to conclude from [Stu] that the defining relations of  $\operatorname{Aut}^+ H_n$  are (the indices are modulo n) :

$$S = E_1 E_2 \dots E_{n-1}, \qquad \widehat{S} = E_{n-1} \dots E_2 E_1, \qquad S^n = 1, \qquad \widehat{S}^n = 1, \qquad S\widehat{S}S\widehat{S} = 1,$$

$$E_i^S = E_{i+1}, \qquad E_i E_{i+1} E_i = E_{i+1} E_i E_{i+1}, \qquad E_i E_j = E_j E_i \text{ if } |i-j| \ge 2$$

(cf. [Bir]). The additional defining relations of Aut  $H_n$  are  $E_i^J = E_{n+1-i}^{-1}$  and  $J^2 = 1$ .

### 5. W. M. Goldman's Theorem

Let  $n \geq 6$  be even. Recall that  $G_n$  denotes the fully characteristic torsion-free subgroup of index 2 in  $H_n$  constituted by the words of even length in the  $r_i$ 's. By Proposition 4.1,  $G_n$  is the fundamental group of a closed orientable Riemann surface of genus  $\frac{n}{2} - 1$ . In this section, we will prove the

**5.1. Theorem** [Gol1, Corollary C]. Let  $\varrho: G_n \to \mathcal{L}$  be a representation. Then  $\varrho \in \mathcal{R}G_n$  if and only if Area  $\rho = \pm 2(n-4)\pi$ .

We are going to explore the ideas developed in the hyperelliptic case. A given representation  $\rho$ :  $G_n \to \mathcal{L}$  defines an action of  $G_n$  on  $\overline{B}W$ . We write gp instead of  $\varrho(g)p$  for all  $g \in G_n$  and  $p \in \overline{B}W$ . Working in terms of the  $r_i$ 's, we are allowed to apply  $\rho$  to any expression of even length in  $r_i$ 's. Hence, the expression  $r_i r_j p$  makes sense, whereas  $r_i p$  does not.

We will deal with a 'fundamental polygon' Q for  $\rho G_n$  that mimics the duplicated fundamental polygon  $P_n$  for the hyperelliptic case, namely,  $Q := P_n \cup \varrho(r_n)P_n$  (see the last picture in the proof of Theorem 3.15). In the hyperelliptic case, the polygon  $P_n$  is generated by the choice of  $p = p_n \in BW$  because it has a single cycle of vertices. The point  $p_{n-1} \in BW$  is given by  $p_{n-1} = \varrho(r_n)p_n$ . Since, in the nonhyperelliptic case, we have no reflection  $\varrho(r_n)$  available and the polygon Q should have two cycles of vertices, we choose two points  $p, q \in \overline{B}W$  that are intended to respectively play the roles of  $p_n, p_{n-1}$ . In this way, for suitable  $w_i \in G_n$ ,



the even vertices of the polygon Q have the form  $w_{2i}p$  and the odd ones, the form  $w_{2i+1}q$ .

The proof of Theorem 5.1 is 'almost' the same as that of Theorem 3.15. We simply adapt the arguments of the latter to the nonhyperelliptic case by avoiding the use of the elements from  $H_n \setminus G_n$ . For instance, Corollary 5.8, Remark 5.9, Remark 5.10, Lemma 5.12, and Lemma 5.13 that we prove below are analogs of the following hyperelliptic assertions: Lemma 3.2, Remark 3.3, Remark 3.4, Lemma 3.9, and Proposition 3.13.

**5.2.** Notation. Denote by S, I, and J the automorphisms of  $H_n$  given by the rules  $Sr_i = r_{i+1}$ ,  $I: h \mapsto h^{r_n}$ , and  $Jr_i := r_{n-i}$ . The same symbols denote the induced automorphisms of  $G_n$ . For  $0 \leq r_n$  $i \leq n-1$ , denote  $v_i := r_i \dots r_2 r_1$  and regard the indices of the  $v_i$ 's modulo n. So,  $v_0 = v_n = 1$ . For  $0 \leq i \leq n-2$ , introduce

$$w_i := v_i$$
 if i is even,  $w_i := v_i r_n$  if i is odd,  $w_{i+n-1} := I(w_i)$ 

and regard the indices of the  $w_i$ 's modulo 2n-2. Clearly,  $w_0 = w_{n-1} = w_{2n-2} = 1$ . Note that  $w_{i+n-1} = I(w_i)$  for all i. As is easy to see, the formula  $w_i = v_i r_n$  works for all odd i such that  $1 \le i \le n-1.$ 

The elementary properties of the  $w_i$ 's that we use in what follows are gathered in the

**5.3. Lemma.** (1)  $w_{i+n-1}^{-1}w_{i+n} = w_{i+1}^{-1}w_i$  for all i. (2)  $J(w_i) = w_{n-1-i}$  for all i. (3)  $S(w_i)w_1 = w_{i+1}$  for all even i such that  $0 \le i \le n-2$ . (4)  $S(w_i) = w_{i+1}$  for all odd i such that  $1 \le i \le n-3$ . (5)  $S(w_i) = w_1w_{i+1}$  for all odd i such that  $n-1 \le i \le 2n-3$ . (6)  $S(w_i)w_1 = w_1w_{i+1}$  for all even i such that  $n \le i \le 2n-4$ . (7)  $r_nr_iw_{i-1} = w_{i+n-1}$  and  $r_nr_iw_i = w_{i+n-2}$  for all  $1 \le i \le n-1$ . (8)  $r_nr_{i+1}w_i = w_{i+n}$ ,  $r_nr_{i+1}w_{i+1} = w_{i+n-1}$ ,  $r_{i+1}r_nw_{i+n-1} = w_{i+1}$ , and  $r_{i+1}r_nw_{i+n} = w_i$  for all  $2 \le i \le n$ .

**Proof.** (1) Let  $0 \le i \le n-2$ . If *i* is even, we have

$$w_{i+n-1}^{-1}w_{i+n} = (r_n w_i r_n)^{-1} r_n w_{i+1} r_n = r_n v_i^{-1} v_{i+1} = r_n v_i^{-1} r_{i+1} v_i = w_{i+1}^{-1} w_i.$$

If i is odd, we have

$$w_{i+n-1}^{-1}w_{i+n} = (r_n w_i r_n)^{-1} r_n w_{i+1} r_n = v_i^{-1} v_{i+1} r_n = v_i^{-1} r_{i+1} v_i r_n = w_{i+1}^{-1} w_i.$$

For  $n-1 \le i \le 2n-3$ , the fact follows by taking inverses in the equalities that are already established for  $0 \le i \le n-2$ .

(2) Let  $0 \le i \le n-2$ . It follows from the relation  $r_n \ldots r_2 r_1 = 1$  that

$$J(w_i) = J(v_i) = r_{n-i} \dots r_{n-2} r_{n-1} = v_{n-1-i} r_n = w_{n-1-i}$$

if i is even and that

$$J(w_i) = J(v_i r_n) = r_{n-i} \dots r_{n-2} r_{n-1} r_n = v_{n-1-i} = w_{n-1-i}$$

if i is odd. Now, for  $n-1 \le i \le 2n-3$ , we obtain

$$J(w_i) = J(r_n w_{i-n+1} r_n) = r_n w_{n-1-i+n-1} r_n = r_n w_{2n-2-i} r_n = w_{3n-3-i} = w_{n-1-i}.$$

(3) The case of i = 0 is immediate. For  $2 \le i \le n-2$ , we have  $S(w_i)w_1 = S(v_i)r_1r_n = v_{i+1}r_n = w_{i+1}$ . (4)  $S(w_i) = S(v_ir_n) = v_{i+1} = w_{i+1}$ .

$$(5) S(w_i) = S(r_n w_{i-n+1} r_n) = S(r_n v_{i-n+1} r_n) = r_1 v_{i-n+2} = r_1 w_{i-n+2} r_n = r_1 r_n w_{i+1} = w_1 w_{i+1}.$$

(6)  $S(w_i)w_1 = S(r_nw_{i-n+1}r_n)w_1 = S(r_nv_{i-n+1})r_1r_n = r_1v_{i-n+2}r_n = r_1w_{i-n+2}r_n = w_1w_{i+1}.$ 

(7) As is easy to see,  $r_i w_{i-1} = w_i r_n$  and  $r_i w_i = w_{i-1} r_n$  for all  $1 \le i \le n-2$ . Therefore,  $r_n r_i w_{i-1} = r_n w_i r_n = I(w_i) = w_{i+n-1}$  and  $r_n r_i w_i = r_n w_{i-1} r_n = I(w_{i-1}) = w_{i+n-2}$ . For i = n-1, we have  $r_n r_{n-1} w_{n-2} = 1 = w_{2n-2}$  and  $r_n r_{n-1} w_{n-1} = r_n r_{n-1} = r_n v_{n-2} r_n = I(w_{n-2}) = w_{2n-3}$  since  $r_n r_{n-1} \dots r_2 r_1 = 1$ ,  $r_{n-1} = r_{n-2} \dots r_2 r_1 r_n$ , and  $w_{n-1} = 1$ .

(8) The first two equalities are in fact shown in (7). The last two equalities follow immediately from the first two  $\blacksquare$ 

Given  $p, q \in \overline{B}W$ , define

Area<sub>n</sub>
$$(p,q;\varrho) :=$$
 Area $(w_0p, w_1q, \dots, w_{n-2}p, w_{n-1}q, \dots, w_{2n-4}p, w_{2n-3}q)$ 

Area<sub>*i*+1</sub>
$$(p,q;\varrho) := \text{Area}_i(p,q;\varrho S).$$

**5.4. Remark.** The relation  $w_{i+n-1} = r_n w_i r_n$  valid for all *i* implies  $\operatorname{Area}_n(p,q;\varrho) = \operatorname{Area}_n(q,p;\varrho I)$ .

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**5.5. Lemma.** Area<sub>n</sub> $(p,q;\varrho) =$ Area<sub>1</sub> $(w_1q,p;\varrho)$ .

**Proof.** By definition,

 $\operatorname{Area}_{1}(w_{1}q, p; \varrho) = \operatorname{Area}\left(S(w_{0})w_{1}q, S(w_{1})p, \dots, S(w_{n-2})w_{1}q, S(w_{n-1})p, \dots, S(w_{2n-4})w_{1}q, S(w_{2n-3})p\right).$ 

By Lemma 5.3 (3-6),

$$\operatorname{Area}_{1}(w_{1}q, p; \varrho) = \operatorname{Area}(w_{1}q, w_{2}p, \dots, w_{n-1}q, w_{1}w_{n}p, \dots, w_{1}w_{2n-3}q, w_{1}w_{2n-2}p).$$

Taking into account that  $w_1w_n = w_1r_nw_1r_n = 1$  and that  $w_0 = w_{n-1} = w_{2n-2} = 1$ , by Remark 2.3, we obtain

 $Area_1(w_1q, p; \varrho) = Area(w_1q, w_2p, \dots, w_{n-1}q, w_1w_np) + Area(w_1w_np, \dots, w_1w_{2n-3}q, w_1w_{2n-2}p, w_1q) = 0$ 

$$= \operatorname{Area}(w_0 p, w_1 q, w_2 p, \dots, w_{n-1} q) + \operatorname{Area}(w_n p, \dots, w_{2n-2} p, q) =$$
  
= 
$$\operatorname{Area}(w_0 p, w_1 q, \dots, w_{n-2} p, w_{n-1} q) + \operatorname{Area}(w_{n-1} q, w_n p, w_{n+1} q, \dots, w_{2n-3} q, w_0 p) =$$
  
= 
$$\operatorname{Area}(w_0 p, w_1 q, \dots, w_{n-2} p, w_{n-1} q, \dots, w_{2n-3} q) = \operatorname{Area}_n(p, q; \varrho) \blacksquare$$

**5.6. Lemma.** Area<sub>n</sub> $(p,q;\varrho)$  is independent of the choice of p and q.

**Proof.** We will show the independence of q. (The independence of p can be shown in a similar way.) Taking c = p in (2.2), we obtain

$$\operatorname{Area}_n(p,q;\varrho) = \sum_{\operatorname{even}\,i} \operatorname{Area}\Delta(p,w_ip,w_{i+1}q) + \sum_{\operatorname{odd}\,i} \operatorname{Area}\Delta(p,w_iq,w_{i+1}p) =$$

(5.7) 
$$= \sum_{\text{even } i} \operatorname{Area} \Delta(q, w_{i+1}^{-1}p, w_{i+1}^{-1}w_ip) + \sum_{\text{odd } i} \operatorname{Area} \Delta(q, w_i^{-1}w_{i+1}p, w_i^{-1}p).$$

Let us show that (5.7) is the area (calculated with respect to the centre q) related to some closed piecewise geodesic path C independent of the choice of q. Denote by  $\xrightarrow{i}$  the side opposite to the vertex q of the *i*th triangle involved in (5.7). This side is oriented with respect to the orientation of the *i*th triangle. The consecutive vertices of C are described by the following list:

$$w_{1}^{-1}p \xrightarrow{0} w_{1}^{-1}w_{0}p = w_{n-1}^{-1}w_{n}p \xrightarrow{n-1} w_{n-1}^{-1}p \xrightarrow{n-2} w_{n-1}^{-1}w_{n-2}p = w_{2n-3}^{-1}w_{0}p \xrightarrow{2n-3} w_{2n-3}^{-1}p \xrightarrow{2n-4} \dots$$

$$\dots \xrightarrow{2j} w_{2j+1}^{-1}w_{2j}p = w_{2j+n-1}^{-1}w_{2j+n}p \xrightarrow{2j+n-1} w_{2j+n-1}^{-1}p \xrightarrow{2j+n-2} w_{2j+n-1}^{-1}w_{2j+n-2}p = w_{2j-1}^{-1}w_{2j}p \xrightarrow{2j-1} \dots$$

$$\dots \xrightarrow{3} w_{3}^{-1}p \xrightarrow{2} w_{3}^{-1}w_{2}p = w_{n+1}^{-1}w_{n+2}p \xrightarrow{n+1} w_{n+1}^{-1}p \xrightarrow{n} w_{n+1}^{-1}w_{n}p = w_{1}^{-1}w_{2}p \xrightarrow{1} w_{1}^{-1}p,$$

where the equalities are provided by Lemma 5.3 (1). In this list, the mentioned sides of even triangles appear in the order

$$\stackrel{0}{\longrightarrow} \quad \stackrel{n-2}{\longrightarrow} \quad \dots \quad \stackrel{2j}{\longrightarrow} \quad \stackrel{2j+(n-2)}{\longrightarrow} \quad \dots \quad \stackrel{n}{\longrightarrow}$$

and the mentioned sides of odd ones, in the order

$$\stackrel{n-1}{\longrightarrow} \quad \stackrel{n-1+(n-2)}{\longrightarrow} \quad \dots \quad \stackrel{2j+1}{\longrightarrow} \quad \stackrel{2j+1+(n-2)}{\longrightarrow} \quad \dots \quad \stackrel{1}{\longrightarrow} .$$

Since n-2 and n-1 are coprime, every side appears exactly once in the list

**5.8.** Corollary. Area<sub>i</sub> $(p,q;\varrho)$  does not depend on the choice of p, q, and  $i \blacksquare$ 

**5.9. Remark.** By Lemma 5.3 (2), Area  $\rho J = -$  Area  $\rho$ .

In the sequel, we assume without loss of generality that Area  $\rho \geq 0$ .

**5.10. Remark.** Take a fixed point  $c = p = q \in \overline{B}W$  of  $\varrho(w_1)$ . It follows from  $w_{n-1} = w_0 = 1$  and  $w_n = w_1^{-1}$  that  $w_1q = w_{n-1}q = w_np = w_0p = c$ . Therefore,

Area  $\Delta(c, w_0 p, w_1 q) = \operatorname{Area} \Delta(c, w_1 q, w_2 p) =$ 

$$= \operatorname{Area} \Delta(c, w_{n-2}p, w_{n-1}q) = \operatorname{Area} \Delta(c, w_{n-1}q, w_n p) =$$

$$= \operatorname{Area} \Delta(c, w_n p, w_{n+1} q) = \operatorname{Area} \Delta(c, w_{2n-3} q, w_0 p) = 0.$$

Hence, Area  $\rho \leq 2(n-4)\pi$ . When Area  $\rho = 2(n-4)\pi$ , we say that Area  $\rho$  is *maximal*. In this case,  $p \in SW$  and the cycles

$$p, w_2 p, w_3 p, \dots, w_{n-2} p, p, w_{n+1} p, w_{n+2} p, \dots, w_{2n-3} p$$

are positive.

**5.11. Remark.** Let  $p_1, p_2, q_2, q_1 \in SW$  be a positive cycle and suppose that some isometry  $h \in \mathcal{L}$  maps  $p_i$  to  $q_i$ , i = 1, 2. Then h is hyperbolic and the cycle  $p_1, s, p_2, q_2, t, q_1$  is positive, where  $s \in SW$  and  $t \in SW$  stand for the repeller and for the attractor of h.

**5.12. Lemma.** Let  $\varrho : G_n \to \mathcal{L}$  be a representation with maximal Area  $\varrho$  and let  $d \in SW$  be a fixed point of  $\varrho(w_1)$ . Then the cycles

 $d, w_2 d, r_3 r_1 d, w_3 d, r_4 r_1 d, w_4 d, \dots, r_{n-3} r_1 d, w_{n-3} d, r_{n-2} r_1 d, w_{n-2} d$ 

and

$$d, w_{n+1}d, r_n r_3 d, w_{n+2} d, r_n r_4 d, w_{n+3} d, \dots, r_n r_{n-3} d, w_{2n-4} d, r_n r_{n-2} d, w_{2n-3} d$$

are positive.

**Proof.** The cycles  $d, w_i d, w_{i+1} d$  and  $d, w_{i+n-1} d, w_{i+n} d$  are positive for all  $2 \le i \le n-3$  by Remark 5.10. Hence, by Lemma 5.3 (8), the cycles  $r_n r_{i+1} d, w_{i+n} d, w_{i+n-1} d$  and  $r_{i+1} r_n d, w_{i+1} d, w_i d$  are positive. In other words, the cycles  $w_{n+i-1}d, r_n r_{i+1} d, w_{n+i} d$  and  $w_i d, r_{i+1} r_n d, w_{i+1} d$  are positive. Since d is a fixed point of  $w_1 = r_1 r_n$ , we have  $r_{i+1} r_1 d = r_{i+1} r_1 r_n d$ . Therefore, the cycles  $w_i d, r_{i+1} r_1 d, w_{i+1} d$  and  $w_{n+i-1} d, r_n r_{i+1} d, w_{n+i} d$  are positive for all  $2 \le i \le n-3$ . By Remarks 5.10 and 3.7, the cycles in Lemma 5.12 are positive  $\blacksquare$ 

**5.13. Lemma.** In the situation of Lemma 5.12, the isometry  $h_i := \varrho(r_i r_{i-1})$  is hyperbolic for all *i* (the indices are modulo *n*). Denote by  $s_{i-1}$  and  $t_i$  the repeller and the attractor of  $h_i$ . Then, for every  $d \in \{s_n, t_1\}$ , the cycle

$$t_1, s_2, w_2d, s_3, t_3, w_3d, s_4, t_4, w_4d, \dots, s_{n-2}, t_{n-2}, w_{n-2}d, t_{n-1}, s_n$$

is positive.



**Proof.** The cycle  $w_{i-2}d$ ,  $r_{i-1}r_1d$ ,  $r_ir_1d$ ,  $w_id$  is positive for all  $3 \leq i \leq n-1$ . Indeed, for  $4 \leq i \leq n-2$ , this follows straightforwardly from Lemma 5.12. For i = 3, the cycle has the form  $d, w_2d, r_3r_1d, w_3d$  because  $w_1d = d$  and  $r_2r_1 = w_2$ . It is positive by Lemma 5.12. The relation  $r_nr_{n-1}\ldots r_2r_1 = 1$  implies  $w_{n-2} = v_{n-2} = r_{n-1}r_n$ . From  $d = w_1d$  and  $w_1 = r_1r_n$ , we obtain  $r_{n-1}r_1d = r_{n-1}r_1r_1r_nd = r_{n-1}r_nd = w_{n-2}d$ . Taking  $w_{n-1} = 1$  into account, we can see that, for i = n - 1, the cycle has the form  $w_{n-3}d, r_{n-2}r_1d, w_{n-2}d, d$ . By Lemma 5.12, it is positive.



The isometry  $h_i$  maps  $r_{i-1}r_1d$  to  $r_ir_1d$  and  $w_{i-2}d$  to  $w_id$  for all  $3 \le i \le n-1$ . By Remark 5.11,  $h_i$  is hyperbolic and the cycle

$$(5.14) w_{i-2}d, s_{i-1}, r_{i-1}r_1d, r_ir_1d, t_i, w_id, 3 \le i \le n-1,$$

is positive.

The cycle  $r_{i-1}r_1d$ ,  $w_{i-1}d$ ,  $r_ir_1d$  is positive for all  $4 \le i \le n-2$  by Lemma 5.12. We can combine this cycle and the cycle (5.14) by Remark 3.7 and obtain the positive cycle  $w_{i-2}d$ ,  $s_{i-1}$ ,  $r_{i-1}r_1d$ ,  $w_{i-1}d$ ,  $r_ir_1d$ ,  $t_i$ ,  $w_id$  for all  $4 \le i \le n-2$ . The first and the second parts of this cycle provide the positive cycles

(5.15) 
$$w_{i-1}d, s_i, r_ir_1d, w_id, \quad 3 \le i \le n-3,$$

(5.16) 
$$w_{i-1}d, r_ir_1d, t_i, w_id, \quad 4 \le i \le n-2.$$

Combining the cycles (5.15) and (5.16) by Remark 3.7, we get the positive cycle

(5.17) 
$$w_{i-1}d, s_i, t_i, w_id, \quad 4 \le i \le n-3.$$

Taking into account that  $w_1d = d$  and  $r_2r_1 = w_2$ , we can see that  $d, s_2, w_2d, r_3r_1d, t_3, w_3d$  and  $w_2d, s_3, r_3r_1d, w_3d$  are the cycles (5.14) and (5.15) with i = 3. Combining these cycles by Remark 3.7 and excluding the term  $r_3r_1d$ , we arrive at the positive cycle

$$(5.18) d, s_2, w_2 d, s_3, t_3, w_3 d$$

As was shown above,  $r_{n-1}r_1d = w_{n-2}d$ . Taking the cycle (5.16) with i = n - 2 and the cycle (5.14) with i = n - 1, we obtain the positive cycles  $w_{n-3}d$ ,  $r_{n-2}r_1d$ ,  $t_{n-2}$ ,  $w_{n-2}d$  and  $w_{n-3}d$ ,  $s_{n-2}$ ,  $r_{n-2}r_1d$ ,  $w_{n-2}d$ ,  $t_{n-1}$ , d since  $w_{n-1} = 1$ . Combining these cycles by Remark 3.7 and excluding the term  $r_{n-2}r_1d$ , we arrive at the positive cycle

$$(5.19) w_{n-3}d, s_{n-2}, t_{n-2}, w_{n-2}d, t_{n-1}, d.$$

The cycle  $d, w_2d, w_3d, \ldots, w_{n-2}d$  is positive by Lemma 5.12. Combining this cycle with the cycles (5.18), (5.17) for all *i*, and (5.19), we get the positive cycle

$$(5.20) d, s_2, w_2d, s_3, t_3, w_3d, s_4, t_4, w_4d, \dots, w_{n-3}d, s_{n-2}, t_{n-2}, w_{n-2}d, t_{n-1}.$$

Shifting the indices, i.e., applying the results already obtained to the representations  $\rho S^{j}$ , we conclude that  $h_{i}$  is hyperbolic for all *i*. So, the points  $s_{n}, t_{1}, s_{1}, t_{2}, s_{n-1}, t_{n}$  make sense.

Since the cycle (5.20) is positive for  $d = t_1$ , the cycle  $t_1, s_2, t_3, s_4$  is positive. Shifting the indices, we conclude that the cycle  $t_{n-1}, s_n, t_1, s_2$  is positive. Combining the positive cycles  $t_1, s_2, t_3, s_4$ , (5.20), and  $t_{n-1}, s_n, t_1, s_2$ , we arrive at the positive cycle in Lemma 5.13

**5.21. Proposition.** Let  $\varrho: G_n \to \mathcal{L}$  be a representation with maximal Area  $\varrho$ . Then the isometries  $h_i := \varrho(r_i r_{i-1})$  and  $h'_i := \varrho(r_n r_i r_{i-1} r_n)$  are hyperbolic for all i (the indices are modulo n). Denote by  $s_{i-1}, s'_{i-1}$  and  $t_i, t'_i$  the repellers and the attractors of  $h_i, h'_i$ , respectively. Then  $s_n = t'_1, t_1 = s'_n$ , and, for every  $d \in \{s_n, t_1\}$ , the cycle

$$t_1, s_2, w_2d, s_3, t_3, w_3d, s_4, t_4, w_4d, \dots, s_{n-2}, t_{n-2}, w_{n-2}d, t_{n-1}, s_n,$$
  
$$s'_2, w_{n+1}d, s'_3, t'_3, w_{n+2}d, s'_4, t'_4, w_{n+3}d, \dots, s'_{n-2}, t'_{n-2}, w_{2n-3}d, t'_{n-1}$$

is positive.

**Proof.** By Lemma 5.13, the isometries  $h_i$ 's are hyperbolic and the cycle

$$t_1, s_2, w_2d, s_3, t_3, w_3d, s_4, t_4, w_4d, \dots, s_{n-2}, t_{n-2}, w_{n-2}d, t_{n-1}, s_n$$

is positive for every  $d \in \{s_n, t_1\}$ . By Remark 5.4 and Corollary 5.8, Area  $\rho I = 2(n-4)\pi$ . By Lemma 5.13 applied to the representation  $\rho I$ , the isometries  $h'_i$ 's are hyperbolic and the cycle

$$t'_1, s'_2, w_{n+1}d, s'_3, t'_3, w_{n+2}d, s'_4, t'_4, w_{n+3}d, \dots, s'_{n-2}, t'_{n-2}, w_{2n-3}d, t'_{n-1}, s'_n$$

is positive for every  $d \in \{s'_n, t'_1\}$  since  $w_{i+n-1} = I(w_i)$  for all *i*. It remains to observe that  $h'_1^{-1} = h_1$  and to combine the above positive cycles

**Proof of Theorem 5.1.** Let us show that Area  $\rho = 2(n-4)\pi$  implies  $\rho \in \mathcal{R}G_n$ .

Denote  $G := G[s_n, t_1]$ ,  $G_i := w_i G$ , and  $G'_i := w_{i+n-1} G$  for all  $2 \le i \le n-2$ . By Proposition 5.21, G is the axis of  $h_1 = h'_1^{-1}$ . Hence, the vertices of  $G_i$  and of  $G'_i$  are respectively of the form  $w_i d$  and  $w_{i+n-1}d$ , where  $d \in \{s_n, t_1\}$ .

Take  $p, q \in G \cap BW$  such that  $p = h_1 q$  and denote by Q the polygon with the successive vertices

$$w_1q, w_2p, w_3q, \dots, w_{n-3}q, w_{n-2}p, w_np, w_{n+1}q, w_{n+2}p, \dots, w_{2n-4}p, w_{2n-3}q$$

and the successive edges  $e_2, e_3, \ldots, e_{n-1}, e'_2, e'_3, \ldots, e'_{n-1}$  such that

(5.22)  $e_i := G[w_{i-1}q, w_i p], \quad e'_i := G[w_{i+n-2}p, w_{i+n-1}q] \quad \text{for even } i, \quad 2 \le i \le n-2,$ 

(5.23) 
$$e_i := G[w_{i-1}p, w_iq], \quad e'_i := G[w_{i+n-2}q, w_{i+n-1}p] \quad \text{for odd } i, \quad 3 \le i \le n-1.$$

(Note that  $w_{n-1}q = w_n p$  and  $w_{2n-2}p = w_1 q$  since  $w_{n-1} = w_{2n-2} = 1$  and  $w_n = w_1^{-1}$ .)

We claim that Q is a fundamental polygon for the group  $\rho G_n$ . Obviously,  $w_i p, w_i q \in G_i$  and  $w_{i+n-1}p, w_{i+n-1}q \in G'_i$  for all  $2 \leq i \leq n-2$ . Also,  $w_n p, w_1 q \in G$  since  $w_n = w_1^{-1}$ ,  $h_1 = \rho(w_1)$ ,  $p = h_1 q$ , and  $p, q \in G$ . Let  $d \in \{s_n, t_1\}$ . Then the cycle in Proposition 5.21 is positive. This implies that G, the  $G_i$ 's, and the  $G'_j$ 's are all disjoint. Therefore, the edges  $e_i$ 's and  $e'_i$ 's are not degenerated and, thus, generate complete geodesics  $\Gamma_i$  and  $\Gamma'_i$ .

Define the arcs

$$A := \{b \in S W \mid \text{the cycle } t_1, b, s_n \text{ is positive}\}, \qquad A' := \{b \in S W \mid \text{the cycle } s_n, b, t_1 \text{ is positive}\}.$$



Let  $A_i \subset A$  and  $A'_i \subset A'$  be the arcs with the same ends as  $G_i$  and  $G'_i$ , respectively. The arcs

 $A', A_2, A_3, \ldots, A_{n-2}$  are disjoint because the cycle in Proposition 5.21 is positive. It is easy to see that the vertices of  $\Gamma_i$  belong to  $A_{i-1}$  and  $A_i$  for all  $3 \leq i \leq n-2$ , that the vertices of  $\Gamma_2$  belong to A' and  $A_2$ , and that the vertices of  $\Gamma_{n-1}$  belong to  $A_{n-2}$  and A'. The only intersections between  $\Gamma_i$ 's are the known intersections between  $\Gamma_{i-1}$  and  $\Gamma_i$ ,  $3 \leq i \leq n-1$ , and a possible intersection between  $\Gamma_2$  and  $\Gamma_{n-1}$ . Nevertheless, the edges  $e_2$  and  $e_{n-1}$  do not intersect. Indeed, it follows from Proposition 5.21 that the cycle  $t_1, w_2d, s_3, w_{n-2}d, s_n$  is positive for every  $d \in \{s_n, t_1\}$ . Since  $s_n$  and  $t_1$  are the repeller and the attractor of  $h_1 = \varrho(w_1), p = h_1q, w_np = q$ , and  $w_1q = p$ , the edges  $e_2$  and  $e_{n-1}$  cannot intersect. Consequently, the edges  $e_2, e_3, \ldots, e_{n-1}$  intersect in the 'prescribed' way and are on the side of the normal vector to G. For similar reasons, the edges  $e'_2, e'_3, \ldots, e'_{n-1}$  intersect in the 'prescribed' way and are on the opposite side of the normal vector to G. In other words, Q is simple.

The polygon Q has 2(n-2) vertices and Area  $\rho = \text{Area } Q = 2(n-4)\pi$ . Therefore, the sum of the interior angles of Q equals  $2(n-3)\pi - \text{Area } P = 2\pi$ . The isometry  $\gamma_i := \rho(r_n r_i)$  maps the edge  $e_i$  onto the edge  $e'_i$  for all  $2 \le i \le n-1$ . This follows from (5.22–23) and from Lemma 5.3 (7).

As is easy to see, the identifications by the  $\gamma_i$ 's produce the only cycle of vertices. By Poincaré's Polyhedron Theorem, Q is a fundamental polygon for the group generated by the  $\gamma_i$ 's and

$$\gamma_{n-1} \dots \gamma_4^{-1} \gamma_3 \gamma_2^{-1} \gamma_{n-1}^{-1} \dots \gamma_4 \gamma_3^{-1} \gamma_2 = 1$$

is a unique defining relation of this group. In other words,  $\rho$  is an isomorphism and, thus,  $\rho \in \mathcal{R}G_n$ .

For the converse, we simply repeat the arguments presented at the end of the proof of Theorem 3.15

**5.24. Remark.** It is easy to verify that the group  $G_n$  admits the generators  $g_{i(i-1)}$  (the indices are modulo n) subject to the defining relations

 $g_{n(n-1)}g_{(n-1)(n-2)}g_{(n-2)(n-3)}\dots g_{32}g_{21}g_{1n} = 1,$ 

 $g_{n(n-1)}g_{(n-2)(n-3)}\dots g_{43}g_{21} = 1, \qquad g_{(n-1)(n-2)}g_{(n-3)(n-4)}\dots g_{32}g_{1n} = 1.$ 

(In terms of  $H_n, g_{i(i-1)} := r_i r_{i-1}$ .)

Let  $\varrho : G_n \to \mathcal{L}$  be a representation. Fix some *i* and suppose that  $g := \varrho(g_{i(i-1)})$  is hyperbolic. For every  $t \in \mathbb{R}$ , define a representation  $\varrho E_i(t)$  as

$$\begin{split} \varrho E_i(t)(g_{(i+1)i}) &:= \varrho(g_{(i+1)i})g^{-2t}, \qquad \varrho E_i(t)(g_{(i-1)(i-2)}) := g^{2t}\varrho(g_{(i-1)(i-2)}), \\ \varrho E_i(t)(g_{j(j-1)}) &:= \varrho(g_{j(j-1)}) \quad \text{for all } j \notin \{i-1, i+1\}. \end{split}$$

If  $\rho$  is induced by some  $\hat{\rho}: H_n \to \mathcal{L}$ , then  $\hat{\rho}(r_i) = R(q_i)$  and  $\hat{\rho}(r_{i-1}) = R(q_{i-1})$  for some  $q_i, q_{i-1} \in BW$ belonging to the axis of g. As is easy to see,  $g^t R(q_i)g^{-t} = R(q_i)g^{-2t}$  and  $g^t R(q_{i-1})g^{-t} = g^{2t}R(q_{i-1})$ . In other words, we obtain an extension of the action of  $\mathcal{E}_n$  on  $\mathcal{R}H_n$  (and on  $\mathcal{H}_n$ ) to that on  $\mathcal{R}G_n$  (and on  $\mathcal{T}_n$ ).

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