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A Haar-like Construction for the Ornstein Uhlenbeck Process

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Abstract The classical Haar construction of Brownian motion uses a binary tree of triangular wedge-shaped functions. This basis has compactness properties which make it especially suited for certain classes of numerical algorithms. We present a similar basis for the Ornstein-Uhlenbeck process, in which the basis elements approach asymptotically the Haar functions as the index increases, and preserve the following properties of the Haar basis: all basis elements have compact support on an open interval with dyadic rational endpoints; these intervals are nested and become smaller for larger indices of the basis element, and for any dyadic rational, only a finite number of basis elements is nonzero at that number. Thus the expansion in our basis, when evaluated at a dyadic rational, terminates in a finite number of steps. We prove the covariance formulae for our expansion and discuss its statistical interpretation and connections to asymptotic scale invariance.

Keywords Ornstein-Uhlenbeck process · Brownian motion · Haar basis

1 Introduction

Random walks and continuous stochastic processes are of fundamental importance in a number of applied areas, including optics [1], chemical physics [2], biophysics [3, 4], biology [5] and finance [6]. The mathematical idealization of the one-dimensionnal continuous random walk, the Wiener process, can be

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expressed in —infinitely— many bases as a sum of random coefficients times basis elements. Unique among these bases, the Haar —or Schauder— basis has three properties that make it particularly suitable for certain numerical computations. First, the basis elements all have *compact support*: the basis elements are nonzero only in open intervals. Second, the support is *increasingly compact*, i.e., the open intervals become smaller for higher indices of the basis elements; in fact, the intervals are nested in binary tree fashion, and have dyadic rational endpoints. Finally, given any dyadic rational, there is a finite number of basis elements which are nonzero at that number, so that evaluation of the Haar expansion at a dyadic rational *terminates* in a finite number of steps known beforehand. These properties can be used to great advantage in algorithms that construct the random walk in a "top-down" fashion, such as dychotomic search algorithms for first passage times.
However, the "plain" Wiener process has limited applicability in the areas mentioned above, so an extension of this construction to more complex stochastic processes is desirable. The naive generalization of the Haar basis construction to other stochastic processes fails to display our three properties. We present a method for constructing a Haar-like basis for the Ornstein-Uhlenbeck process which preserves these properties. The basis is therefore useful for advanced numerical computations: a fast dichotomic search algorithm for first passage time computations shall be presented elsewhere. The method we present is also amenable to further generalizations to other stochastic processes.
This paper is organized as follows. We first review some background in stochastic processes and basis expansions. Then we review the well-known decomposition of a Wiener process in a basis of functions derived from the Haar system. In the third section, we give a statistical interpretation of such a construction, which leads us to propose a basis for the Ornstein-Uhlenbeck process. In the fourth section, we prove that the Ornstein-Uhlenbeck process is correctly represented as a discrete process in the proposed basis. In the last section, we further the statistical interpretation and its connection to scale invariance and Markovian properties.

2 Background on Stochastic Processes

The Wiener process and the Ornstein-Uhlenbeck process are continuous stochastic processes; we specify this class of process through the Langevin equation

$$\begin{cases} \dot{x} = f(x,t) + \eta(t) \\ x(t_0) = x_0,\ t \in [t_0, T] \end{cases} , \tag{1}$$

where f is a deterministic function and $\eta(t)$ describes the stochastic forcing. Equation (1) is a first order stochastic differential equation and its connection to the Fokker-Planck equation has been extensively studied [1,2]. We only consider here the case where the noise is white and Gaussian: $\eta(t)$ are realizations of independent identically distributed Gaussian variables η_t, with time correlations satisfying

$$\langle\, \eta_t \cdot \eta_s \,\rangle = \Gamma \cdot \delta(s-t) \,,$$

where δ is the Dirac distribution.
We denote by ω a given realization of the stochastic forcing: the collection of all the values $\{\eta(t)\}_{t\in[t_0,T]}$ in an interval. The set of ω values defines the sample space Ω and the probability of occurence of a sequence ω in Ω is determined by the joint probability of $\{\eta_t\}_{t\in[t_0,T]}$. With this notation, we can introduce the general solution of the stochastic system as the stochastic process X_t. For a given realization of the noise ω, there is a unique solution to (1) called a sample path: neglecting to notate the dependence on the initial condition, we write $X_t(\omega)$ the value of this sample path at time t. Since each sample path $\{X_t(\omega)\}_{t\in[t_0,T]}$ occurs with the same probability as its matching sequence ω in the sample space Ω, the value $X_t(\omega)$ can be seen as the outcome of a random function X_t defined on Ω. X_t is the stochastic process solution of (1) and has several important properties: it is a continuous process as it is defined for a continuous index set $[t_0, T]$; it is a Gaussian process, as it integrates contributions of Gaussian variables; and, being a Markovian process, the value of X_t only depends on $\{\eta(u)\}_{u\in[0,t]}$, the sequence of realizations preceding t. Two special forms of f shall concern us. When f is zero, the process is called the Wiener process W_t; when f is linear in x, the process is the Ornstein-Uhlenbeck process U_t. Due to the relative simplicity of both situations, the probability laws of the processes (i.e., the Green functions of the associated Fokker-Planck equations) are known analytically. If a Wiener process is at x_0 at time $t = t_0$, the probability of finding the process in x at time t is

$$\mathbf{P}(W_t{=}x\,|W_{t_0}{=}x_0) = \frac{1}{\sqrt{2\pi}\cdot {}_W\sigma_t}\cdot\exp\left(-\frac{(x-x_0)^2}{2\cdot {}_W\sigma_t^2}\right) \tag{2}$$

with a variance ${}_W\sigma_t^2 = \Gamma\cdot(t-t_0)$. For the Ornstein-Uhlenbeck process, a similar result holds

$$\mathbf{P}(U_t{=}x\,|U_{t_0}{=}x_0) = \frac{1}{\sqrt{2\pi}\cdot {}_U\sigma_t}\cdot\exp\left(-\frac{\left(x-x_0\,e^{-\alpha(t-t_0)}\right)^2}{2\cdot {}_U\sigma_t^2}\right) \tag{3}$$

with a variance ${}_U\sigma_t^2 = \frac{\Gamma}{2\alpha}\cdot\left(1-e^{-\alpha(t-t_0)}\right)$. The previous expressions describe the statistics of W_t and U_t, which will be called X_t when collectively designated. Continuous processes require an infinite number of random variables, and establishing results about them is quite difficult. Consider for example the Ornstein-Uhlenbeck process, widely used from finance to neuroscience: finding analytically the first-passage times distribution with a fixed threshold proves a surprisingly intricate question in this situation [8,9,10], and numerically, sample paths are only approximated by stochastic Euler methods, with integration schema of low efficiency [11,12,13]. Abating these difficulties for the Ornstein-Uhlenbeck process is the motivation for this paper.
To circumvent the problem, it is advantageous to represent a continuous process as a discrete process. Conspicuously enough, a discrete process has a countable index set of random variables. At stake is to write a Gaussian process X_t as a convergent series of random functions $f_n\cdot\xi_n$, where f_n is a deterministic function and ξ_n a Gaussian variable of law $\mathcal{N}(0,1)$ (i.e.with null

mean and unitary variance). Assuming the coefficients of the decomposition to be included in the definition of f_n, the identity

$$X_t = \sum_{n=0}^{\infty} f_n(t) \cdot \xi_n = \lim_{N \to \infty} \sum_{n=0}^{N} f_n(t) \cdot \xi_n$$

shows X_t as the limit of a sequence of finite processes $\sum_{n=0}^{N} f_n(t) \cdot \xi_n$. Depending on the nature of the convergence, this may result in two advantages. Analytically, it is generally more tractable to prove mathematical results on finite combination of simple random functions and then to extend them to a limit random process. Numerically, the quantity $\sum_{n=0}^{N} f_n(t) \cdot \xi_n$ can be accurately computed and gives a correct approximation of the process at any level of precision.

3 The Haar construction of the Wiener process

The Haar system is the set of functions $h_{n,k}$ in $L^2([0,1])$ defined by

$$h_{n,k}(t) = \begin{cases} 2^{\frac{n-1}{2}} \text{ if } (2k)\,2^{-n} \leq t \leq (2k+1)\,2^{-n}\,, \\ -2^{\frac{n-1}{2}} \text{ if } (2k+1)\,2^{-n} \leq t \leq 2\,(k+1)\,2^{-n}\,, \\ 0 \quad \text{otherwise}\,. \end{cases}$$

for $n \geq 1$ with the addition of the function $h_{0,0}(t) = 1$ on $[0,1]$. The Haar system has a several interesting properties. First, the functions $h_{n,k}$ form a complete orthonormal basis of $L^2([0,1])$ for the scalar product $(f,g) = \int_0^1 f(t)g(t)dt$. Second, each element $h_{n,k}$ has a compact support

$$S_{n,k} = \left[k \cdot 2^{-n+1}, (k+1)2^{-n+1}\right]$$

and, for a given n, the collection of supports $S_{n,k}$ represents a partition of $[0,1]$. Third, the functions $h_{n,k}$ build up a wavelet basis of $L^2([0,1])$, since we have the scale-invariant construction rule

$$h_{n,k}(t) = 2^{\frac{n-1}{2}} \cdot h_{1,0}(2^{n-1}t - k)\,. \tag{4}$$

Such properties prove useful to decompose simple Gaussian process as related in the following. Consider the Wiener process W_t as the stochastic integral of the independent random variables η_t which follow a normal law $\mathcal{N}(0, \sqrt{I})$. We introduce the associated Gaussian white noise process formally defined as $\frac{dW}{dt}$ on Ω. A sample path $\frac{dW_t}{dt}(\omega)$ is almost surely an element of $L^2([0,1])$ for a given noise realization $\omega = \{\eta(t)\}_{t \in [0,1]}$. We write its decomposition on the Haar basis

$$\left(\frac{dW_t}{dt}\right)(\omega) = \sum_{n=0}^{\infty} \sum_{0 \leq k < 2^{n-1}} c_{n,k}(\omega) h_{n,k}(t)\,,$$

introducing $c_{n,k}(\omega)$ the component of $\frac{dW_t}{dt}(\omega)$ in the direction of $h_{n,k}$. Each sequence ω can yield different coefficients $c_{n,k}(\omega)$ and their values appear as the outcome of a random variable $c_{n,k}$ defined on the sample space Ω

$$c_{n,k} = \int_0^1 h_{n,k}(t)\frac{dW_t}{dt}\,dt = \int_0^1 h_{n,k}(t)dW_t\ . \tag{5}$$

In our specific case of Gaussian uncorrelated noise, stochastic integration shows that the random variables $c_{n,k}$ are all independent and identically distributed following the law $\mathcal{N}(0,\sqrt{\Gamma})$. The white noise process $\frac{dW_t}{dt}$ is then expressed as a discrete process on Ω by

$$\frac{dW_t}{dt} = \sum_{n=0}^{\infty}\sum_{0\leq k<2^{n-1}} \sqrt{\Gamma}\cdot h_{n,k}(t)\cdot\xi_{n,k}\ ,$$

where the $\xi_{n,k}$ are independent and distributed with law $\mathcal{N}(0,1)$.
The use of Haar functions to decompose white Gaussian noise directly suggests a corresponding result for the Wiener process. The Wiener process W_t is the stochastic integral of the Gaussian white noise process $\frac{dW_t}{dt}$, which leads to introduce the integrals of $h_{n,k}$ as candidates to build a basis of functions for the Wiener process [14,15]. We note this set of integral functions as

$$\Psi_{n,k}(t) = \sqrt{\Gamma}\cdot\int_0^1 \chi_{[0,t]}(u)h_{n,k}(u)\,du\ , \tag{6}$$

with the help of the indicator functions given by

$$\chi_{[0,t]}(u) = \begin{cases} 1 \text{ if } 0\leq u\leq t \\ 0 \text{ otherwise} \end{cases}\ .$$

The first elements of the so-defined basis are shown on figure 1. The question is then to know whether the process W_t^N defined as the finite sum of random functions

$$W_t^N(\omega) = \sum_{n=0}^{N}\sum_{0\leq k<2^{n-1}} \Psi_{n,k}(t)\cdot\xi_{n,k}(\omega)$$

converges toward a Wiener process. It can be shown that the series converges normally to a limit process almost surely on Ω [14,15]. Even though we have not proven its Wiener process nature yet, we refer to this limit as W_t. Due to the normal convergence, W_t is continuous in t and being a sum of Gaussian variables, it is a Gaussian process. Therefore, showing that W_t is a Wiener process only amounts to demonstrate it has the same law of covariance as a Wiener process [14,15], i.e. $\langle W_t\cdot W_s\rangle = \Gamma\cdot\min(t,s)$, where $\min(t,s)$ is the minimum of t and s. In other words, we need to evaluate the quantity

$$\langle W_t\cdot W_s\rangle = \lim_{N\to\infty}\langle W_t^N\cdot W_s^N\rangle = \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{0\leq k<2^{n-1}} \Psi_{n,k}(t)\cdot\Psi_{n,k}(s)$$

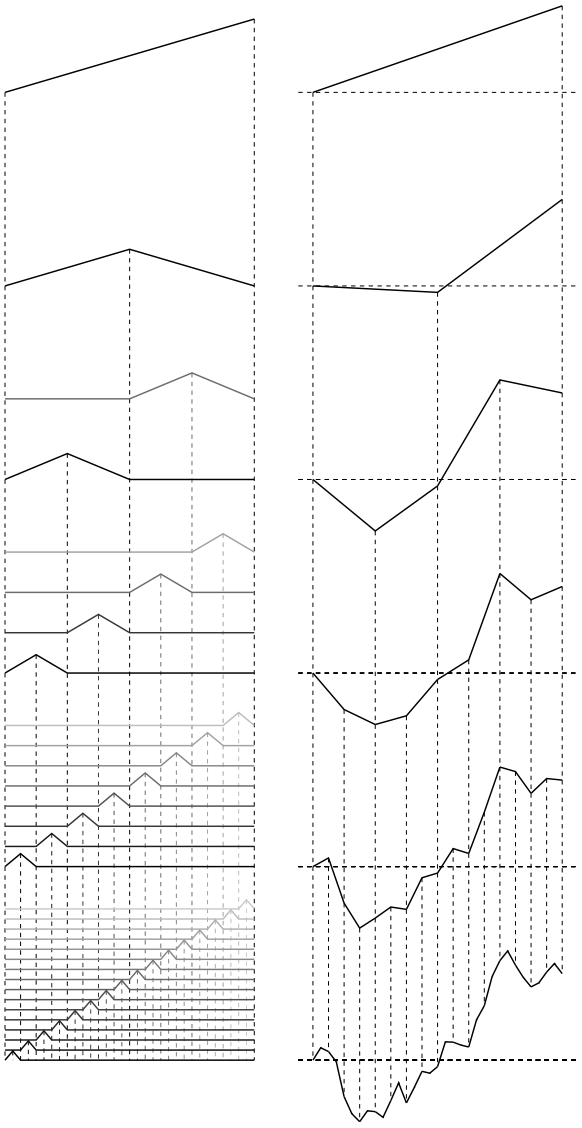

Fig. 1 In the left column, the elements of the basis $\Psi_{n,k}$ are represented for each rank n with $0 \leq n < 6$. In the right column, the partial sums $W^n(\omega)$ are shown for a given set of realizations ω. Note that each element $\Psi_{n,k}$ has a compact support delimited by dyadic numbers in $D_n = \{k2^{-n} \,|\, 0 \leq k \leq 2^n\}$ and that all $\Psi_{n',k}$ is zero on D_n for $n' > n$.

which entails the calculation of a rather tedious series. Indeed, for a given t, at each step n, there is only one k for which $\Psi_{n,k}(t)$ is nonzero, and expressing the series analytically results in a complicated operation. One way to overcome the issue is to notice that the expected covariance result can be expressed in terms of

$$\min(t,s) = \int_0^1 \chi_{[0,t]}(u)\chi_{[0,s]}(u)\,du\,. \tag{7}$$

The right term of (7) is actually the scalar product of the functions $\chi_{[0,t]}$ and $\chi_{[0,s]}$ on $L^2([0,1])$. The key point is then to introduce the decomposition in the Haar orthonormal basis to write the scalar product of two given functions as

$$\int_0^1 f(t)g(t)\,dt \;=\; \sum_{\substack{n\geq 0\\ 0\leq k<2^{n-1}}} \int_0^1 f(u)h_{n,k}(u)\,du \int_0^1 g(u)h_{n,k}(u)\,du\,. \tag{8}$$

When applied to the indicator functions of interest $\chi_{[0,t]}$ and $\chi_{[0,s]}$, the relation (8) leads to

$$\begin{aligned}\Gamma\cdot\min(t,s) &= \Gamma\cdot\int_0^1 \chi_{[0,t]}(u)\chi_{[0,s]}(u)\,du\\ &= \sum_{n=0}^{\infty}\sum_{0\leq k<2^{n-1}} \Psi_{n,k}(t)\Psi_{n,k}(s)\,,\end{aligned}$$

since the definition (6) describes $\Psi_{n,k}(t)$ as the coefficient relative to $h_{n,k}$ in the decomposition of $\chi_{[0,t]}$ on the Haar system. We can finally recap the result

$$\begin{aligned}\langle W_t\cdot W_s\rangle &= \lim_{N\to\infty}\langle W_t^N\cdot W_s^N\rangle\\ &= \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{0\leq k<2^{n-1}} \Psi_{n,k}(t)\cdot\Psi_{n,k}(s) \quad = \quad \Gamma\cdot\min(t,s)\;,\end{aligned}$$

establishing the discrete description of the Wiener process as a normally convergent series of terms $\Psi_{n,k}\cdot\xi_{n,k}$, where $\Psi_{n,k}$ is a Haar-derived function and $\xi_{n,k}$ a random variable of normal law $\mathcal{N}(0,1)$.

4 Comparison of the Wiener and Ornstein-Uhlenbeck processes

We recall that the Langevin equation (1) can be solved by quadratures in simple cases. If the process is at x_0 when $t=0$, the Ornstein-Uhlenbeck process U_t is expressed

$$U_t = x_0\,e^{-\alpha t} + \int_0^t e^{\alpha(u-t)}\eta(u)\,du\,, \tag{9}$$

as opposed to the Wiener process W_t in the same conditions

$$W_t = x_0 + \int_0^t \eta(u)\, du\ . \tag{10}$$

The comparison of definitions (9) and (10) explains why finding a basis of decomposition for U_t stumbles on a several difficulties. First the process U_t is not anymore a simple integral of white Gaussian noise, which is naturally described as a discrete process. Second, there is no more scale invariance, implying that a putative basis of decomposition is not to be thought of as wavelets. Finally, the exponential factor in (9) indicates that the process U_t does not sum the η_s evenly; their contribution depends on the position of s compared to t. As a consequence, the presence of these correlations makes it unlikely for the basis to conserve any orthogonality properties.
Yet, as noticeable in figure 2, the examination of a sample path $\{U_t(\omega)\}_{t\in[0,1]}$ reveals the scale-invariant behavior of a Wiener process for asymptotically small time scale as well as for asymptotically small α. It means that the basis of decomposition $\Psi_{n,k}$ for the Wiener process is asymptotically valid to describe U_t at fine scale. This observation suggests that, upon slight alteration of its analytical expression, the Haar derived basis $\Psi_{n,k}$ can give rise to a basis $\Phi_{n,k}$ adapted to the Ornstein-Uhlenbeck process. The change in the analytical expression of $\Psi_{n,k}$ should be consistent with the previously mentioned difficulties, preventing its formulation to be scale invariant or orthogonal. Under this restraint, the fundamental property that each element $\Psi_{n,k}$ exhibits a compact support of the form $S_{n,k}$ should be preserved in the expression of $\Phi_{n,k}$.
To carry out this program, the key point is to consider $\mathbf{P}(X_{t_y}{=}\, y\,|X_{t_x}{=}\, x, X_{t_z}{=}\, z)$ with $t_x < t_y < t_z$, the probability law of X_t knowing its values x and z at two framing times t_x and t_z. Because X_t is a markovian process, a sample path $\{X_t(\omega)\}_{t\in[0,1]}$ which originates from x and joins z through y is just the junction of two independent paths: a path originating in x going to y and a path originating from y going to z. Assuming conditional knowledge of its origin x, the probability of such a compound path is the product of the probability of the two elementary paths with conditional knowledge of their respective origins x and y. Therefore, after normalization by the absolute probability for a path to go from x to y, $\mathbf{P}(X_{t_y}{=}\, y\,|X_{t_x}{=}\, x, X_{t_z}{=}\, z)$ is expressed in the following expression

$$\mathbf{P}(X_{t_y}{=}\, y\,|X_{t_x}{=}\, x, X_{t_z}{=}\, z) \quad = \quad \frac{\mathbf{P}(X_{t_y}{=}\, y\,|X_{t_x}{=}\, x)\cdot \mathbf{P}(X_{t_z}{=}\, z\,|X_{t_y}{=}\, y)}{\mathbf{P}(X_{t_z}{=}\, z\,|X_{t_x}{=}\, x)}\ . \tag{11}$$

It is now a simple matter of calculation to compute the distribution of X_{t_y} knowing $X_{t_x}{=}\, x$ and $X_{t_z}{=}\, z$ with the analytical expression of the probability $\mathbf{P}(X_{t_y}{=}\, y\,|X_{t_x}{=}\, x)$. In the case of a Gaussian process, it is expected to follow a normal law, which we refer to as $\mathcal{N}({}_P\mu(t_y), {}_P\sigma(t_y))$. For the Wiener process, using expression (2) for $\mathbf{P}(X_{t_y}{=}\, y\,|X_{t_x}{=}\, x)$, the mean value ${}_W\mu(t_y)$ and the variance ${}_W\sigma(t_y)^2$ result in

$${}_W\mu(t_y) = \frac{t_z - t_y}{t_z - t_x}\cdot x + \frac{t_y - t_x}{t_z - t_x}\cdot z\ , \tag{12}$$

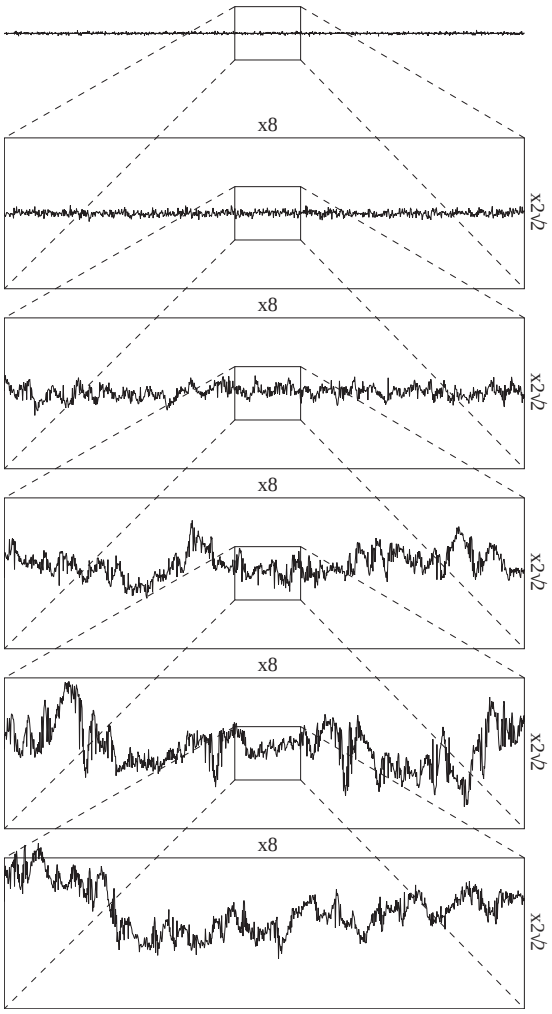

Fig. 2 A sample path $U_t(\omega)$ is represented at different magnifications following the scale invariance of a Wiener process: the vertical zooming factor is the square root of the horizontal factor. Note that the sample path $U_t(\omega)$ behaves as a Wiener process at small scales.

$$ {}_W\sigma(t_y)^2 = \Gamma \cdot \frac{(t_y - t_x)(t_z - t_y)}{(t_z - t_x)} \,. \tag{13} $$

For the Ornstein-Uhlenbeck process, using expression (3) for $\mathbf{P}(X_{t_y}{=}y \,|X_{t_x}{=}x)$ similarly yields the mean ${}_U\mu(t_y)$ and the variance ${}_U\sigma(t_y)^2$:

$$ {}_U\mu(t_y) = \frac{\sinh\big(\alpha(t_z - t_y)\big)}{\sinh\big(\alpha(t_z - t_x)\big)} \cdot x + \frac{\sinh\big(\alpha(t_y - t_x)\big)}{\sinh\big(\alpha(t_z - t_x)\big)} \cdot z \,, \tag{14} $$

$$ {}_U\sigma(t_y)^2 = \frac{\Gamma}{2\alpha} \cdot \frac{2 \cdot \sinh\big(\alpha(t_y - t_x)\big) \cdot \sinh\big(\alpha(t_z - t_y)\big)}{\sinh\big(\alpha(t_z - t_x)\big)} \,. \tag{15} $$

In the limit of very short time scale or vanishing α, we notice that ${}_U\mu(t_y)$ and ${}_U\sigma(t_y)^2$ approximate ${}_W\mu(t_y)$ and ${}_W\sigma(t_y)^2$.
We note D_N the set of reals $\{k2^{-N} \,|\, 0 \leq k \leq 2^N\}$ and we have $\{0,1\} = D_0 \subset D_1 \subset \cdots \subset D_N$ a growing sequence of sets with limit ensemble $\mathcal{D}$ the set of dyadic points in $[0,1]$. Assuming we know the values of the process on a subset of dyadic points D_N, we can construct the conditional average $\langle P_t \rangle_{D_N}$, which is the most probable outcome of X_t knowing its values on D_N. For a Wiener process, (12) shows that $\langle W_t \rangle_{D_N}$ is a piece-wise linear function of t interpolating each points of D_N; whereas for an Ornstein-Uhlenbeck process, (14) depicts $\langle U_t \rangle_{D_N}$ as a succession of catenaries joining successive points of D_N. With $0 \leq k < 2^{-N}$, if $t_x = k2^{-N}$ and $t_z = (k+1)2^{-N}$ are the two successive points of D_N framing t, the average $\langle X_t \rangle_{D_N}$ is only conditioned by $X_{t_x} = x$ and $X_{t_z} = z$. For the sake of simplicity, we write

$$ \langle X_t \rangle_{D_N} = \langle X_t \rangle_{x,z} = {}_X\mu_{t_x,t_z}(t,x,z) \stackrel{def}{=} {}_X\mu^{N,k}(t) \,, \tag{16} $$

where the conditional dependency upon x and z is implicit in ${}_X\mu^{N,k}$. We want to investigate the change in the estimation of P_t due to the conditional knowledge of its value on the dyadic set D_{N+1}. In that perspective, we exemplified the conditional average $\langle X_t \rangle_{D_{N+1}}$ on $[t_x, t_z]$ where the estimation of X_t is now dependent upon the value $X_{t_y} = y$ with t_y the midpoint of t_x and t_z:

$$ \begin{aligned} \langle X_t \rangle_{D_{N+1}} = \langle X_t \rangle_{x,y,z} \;&=\; \begin{cases} \langle X_t \rangle_{x,y} & \text{if } t_x \leq t \leq t_y \,, \\ \langle X_t \rangle_{y,z} & \text{if } t_y \leq t \leq t_z \,, \end{cases} \\ &\stackrel{def}{=} {}_X\nu^{N,k}(t,y) \,. \end{aligned} \tag{17} $$

We remark that, being a function of y, the conditional average $\langle X_t \rangle_{x,y,z}$ determines a random function ${}_X\nu^{N,k}(t, Y_{N,k})$, where the short notation $Y_{N,k}$ indicates the Gaussian variable X_{t_y} knowing $X_{t_x} = x$ and $X_{t_z} = z$. The probability distribution of $Y_{N,k}$ follows the law $\mathcal{N}({}_X\mu(t_y), {}_X\sigma(t_y))$ and it gives, through the function ${}_X\nu^{N,k}$, the random contribution of ignoring $X_{t_y} = y$ when one estimates the process knowing its values on t_x and t_z.
The results above allows to gain insight in the building of a Wiener process W_t as the converging series of random functions $\Psi_{n,k} \cdot \xi_{n,k}$. It is easy to see from the definition (4) that $\Psi_{n,k}$ is linear between any two points in D_n for $n \leq N$ and that $\Psi_{n,k}$ is zero on D_n for every $n > N$. In other words, the partial sum

$$ W_t^N = \sum_{n=0}^{N} \sum_{0 \leq k < 2^{n-1}} \Psi_{n,k}(t) \cdot \xi_{n,k} \quad \text{for} \quad t \in D_N $$

coincide with W_t on D_N and more generally with $\langle X_t \rangle_{D_N}$ on $[0,1]$. Identifying partial sums with conditional averages, it is then straightforward to express

the component $\Psi_{N+1,k}(t)\cdot\xi_{N+1,k}$ in the decomposition of W_t

$$\begin{aligned}\Psi_{N+1,k}(t)\cdot\xi_{N+1,k} &= W_t^{N+1} - W_t^N \\ &= \langle W_t\rangle_{D_{N+1}} - \langle W_t\rangle_{D_N} \\ &= {}_W\nu^{N,k}(t,Y_{N,k}) - {}_W\mu^{N,k}(t)\,,\end{aligned} \tag{18}$$

bearing in mind the previous definitions for which $[t_x,t_z]=\left[k2^{-N},(k+1)2^{-N}\right]$ is the support $S_{N+1,k}$ of $\Psi_{N+1,k}$. The tight connection between $\xi_{N+1,k}$ and $Y_{N,k}$ is made obvious: if one knows the values of the process on D_N, the random contribution of $\sum_k \Psi_{N+1,k}(t)\cdot\xi_{N+1,k}$ conveys the uncertainty about W_t that is discarded by the knowledge of its values on $D_{N+1}\setminus D_N$.
We are now in a position to complete our program: continuing the identification of partial sums and conditional average for the Ornstein-Uhlenbeck process U_t, it is direct to propose a basis of decomposition

$$\Phi_{N+1,k}(t)\cdot\xi_{N+1,k} = {}_U\nu^{N,k}(t,Y_{N,k}) - {}_U\mu^{N,k}(t) \tag{19}$$

with the adapted definitions of ${}_U\mu^{N,k}$ and ${}_U\nu^{N,k}$ on $S_{N+1,k}$, the support of the investigated functions $\Phi_{N+1,k}$. We underline that the notation $Y_{N,k}$ refers here to the random process U_t at the midpoint of the support $t=(2k+1)2^{-(N+1)}$ knowing its values on the extremities.

5 A discrete basis of functions to generate the Ornstein-Uhlenbeck process

In view of representing an Ornstein-Uhlenbeck process as a discrete process, the comparison with a Wiener process suggests a candidate basis of decomposition of the form $\Phi_{n,k}\cdot\xi_{n,k}$, the variable $\xi_{n,k}$ following the law $\mathcal{N}(0,1)$. The deterministic function $\Phi_{n,k}$ is defined with support $S_{n,k}=\left[k\cdot2^{-n+1},(k+1)2^{-n+1}\right]$ for $n>0$ with $0\le 2k<2^n$. We use expressions (14) and (15) to make explicit the formulation of $\Phi_{n,k}$ in relation (19) and we obtain

$$\Phi_{n,k}(t)=\begin{cases}\sqrt{\dfrac{\Gamma}{\alpha}}\cdot\dfrac{\sinh\left(\alpha\left(t-2k\cdot2^{-n}\right)\right)}{\sqrt{\sinh\left(\alpha2^{-n+1}\right)}} \\ \text{if } (2k)\,2^{-n}\le t\le(2k+1)\,2^{-n}\,, \\ \sqrt{\dfrac{\Gamma}{\alpha}}\cdot\dfrac{\sinh\left(\alpha\left(2\,(k+1)\,2^{-n}-t\right)\right)}{\sqrt{\sinh\left(\alpha2^{-n+1}\right)}} \\ \text{if } (2k+1)\,2^{-n}\le t\le 2\,(k+1)\,2^{-n}\,, \\ 0 \quad \text{otherwise}\,.\end{cases} \tag{20}$$

Without any further comment, the element $\Phi_{0,0}$ is defined as

$$\Phi_{0,0}(t)=\sqrt{\frac{\Gamma}{\alpha}}\cdot\frac{e^{-\frac{\alpha}{2}}\sinh(\alpha t)}{\sqrt{\sinh\alpha}}\,, \tag{21}$$

a choice we will explain in the following section. The first elements $\Phi_{n,k}$ are shown in figure 3. As expected, they are only asymptotically scale-invariant but they exhibit the desirable property of being compactly supported on $S_{n,k}$, the interval between two dyadic points $k{\cdot}2^{-n+1}$ and $(k{+}1)2^{-n+1}$.
To validate the decomposition of an Ornstein-Uhlenbeck process U_t on the set of functions $\Phi_{n,k}$, we need to study the convergence of the partial sums

$$U_t^N(\omega) = \sum_{n=0}^{N} \sum_{0 \le k < 2^{n-1}} \Phi_{n,k}(t) \cdot \xi_{n,k}(\omega) \, .$$

As each function $\Phi_{n,k}$ is dominated by the Haar-derived element $\Psi_{n,k}$, the normal convergence of the series $W_t^N(\omega)$ in (7) entails the normal convergence of $U_t^N(\omega)$ almost surely on the sample space Ω. With anticipation of its Ornstein-Uhlenbeck nature, we denote U_t the corresponding limit process. The normal convergence causes U_t to be continuous and, being a sum of Gaussian variables, a Gaussian process. Therefore, proving that U_t is an Ornstein-Uhlenbeck process just requires us to show that the covariance of U_t satisfies

$$\langle U_t \cdot U_s \rangle = \frac{\Gamma}{2\alpha} \cdot e^{-\alpha(t+s)} \left(e^{2\alpha(t \wedge s)} - 1 \right) \, . \tag{22}$$

To establish this relation, we need to evaluate the covariance of U_t as the limit covariance of the partial sums

$$\langle U_t^N \cdot U_s^N \rangle = \sum_{n=0}^{N} \sum_{0 \le k < 2^{n-1}} \Phi_{n,k}(t) \cdot \Phi_{n,k}(s) \, .$$

It is possible to simplify the above expression, even though the functions $\Phi_{n,k}$ are not orthogonal. For each given n, the disjoint supports of $\Phi_{n,k}$ forms a partition of $[0,1]$ as a collection of segments $S_{n,k}$ of equal length 2^{-n+1}. Considering a real t, there is only one sequence of indexes k_n such that t belongs to each support of S_{n,k_n}. The succession of k_n represents t as the intersection of decreasing dyadic segments $\cap_{n=0}^{\infty} S_{n,k_n}$, which can be explained in terms of the binary representation $t = \sum_1^{\infty} a_i 2^{-i}, a_i \in \{0,1\}$, if we exclude inappropriate infinite developments. Bearing in mind the system of indexing for $S_{n,k}$, a simple recurrence argument leads to the expression of k_n corresponding to a given t in its binary representation

$$k_n = \frac{1}{2} \cdot \sum_{i=1}^{n-1} a_i 2^{n-i} \, . \tag{23}$$

We are now in a position to write the reduced expression of the partial sums

$$U_t^N(\omega) = \sum_{n=0}^{N} \Phi_{n,k_n}(t) \cdot \xi_{n,k_n}(\omega)$$

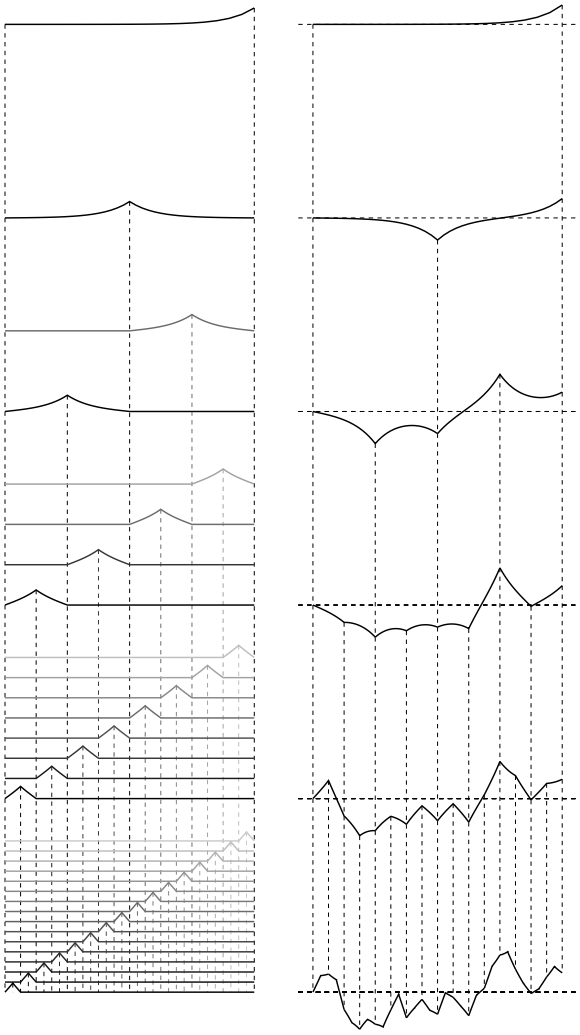

Fig. 3 In the left column, the elements of the basis $\Phi_{n,k}$ are represented for each rank n with $0 \leq n < 6$. In the right column, the conditional Ornstein-Uhlenbeck process $\langle U_t \rangle_{D_n}$ is shown for a given set of realizations on D_n. Once more, note that each element $\Psi_{n,k}$ has a compact support delimited by dyadic numbers in $D_n = \{k2^{-n} \,|\, 0 \leq k \leq 2^n\}$ and that all $\Phi_{n',k}$ is zero on D_n for $n' > n$.

where the terms $\Phi_{k_n,n}(t)$ is made explicit using the previous formulation of k_n in the definition (20)

$$\Phi_{n,k_n}(t) = \begin{cases} \sqrt{\dfrac{\Gamma}{\alpha}} \cdot \dfrac{\sinh\left(\alpha \sum_{n+1}^{\infty} a_i 2^{-i}\right)}{\sqrt{\sinh\left(\alpha 2^{-n+1}\right)}} & \text{if } a_n = 0\,, \\ \sqrt{\dfrac{\Gamma}{\alpha}} \cdot \dfrac{\sinh\left(\alpha \sum_{n+1}^{\infty} (1-a_i)\, 2^{-i}\right)}{\sqrt{\sinh\left(\alpha 2^{-n+1}\right)}} & \text{if } a_n = 1\,. \end{cases}$$

Informed by these preliminaries, we shall carry out the calculation of the covariance. The reduced formulation of partial sums allows us to write

$$\langle U_t^N \cdot U_s^N \rangle = \sum_{n=1}^{N} \Phi_{n,k_n}(t)\Phi_{n,l_n}(s) + \Phi_{0,0}(t)\Phi_{0,0}(s)\,. \tag{24}$$

where the indexes k_n and l_n designate the sequence of functions $\Phi_{k_n,n}$ and $\Phi_{l_n,n}$ whose supports contain t and s respectively. When t and s are distinct, we notice that for $n > 1 - \log_2 |t-s|$, the supports $S_{k_n,n}$ and $S_{l_n,n}$ containing t and s respectively are disjoint, so that the cross-products $\Phi_{k_n,n}(t)\Phi_{l_n,n}(s)$ cancel out if n is large enough. It is then possible to write expression (24) as a finite sum where the terms $\Phi_{k_n,n}(t)$ and $\Phi_{l_n,n}(s)$ are specified due to the binary representations $t = \sum_1^{\infty} a_i 2^{-i}$ and $s = \sum_1^{\infty} b_i 2^{-i}$. We specify that we only consider proper binary representations, that is, the binary representation of dyadic points is chosen in its finite form. For the sake of simplicity, we assume that $t < s$. Formulated in the binary representation, the order $t < s$ is equivalent to the existence of a natural $N_0 > 0$ such that $a_n = b_n$ as long as $n < N_0$ and $a_{N_0} < b_{N_0}$, that is $a_{N_0} = 0$ and $b_{N_0} = 1$. With the preceding remarks, it is clear that $S_{k_n,n}$ and $S_{l_n,n}$ are disjoint for $n > N_0$ and we can write the covariance of U_t^N for $N > N_0$ in the explicit form

$$\langle U_t^N \cdot U_s^N \rangle_{N>N_0} = \frac{\Gamma}{2\alpha} \left(\sum_{n=1}^{N_0} \frac{2 \cdot u_n}{\sinh\left(\alpha 2^{-n+1}\right)} + e^{-\alpha} \frac{2 \sinh(\alpha t) \sinh(\alpha s)}{\sinh \alpha} \right) \tag{25}$$

The variable u_n apparent in (25) represents for $n < N_0$ the numerator of the cross-products $\Phi_{k_n,n}(t)\Phi_{l_n,n}(s)$ when the extension a_n and b_n coincide

$$u_n = \begin{cases} \sinh\left(\alpha \sum_{n+1}^{\infty} a_i 2^{-i}\right) \sinh\left(\alpha \sum_{n+1}^{\infty} b_i 2^{-i}\right) \\ \text{if} \quad a_n = b_n = 0\,, \\ \sinh\left(\alpha \sum_{n+1}^{\infty} (1-a_i)\, 2^{-i}\right) \sinh\left(\alpha \sum_{n+1}^{\infty} (1-b_i)\, 2^{-i}\right) \\ \text{if} \quad a_n = b_n = 1\,, \end{cases}$$

As for the limit case $n = N_0$, u_{N_0} expresses the numerator of the cross-product $\Phi_{k_{N_0},N_0}(t)\Phi_{l_{N_0},N_0}(s)$ with $a_{N_0} = 0$ and $b_{N_0} = 1$

$$u_{N_0} = \sinh\left(\alpha \sum_{N_0+1}^{\infty} a_i 2^{-i}\right) \sinh\left(\alpha \sum_{N_0+1}^{\infty} (1-b_i)\, 2^{-i}\right).$$

At that point, the explicit form of the covariance (25) results in a rather complicated combination of hyperbolic functions. Fortunately enough, we can resort to using remarkable identities to simplify its expression. The solution actually lies in the consideration of the quantity

$$v_n = \sinh\left(\alpha \sum_n^\infty a_i 2^{-i}\right) \cdot \sinh\left(\alpha \sum_n^\infty (1-b_i)\, 2^{-i}\right). \tag{26}$$

We show in the supplementary materials that, as long as $n < N_0$, v_n verifies the recurrence relation

$$v_n = 2\cosh(\alpha 2^n) \cdot v_{n+1} + u_n\,. \tag{27}$$

We can express u_n in terms of v_n and $v_n + 1$ to compute the following series by cancellation term by term

$$\sum_{n=1}^{N_0-1} \frac{u_n}{\sinh(\alpha 2^{-n+1})} = \frac{v_1}{\sinh(\alpha/2)} - \frac{v_{N_0}}{\sinh(\alpha 2^{-N_0+1})}\,. \tag{28}$$

Remembering that $a_{N_0} = 0$ and $b_{N_0} = 1$, we remark that $v_{N_0} = v_{N_0+1} = u_{N_0}$ so that the insertion of (28) in expression (25) caused the remaining terms in u_{N_0} to cancel out. It is then straightforward to write the covariance

$$\langle U_t^N \cdot U_s^N \rangle_{N>N_0} = \frac{\Gamma}{2\alpha}\left(\frac{2\cdot v_1}{\sinh(\alpha/2)} + e^{-\alpha}\frac{2\sinh(\alpha t)\sinh(\alpha s)}{\sinh\alpha}\right), \tag{29}$$

We observed that the definition of v_1 invokes the full binary representations of t and s so that we have $v_1 = \sinh(\alpha t)\sinh(\alpha(1-s))$. After a several manipulations, expression (29) finally yields

$$\langle U_t^N \cdot U_s^N \rangle_{N>N_0} = \frac{\Gamma}{2\alpha} \cdot e^{-\alpha(t+s)}\left(e^{2\alpha t} - 1\right),$$

which is the expected result for the covariance of an Ornstein-Uhlenbeck process (22) given that $t = \min(t, s)$ as $t < s$.
Regarding the calculation of the variance when $t = s$, the series of cross-products $\Phi_{k_n,n}(t)\Phi_{l_n,n}(s) = \Phi^2_{k_n,n}(t)$ becomes infinite, but fortunately the recurrence relation (27) is then valid for every $n > 0$. As the quantity v_n vanishes when n grows to infinity, the cancellation term by term is still effective to compute the series in (25). It leads to the expected variance expression for an Ornstein-Uhlenbeck process ${}_U\sigma_t^2 = \frac{\Gamma}{2\alpha}\cdot(1 - e^{-\alpha t})$.
We finally recap the result for any t and s without assuming any order

$$\langle U_t \cdot U_s \rangle = \lim_{N\to\infty} \langle U_t^N \cdot U_s^N \rangle = \frac{\Gamma}{2\alpha} \cdot e^{-\alpha(t+s)}\left(e^{2\alpha(t\wedge s)} - 1\right).$$

It proves the discrete description of an Ornstein-Uhlenbeck processes as the normally convergent series of random functions $\Phi_{n,k} \cdot \xi_{n,k}$, where $\Phi_{n,k}$ is a deterministic function defined in (6) and $\xi_{n,k}$ a random variable of normal law $\mathcal{N}(0, 1)$.

6 Representation as a bi-infinite sum of random functions

Whether standing for a Wiener process or an Ornstein-Uhlenbeck process, X_t can be decomposed in a discrete basis of functions $f_{n,k}$, where $f_{n,k}$ is a generic notation for the deterministic functions $\Psi_{n,k}$ and $\Phi_{n,k}$. It suggests to consider the process X_t as a recurrence construction, a view that explains how to chose the first element of the basis $f_{0,0}$.
Imagine we want to build a sample path of the continuous process X_t starting with the prior knowledge of its values on the dyadic set D_N. To proceed at the next stage $N+1$, we need to establish the values of X_t on $D_{N+1} \setminus D_N$, which implies the drawing of as many Gaussian random variables as there are points in this set. If we consider a given time t, there exists a unique k such that $k2^{-N} \leq t < (k+1)2^{-N}$ and we know that the collection of segments $S_{N+1,k} = [k2^{-N}, (k+1)2^{-N}]$ for $0 \leq k < 2^N$ defines a partition of $[0,1]$. We also remark that $t_{N+1,k} = (2k+1)2^{-(N+1)}$ is the only point of $D_{N+1} \setminus D_N$ in $S_{N+1,k}$ and consequently, we note $\zeta_{N+1,k}$ the Gaussian drawing occurring there. According to the results exposed in the second section, we posit

$$\zeta_{N+1,k} = {}_x\sigma(t_{N+1,k}) \cdot \xi_{N+1,k} + {}_x\mu(t_{N+1,k}) \, .$$

where $\xi_{N+1,k}$ is of normal law $\mathcal{N}(0,1)$. Repeating such a construction for $n > N+1$ leads us to evaluate the sample path on the whole set of dyadic points $\mathcal{D}$. As $\mathcal{D}$ is dense in $[0,1]$, the complete path is naturally obtained by continued extension.
To construct a process rather than a sample path, we need to formulate the above recurrence argument in terms of random functions. We consider the conditional average $\langle X_t \rangle_{D_N}$, which is also the most probable outcome of X_t knowing its values on D_N. As X_t is a Markov process, the change in the estimation $\langle X_t \rangle_{D_{N+1}} - \langle X_t \rangle_{D_N}$ only depends on the outcome of $\xi_{N+1,k}$ when restricted on the support $S_{n+1,k}$. Due to the simplicity of the situation, it is possible to find an analytical expression of the form $f_{N+1,k} \cdot \xi_{n,k}$ to describe $\langle X_t \rangle_{D_{N+1}} - \langle X_t \rangle_{D_N}$ on $S_{N+1,k}$. With the help of the so-defined functions $f_{n,k}$, we can introduce the partial sum

$$X_t^N = \sum_{n=0}^{n=N} \sum_{0 \leq k < 2^{n-1}} f_{n,k}(t) \cdot \xi_{n,k}$$

and relate it to the conditional average $\langle P_t \rangle_{D_N}$ considered as a random variable. By definition of f_{N+1}, if P_t^N coincides with $\langle X_t \rangle_{D_N}$, P_t^{N+1} equals $\langle X_t \rangle_{D_{N+1}}$ at next step. Continuing this identification for $n > N+1$ shows that the limit process $\lim_{N \to \infty} P_t^N$ agrees with the corresponding Ornstein-Uhlembeck process on the index set $\mathcal{D}$. Dealing with continuous processes, the density of $\mathcal{D}$ in $[0,1]$ allows us to extent the results on $[0,1]$. Incidentally, we have an interpretation for the statistical contribution of a component $f_{n,k}(t) \cdot \xi_{n,k}$. At each step N,the function $\sum_k f_{N+1,k}(t) \cdot \xi_{N+,k}$ represents the uncertainty about P_t that is discarded by the knowledge of its values on $D_{N+1} \setminus D_N$.

To validate the recurrence argument, it now remains to verify the initial statement

$$P_t^0 = f_{0,0}(t) \cdot \xi_{0,0} = \langle P_t \rangle_{D_0} .$$

Actually, the need to satisfy this prerequisite enforces how to set the expression of $f_{0,0}$. The conditional average $\langle X_t \rangle_{D_0}$ is a function of the value of X_t on $D_0 = \{0, 1\}$. By construction the value of X_t in 0 is assumed to be zero. We note $Z_{0,0}$ the random function X_1 knowing $X_0 = 0$, and we recall that its statistics is given by relations (2) for a Wiener process and (3) for an Ornstein-Uhlenbeck process respectively. With the notation of the second section, we write $\langle X_t \rangle_{D_0}$ as a function of $Z_{0,0}$

$$\langle X_t \rangle_{D_0} = {}_x\mu_{0,1}(t, 0, Z_{0,0}) .$$

It defines a Gaussian random function $\langle X_t \rangle_{D_0}$ of the form $f_{0,0} \cdot \xi_{0,0}$. The dependency of its variance upon t yields the expression of the deterministic part $f_{0,0}$

$$f_{0,0}(t) = \sqrt{\langle \langle X_t \rangle_{D_0}^2 \rangle} . \tag{30}$$

When applied to the Wiener process, the above relation gives the right expression for $\Psi_{0,0}$; when applied to the Ornstein-Ulenbeck process, it gives the already mentioned expression of $\Phi_{0,0}$.
Now, we further this recurrence description to show why X_t is naturally represented as a bi-infinite series of random functions. In that perspective, we extend the definition of the dyadic sets to $D_N = \{k2^{-N} \,|\, k \in \mathbb{Z}\}$ and we have $D_{-N} = 2^N\mathbb{Z} \subset \cdots \subset D_0 = \mathbb{Z} \subset \cdots \subset D_N = 2^{-N}\mathbb{Z}$. If we restrain the description of P_t to the index set $[0, 1]$, the argument to set the initial step of the recurrence allows us to define a function $f^\star_{-N,0}$ so that $f^\star_{-N,0} \cdot \xi_{-N,0} = \langle X_t \rangle_{D_{-N}}$ on $[0, 1]$. The only requirement to adjust (30) is that $]0, 1[$ has no points in D_{-N}. The usual recurrence construction is then easily adapted to build the process P_t on that segment: for $n > -N$, the analytical expressions of $f^\star_{n,k}$ are simple extensions of the usual formulas $f_{n,k}$. In the case of a Wiener process, we make explicit the functions $\Psi^\star_{n,k}$

$$\Psi^\star_{n,k}(t) = \begin{cases} \sqrt{\dfrac{\Gamma}{2^{-n+1}}} \cdot \left(t - 2k \cdot 2^{-n}\right) \\ \text{if } (2k)\, 2^{-n} \leq t \leq (2k+1)\, 2^{-n} , \\ \\ \sqrt{\dfrac{\Gamma}{2^{-n+1}}} \cdot \left(2\,(k+1)\, 2^{-n} - t\right) \\ \text{if } (2k+1)\, 2^{-n} \leq t \leq 2\,(k+1)\, 2^{-n} , \\ \\ 0 \quad \text{otherwise} . \end{cases}$$

for $n > -N$, and the first element $\Psi^\star_{-N,0}$

$$\Psi^\star_{-N,0}(t) = \sqrt{\frac{\Gamma}{2^N}} \cdot t . \tag{31}$$

In the case of an Ornstein-Uhlenbeck process, we similarly write the functions $\Phi^\star_{n,k}$

$$\Phi^\star_{n,k}(t) = \begin{cases} \sqrt{\dfrac{\Gamma}{\alpha}} \cdot \dfrac{\sinh\left(\alpha\left(t - 2k\cdot 2^{-n}\right)\right)}{\sqrt{\sinh\left(\alpha 2^{-n+1}\right)}} \\ \text{if } (2k)\,2^{-n} \leq t \leq (2k+1)\,2^{-n}\,, \\ \sqrt{\dfrac{\Gamma}{\alpha}} \cdot \dfrac{\sinh\left(\alpha\left(2\,(k+1)\,2^{-n} - t\right)\right)}{\sqrt{\sinh\left(\alpha 2^{-n+1}\right)}} \\ \text{if } (2k+1)\,2^{-n} \leq t \leq 2\,(k+1)\,2^{-n}\,, \\ 0 \quad \text{otherwise}\,. \end{cases}$$

for $n > -N$, and the first element $\Phi^\star_{-N,0}$

$$\Phi^\star_{-N,0}(t) = \sqrt{\frac{\Gamma}{\alpha}} \cdot \frac{e^{-\alpha 2^{N-1}} \sinh\left(\alpha t\right)}{\sqrt{\sinh \alpha 2^N}}\,. \tag{32}$$

As apparent in (31) and (32), the upper bound of $f_{-N,0}$ on $[0,1]$ is exponentially decreasing toward zero when N goes to infinity. The exponential uniform convergence of $f_{-N,0}$ toward zero prescribes to represent the process X_t as a bi-infinite series of random functions $f^\star_{n,k} \cdot \xi^\star_{n,k}$

$$X^{\star N}_t = \sum_{n=1}^{N} f^\star_{n,k_n}(t)\xi^\star_{n,k_n} + \sum_{n=-N}^{0} f^\star_{n,0}(t)\xi^\star_{n,0}\,.$$

In the partial sum $X^{\star N}_t$, the index k_n refers to the unique functions $f^\star_{n,k}$ whose support contains t for a fixed n. In the case of $n \leq 0$, this index is constantly set to zero. The two series in $X^{\star N}_t$ are normally convergent on $[0,1]$. To verify the cogency of the bi-infinite decomposition, it is enough to demonstrate that the covariance of the partial sum

$$\langle X^{\star N}_t \cdot X^{\star N}_s \rangle = \sum_{n=1}^{N} f^\star_{n,k_n}(t) f^\star_{n,l_n}(s) + \sum_{n=-N}^{0} f^\star_{n,0}(t) f^\star_{n,0}(s)\,.$$

converges toward the expected covariance of the process X_t. This amounts to show that the right series in the above expressions equals to $f_{0,0}(t) \cdot f_{0,0}(s)$. In the case of a Wiener process, a direct calculation leads to

$$\sum_{n=-N}^{0} \Psi^\star_{n,0}(t)\Psi^\star_{n,0}(s) = \sum_{n=0}^{N} \frac{\Gamma}{2^{n+1}} \cdot ts = \Gamma \cdot ts\,,$$

which is the exact expression of $\Psi_{n,0}(t)\Psi_{n,0}(s)$. In the case of an Ornstein-Uhlenbeck process, a similar calculation yields

$$\sum_{n=-N}^{0} \Phi^{\star}_{n,0}(t)\Phi^{\star}_{n,0}(s) = \sum_{n=0}^{N} \frac{\Gamma}{\alpha} \cdot \frac{\sinh(\alpha t)\sinh(\alpha s)}{\sinh(\alpha 2^{n+1})}$$
$$= \frac{\Gamma}{\alpha} \cdot e^{-\alpha} \frac{\sinh(\alpha t)\sinh(\alpha s)}{\sinh \alpha} ,$$

which consistently equals to $\Phi_{0,0}(t)\Phi_{0,0}(s)$. The previous derivation requires the use of the identity

$$\sum_{n=1}^{\infty} \frac{1}{\sinh(\alpha 2^n)} = \frac{e^{-\alpha}}{\sinh \alpha}$$

proven in the supplementary materials.
The representation of a Wiener process as a bi-infinite series of random functions stems from its scale-invariance. During the construction process, the values of W_t on $D_{N+1} \backslash D_N$ only depends upon the previous drawings on D_N. Initial drawings on the asymptotic bounds of $\{-\infty, 0, \infty\} = \lim_{N\to-\infty} D_N$ only produce vanishing correlations for later stage. At any finite step $N+1$ in $\mathbb{Z}$, evaluating W_t on $D_{N+1} \backslash D_N$ is a completely scale-invariant operation, which justifies a common analytical expression for the elements $\Psi_{n,k}$. For nonzero coefficient α, the linear component in the Langevin equation precludes any scale-invariance for an Ornstein-Uhlenbeck process U_t. In that respect, it is quite remarkable that we can decompose U_t in a basis of functions $\Phi_{n,k}$ defined with a unique analytical expression. If $n > \log_2 \alpha$ the functions $\Phi_{n,k}$ are well approximated by the functions $\Psi_{n,k}$ showing the typical scale-invariance of a Wiener process; if $n < \log_2 \alpha$ the functions $\Phi_{n,k}$ are exponential attenuation of the functions $\Psi_{n,k}$, with constant extremal value $\sqrt{\Gamma/2\alpha}$. As is obvious, the parameter α can be interpreted in terms of a characteristic time. For time-scales smaller than $2/\alpha$, the components $\Phi_{n,k}$ with intersecting supports add to each others so that the resulting process displays the same correlations as a Wiener process. For time-scales larger than $2/\alpha$, the components $\Phi_{n,k}$ add independently because the exponential attenuation causes all the components but one to be negligible on intersecting supports.

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